An Introduction to Causal Fermion Systems

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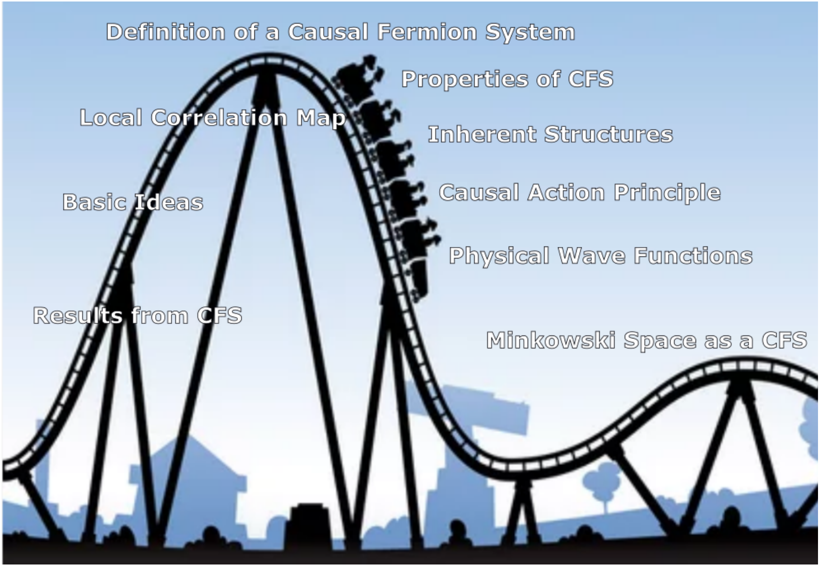

Definition of a Causal Fermion System
Properties of CFS
Local Correlation Map
Inherent Structures
Causal Action Principle
Basic Ideas
Physical Wave Functions
Results from CFS
Minkowski Space as a CFS

Goal

My goals for this talk are:

- Summarize the successes of CFS.
- Recall the structures available to us in the existing theories.
- Introduce the basic definitions of the theory.
- Sketch how to relate them to the structures we know and love.

What is a causal fermion system?

- A new candidate for a unifying theory.
- novel mathematical model of spacetime
- physical equations are formulated in generalized spacetimes
- Different limiting cases:
  - Continuum limit: Quantized fermionic fields interacting via classical bosonic fields
  - QFT limit: fermionic and bosonic quantum fields

Results from Theory of Causal Fermion Systems

- All physical structures are encoded in a single object.
- Standard Model gauge group and it's classical field equations in linear perturbation theory of Minkowski space.
- For that to work it requires at least three generations of fermions thereby explaining one parameter of the Standard Model.
- Quantum Field Theory in non-linear perturbation theory of Minkowski space.
- Einstein equations as a third order effect
  $\Rightarrow$ Explains weakness of gravity

Which Structures Do We Have Available In Spacetime?

- Starting point: Consider wave functions in spacetime.
  - Canonically ψ describes quantum mechanical particle. (only wave character, no point particle)
  - Dynamics as described in the simplest case by Schrödinger equation (or Dirac equation, scalar wave equation, ...).
- Vector ψ in a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.
- This is not quite the right description:
  - Phase has no significance: $\psi \to e^{i\Lambda}\psi$
    instead of ψ consider ray generated by ψ
  - Local gauge invariance

$$\psi(t, \vec{x}) \to e^{i\Lambda(t,\vec{x})}\, \psi(t, \vec{x})$$

    Therefore, only $|\psi(t, \vec{x})|^2$ is of physical significance; interpretation: probability density

Which Structures Do We Have Available In Spacetime?

- Thus: Consider $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$ of all wave functions as the starting point.
- General question: Suppose we know $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$ for all the wave functions of the system, what can we say about the spacetime structures (causality, metric, fields, ...)
- Try to probe spacetime by looking at $|\psi(\mathrm{t}, \vec{\mathrm{x}})|^2$.

  Here "probing" should be thought of as a mathematical operation; no collapse of the wave function involved.

Which Structures Do We Have Available In Spacetime?

- Begin in Minkowski space (usual spacetime structures)

$$x = (t, \vec{x}), \qquad t \in \mathbb{R},\ \vec{x} \in \mathbb{R}^3$$

  (later curved spacetime)

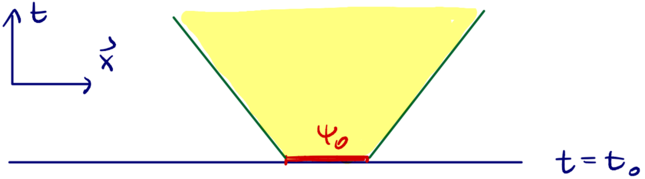
- Consider scalar particle (no spin)

$$|\psi(x)|^2 \qquad \text{(local density)}$$

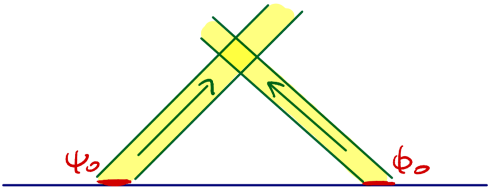
What Information Is Encoded In This Structure?

First step: Allow for preparation of the "initial state" at time t.

- Allows for detecting the causal structure of spacetime:

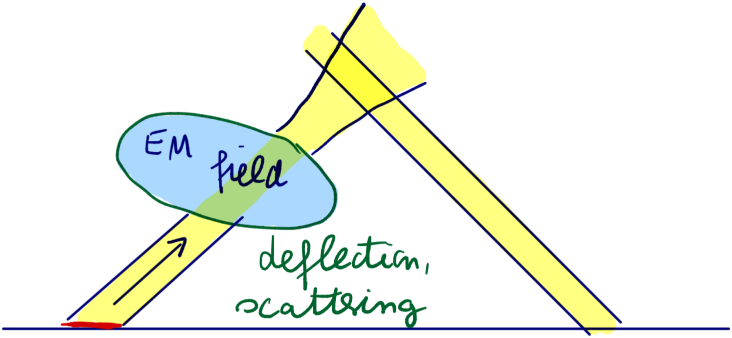


- Allows for recovering the full metric information: consider massive situation



Which spacetime structures are fundamental?

- Allows for detecting an electromagnetic field:

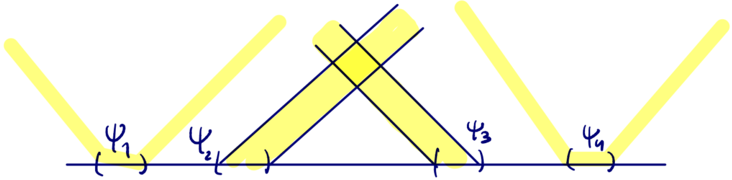


- ..., ...

What Information Is Encoded In This Structure?

Second step: Do not allow for preparation of the "initial state". Instead: Get by with the wave function already present.

- Probing still works, provided that there are "sufficiently many" wave functions around.



- The more wave functions there are, the more information we have on spacetime (spacetime resolution).

Formalize this idea: The local correlation operator

- Consider wave functions $\psi_1, \ldots, \psi_f : \mathscr{M} \to \mathbb{C}$ (with $f < \infty$)
- Are vectors in a Hilbert space, orthonormalize,

$$\langle \psi_k | \psi_l \rangle = \delta_{kl} \,,$$

gives f-dim Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$.

basic object: for any lattice point x introduce

local correlation operator $F(x) : \mathcal{H} \to \mathcal{H}$

- define matrix elements by

$$(F(x))^j_k = \overline{\psi_j(x)} \psi_k(x)$$

basis invariant:

$$\langle \psi, F(x)\, \phi \rangle_{\mathcal{H}} = \overline{\psi(x)} \phi(x) \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

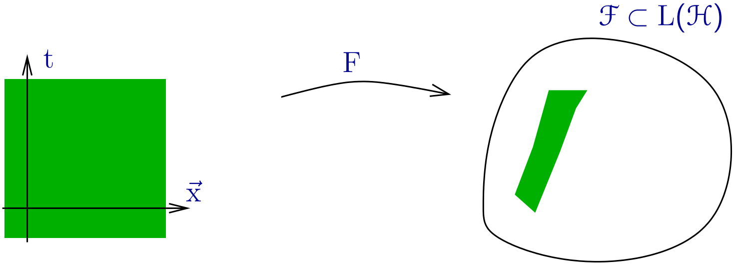
- Hermitian matrix = symmetric operator
- Has rank at most one, is positive semi-definite

$$F(x) = e^* e \qquad \text{with} \qquad e : \mathcal{H} \to \mathbb{C} \,, \quad \psi \mapsto \psi(x)$$

The local correlation map

$$F(x) \in \mathcal{F} := \{F \text{ rank at most one, positive semi-definite}\}$$

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset L(\mathcal{H})$

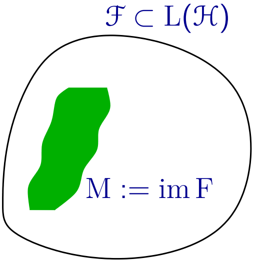


- The right side contains all the information which can be retrieved from the ensemble of wave functions.
- We consider the objects on the right as the basic physical objects.

Key Idea: Spacetime as the set of all local correlation operators

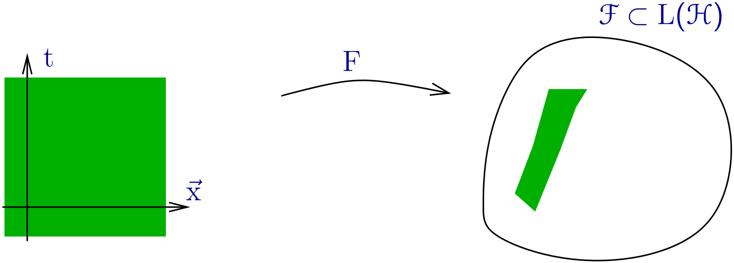
General strategy:

- Treat objects on the left as effective description (spacetime, matter fields, ...)
- Formulate a fundamental theory with the objects on the right (only local correlation operators).



A volume measure on spacetime

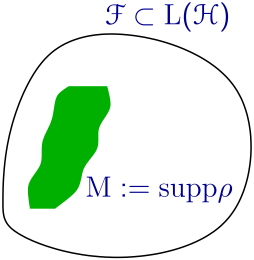
- Adding a key structure: Volume measure on spacetime.



Take push-forward measure of $F : \mathscr{M} \to \mathcal{F}$,

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Spacetime



- image of F recovered as the support of the measure,

$$M := \text{supp}\, \rho = \big\{ F \in \mathcal{F} \mid \rho(\Omega) \neq 0 \text{ for every open neighborhood } \Omega \text{ of } x \big\}$$

Let's Introduce One More Spin

Let $(\mathscr{M}, g)$ be a Lorentzian space-time,
for simplicity 4-dimensional, globally hyperbolic,
then automatically spin,

$$(S\mathscr{M}, \prec .|. \succ) \qquad \text{spinor bundle}$$

- $S_x\mathscr{M} \simeq \mathbb{C}^4$
- spin scalar product

$$\prec .|. \succ_x \;:\; S_x\mathscr{M} \times S_x\mathscr{M} \to \mathbb{C}$$

is indefinite of signature (2,2)

$$(\mathcal{D} - m)\psi_m = 0 \qquad \text{Dirac equation}$$

Let's Introduce One More Spin

- Cauchy problem well-posed, global smooth solutions (for example symmetric hyperbolic systems)
- finite propagation speed

$C^\infty_{sc}(\mathscr{M}, S\mathscr{M})$ spatially compact solutions

$$(\psi_m|\phi_m)_m := 2\pi \int_{\mathscr{N}} \prec \psi_m | \nu\!\!\!/ \phi_m \succ_x \, d\mu_{\mathscr{N}}(x) \quad \text{scalar product}$$

completion gives Hilbert space $(\mathcal{H}_m, (.|.)_m)$

Let's Introduce One More Spin

- Choose $\mathcal{H}$ as a subspace of the solution space,

$$\mathcal{H} = \overline{\operatorname{span}(\psi_1, \ldots, \psi_{\mathrm{f}})}$$

- To $\mathrm{x} \in \mathbb{R}^4$ associate a local correlation operator

$$\langle \psi | F(\mathrm{x}) \phi \rangle = - \prec \psi(\mathrm{x}) | \phi(\mathrm{x}) \succ_{\mathrm{x}} \qquad \forall \psi, \phi \in \mathcal{H}$$

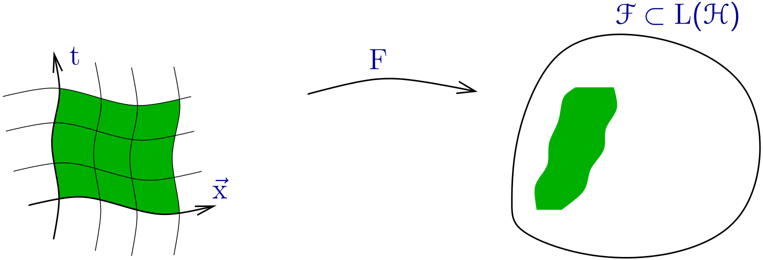
Is symmetric, rank ≤ 4
at most two positive and at most two negative eigenvalues

Let's Introduce One More Spin

- Thus $F(x) \in \mathcal{F}$ where
  $\mathcal{F} := \Big\{ F \in L(\mathcal{H})$ with the properties:
  - ▷ F is symmetric and has rank ≤ 4
  - ▷ F has at most 2 positive and at most 2 negative eigenvalues $\Big\}$

Let's Introduce One More Spin

We obtain mapping $\quad x \mapsto F(x) \in \mathcal{F} \subset L(\mathcal{H})$



Take push-forward measure

$$\rho := F_*(\mu_{\mathscr{M}}) \qquad \text{(i.e. } \rho(\Omega) := \mu_{\mathscr{M}}\big(F^{-1}(\Omega)\big)\text{)}$$

Causal Fermion Systems

Definition (Causal fermion system)

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be Hilbert space

Given parameter $n \in \mathbb{N}$ ("spin dimension")

$\mathcal{F} := \Big\{ x \in L(\mathcal{H})$ with the properties:

- x is symmetric and has finite rank
- x has at most n positive and at most n negative eigenvalues $\Big\}$

ρ a measure on $\mathcal{F}$

Comparison

Classical even dimensional tangent bundle:

- smooth 2n - dimensional manifold
- canonical projection that "assigns" an n-dimensional vector space to every point. These vector spaces are all isomorphic but independent.
- metric defines: 1) causal structure 2) connection 3) measure supported on entire manifold.

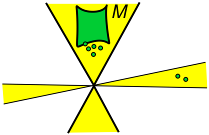
Properties of $\mathcal{F}$

- (infinite dimensional) operator manifold
- every spacetime point comes with a 2n-dimensional vector space. These vector spaces are not independent.
- The measure is the central actor, supported on a low dimensional subset.
- Causal structure and a connection can be defined through the properties of operator products.

Inherent Structures

Let $(\rho, \mathcal{F}, \mathcal{H})$ be a causal fermion system,
Then a space-time can be defined by $\mathrm{M} := \mathrm{supp}\rho$.

Space-time points are linear operators on $\mathcal{H}$

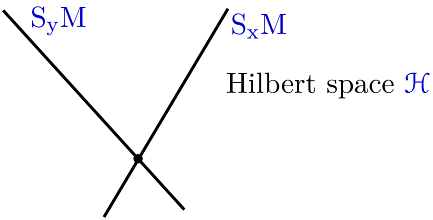


- For $\mathrm{x} \in \mathrm{M}$, consider eigenspaces of x.
- For $\mathrm{x}, \mathrm{y} \in \mathrm{M}$ consider
  - consider operator products xy
  - project eigenspaces of x to eigenspaces of y

Inherent structures of a causal fermion system

- Spinors

$$S_x M := x(\mathcal{H}) \subset \mathcal{H} \qquad \text{“spin space”}, \; \dim S_x M \leq 2n$$

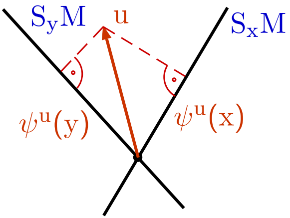


Inherent structures in spacetime

- Physical wave functions

$$\psi^{\mathrm{u}}(\mathrm{x}) = \pi_{\mathrm{x}}\, \mathrm{u} \;\; \text{with } \mathrm{u} \in \mathcal{H} \qquad \text{physical wave function}$$

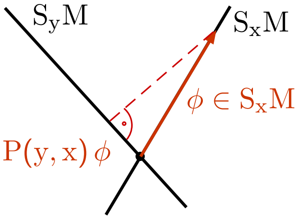
$$\pi_{\mathrm{x}} : \mathcal{H} \to \mathcal{H} \qquad \text{orthogonal projection on } \mathrm{x}(\mathcal{H})$$



Inherent structures in spacetime

- The kernel of the fermionic projector:

$$P(y,x) = \pi_y \, x|_{S_xM} \; : \; S_xM \to S_yM$$



$$P(y,x) = -\sum_{i=1}^{f} |\psi^{e_i}(y) \succ \prec \psi^{e_i}(x)| \qquad \text{where } (e_i) \text{ ONB of } \mathcal{H}$$

Inherent structures in spacetime

- Geometric structures
  - $P(x, y) : S_yM \to S_xM$ yields relations between spin spaces. Using a polar decomposition (..., ...) one gets:

    $$D_{x,y} : S_yM \to S_xM \text{ unitary} \qquad \text{“spin connection”}$$

  - tangent space T_x, carries Lorentzian metric,

    $$\nabla_{x,y} : T_y \to T_x \qquad \text{corresponding “metric connection”}$$

  - holonomy of connection gives curvature

    $$R(x, y, z) = \nabla_{x,y} \nabla_{y,z} \nabla_{z,x} : T_x \to T_x$$

Causal structure

Let $x, y \in \mathcal{F}$. Then

$x \cdot y \in \mathrm{L}(H)$ has non-trivial complex eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$

Definition (causal structure)

The points $x, y \in \mathcal{F}$ are called

$$\begin{cases} \text{spacelike separated} & \text{if } |\lambda_j^{xy}| = |\lambda_k^{xy}| \text{ for all } j, k = 1, \ldots, 2n \\ \text{timelike separated} & \text{if } \lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \text{ are all real} \\ & \text{and } |\lambda_j^{xy}| \neq |\lambda_k^{xy}| \text{ for some } j, k \\ \text{lightlike separated} & \text{otherwise} \end{cases}$$

Remarks

For points $x, y \in \mathcal{F}$ their product $x \cdot y$ is not necessarily in $\mathcal{F}$ but it still has finite rank $\leq 2n$

For arbitrary points $x, y \in \mathcal{F}$ we have in general that $x \cdot y \neq y \cdot x$. So spacetime points do not necessarily commute.

A point x is timelike separated from itself.

Causal action principle

Lagrangian $\mathcal{L}[A_{xy}] = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 \geq 0$

Action $\mathcal{S} = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}[A_{xy}] \, d\rho(x) \, d\rho(y) \in [0, \infty]$

Minimize $\mathcal{S}$ under variations of ρ, with constraints

volume constraint: $\rho(\mathcal{F}) = \text{const}$

trace constraint: $\int_{\mathcal{F}} \text{tr}(x) \, d\rho(x) = \text{const}$

boundedness constraint: $\iint_{\mathcal{F}\times\mathcal{F}} \sum_{i=1}^{2n} |\lambda_i^{xy}|^2 \, d\rho(x) \, d\rho(y) \leq C$

Minimizer

Definition (Minimizer)

ρ is a minimizer if

$$\mathcal{S}[\tilde{\rho}] - \mathcal{S}[\rho] \geq 0$$

for all $\tilde{\rho}$ with

$$|\tilde{\rho} - \rho| < \infty \qquad (\tilde{\rho} - \rho)\mathcal{F} = 0.$$

$$\ell(\mathrm{x}) = \int_{\mathrm{M}} \mathcal{L}(\mathrm{x}, \mathrm{y})\, \mathrm{d}\rho(\mathrm{y})$$

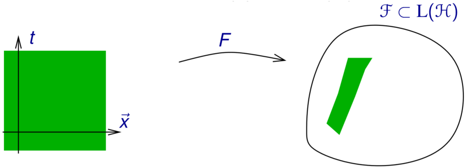
Lemma (Euler-Lagrange equations)

$$\ell|_{\mathrm{M}} = \inf_{\mathcal{F}} \ell =: \mathrm{c}$$

Local Correlation Map

We can define a mapping

$$F[g_{\mu\nu}, A_\mu, \ldots] : \mathcal{M} \mapsto \mathfrak{F} \qquad \subset L(\mathcal{H})$$
$$x \mapsto F[g_{\mu\nu}, A_\mu, \ldots](x)$$

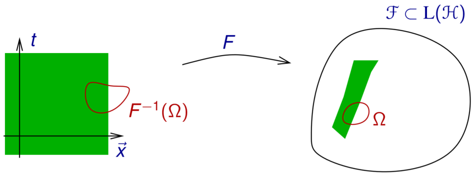


Concept:

- Left side: (approximate) effective description
- Right side: fundamental description of the physical system.
- Map allows us to work with familiar structures on the left.

Local Correlation Map

One more thing: The space-time volume



$$\rho(\Omega) := \int_{F^{-1}(\Omega)} d^4x = \mu\big(F^{-1}(\Omega)\big)$$

- push-forward measure, is measure on $\mathcal{F}$.
- image of F recovered as the support of the measure,

$$M := \text{supp}\ \rho = \big\{x \in \mathcal{F} \mid \rho(\Omega) \neq 0 \text{ for every open neighborhood } \Omega \text{ of } x\big\}$$

Realization of the Hilbertspace in Spacetime

Apriori $\mathcal{H}$ is an abstract Hilbert space.
Physical wave functions

$$\psi(\mathrm{x}) = \pi_{\mathrm{x}}\, \psi \quad \text{with } \psi \in \mathcal{H}$$

plus local correlation map

$$\begin{aligned} \mathrm{F}[\mathrm{g}_{\mu\nu}, \mathrm{A}_{\mu}, \ldots] : \mathcal{M} \mapsto \mathcal{F} \qquad & \subset \mathrm{L}(\mathcal{H}) \\ \mathrm{x} \mapsto \mathrm{F}[&\mathrm{g}_{\mu\nu}, \mathrm{A}_{\mu}, \ldots](\mathrm{x}) \end{aligned}$$

allow to work with suitable function spaces in $\mathcal{M}$.

How To Work With CFS (Wish)

Find a minimizing measure of the causal action

$\Downarrow$

Determine all physical wave functions for a basis of $\mathcal{H}$

$\Downarrow$

Find a spacetime and matter configuration that gives an approximate effective description for the causal structure of the minimizer and gives rise to equations satisfied by the physical wave functions.

How To Work With CFS (Reality)

Choose a spacetime and matter fields configuration of interest (which do not need to obey the usual physical equations).

$\Downarrow$

Select a set of functions in the spacetime at hand (obeying certain interesting conditions) as a physical wave function representation of a basis of $\mathcal{H}$

$\Downarrow$

Find a way to represent this configuration as a CFS

$\Downarrow$

Verify whether this system is a minimizer of the causal action principle in a suitable sense

$\Downarrow$

Obtain restrictions on your spacetime and matter fields from the requirement that the configuration has to be a minimizer of the causal action principle

Dirac spinors in Minkowski space

Space-time is Minkowski space, signature $(+ - - -)$

Space-time point $x \in \mathbb{R}^4$, need to associate operator $F(x)$

Physical wave function representation of $\mathcal{H}$ allows us to work with functions in $\mathbb{R}^4$.

- free Dirac equation $\quad (i\gamma^k \partial_k - m)\,\psi = 0$
- probability density $\psi^\dagger \psi = \overline{\psi}\gamma^0\psi$,

  gives rise to a scalar product:

$$\langle \psi | \phi \rangle = \int_{t=\text{const}} (\overline{\psi}\gamma^0\phi)(t, \vec{x})\, d\vec{x}$$

  time independent due to current conservation

Dirac spinors in Minkowski space

- Consider a collection of one-particle wave functions

$$\mathcal{H} := <\psi_1, \ldots, \psi_f> \qquad \text{Hilbert space}$$

$$b_x(\psi, \phi) = -\overline{\psi(x)}\phi(x)$$

$$\langle \psi | F(x) \phi \rangle = -\overline{\psi(x)}\phi(x) \qquad \forall \psi, \phi \in \mathcal{H}$$

local correlation operator, is self-adjoint operator in $L(\mathcal{H})$

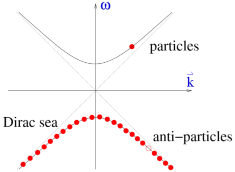
- Thus $F(x) \in \mathcal{F}$ where

$\mathcal{F} := \Big\{ F(x) \in L(\mathcal{H})$ with the properties:
 ▹ $F(x)$ is self-adjoint and has rank 4
 ▹ $F(x)$ has 2 positive
 and 2 negative eigenvalues $\Big\}$

The causal action principle in the continuum limit

Specify vacuum:

- Choose $\mathcal{H}$ as the space of all negative-energy solutions, hence "Dirac sea"



- Introduce regularization:

Fixes length scale ε

Summary

spacetime and matter described by a single object:

a measure ρ on $\mathcal{F}$, the universal measure

- space-time $M := \text{supp}\,\rho$
- causal relations given by spectrum of $x \cdot y$ for $x, y \in M$

dynamics described by causal action principle:

- minimize $\mathcal{S}$ by varying ρ
- implicitly varies space-time and all structures therein

applications:

- describes fundamental forces of nature
- approach for unification of gravitation and the standard model.
- Potentially gives rise to new mechanism for fermiogenesis.

Thank You

Thank you for your attention.

For a more in depth introduction see the website
causal-fermion-system.com