Dispersive Determination of $\eta^{(\prime)}$ Transition Form Factors

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Precision tests of fundamental physics with light mesons
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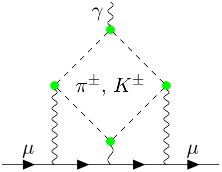
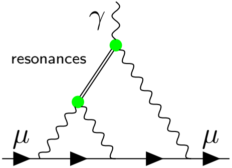
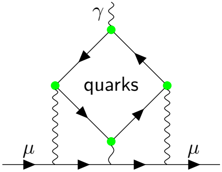
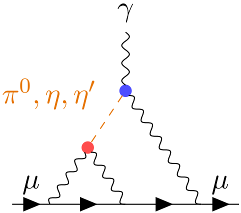
Hadronic light-by-light scattering







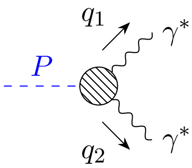
- Pre-2014 HLbL estimate based on hadronic models
- Model-independent dispersive approach: relate contributions to observables like form factors Colangelo et al. 2014
- Pseudoscalar pole contribution (π^0 dominant Hoferichter et al. 2018):
  - (blue dot) : Singly-virtual transition form factor (TFF)
  - (red dot) : Doubly-virtual TFF



η and η' transition form factors

- Pseudoscalar ($P = \pi^0,\ \eta,\ \eta'$) transition form factors defined by

$$i\int \mathrm{d}^4x\, e^{iq_1\cdot x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|P(q_1+q_2)\rangle$$
$$= \epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta F_{P\gamma^*\gamma^*}(q_1^2,q_2^2)$$



- Normalization related to di-photon decays governed by chiral anomaly:

$$\Gamma(P\to\gamma\gamma) = \frac{\pi\alpha_{\mathrm{em}}^2 M_P^3}{4}\left|F_{P\gamma^*\gamma^*}(0,0)\right|^2$$

- For pion: low-energy theorem predicts its value

  Bell, Jackiw 1969; Adler 1969; Bardeen 1969

- For η and η': complicated by η–η' mixing

  Feldmann, Kroll, Stech 1998–2000;

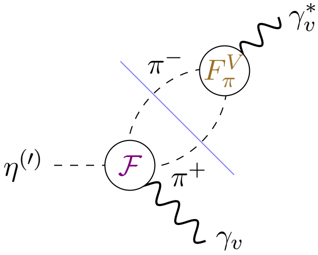
  Escribano, Gonzàlez-Solís, Masjuan, Sánchez-Puertas 2016

  on lattice: cf. Konstantin's talk on Monday

Transition form factor $\eta^{(\prime)} \to \gamma\gamma^*$

Isospin decomposition: $F^{\text{disp}}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\eta^{(\prime)}}_{vv}(q_1^2, q_2^2) + F^{\eta^{(\prime)}}_{ss}(q_1^2, q_2^2)$

Reconstruction from the lowest-lying hadronic states:



Isovector part:

- largest contribution: $\pi^+\pi^-$ intermediate state
- Dispersively combine data on $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$

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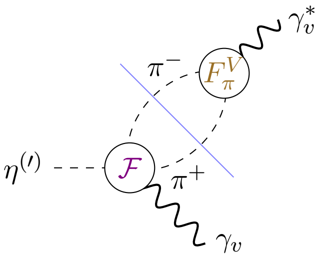


Isovector part:

- largest contribution: $\pi^+\pi^-$ intermediate state
- Dispersively combine data on $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$

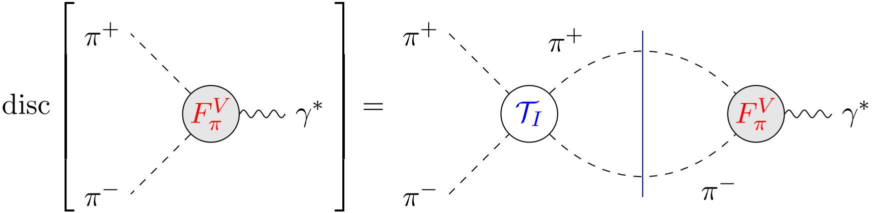
Isoscalar part:

- Dominated by narrow resonances: ω & ϕ Hanhart et al. 2013
- Employ VMD and fix couplings by exp. det. decay widths for
  - $\omega \to \eta\gamma$
  - $\eta' \to \omega\gamma$
  - $\phi \to \eta^{(\prime)}\gamma$
  - $\omega,\ \phi \to e^+e^-$

η: strong cancellation between ω and ϕ

η': isoscalar contribution more significant than for η (*e.g.* in norm $\sim 20\,\%$)

Pion vector form factor



- solution of discontinuity equation: Omnès 1958

$$F_\pi^V(s) = R(s)\Omega(s)\ , \quad \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta_1^1(\omega)}{\omega(\omega-s)}\mathrm{d}\omega\right)$$

- δ_1^1: $I=1$ $\pi\pi$ P-wave phase shift, $R(s) = (1+\alpha_\pi s)$

$$F_\pi^{V,e^+e^-}(s) = \left(1+\epsilon_{\rho\omega}\frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right)F_\pi^V(s)$$

Factorization breaking in the η and η' TFFs

- Past approaches: Application of VMD form factor in the low-energy regime

$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \frac{1}{q_1^2 - M_V^2} \times \frac{1}{q_2^2 - M_V^2}$$

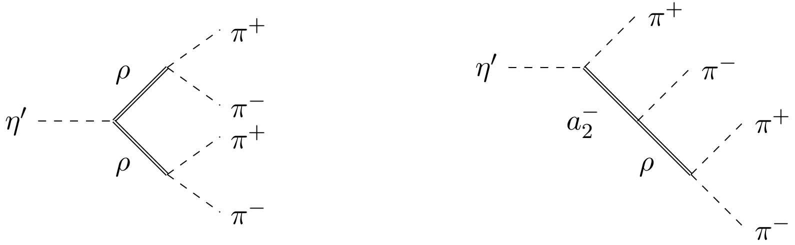
- For high energies ($|q_1^2|, |q_2^2| \to \infty$) pQCD predicts Walsh, Zerwas 1972

$$F_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) \propto \frac{1}{q_1^2 + q_2^2}$$

- No factorization in the singly-virtual TFFs present
- Model-independent description of intermediate energy regime with factorization breaking of paramount importance for control over uncertainties
- Recent exp. study (BaBar 2018) showed for $|q_1^2| = |q_2^2| \in [6.5,\, 45]$GeV^2 VMD factorization is breaking down

Formalism for doubly-virtual representations

- Start from $\eta' \to 2(\pi^+\pi^-)$ amplitude
  - describe decay via two rho resonances by hidden local symmetry (HLS) model Guo, Kubis, Wirzba 2012
  - left-hand-cut contribution due to a_2 exchange by phenomenological Lagrangian models



Final-state interaction

- in HLS amplitude: introduce pair-wise pion rescattering by replacing ρ propagators by Omnès functions
- in a_2 exchange amplitude $\Rightarrow$ inhomogenous Omnès problem

Solution strategies for inhom. Omnès problem

Coupled integral equation(s):

- $1 \to 3$ decay amplitude:

$$A(s) = \Omega(s)\left[P_n(s) + \frac{s^n}{\pi}\int \mathrm{d}\mu(x)\,\frac{\hat{A}(x)}{x - s - i\epsilon}\right]$$

- with 'hat'-function given by angular averages:

$$\hat{A}(x) = \frac{1}{2\kappa}\sum_{\ell} C_\ell(x,\kappa)\int_{-1}^{1}\mathrm{d}z\, z^\ell A(h(x,z))$$

- with Kacser function

$$\kappa(x, M_1^2, M_2^2) = \sqrt{1 - 4M_3^2/x}\sqrt{x - (M_1 - M_3)^2}\sqrt{x - (M_1 + M_3)^2}$$

- and path $h(x,z) = \left[M_1^2 + 3M_3^2 - x + z\kappa(x, M_1^2, M_3^2)\right]/2$
- κ adds complex analytic structure with branch points and cuts

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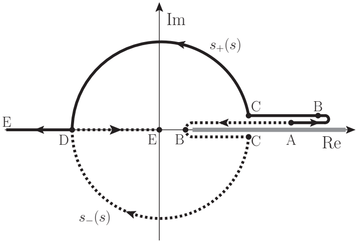
- 'Classic' strategy by

  Khuri and Treiman, 1960

  cf. talks of Igor and Miguel on Monday

  - deform path of angular integral to avoid crossing branch cuts
  - applied to *e.g.* $\omega/\phi \to 3\pi$

  ⇑Niecknig, Kubis, Schneider 2012 ⇒



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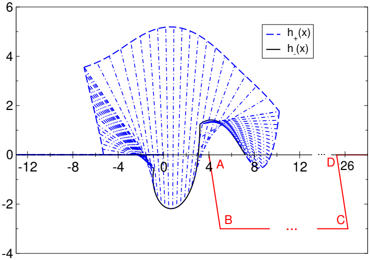
- Alternative approach by Gasser and Rusetsky, 2018
  - ▶ deform path of dispersion integral
  - ▶ applied by them to $\eta \to 3\pi \Rightarrow$

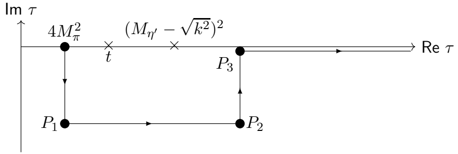


Inhomogeneous Omnès problem in $\eta' \to 2(\pi^+\pi^-)$

- Solution (P-wave) expressed in twice subtracted dispersion integral

$$\tilde{\mathcal{F}}(t,k^2) = \left[P(t) + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}\tau}{\tau^2} \frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|} \right] \Omega(t)$$

- Inhomogeneity $\hat{G}$ known for phenomenological model, but challenges direct evaluation due to singularity structure



- deform path of integration into complex plane (inspired by ideas of Gasser, Rusetsky 2018)

Strategy for going into the complex plane

Dispersion integral:

$$\int_\Gamma \frac{\mathrm{d}\tau}{\tau^2} \frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|}$$

1. deal with phase shift for complex arguments

Strategy for going into the complex plane

Dispersion integral:

$$\int_\Gamma \frac{\mathrm{d}\tau}{\tau^2} \frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|}$$

1. deal with phase shift for complex arguments
2. analytically continue inhomogeneity function

Strategy for going into the complex plane

Dispersion integral:

$$\int_{\Gamma} \frac{\mathrm{d}\tau}{\tau^2} \frac{\hat{G}(\tau, k^2) \sin \delta_1^1(\tau)}{(\tau - t - i\epsilon)|\Omega(\tau)|}$$

1. deal with phase shift for complex arguments
2. analytically continue inhomogeneity function
3. identify appropriate integration paths, *i.e.* avoid critical regions

Treatment of scattering phase in complex plane

- express $\pi\pi$ P-wave phase shift δ_1^1 in terms of scattering amplitude

Niehus, Hoferichter, Kubis 2019

$$\frac{\hat{G}(\tau,k^2)\sin\delta_1^1(\tau)}{\tau^2(\tau-t-i\epsilon)|\Omega(\tau)|} = \frac{\hat{G}(\tau,k^2)}{\tau^2(\tau-t-i\epsilon)}\Omega^{-1}(\tau)\sigma_\pi(\tau)t_1^1(\tau)$$

- utilize inverse amplitude method (unitarized ChPT to $\mathcal{O}(p^4)$)

Dobado, Herrero, Peláez, Truong 1990–96

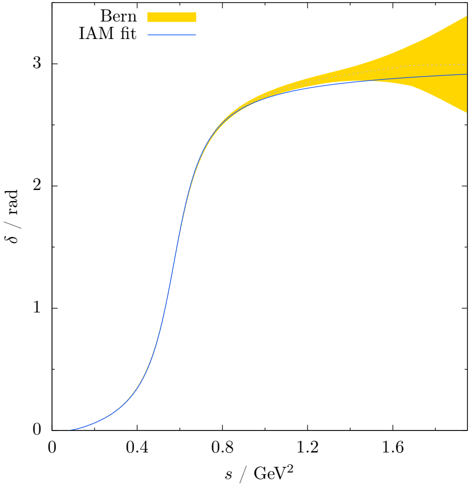
$$t_1^1(s) = \frac{(t_2(s))^2}{t_2(s)-t_4(s)}$$

- supplement by hand by N^{2}LO inspired terms to fix asymptotic behavior

$$t_4(s) \mapsto t_4(s) + \frac{t_2(s)}{48\pi^2F_0^2}(\hat{l}_s s^2 + \hat{l}_\pi M_\pi^4)$$

Fixing the IAM representation

- fix LEC $\bar{l}_2 - \bar{l}_1$ and parameters $\hat{l}_s$, $\hat{l}_\pi$ by fitting phase of amplitude to Roy-equation analysis Caprini, Colangelo, Leutwyler 2011



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- extract ρ-pole position on second Riemann sheet of amplitude

$$0 \stackrel{!}{=} 1 - t_1^1(s_{\text{pole}}) \sqrt{\frac{16M_\pi^2}{s_{\text{pole}}} - 4}$$

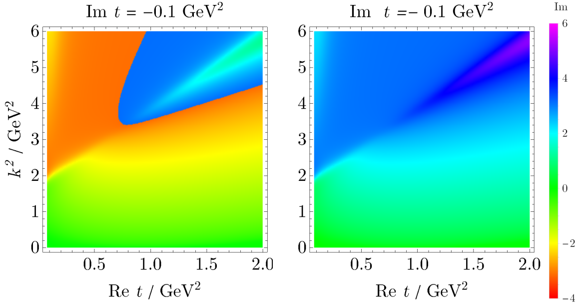
	M_ρ / MeV	Γ_ρ / MeV
$s_{\text{cut}} = 1\,\text{GeV}^2$	759	140
$s_{\text{cut}} = 1.69\,\text{GeV}^2$	759	139
Madrid (GKPY)	763.7(16)	146.4(22)
Bern (Roy)	762.4(18)	145.2(28)

Analytic continuation of inhomogeneity

Dispersion integral over inhomogenity function $\hat{G}(t,k^2)$:

$$\int \mathrm{d}t\, \frac{\hat{G}(t,k^2)}{t^2(t-x-i\epsilon)}\Omega^{-1}(t)\sigma_\pi(t)t_1^1(t)$$

- κ: branch cuts $t \in [0,\, 4M_\pi^2]$ and $t \in [(M_{\eta^{(\prime)}} - \sqrt{k^2})^2,\, (M_{\eta^{(\prime)}} + \sqrt{k^2})^2]$
- simultaneous continuation in both t and k^2 challenging
  - analytically continue one variable at a time into the complex plane
- $\hat{G}(t,k^2) \supset \log \frac{y+1}{y-1}$, with $y(t,k^2) \propto 1/\kappa(t,k^2)$
  - crucial to take $\log z(t,k^2) = \log|z| + i \arg z + 2\pi i \theta(t,k^2)$

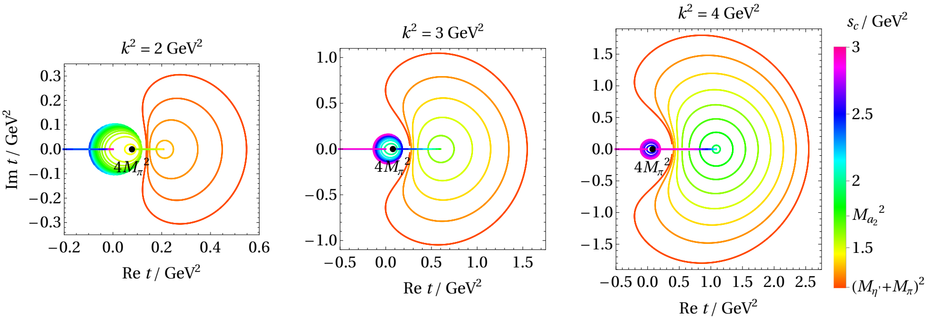


Identification of critical regions

- need to identify regions that integration path should avoid

Gasser, Rusetsky 2018

- integration over Mandelstam $t \to$ avoid cut in s-channel $s_c \in [(M_{\eta^{(\prime)}} + M_\pi)^2, \infty)$
  - solving $s(t,z) = s_c$ for t yields three indep. solutions
  - 1 solution real; remaining two: complex conjugates of each other



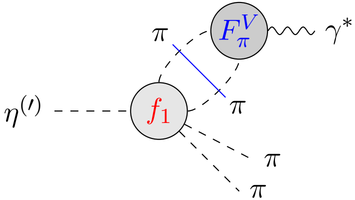
Intermediate step $\eta^{(\prime)}(q) \to \pi^+(p_1)\pi^-(p_2)\gamma^*(k)$

- amplitude in terms of scalar fn.: $\mathcal{M} = e\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu*}(k)p_1^\nu p_2^\alpha q^\beta \mathcal{F}^{\eta^{(\prime)}\to\pi\pi\gamma}$
- (dominant) P-wave contribution from unitarity condition:

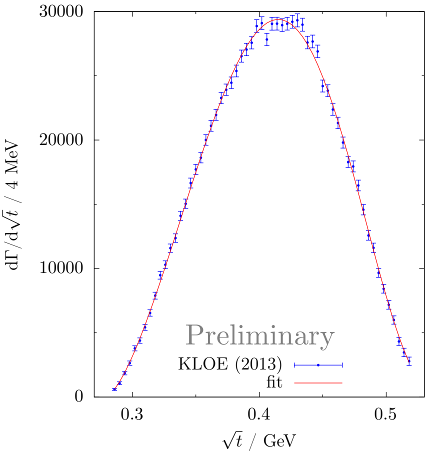
$$\mathcal{F}^{\eta^{(\prime)}\to\pi\pi\gamma}(t,k^2) = \frac{-1}{48\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}x \, \frac{x\sigma_\pi^3(x)(F_\pi^V(x))^*\left[f_1(t,x)\Omega(x) + f_1(x,t)\Omega(t)\right]}{x - k^2 - i\epsilon}$$

- with $f_1(t,k^2) = \left[P(t) + \frac{t^2}{\pi}\int \mathrm{d}\tau \, \frac{\hat{G}(\tau,k^2)\sigma_\pi(\tau)t_1^1(\tau)}{\tau^2(\tau - x - i\epsilon)\Omega(\tau)}\right]\Omega(t) + \hat{G}(t,k^2)$
- fix subtraction constants by fitting to data for real photon case ($k^2 = 0$)

η: KLOE 2013; η': BESIII 2018

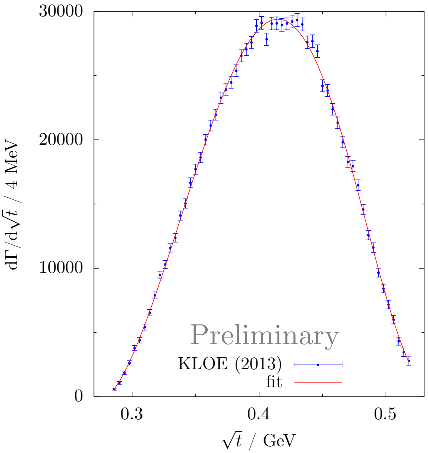


Fixing subtraction constants by $\eta^{(\prime)} \to \pi\pi\gamma$ fits



for η decay

Fixing subtraction constants by $\eta^{(\prime)} \to \pi\pi\gamma$ fits

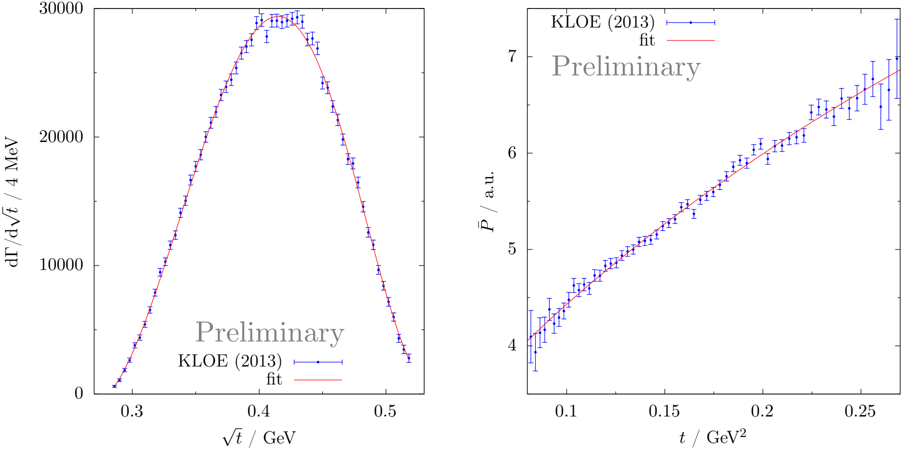


$$\bar{P}(t) = \left[\frac{1}{\Gamma_0|\Omega(t)|^2}\frac{\mathrm{d}\Gamma(\eta^{(\prime)}\to\pi^+\pi^-\gamma)}{\mathrm{d}t}\right]^{1/2}$$

- divide by phase space and Omnès function

for η decay

Fixing subtraction constants by $\eta^{(\prime)} \to \pi\pi\gamma$ fits



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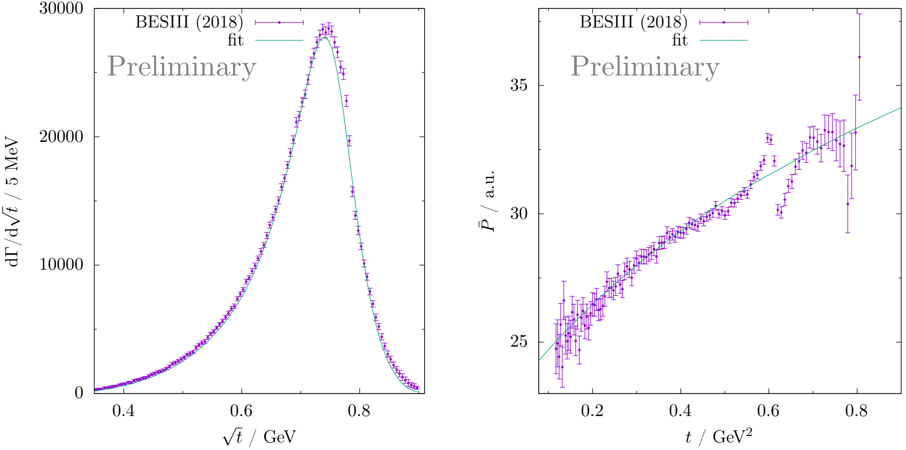
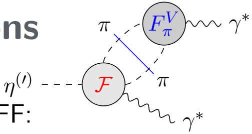


for η' decay

Doubly-virtual representations



- application of yet another dispersion relation:
  - doubly-virtual representation of isovector part of TFF:

$$F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{96\pi^2}\int_{4M_\pi^2}^{\infty} \mathrm{d}y\, \frac{y\sigma_\pi^3(y)(F_\pi^V(y))^*\mathcal{F}^{\eta^{(\prime)}\to\pi\pi\gamma}(x, q_2^2)}{y - q_1^2 - i\epsilon}$$

- f_1 from solution of inhomogeneous Omnès problem
  - encodes non-factorization in TFF
  - subtraction constants fixed by fit to $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ data

KLOE 2013, BESIII 2018

- unsubtracted dispersion relation (in q_i^2) necessary for desired (singly-virtual) space-like asymptotics

$$\lim_{Q^2\to\infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma}(-Q^2, 0) = \text{const.}$$

- Bose symmetry respected $F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_2^2, q_1^2)$

Putting the pieces together

Construct TFF from four ingredients:

$$F_{\eta^{(\prime)}\gamma^*\gamma^*} = F^{(I=1)}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{(I=0)}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{\text{eff}}_{\eta^{(\prime)}\gamma^*\gamma^*} + F^{\text{asym}}_{\eta^{(\prime)}\gamma^*\gamma^*}$$

Isospin 1

- Dispersive piece: offers low-energy description
- reproduces low-energy cuts and singularities
  - additionally, left-hand cut contribution

Isospin 0

- Description of narrow low-energy resonances

Effective Pole Term

- Parameterize higher intermediate states
- Full saturation of normalization sum rule
- Describe high-energy singly-virtual data

pQCD piece

- Induces leading-twist behavior of TFF ($\mathcal{O}(1/Q^2)$ asymptotics)

Effective Pole Term

Isoscalar piece

- channel dominated by narrow resonances $V \in \{\omega, \phi\}$

$$F^{(I=0)}_{\eta^{(\prime)}}(q_1^2, q_2^2) = F^{\text{exp}}_{\eta^{(\prime)}}(0,0) \sum_V \frac{w_{\eta^{(\prime)} V\gamma} M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

Effective Pole Term

- Normalization of isovector + isoscalar part should be compared to exp. det. decay width $\Gamma(\eta^{(\prime)} \to \gamma\gamma)$ ($\Leftrightarrow F^{\text{exp}}_{\eta^{(\prime)}}(0,0)$)

$\Rightarrow$ Introduce an effective pole term

$$F^{\text{eff}}_{\eta^{(\prime)}\gamma^*\gamma^*}(q_1^2, q_2^2) = g_{\text{eff}} F^{\text{exp}}_{\eta^{(\prime)}}(0,0) \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)}$$

- g_{eff} fixed by fulfilling norm
- M_{eff} fixed by fit to singly-virtual space-like data above $5\,\text{GeV}^2$

BaBar; CELLO; CLEO; L3

Doubly-virtual asymptotics

- Make use of analogy to pion transition form factor

  Hoferichter, Hoid, Kubis, Leupold, Schneider 2018

- From pQCD considerations:

$$F^{\mathrm{asym}}_{\eta^{(\prime)}\gamma^\star\gamma^\star}(q_1^2, q_2^2) = F^{(\prime)}_{\mathrm{asym}} \int_{s_m}^{\infty} \mathrm{d}x \, \frac{q_1^2 q_2^2}{(x - q_1^2)^2 (x - q_2^2)^2}$$

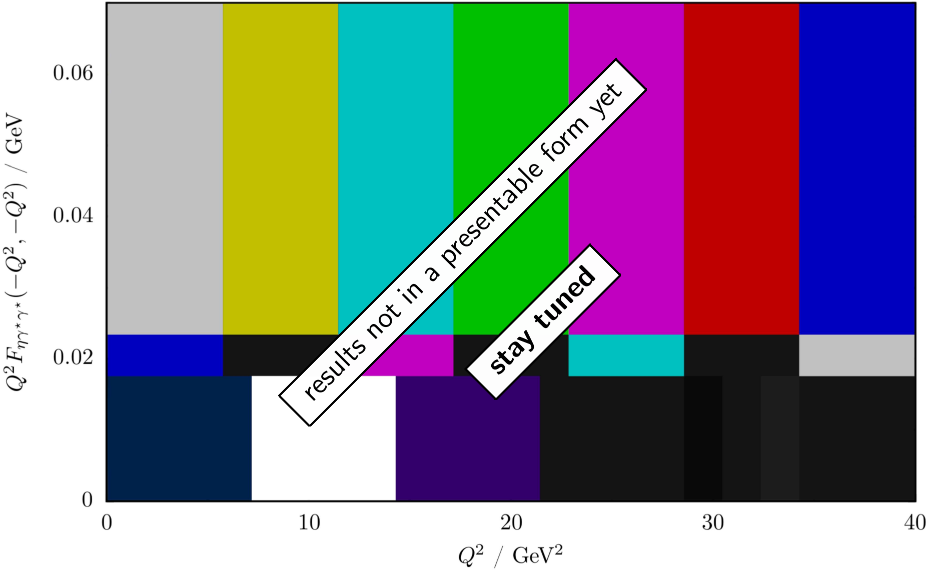
- Does not contribute to singly-virtual kinematics
- Complications due to η/η'-mixing Escribano et al. 2014-2016
- However $F^{(\prime)}_{\mathrm{asym}}$ fixed by Brodsky-Lepage-like limit:

  Brodsky, Lepage 1979-1981

$$\lim_{Q^2\to\infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, 0) = F^{(\prime)}_{\mathrm{asym}} \qquad (= 2F_\pi \text{ for } \pi^0)$$

- By adding pQCD piece TFF fulfills operator product expansion constraint: Nesterenko, Radyushkin 1983; Novikov et al. 1984; Manohar 1990

$$\lim_{Q^2\to\infty} Q^2 F_{\eta^{(\prime)}\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{1}{3} F^{(\prime)}_{\mathrm{asym}} \qquad \left(= \frac{2F_\pi}{3} \text{ for } \pi^0\right)$$



0.06
0.04
0.02
0
$Q^2 F_{\eta\gamma^\star\gamma^\star}(-Q^2, -Q^2)$ / GeV
0
10
20
30
40
Q^2 / GeV^2
results not in a presentable form yet
stay tuned

Summary and outlook

- Construction of doubly-virtual representation of η and η' TFFs with dispersive methods
- starting point: decay into $2(\pi^+\pi^-)$
  - successive application of dispersion-relations
  - non-factorizing effects included via a_2-exchange model
- solved the inhomogeneous Omnès problem at hand
  - several hurdles to overcome: deformation of integration path into complex plane
- fixed subtraction constants via fits to $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ data
- construction of final representation analogous with additional components to dispersive π^0 TFF construction
- aim: extraction of $\eta^{(\prime)}$ pole contribution to HLbL in $(g-2)_\mu$ with fully controlled uncertainties
  - possible application to determination of rare decay rates $\eta^{(\prime)} \to \ell^+\ell^-$ within SM