η, η' mixing from the lattice

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Precision tests of fundamental physics with light mesons

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Introduction

Quarks cannot be observed directly but are bound in hadrons (at low energies):

- The lightest hadrons $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η ("octet mesons") have masses from 135 MeV to 548 MeV.
- In addition there is a "flavor-singlet", the η'.
- For exact flavor symmetry ($m_u = m_d = m_s$) all 9 mesons should have the same mass.

However: $M_{\eta'} \approx 958\,\text{MeV} \gg M_{octet}$

Theoretical solution to this puzzle in QCD:

- Large mass of the η' is caused by the QCD vacuum structure and the $U(1)_A$ anomaly.
  Weinberg (1975), Belavin et al. (1975), t'Hooft (1976), Witten (1979), Veneziano (1979)
- The $U(1)$ axial current is anomalously broken, i.e. even for $m_q = 0$:
  Adler (1969), Jackiw and Bell (1969)

$$\partial_\mu A^0_\mu = \frac{N_f g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \neq 0$$

- Instantons with non-trivial topology provide non-perturbative explanation.
  Belavin et al. (1975) , t'Hooft (1976)
- The flavor-singlet η' remains massive as $m_l, m_s \to 0$.

Does lattice QCD reproduce the large η' mass from first principles?

Introduction

For exact $SU(3)$ flavor symmetry one expects

- Flavor octet state $|\eta_8\rangle = \frac{1}{\sqrt{6}}(|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle)$ (Pseudo-Goldstone boson)
- Flavor singlet state $|\eta_0\rangle = \frac{1}{\sqrt{3}}(|\bar{u}u\rangle + |\bar{d}d\rangle + |\bar{s}s\rangle)$ (related to $U(1)_A$ anomaly)

However, $SU(3)$ flavor symmetry is broken by large $m_s \gg m_u \approx m_d \equiv m_l$:

- Physical η, η' states are not flavor eigenstates but **mixtures**, e.g.

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi_l & -\sin\phi_s \\ \sin\phi_l & \cos\phi_s \end{pmatrix} \begin{pmatrix} |\eta_l\rangle \\ |\eta_s\rangle \end{pmatrix}$$

  in the **quark flavor basis** $|\eta_l\rangle = \frac{1}{\sqrt{2}}(|\bar{u}u\rangle + |\bar{d}d\rangle)$, $|\eta_s\rangle = |\bar{s}s\rangle$.

- In nature further mixing possible, e.g. with π^0 ($m_u \neq m_d$), η_c (including c quark)
- How did nature arrange the mixing pattern?

Use lattice QCD to determine the mixing parameters.

Outline

1. η,η' in Lattice QCD
2. Lattice setup
3. Physical extrapolations for $M_{\eta,\eta'}$ and mixing parameters
4. Model averages and (**preliminary!**) results

Contributions by many collaborators over the years:

Krzysztof Cichy
Elena Garcia-Ramos
Karl Jansen
Bastian Knippschild
Liuming Liu
Marcus Petschlies *
Siebren Reker
Urs Wenger *
Falk Zimmermann

Petros Dimopoulos
Christopher Helmes
Christian Jost
Bartosz Kostrzewa*
Chris Michael
Ferenc Pittler *
Carsten Urbach *
Markus Werner
...

* people involved in the current analysis (this work)

Extended Twisted Mass Collaboration

η, η' on the lattice

Information on masses and mixing is encoded in (expectation values of) meson two-point correlation functions:

$$\mathcal{C}_{ij}(t) \sim \sum_{\vec{x}} \langle 0|\, O_i(x)\, O_j^{\dagger}(0)\, |0\rangle$$

- For η, η' use local, pseudoscalar interpolating operators $O_{i,j}$:

$$\eta_l = \frac{1}{\sqrt{2}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d)\,, \qquad \eta_s = \bar{s}i\gamma_5 s\,, \qquad \eta_c = \bar{c}i\gamma_5 c$$

- For e.g. $i=j$: $\quad \mathcal{C}_{ii}(t) = \sum_n \frac{|\langle 0|O_i|n\rangle|^2}{2M_n}\exp(-M_n t) \overset{t\gg 0}{\rightarrow} \frac{|\langle 0|O_i|\eta\rangle|^2}{2M_\eta}\exp(-M_\eta t)$

  $\rightarrow$ Ground state mass M_η can be extracted directly at sufficiently large t.

  $\rightarrow$ Decay constants / mixing parameters related to physical amplitudes $A_i^n = \langle 0|\, O_i\, |n\rangle$.

- Information on higher states (η') from solving GEVP: $\quad \mathcal{C}(t)v^{(n)}(t,t_0) = \lambda^{(n)}(t,t_0)\mathcal{C}(t_0)v^{(n)}(t,t_0)$

  $\rightarrow$ Eigenvalues $\lambda^{(n)}(t,t_0)$ give mass of n-th state at $t \gg 0$.

  $\rightarrow$ Eigenvectors $v^{(n)}(t,t_0)$ carry information on physical amplitudes $A_{l,s,c,\dots}^{\eta,\eta',\dots}$.

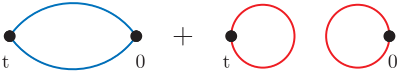
Quark disconnected diagrams

- Consider $O_i = O_j = \eta_l$:

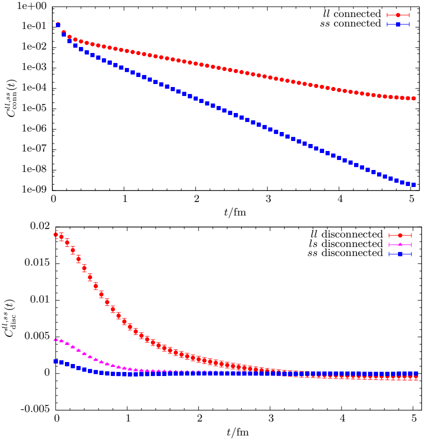
$$\mathcal{C}_{ll}(t) \sim \sum_{\vec{x}} \langle 0|\, \bar{u}(x) i\gamma_5 u(x) \bar{u}(0) i\gamma_5 u(0)\, |0\rangle$$

$$\sim \mathrm{tr}\left[D^{-1}_{0t}\gamma_5 D^{-1}_{t0}\gamma_5\right] + \mathrm{tr}\left[D^{-1}_{tt}\gamma_5\right] \mathrm{tr}\left[D^{-1}_{00}\gamma_5\right]$$

- Quark connected and disconnected pieces:



- Lattice Dirac operator D_{xy} is a very large $(3 \cdot 4 \cdot L^3 \cdot T) \times (3 \cdot 4 \cdot L^3 \cdot T)$ – matrix



Quark-connected and disconnected correlators;
$M_\pi = 139\,\mathrm{MeV},\ a = 0.080\,\mathrm{fm}$

- Disconnected diagrams need **all-to-all** propagator D^{-1}_{xx} $\Rightarrow$ prohibitively expensive
- Use stochastic method + one-end trick instead

Quark disconnected diagrams

- Consider $O_i = O_j = \eta_l$:

$$\mathcal{C}_{ll}(t) \sim \sum_{\vec{x}} \langle 0| \bar{u}(x) i\gamma_5 u(x) \bar{u}(0) i\gamma_5 u(0) |0\rangle$$

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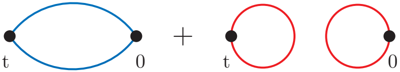

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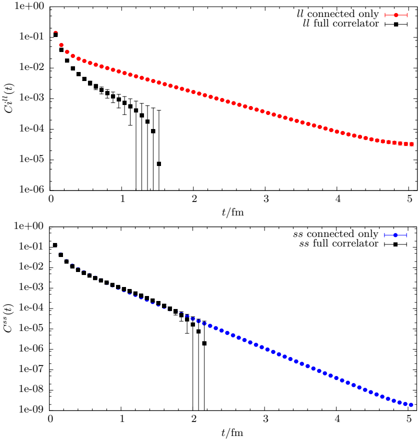


Quark-connected vs. full correlators; $M_\pi = 139\,\mathrm{MeV}$, $a = 0.080\,\mathrm{fm}$

→ **Severe signal-to-noise problem; signal typically lost at $t \gtrsim 1\,\mathrm{fm}$ ...**

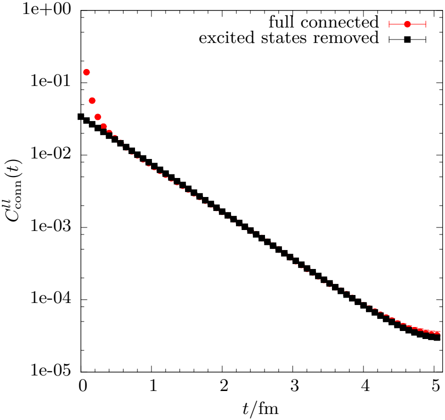
How to tackle the signal-to-noise problem?

Assumption:

Disconnected diagrams couple only to η, η'.

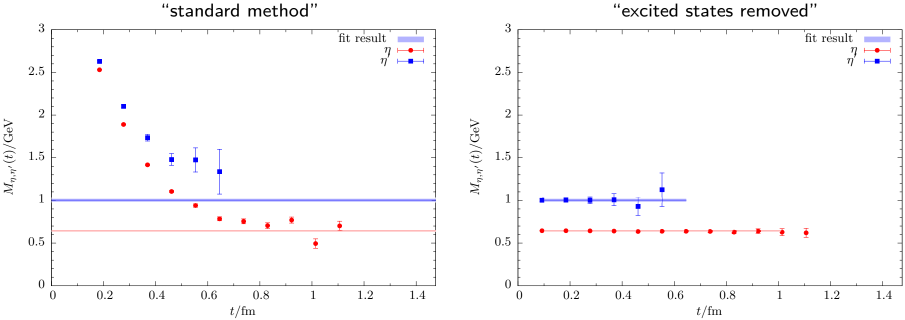
- No signal-to-noise problem in quark-connected contribution.
- Replace connected contributions by respective ground state contributions.
  PRD 64 (2001), 114509, EPJ C58 (2008), 261-269
  PRL 111 (2013) 18, 181602
- Charm quark contributions are neglected (they are very small)

If this assumption is correct we should see a plateau at very small values of t/a ...



Connected contribution with and w/o excited states
$M_\pi = 270\,\mathrm{MeV}$, $a = 0.78\,\mathrm{fm}$

Removal of excited state



- **... We observe plateaux in both states starting at very small values of t.**
- M_η agrees very well with asymptotic behavior of $M_\eta(t)$ from standard method.
- Significant improvement in the statistical error for $M_{\eta'}$.
- Can check validity of assumption from Monte-Carlo data.

Topological finite volume effect (I)

In **finite volume** and for **fixed top. charge** Q_t one finds

$$\langle \omega(x)\omega(0)\rangle_{Q_t=\text{fixed}} \to \frac{1}{V}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right) + \dots ,$$

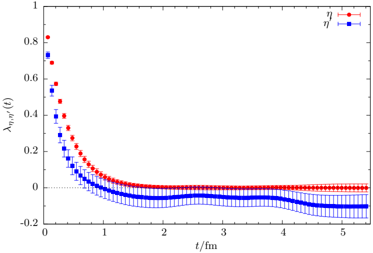
for correlators of winding number densities $\omega(x)$ at large $|x|$.

S. Aoki et al., Phys.Rev. D76, 054508 (2007)

$\Rightarrow$ Expect constant offset in η' (η) correlator **at large** t:

$$< \lambda^{\eta'}(t) >_{Q_t=\text{fixed}} \to \sim \frac{a^5}{T}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right) .$$

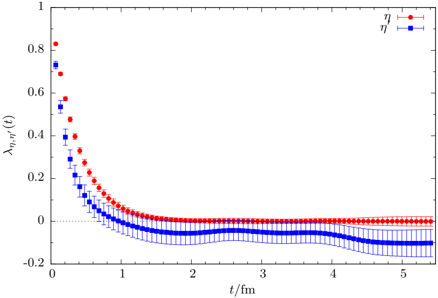
G. S. Bali et al., Phys.Rev. D91 (2015) 1, 014503



η,η' principal correlators
$M_\pi = 137\,\mathrm{MeV},\ a = 0.068\,\mathrm{fm},\ L \approx 5.4\,\mathrm{fm}$

- Always present for finite volume + finite statistics.
- Often masked by statistical point errors!
- **(Correlated) Noise in η'-signal largely due to fluctuation + autocorrelation of this constant.**

Topological finite volume effect (II)



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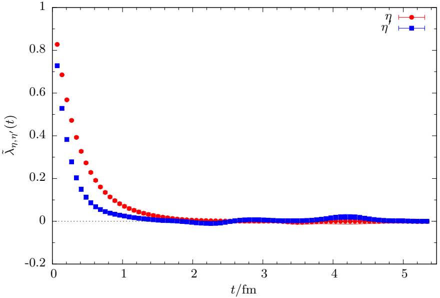

Same, but from time-derivative

Simple but efficient way to correct for this effect:

- Remove constant using discrete derivative correlator:

$$\mathcal{C}(t) \to \tilde{\mathcal{C}}(t) = \mathcal{C}(t) - \mathcal{C}(t+\Delta t)$$

- Resulting data are much less correlated.
- Further analysis (GEVP, physical extrapolation) can be carried out in the standard way.

How to obtain physical results?

- **Fix bare parameters (a, m_l, m_s,...):**
  - Use known hadronic quantities (e.g. M_π^{phys}, M_K^{phys}, ...) $\rightarrow$ Further observables are predictions.
- **Control discretization effects:**
  - Simulate at different (small) values of a.
  - Perform continuum extrapolation.
  - With modern LQCD calculations lattice artifacts are typically $\propto a^2$.
- **Correct for unphysical quark masses:**
  - Simulate at several light and strange quark masses, or tune $m_s = m_s^{\mathrm{phys}}$
  - Perform chiral extrapolation.
  - State-of-the-art lattice simulations include physical quark masses.
- **Control finite volume effects:**
  - Simulate several physical volumes.
  - Perform infinite volume extrapolation OR correct for finite volume effects (if possible)

Lattice setup

ID	β	a/fm	T/a	L/a	M_π / MeV	$M_\pi L$	$N_{\rm conf}$	$\Delta N_{\rm conf}$
cA211.12.48	1.726	0.0922	96	48	172	3.85	287	4
cA211.15.48			96	48	191	4.27	1853	2
cA211.30.32			64	32	268	4.01	1261	4
cA211.40.24			48	24	311	3.48	1320	2
cA211.53.24			48	24	357	4.00	624	8
cB211.072.64	1.778	0.0800	128	64	139	3.62	779	4
cB211.14.64			128	64	193	5.01	456	4
cB211.25.48			96	48	257	5.01	574	2
cB211.25.32			64	32	260	3.37	989	1
cB211.25.24			48	24	271	2.64	654	4
cC211.06.80	1.836	0.0684	160	80	137	3.80	738	4
cC211.20.48			96	48	248	4.12	611	4
cD211.054.96	1.900	0.0573	192	96	139	3.89	492	2

- Gauge configurations generated by the Extended Twisted Mass Collaboration (ETMC).
- $N_f = 2+1+1$ flavors of Wilson Clover twisted-mass sea quarks. *PRD 98 (2018) 054518* *PRD 104 (2021) 074520*
- **Automatic $O(a)$ improvement at maximal twist.** *JHEP 08 (2004) 007* *JHEP 10 (2004) 070*
- Degenerate light quark doublet, i.e. $m_u^{\rm sea} = m_d^{\rm sea}$.
- Non-degenerate heavy quark doublet with $m_s^{\rm sea} = \rm phys$, $m_c^{\rm sea} = \rm phys$.
- New action and (much) more chiral + finer ensembles compared to "old" analysis (2012–2018)
  JHEP 11 (2012) 048, PRL 111 (2013) 18, 181602, JHEP 09 (2015) 020, PRD 97 (2018) 5, 054508

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- **Ensembles cover four values of the lattice spacing a**
  $\rightarrow$ continuum extrapolation
- Various physical volumes with $L \approx 2 \ldots 5.5\,\mathrm{fm}$, $2.6 \leq M_\pi L \leq 5.0$.
  $\rightarrow$ extrapolation to infinite volume / check for finite size effects.
- Pion masses from $\sim 140\,\mathrm{MeV}$ to $\sim 360\,\mathrm{MeV}$; six boxes with $M_\pi < 200\,\mathrm{MeV}$.
  $\rightarrow$ chiral extrapolation and checking its convergence
- Three boxes at physical quark masses.

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- **Three boxes at physical quark masses.**

Mixed action

In this study we use a **mixed action setup**

$$\mathcal{S} = \mathcal{S}_G[U] + \mathcal{S}_F^{\mathrm{sea}}[\psi_l, \psi_h, U] + \mathcal{S}_F^{\mathrm{val}}[\{q_f, q_f'\}, U] + \mathrm{ghosts}\,,$$

where

- $\mathcal{S}_G[U]$ is the Iwasaki gauge action. *Nucl. Phys. B 258 (1985) 141*
- $\mathcal{S}_F^{\mathrm{sea}}[\psi_l, \psi_h, U]$ is the Wilson clover twisted-mass action with $\psi_l = (u_{\mathrm{sea}}, d_{\mathrm{sea}})^T$ and $\psi_h = (c_{\mathrm{sea}}, s_{\mathrm{sea}})^T$.
- $\mathcal{S}_F^{\mathrm{val}}[\{q_f, q_f'\}, U]$ is the Osterwalder-Seiler action for quark flavors $f = u, d, s, \ldots$. *Annals Phys. 110 (1978) 440*

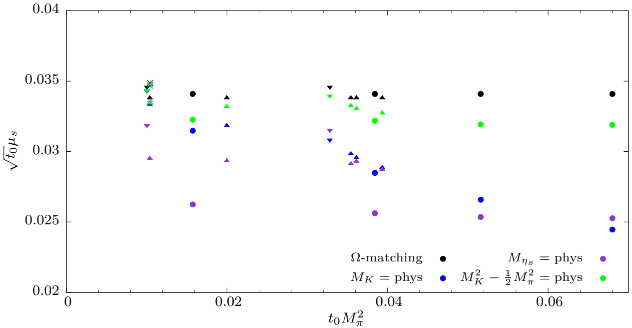
This setup has several advantages over a unitary setup:

- Avoids strange-charm mixing through cutoff effects.
- One-end trick variance reduction can be used for heavy quarks (i.e. for the strange).
- Preserves automatic $\mathcal{O}(a)$-improvement.

Physical results require fixing μ_s by matching with an observable and taking the continuum limit:

- Compute all the desired observables for a set of μ_s values at each β.
- Perform linear interpolation of observables to target μ_s value.

μ_s–matching



We employ four choices for the μ_s matching conditions

1. "Ω–matching": $m_\Omega(M_\pi^{\text{phys}}, \beta) = m_\Omega^{\text{phys}} \to \mu_s(\beta)$.
2. "kaon– matching": $M_K(M_\pi, \beta) = M_K^{\text{phys}} \to \mu_s(M_\pi, \beta)$.
3. "η_s–matching": $M_{\eta_s}(M_\pi, \beta) = M_{\eta_s}^{\text{phys}} \to \mu_s(M_\pi, \beta)$ using $M_{\eta_s}^{\text{phys}} = 689.89\,\text{MeV}$. *Nature 593 (2021) 7857, 51-55*
4. Using LO χPT proxy for μ_s, i.e. $M_K^2 - \frac{1}{2}M_\pi^2 = \text{phys} \to \mu_s(M_\pi, \beta)$.

Different choices lead to different slopes $\sim a^2$ for continuum extrapolation (+ higher order effects).

Mixing observables

Decay constants f_{P}^{i} are defined from axial-vector matrix elements (amplitudes)

$$\langle 0| A_\mu^i |\mathrm{P}(p)\rangle = i f_{\mathrm{P}}^{i} p_\mu \,, \qquad \mathrm{P} = \eta, \eta' \,,$$

On the lattice: **quark flavor basis** (i=l,s) is a more "natural" choice

$$A_\mu^l = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d) \,, \qquad A_\mu^s = \bar{s}\gamma_\mu\gamma_5 s \,.$$

η and η' are not flavor eigenstates; most general parametrization:

$$\begin{pmatrix} f_\eta^l & f_\eta^s \\ f_{\eta'}^l & f_{\eta'}^s \end{pmatrix} = \begin{pmatrix} f_l \cos\phi_l & -f_s \sin\phi_s \\ f_l \sin\phi_l & f_s \cos\phi_s \end{pmatrix}$$

From χPT one expects $|\phi_l - \phi_s|$ to be small, i.e. $\frac{|\phi_l - \phi_s|}{|\phi_l + \phi_s|} \ll 1$

- $|\phi_l - \phi_s| \sim \frac{1}{N_c}$ is OZI-suppressed, unlike e.g. $|\phi_0 - \phi_8|$ which receives $SU(3)_F$-breaking contributions.
- Small difference in one basis does **NOT** imply small difference in another basis!

$\Rightarrow$ **Approximate, single angle $\phi \approx \phi_l \approx \phi_s$ (FKS) scheme:** $\tan^2(\phi) = -\frac{f_l^{\eta'} f_s^{\eta}}{f_l^{\eta} f_s^{\eta'}}$.

Mixing observables

Axial vector matrix elements are very noisy $\Rightarrow$ Difficult to directly determine $\phi_{l,s}$ and $f_{l,s}$.

Consider pseudoscalar matrix elements

$$h^i_{\mathrm{P}} = 2m_i < 0|P^i|\mathrm{P} >, \quad \mathrm{P} = \eta, \eta',$$

which can be related to axial vector ones via the **anomaly equation** using χPT:

$$\begin{pmatrix} h^l_\eta & h^s_\eta \\ h^l_{\eta'} & h^s_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mathrm{diag}\left(f_l M_\pi^2, f_s\left(2M_K^2 - M_\pi^2\right)\right).$$

Th. Feldmann et al., PRD 58 (1998), 114006

Th. Feldmann et al., Phys.Lett. B449 (1999) 339-346

- This expression holds to LO χPT.
- **Scale dependence due to Z_A^0 is neglected**; expect deviations at high(er) energies.
  $\rightarrow$ cf. recent study by the Regensburg group (RQCD) on CLS ensembles *JHEP 08 (2021) 137*
- ϕ does not depend on renormalization at all, $f_{l,s}$ depend only on Z_P/Z_S.
- Can check whether $|\phi_l - \phi_s|$ is small!

Chiral, continuum and finite size (CCF) fit models

We use the following, basic ansätze for physical extrapolations:

$$M_\eta^2 = A_0(m_l + 2m_s) + B_0 a^2 \,, \tag{1}$$

$$M_{\eta'}^2 = \mathring{M}_0^{SU(3)} + A_1(2m_l + m_s) + B_1 a^2 \,, \tag{2}$$

$$\phi = \mathrm{atan}(\sqrt{2}) + A_2(m_l - m_s) + B_2 a^2 \,, \tag{3}$$

$$f_{\{l,s\}} = \mathring{f}_{\{l,s\}}^{SU(2)} + A_{\{3,4\}} m_l + B_{\{3,4\}} a^2 \,, \tag{4}$$

$$f_{\{l,s\}}/f_{\{\pi,K\}} \equiv R_{l,s} = \mathring{R}_{l,s}^{SU(2)} + A_{\{5,6\}} m_l + B_{\{5,6\}} a^2 \,, \tag{5}$$

- where $m_l \cong M_\pi^2$ and $m_s \cong (2M_K^2 - M_\pi^2)$,
- and $\mathring{M}_0^{SU(3)}$, $\mathring{f}_{l,s}^{SU(2)}$, $\mathring{R}_{l,s}^{SU(2)}$, A_i, B_i are free parameters of the fits.

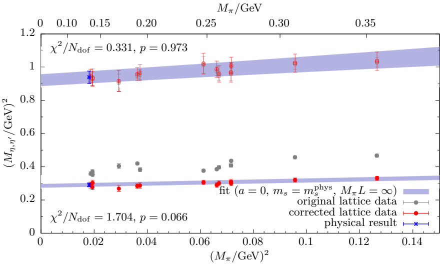
Additional fit models are obtained by any combination of the following changes:

- including a term $C_i \frac{M_\pi^2}{\sqrt{M_\pi L}} e^{-M_\pi L}$ (not for M_η^2; $M_{\pi,K,\eta}$ are explicitly FS-corrected Nucl.Phys.B 721 (2005) 136-174)
- including a term $D_i M_\pi^4$,
- with and w/o explicit m_s dependence (where applicable),
- applying a data cut $\in \{\text{no cut},\ M_\pi < 270\,\mathrm{MeV},\ a < 0.08\,\mathrm{fm},\ M_\pi L > 3.5\}$.

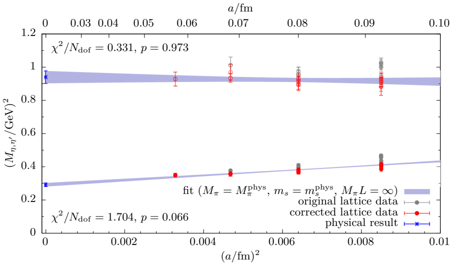
All statistical errors are computed using the non-parametric (binned) bootstrap $N_B = 10000$.

Physical extrapolations – M_η, $M_{\eta'}$

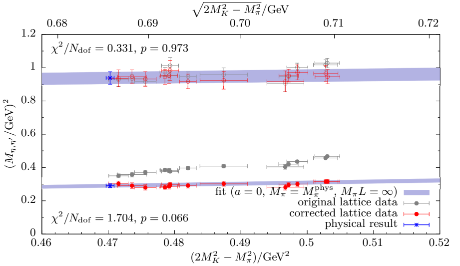
light chiral extrapolation



continuum extrapolation



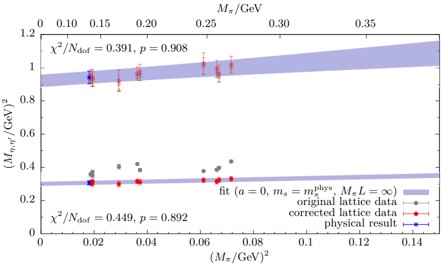
- Light chiral extrapolation very mild.
- Steep continuum extrapolation for M_η, i.e. up to $\sim 30\%$ corrections.
- $M_{\eta'}$ tends to be overfitted.
- **Physical results in good agreement with experiment.**
- Statistically precise results, i.e. $\Delta M_\eta / M_\eta \sim 1\%..2\%$, $\Delta M_{\eta'} / M_{\eta'} \sim 2\%..3\%$



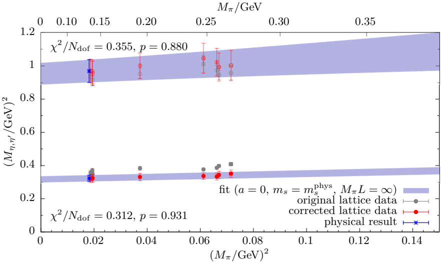
strange chiral extrapolation

Data cuts – M_η, $M_{\eta'}$

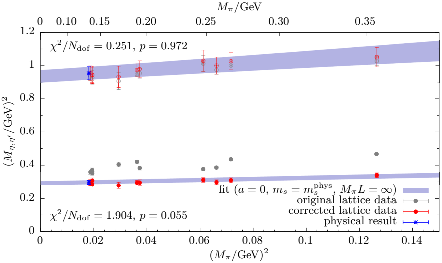
$M_\pi < 270\,\mathrm{MeV}$



$a \lesssim 0.08\,\mathrm{fm}$



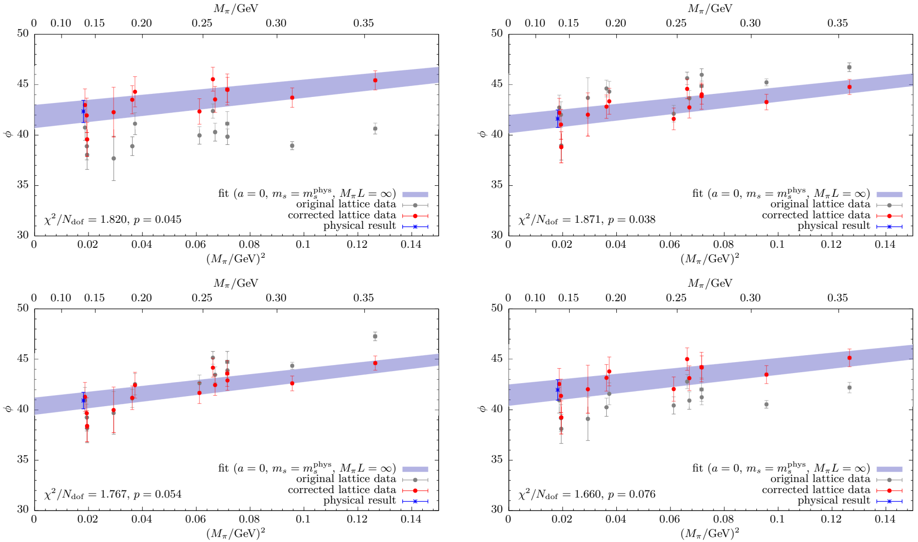
- Cuts in M_π, a typically reduce χ^2/N_{dof} for M_η
- Here: $M_\pi L$ cut has virtually no effect.
  $\rightarrow$ similar for adding an explicit FS term
- Physical results remain very stable.
- Statistical errors may increase (as expected).



$M_\pi L > 3.5$

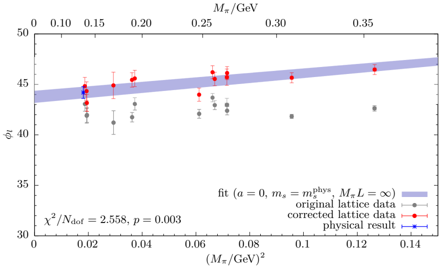
Physical extrapolation for ϕ – comparison of μ_s–matchings

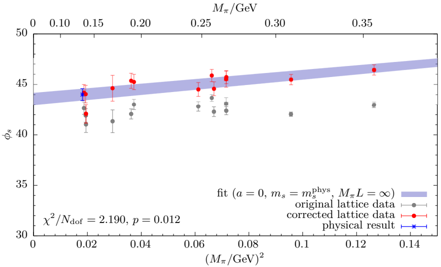
Ω–matching | kaon matching



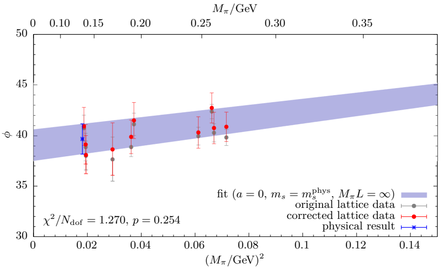
η_s–matching | $M_K^2 - \frac{1}{2}M_\pi^2 = \mathrm{phys}$ matching

Physical extrapolations for ϕ, ϕ_l, ϕ_s

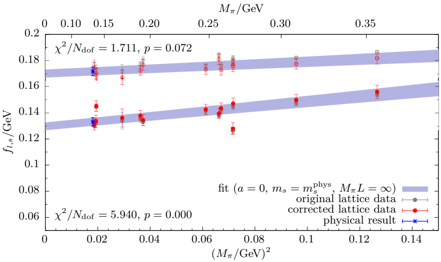


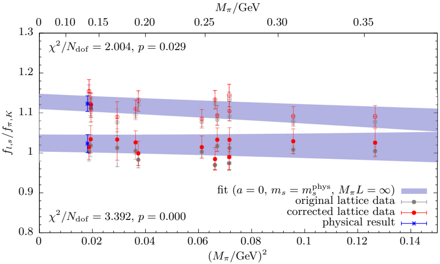


- Data for ϕ_l, ϕ_s even more precise than for ϕ.
- Results for ϕ_l and ϕ_s almost identical.
- ϕ and $\phi_{l,s}$ differ by 1σ–2σ.
- Some tension fitting full set of data with basic fit ansatz
- χ^2/N_{dof} improved by e.g. applying cut in M_π.



Decay constant parameters f_l, f_s

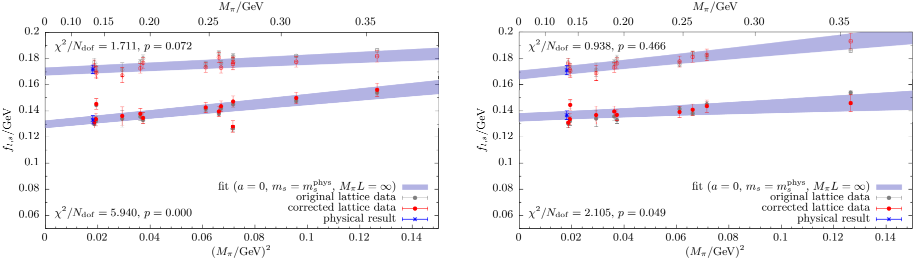




- Data for f_s tends to be more statistically precise; less fluctuations.
- Fits generally work much better for f_s. $\rightarrow$ already seen in "old" analysis *PRD 96 (2018) 5, 054508*

  However: data more chiral, i.e. $M_\pi \in \{140...360\,\mathrm{MeV}\}$ vs $M_\pi \in \{230...500\,\mathrm{MeV}\}$)
- Ratios f_l/f_π and f_s/f_K may cancel some quark mass and FS effects.

Decay constant parameters f_l, f_s



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- Ratios f_l/f_π and f_s/f_K may cancel some quark mass and FS effects.
- **Applying cuts, e.g. $M_\pi L > 3.5$ and including finite size term $\sim e^{-M_\pi L}$ improves fits.**

Model averaging

We assign a weight to each fit *Phys. Rev. D 103,114502 (2021)*

$$w_i \sim e^{-\frac{1}{2}\left(\chi^2+2(N_{\rm para}-N_{\rm prio})-2N_{\rm data}\right)}, \quad N_{\rm prio}=0\,.$$

Central value and total err for an observable y are given by median and the 16% and 84% percentiles of the CDF

$$CDF(y,\lambda)=\int_{-\infty}^{y} d\tilde{y}\sum_i w_i N(\tilde{y},m_i,\sigma_i\sqrt{\lambda})\,.$$

Statistical ($\sigma_{\rm stat}$) and systematic ($\sigma_{\rm sys}$) errors are separated by solving

$$\lambda\sigma^2_{\rm stat}+\sigma^2_{\rm sys}=\left(\frac{y_{\rm hi}-y_{\rm lo}}{2}\right)^2 \quad \text{where} \quad CDF(y_{\rm hi},\lambda)=0.84\,, \quad \text{and} \quad CDF(y_{\rm lo},\lambda)=0.16\,,$$

where λ is used to rescale the statistical errors *Nature 593 (2021) 7857, 51-55*

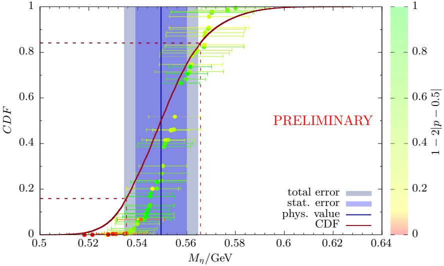
The set of models for each observable is given by

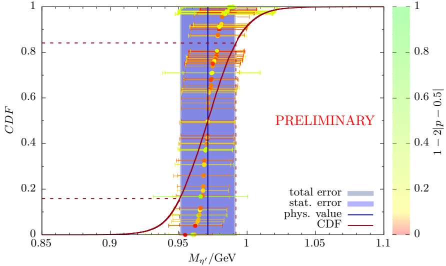
$$\{\mu_s\text{-matchings}\}\bigotimes\{\text{CCF models}\}\bigotimes\{\text{data cuts}\} \tag{6}$$

Typically, this yields $\mathcal{O}(100)$ models per observable:

$$\#\{\mu_s\text{-matchings}\}=4\,, \qquad 4\le\#\{\text{CCF models}\}\le 8\,, \qquad \#\{\text{data cuts}\}=4\,.$$

Masses





Physical results: $M_\eta = 549(11)_{\rm stat}(11)_{\rm sys}\,{\rm MeV}, \quad M_{\eta'} = 971(19)_{\rm stat}(06)_{\rm sys}\,{\rm MeV}$

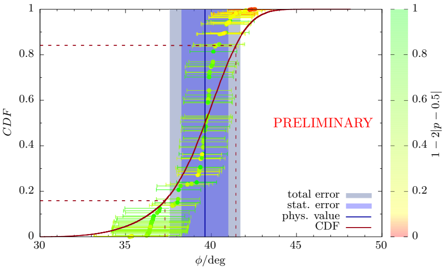
- Agreement with experiment ($M_\eta^{\rm exp} = 547.862(17)\,{\rm MeV}$, $M_{\eta'}^{\rm exp} = 957.78(6)\,{\rm MeV}$).
- Error on M'_η improved by factor ~ 3 compared to our previous results

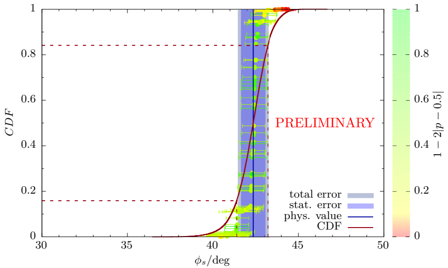
$$M_\eta = 557(11)_{\rm stat}(03)_{\chi PT}\ {\rm MeV}, \quad M'_\eta = 911(64)_{\rm stat}(03)_{\chi PT}\ {\rm MeV}$$

PRD 97 (2018) 5, 054508
PRL 111 (2013) 18, 181602

- Improved control over systematic effects (chiral + continuum + FS).
- Scale is set using $\sqrt{t_0^{\rm phys}} = 0.14436(61)\,{\rm fm}$. *PRD 104 (2021) 7, 074520*

Mixing angle(s)





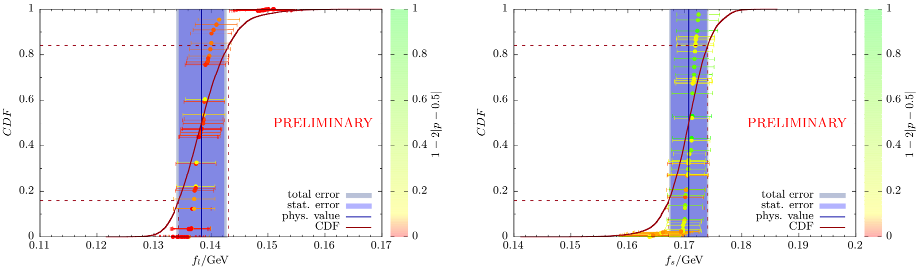
Physical results: $\phi = 39.6(1.4)_{\text{stat}}(1.5)_{\text{sys}}^\circ$, $\phi_l = 42.3(0.7)_{\text{stat}}(0.7)_{\text{sys}}^\circ$, $\phi_s = 42.3(0.7)_{\text{stat}}(0.5)_{\text{sys}}^\circ$

- $\Delta\phi$ improved by factor ~ 1.5 compared to old result $\phi = 38.8(2.2)_{\text{stat}}(2.4)_{\chi PT}^\circ$. *PRD 97 (2018) 5, 054508*
- Value for ϕ mostly in agreement with pheno determinations, e.g.

	ϕ_l	ϕ_s	
R. Escribano et al. (2016)	$39.6(2.3)^\circ$	$40.8(1.8)^\circ$	*PRD 94 (2016), 054033*
R. Escribano et al. (2015)	$39.3(1.2)^\circ$	$39.2(1.2)^\circ$	*EPJC 75, 414 (2015)*
Th. Feldmann (2000)	$39.3(1.0)^\circ$	$39.3(1.0)^\circ$	*Int. J. Mod. Phys. A 15 (2000)*

- Slight tension with RQCD results $\phi_l(\mu = 1\,\text{GeV}) = 38.3(1.8)^\circ$ and $\phi_s(\mu = 1\,\text{GeV}) = 36.8(1.6)^\circ$, possibly due to (neglecting) scale dependence. *JHEP 08 (2021) 137*

Decay constant parameters



Physical results: $f_l = 138.3(4.0)_{\rm stat}(1.8)_{\rm sys}\,\mathrm{MeV}, \quad f_s = 170.7(3.2)_{\rm stat}(1.2)_{\rm sys}\,\mathrm{MeV}$

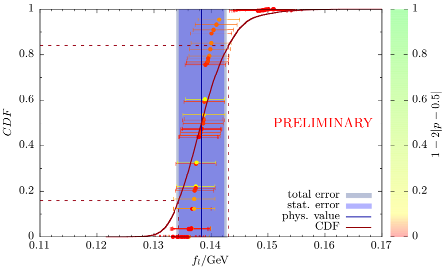
- f_l increased, f_s decreased compared to 2018 analysis, i.e.

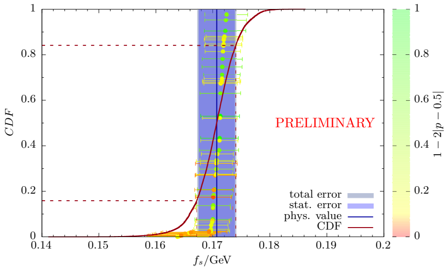
$$f_l = 125(5)_{\rm stat}(6)_{\chi PT}\,\mathrm{MeV}, \quad f_s = 178(4)_{\rm stat}(1)_{\chi PT}\,\mathrm{MeV}$$

- Improved control over systematic effects of physical extrapolations; particularly for f_l.
- Physical extrapolation of ratios: $f_l/f_\pi = 1.057(28)_{\rm stat}(27)_{\rm sys}$ and $f_s/f_K = 1.105(20)_{\rm stat}(13)_{\rm sys}$

$$\Rightarrow \quad f_l = 137.6(3.6)_{\rm stat}(3.5)_{\rm sys}\,\mathrm{MeV}, \quad f_s = 172.0(3.1)_{\rm stat}(2.3)_{\rm sys}\,\mathrm{MeV}$$

Decay constant parameters





Physical results: $f_l = 138.3(4.0)_{\mathrm{stat}}(1.8)_{\mathrm{sys}}\,\mathrm{MeV}, \quad f_s = 170.7(3.2)_{\mathrm{stat}}(1.2)_{\mathrm{sys}}\,\mathrm{MeV}$

- Errors on f_s quite competitive; new analysis in better agreement with pheno results for f_l

	f_l	f_s	
R. Escribano et al. (2016)	134.2(5.2) MeV	177.2(5.2) MeV	*PRD 94 (2016), 054033*
R. Escribano et al. (2015)	139.6(12.7) MeV	181.0(18.3) MeV	*EPJC 75, 414 (2015)*
Th. Feldmann (2000)	139.3(2.5) MeV	174.5(7.8) MeV	*Int. J. Mod. Phys. A 15 (2000)*
G. Bali et al., RQCD (2021)	129.7(4.7) MeV	179.1(6.1) MeV	*JHEP 08 (2021) 137*

- RQCD results at $\mu = 1\,\mathrm{GeV}$ compatible within 1σ to 2σ.

Summary and outlook

Lattice study of η, η' with $N_f = 2+1+1$ dynamical quark flavors:

- **Physical extrapolations for all observables with controlled systematics**.
- Three boxes at physical quark mass; generally much more chiral ensembles.
- Reduced statistical and systematic (!) errors compared to older analysis (2012–2018).
- Mass of η' reproduced from first principles with 2% error; $M_{\eta,\eta'}$ in agreement with experiment.
- Precise results for (FKS scheme) mixing parameters ϕ, f_l and f_s in good agreement with phenomenology.

Possible future plans:

- **Study scale dependence of mixing parameters.**
  $\rightarrow$ Requires computation of $Z_A^0(\mu)$
  $\rightarrow$ Need different strategy to extract axialvector matrix elements.
- Add ensembles with different m_s^{sea}
- Further increase statistics for physical quark mass ensembles (?)
  $\rightarrow$ Direct continuum extrapolations.
  $\rightarrow$ Remove need of chiral extrapolation entirely.