Pion transition form factor and $\pi^0 \to e^+e^-$

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Outline

Introduction & motivation

Pion transition form factor

$\pi^0 \to e^+e^-$

Conclusion and outlook

Introduction & motivation

Neutral pion main decay modes: Zyla et al., 2020

Decay modes	$\pi^0 \to \gamma\gamma$	$\pi^0 \to e^+e^-\gamma$	$\pi^0 \to e^+e^-e^+e^-$	$\pi^0 \to e^+e^-$
Branching ratios	98.823%	1.174%	3.34×10^{-5}	6.46×10^{-8}

talk by Kampf

- $\pi^0 \to \gamma\gamma$: Adler–Bell–Jackiw anomaly
- $\pi^0 \to e^+e^-\gamma$: Dalitz decay
- $\pi^0 \to e^+e^-e^+e^-$: double Dalitz decay
- $\pi^0 \to e^+e^-$: rare decay, loop- and helicity-suppressed

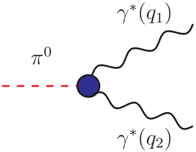
$\Rightarrow$ All listed decays described by pion transition form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

Introduction & motivation

Pion transition form factor (TFF) $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$:

- Defined by the matrix element of two electromagnetic currents $j_\mu(x)$

$$i\int \mathrm{d}^4x\, e^{iq_1\cdot x}\, \langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(q_1+q_2)\rangle$$
$$= \epsilon_{\mu\nu\rho\sigma}\, q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$



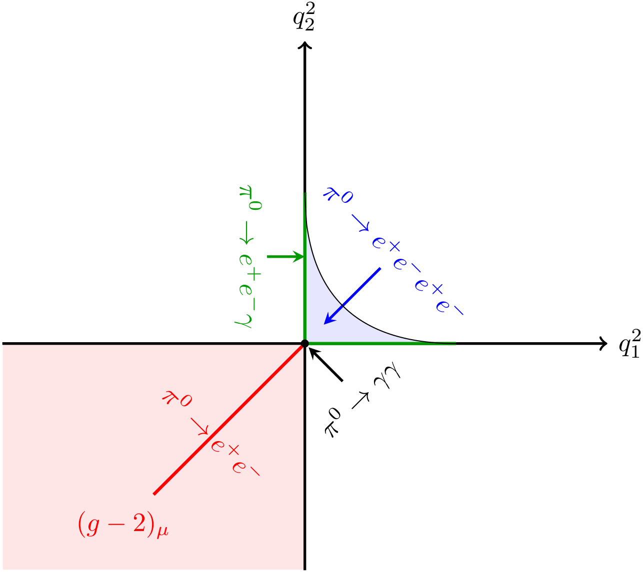
- Normalization fixed by the Adler–Bell–Jackiw anomaly:

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}$$

$F_\pi = 92.28(10)\,\mathrm{MeV}$: pion decay constant — Zyla et al., 2020

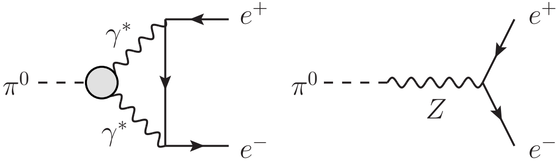
Introduction & motivation

$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ kinematic regions:



Introduction & motivation

Leading Standard-Model contributions to $\pi^0 \to e^+e^-$:



Introduction & motivation

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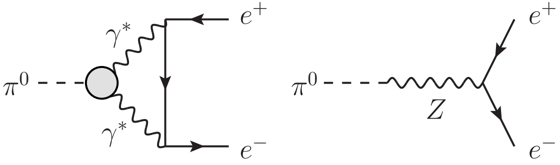


- QED loop contribution dominates

$$\frac{\text{BR}[\pi^0 \to e^+e^-]}{\text{BR}[\pi^0 \to \gamma\gamma]} \sim \left(\frac{\alpha}{\pi}\right)^2 \frac{m_e^2}{M_{\pi^0}^2} \pi^2 \log^2 \frac{m_e}{M_{\pi^0}} \sim \mathcal{O}(10^{-8})$$

Drell, 1959

Introduction & motivation

Experiment: Abouzaid et al., 2006

$$\mathrm{BR}[\pi^0 \to e^+e^-(\gamma), x_D > 0.95]\big|_{\mathsf{KTeV}} = 6.44(25)(22) \times 10^{-8}$$

$$x_D = \frac{m^2_{e^+e^-}}{M^2_{\pi^0}} = 1 - 2\frac{E_\gamma}{M_{\pi^0}}$$

- With old radiative corrections Bergström, 1982

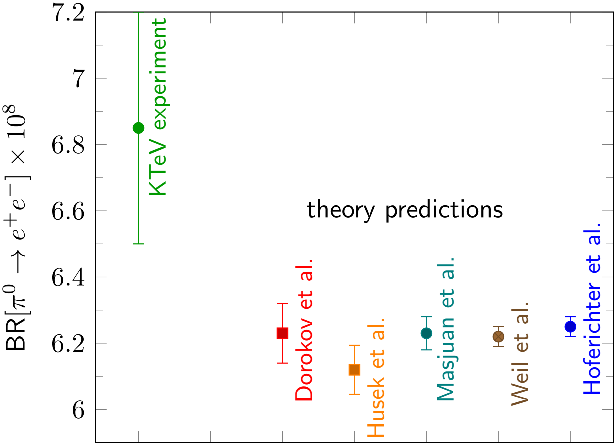
$$\mathrm{BR}[\pi^0 \to e^+e^-]\big|_{\mathsf{KTeV}} = 7.48(29)(25) \times 10^{-8}$$

- With reexamined radiative corrections Vaško, Novotný, 2011, Husek et al., 2014

$$\boxed{\mathrm{BR}[\pi^0 \to e^+e^-]\big|_{\mathsf{KTeV}} = 6.85(27)(23) \times 10^{-8}}$$

Introduction & motivation

Experiment vs theory:



- $\sim 2\sigma$ discrepancy between experiment and theory

Pion transition form factor

We build the form factor double-spectral representation:

Hoferichter et al., 2018

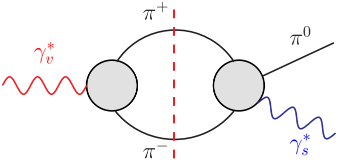
$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F^{\mathsf{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\mathsf{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

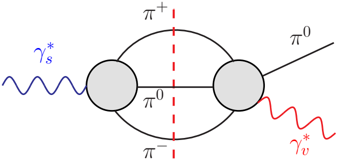
- Reconstructed from the lowest-lying singularities 2π and 3π
- Fulfills the asymptotic constraints at $\mathcal{O}\big(1/Q^2\big)$
- Suitable for muon $g_\mu - 2$ & $\pi^0 \to e^+e^-$ loop-integral evaluation

Pion transition form factor

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$





Pion transition form factor

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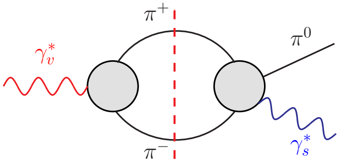
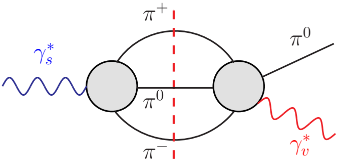


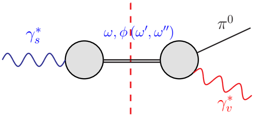


Isovector photon: 2 pions

- $\gamma_v^* \to \pi^+\pi^- \to \gamma_s^*\pi^0$
- $\mathrm{disc} \propto$ pion vector form factor $\times\ \gamma_s^* \to 3\pi$ amplitude

Isoscalar photon: 3 pions

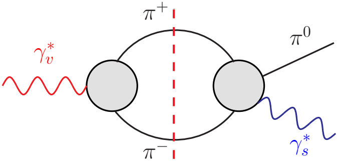
- $\gamma_s^* \to \pi^+\pi^-\pi^0 \to \gamma_v^*\pi^0$
- Dominated by resonances ω, ϕ, ω', & ω''

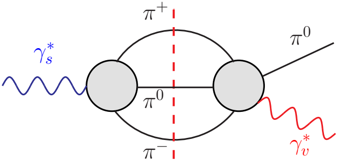


Pion transition form factor

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$





Building blocks of the dispersive treatment:

- Pion vector form factor $F_\pi^V(s)$
- Partial wave amplitude $f_1(s, q^2)$ for the $\gamma_s^*(q) \to \pi^+\pi^-\pi^0$ reaction

Pion transition form factor

Dispersive form factor:

$$F^{\mathsf{disp}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{s_{\mathsf{iv}}} \mathsf{d}x \int_{s_{\mathsf{thr}}}^{s_{\mathsf{is}}} \frac{\rho^{\mathsf{disp}}(x,y)\,\mathsf{d}y}{\left(x+Q_1^2\right)\left(y+Q_2^2\right)},$$

$$\rho^{\mathsf{disp}}(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}}\mathsf{Im}\left[\left(F_\pi^V(x)\right)^* f_1(x,y)\right] + [x\leftrightarrow y]$$

$F_\pi^V(s)$ pion vector form factor, $f_1(s,q^2) = \gamma_s^*(q) \to 3\pi$ P-wave amplitude

Pion transition form factor

Effective pole term:

$F^{\text{disp}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ fulfills the chiral anomaly $F_{\pi\gamma\gamma}$ by around 90%

⇒ Introduce an effective pole term

$$F^{\text{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi} \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)}$$

g_{eff} fixed by fulfilling the chiral anomaly

$g_{\text{eff}} \sim 10\%$, ⇒ small

M_{eff} fit to singly-virtual data excluding BaBar above $5\,\text{GeV}^2$ Gronberg et al., 1998, Aubert et al., 2009, Uehara et al., 2012

$M_{\text{eff}} \sim 1.5$–$2\,\text{GeV}$, ⇒ reasonable

Pion transition form factor

Asymptotically, $F_{\pi^0\gamma^*\gamma^*}$ should fulfill — Brodsky, Lepage, 1979-1981

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3}\int_0^1 \mathrm{d}x \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2} + \mathcal{O}\left(\frac{1}{q_i^4}\right),$$

Pion distribution amplitude $\phi_\pi(x) = 6x(1-x) + \cdots$

Brodsky–Lepage (BL) limit:

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma}(-Q^2, 0) = \frac{2\,F_\pi}{Q^2}$$

Operator product expansion (OPE): — Nesterenko, Radyushkin, 1983, Novikov et al., 1984, Manohar, 1990

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{2\,F_\pi}{3\,Q^2}$$

Pion transition form factor

Rewrite the asymptotic form into a double-spectral representation:

$$F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty}\int_{s_\mathrm{m}}^{\infty} \mathrm{d}x\mathrm{d}y \frac{\rho^{\mathsf{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)},$$

$$\rho^{\mathsf{asym}}(x,y) = -2\pi^2 F_\pi xy\delta''(x-y)$$

This defines the asymptotic contribution:

$$F^{\mathsf{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = 2F_\pi \int_{s_\mathrm{m}}^{\infty} \mathrm{d}x \frac{q_1^2 q_2^2}{(x-q_1^2)^2(x-q_2^2)^2}$$

- Does not contribute for the singly-virtual kinematics
- Restores the asympotics for singly-/doubly-virtual kinematics

Pion transition form factor

Uncertainty estimates:

- The uncertainty in $F_{\pi\gamma\gamma}$ at 0.8% from PrimEx-II — Larin et al., 2020
  - Varying the coupling g_{eff} — talk by Gan
- Dispersive uncertainties estimated by
  - Varying the cutoffs between 1.8 and $2.5\,\text{GeV}$
  - Different $\pi\pi$ phase shifts — Caprini et al., 2012, García-Martín et al., 2011, Schneider et al., 2012
  - Different representations of $F_\pi^V(s)$
  - Different conformal polynomial fits to $e^+e^- \to 3\pi$

Pion transition form factor

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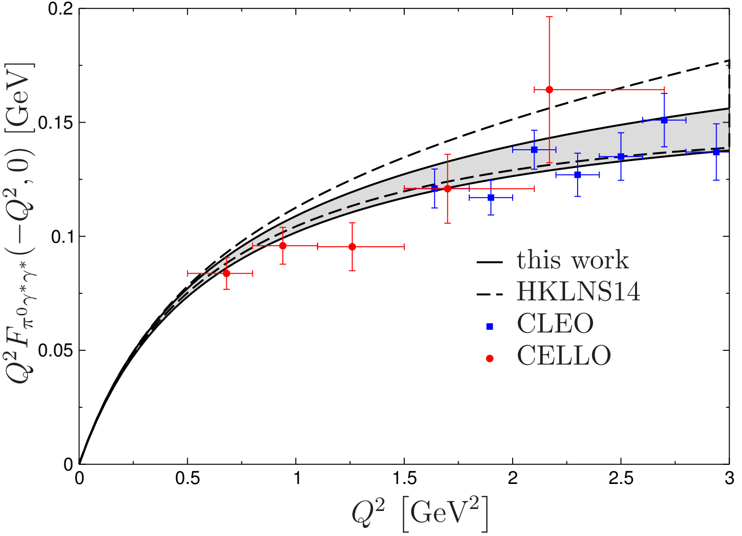
  - Different representations of $F_\pi^V(s)$
  - Different conformal polynomial fits to $e^+e^- \to 3\pi$
- BL limit uncertainty by $^{+20}_{-10}\%$ — Aubert et al., 2009, Uehara et al., 2012
  - Varying the mass parameter M_{eff}
  - Completely covers 3σ band
- Asymptotic part $s_{\text{m}} = 1.7(3)\,\text{GeV}^2$ — Khodjamirian, 1999, Agaev et al., 2011, Mikhailov et al., 2016
  - Expected from light-cone sum rules

Pion transition form factor

Singly-virtual space-like transition form factor in $Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2, 0)$:



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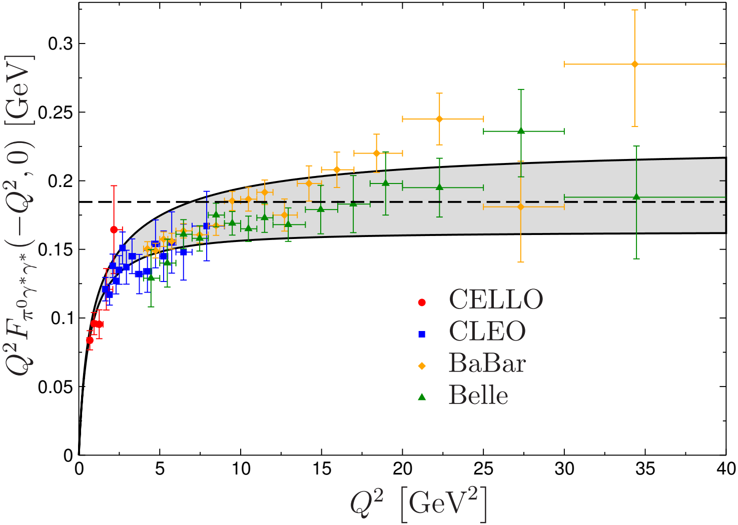
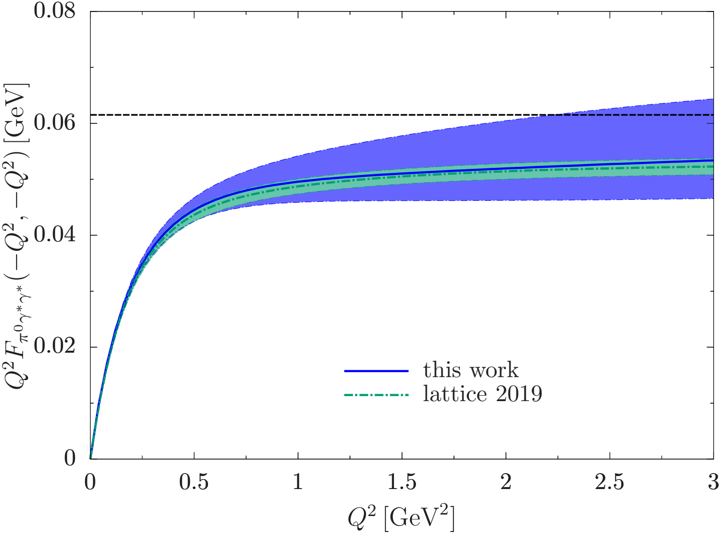


Pion transition form factor

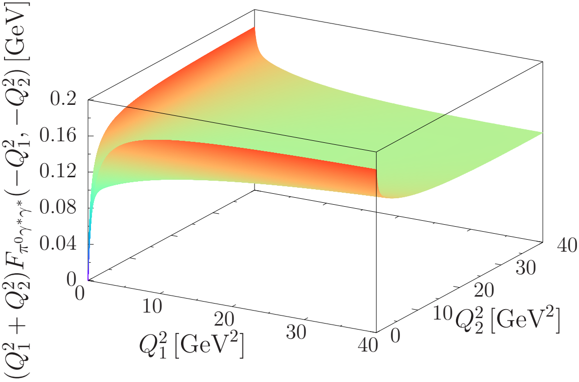
Diagonal form factor in $Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$ in comparison to lattice:

Gérardin et al., 2019



Pion transition form factor

$(Q_1^2+Q_2^2)F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$ as a function of Q_1^2 and Q_2^2:



- $1/Q_i^2$ behavior in the entire domain of space-like virtualities

  ⇒ Hard to obtain in resonance models

Pion transition form factor

Connections to $g_\mu - 2$:

- Hadronic light-by-light scattering — Hoferichter et al., 2018

$$a_\mu^{\pi^0\text{-pole}} = 63.0(0.9)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\left({}^{2.2}_{1.4}\right)_{\text{BL}}(0.6)_{\text{asym}} \times 10^{-11} = 63.0^{+2.7}_{-2.1} \times 10^{-11}$$

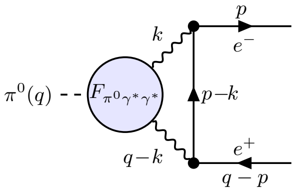
- Hadronic vacuum polarization

$a_\mu^{3\pi} \times 10^{10}$ below $1.8\,\text{GeV}$ in WP 2020		
Davier et al., 2019	Keshavarzi et al., 2019	Hoferichter et al., 2019
46.21 ± 1.45	46.63 ± 0.94	46.16 ± 0.82

  - $\pi^0\gamma$ — Hoid et al., 2020

$\pi^0 \to e^+e^-$

Reduced amplitude $\mathcal{A}(q^2)$:



$$\frac{\mathsf{BR}[\pi^0 \to e^+e^-]}{\mathsf{BR}[\pi^0 \to \gamma\gamma]} = 2\sigma_e(q^2)\left(\frac{\alpha}{\pi}\right)^2 \frac{m_e^2}{M_{\pi^0}^2}\left|\mathcal{A}(q^2)\right|^2$$

$$q^2 = M_{\pi^0}^2,\ \sigma_e(q^2) = \sqrt{1 - \frac{4m_e^2}{q^2}}$$

$$\mathcal{A}(q^2) = \frac{2i}{\pi^2 q^2}\int \mathsf{d}^4k \frac{q^2k^2 - (q\cdot k)^2}{k^2(q-k)^2[(p-k)^2 - m_e^2]} \times \tilde{F}_{\pi^0\gamma^*\gamma^*}\big(k^2,(q-k)^2\big)$$

$\tilde{F}_{\pi^0\gamma^*\gamma^*}\big(k^2,(q-k)^2\big) = F_{\pi^0\gamma^*\gamma^*}\big(k^2,(q-k)^2\big)/F_{\pi\gamma\gamma}$, normalized pion TFF

$\pi^0 \to e^+e^-$

Imaginary part from the $\gamma\gamma$ cut:

$$\mathrm{Im}\,\mathcal{A}(q^2) = \frac{\pi}{2\sigma_e(q^2)} \log\left[\frac{1-\sigma_e(q^2)}{1+\sigma_e(q^2)}\right] = -17.52$$

$\mathrm{Re}\,\mathcal{A}(q^2)$? $\Rightarrow$ need to perform the integral with the form factor

- $\mathcal{A}^{\text{eff}}$: standard reduction Passarino, Veltman, 1979, 't Hooft, Veltman, 1979
- $\mathcal{A}^{\text{asym}}$: integration by parts Chetyrkin, Tkachov, 1981
- $\mathcal{A}^{\text{disp}}$: integration-kernel method Masjuan, Sánchez-Puertas, 2016

$\pi^0 \to e^+e^-$

For the dispersive part we can write Masjuan, Sánchez-Puertas, 2016

$$\mathcal{A}^{\text{disp}}(q^2) = \frac{2}{\pi^2} \int_{4M_\pi^2}^{s_{\text{iv}}} \text{d}x \int_{s_{\text{thr}}}^{s_{\text{is}}} \text{d}y \frac{\tilde{\rho}^{\text{disp}}(x,y)}{xy} K(x,y)$$

Integration kernel

$$K(x,y) = \frac{2i}{\pi^2 q^2} \int \text{d}^4k \frac{q^2k^2 - (q\cdot k)^2}{k^2(q-k)^2[(p-k)^2 - m_e^2]} \times \frac{xy}{(k^2-x)[(q-k)^2-y]}$$

- Checked with several standard techniques

$\pi^0 \to e^+e^-$

Full decomposition:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{\gamma^*\gamma^*} = 10.16 = 9.18_{\text{disp}} + 1.08_{\text{eff}} - 0.10_{\text{asym}}$$

Dispersion relation in q^2: Dorokhov et al., 2007

$$\text{Re}\,\mathcal{A}(q^2) = \mathcal{A}(0) + \frac{1}{\sigma_e(q^2)}\left[\text{Li}_2\big[-y_e(q^2)\big] + \frac{1}{4}\log^2\left[y_e(q^2)\right] + \frac{\pi^2}{12}\right]$$

$$\mathcal{A}(0) \approx 3\log\frac{m_e}{\mu} - \frac{3}{2}\left[\int_0^{\mu^2} \mathrm{d}t\,\frac{\tilde{F}_{\pi^0\gamma^*\gamma^*}(-t,-t)-1}{t} + \int_{\mu^2}^{\infty} \mathrm{d}t\,\frac{\tilde{F}_{\pi^0\gamma^*\gamma^*}(-t,-t)}{t}\right] - \frac{5}{4}$$

$$\text{Re}\,\mathcal{A}(q^2)|_{\gamma^*\gamma^*} \approx 10.00(4)_{\text{disp}}(8)_{\text{BL}}(2)_{\text{asym}}$$

$\Rightarrow$ Not enough precision!

$\pi^0 \to e^+e^-$

Long-range contribution from the final representation:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{\gamma^*\gamma^*} = 10.16(5)_{\text{disp}}(8)_{\text{BL}}(2)_{\text{asym}}$$

Z-boson contribution:

$$\text{Re}\,\mathcal{A}(q^2)\big|_{Z} = -\frac{F_\pi G_F}{\sqrt{2}\,\alpha^2 F_{\pi\gamma\gamma}} = -0.05(0)$$

Final Standard-Model prediction:

$$\boxed{\begin{aligned}\text{Re}\,\mathcal{A}(q^2)\big|_{\text{SM}} &= 10.11(10)\\ \text{BR}[\pi^0 \to e^+e^-]\big|_{\text{SM}} &= 6.25(3)\times 10^{-8}\end{aligned}}$$

- Fully controlled uncertainty estimates
- Mild 1.8σ tension with experiment

$\pi^0 \to e^+e^-$

Comparison to lattice result: Christ et al., 2022

$$\mathrm{Re}\,\mathcal{A} = 18.60(1.19)_{\mathrm{stat}}(1.04)_{\mathrm{sys}}\mathrm{eV}$$

$$\mathrm{Im}\,\mathcal{A} = 32.59(1.50)_{\mathsf{stat}}\ (1.65)_{\mathsf{sys}}\ \mathrm{eV}$$

$$\frac{\mathrm{Re}\,\mathcal{A}}{\mathrm{Im}\,\mathcal{A}} = 0.571(10)_{\mathsf{stat}}\ (4)_{\mathsf{sys}}$$

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = 6.60(0.61)_{\mathsf{stat}}\ (0.67)_{\mathsf{syst}}\ \mathrm{eV}$$

Using experimental decay width $\Gamma\left(\pi^0 \to \gamma\gamma\right)$:

$$\mathrm{Re}\,\mathcal{A} = 20.2(0.4)_{\mathsf{stat}}\ (0.1)_{\mathrm{syst}}(0.2)_{\mathrm{expt}}\mathrm{eV}$$

$$\mathsf{BR}[\pi^0 \to e^+e^-] = 6.22(5)_{\mathsf{stat}}\ (2)_{\mathrm{syst}} \times 10^{-8}$$

$\pi^0 \to e^+e^-$

Beyond the Standard-Model scenerios with light axial-vector Z' or pseudoscalar a:

$$\mathcal{L}_{\mathsf{BSM}} = \sum_{f=e,u,d} \bar{f}\Big(c_A^f \gamma^\mu \gamma_5 Z'_\mu + c_P^f i\gamma_5 a\Big) f$$

Limit from $\pi^0 \to e^+e^-$:

$$-\frac{(c_A^u - c_A^d)c_A^e}{M_{Z'}^2} = (-280)^{+160}_{-150}\ \mathsf{TeV}^{-2}, \quad \frac{(c_P^u - c_P^d)c_P^e}{m_a^2 - q^2} = (-0.108)^{+0.062}_{-0.057}\ \mathsf{TeV}^{-2}$$

Contributions to electron $g-2$:

$$a_e^A = -\frac{(c_A^e)^2 m_e^2}{4\pi^2 M_{Z'}^2}\int_0^1 \mathsf{d}x\, \frac{2x^3 m_e^2 + x(1-x)(4-x)M_{Z'}^2}{m_e^2 x^2 + M_{Z'}^2(1-x)}$$

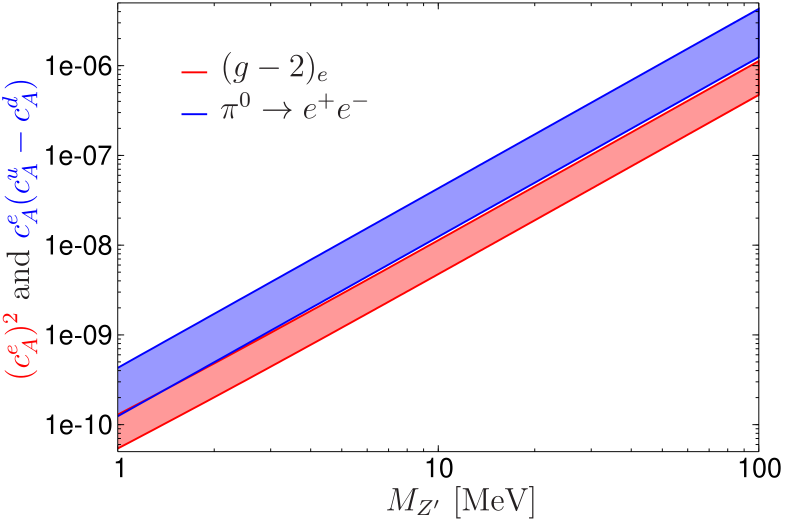
$$a_e^P = -\frac{(c_P^e)^2 m_e^2}{8\pi^2}\int_0^1 \mathsf{d}x\, \frac{x^3}{m_e^2 x^2 + m_a^2(1-x)}$$

Constraints from Cs atom interferometry ($a_e^A \& a_e^P < 0$): Parker et al., 2018

$$\Delta a_e[\mathsf{Cs}] = a_e^{\mathsf{exp}} - a_e^{\mathsf{SM}}[\mathsf{Cs}] = -0.88(36)\times 10^{-12}$$

$\pi^0 \to e^+e^-$

Constraints for axial-vector Z':

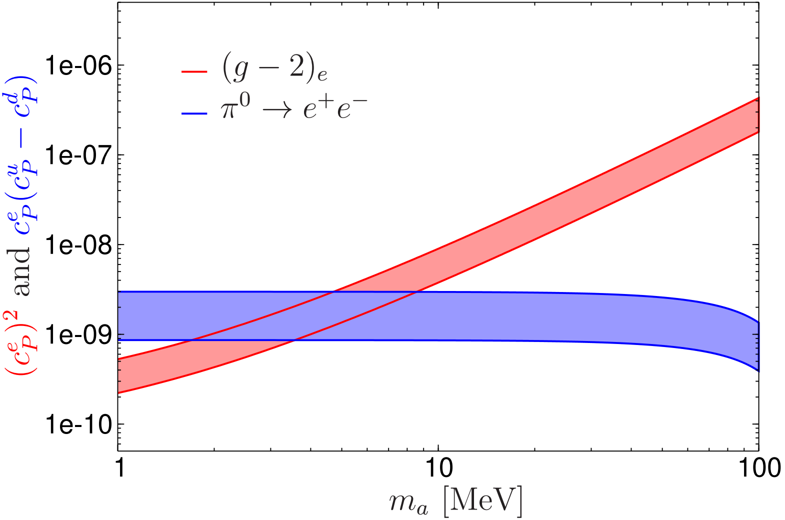


- Revises the previously preferred region

Parker et al., 2018

$\pi^0 \to e^+e^-$

Constraints for axion-like particle a:



Conclusion and outlook

- Pion transition form factor
- $\pi^0 \to e^+e^-$ decay
  - Reduced amplitude with form factor representation
  - Standard-Model prediction with 0.5% precision
  - Constraints on physics beyond the Standard Model
- Experiment and lattice progress — NA62, Christ et al., 2022
- Similar analysis for other pseudoscalar decays

Much obliged for your attention!

"Rare is the union of beauty and purity."

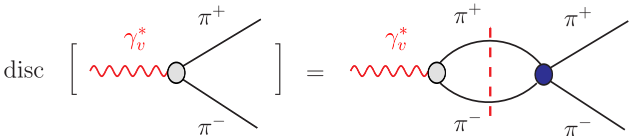
Juvenal

Backup

Theory predictions	BR$[\pi^0 \to e^+e^-] \times 10^8$
Dorokhov et al.	$6.23(9)$
Husek et al.	$6.12(7)$
Masjuan et al.	$6.23(5)$
Weil et al.	$6.22(3)$
This work	$6.25(3)$

Backup

Pion vector form factor $F_\pi^V(s)$:



disc $F_\pi^V(s) = 2i\,\mathrm{Im}\,F_\pi^V(s) = 2iF_\pi^V(s)\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}\theta(s-4M_\pi^2)$

Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

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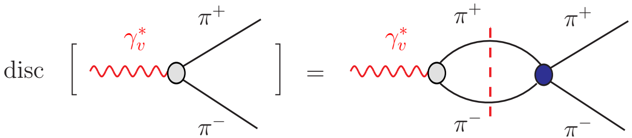

$\text{disc}\, F_\pi^V(s) = 2i\, \text{Im}\, F_\pi^V(s) = 2i F_\pi^V(s) \sin\delta_1^1(s)\, e^{-i\delta_1^1(s)} \theta(s-4M_\pi^2)$

Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

Solution:

$$F_\pi^V(s) = P(s)\Omega(s), \quad \Omega(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty \mathrm{d}s' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

- $\Omega(s)$ is the Omnès function Omnès, 1958
- $P(s)$ polynomial, $P(0) = 1$ from charge conservation
- $\pi\pi$ P-wave phase shift $\delta_1^1(s)$ from Roy equations

Backup

The $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ decay amplitude $\mathcal{F}(s,t,u;q^2)$:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s,t,u;q^2)$$

$q = p_+ + p_- + p_0$; s, t & u are Mandelstam variables

Decompose into single-variable functions:

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

Normalization from the Wess–Zumino–Witten (WZW) anomaly:

$$\mathcal{F}(0,0,0;0) = \frac{1}{4\pi^2 F_\pi^3} \equiv F_{3\pi}$$

$F_\pi = 92.28(10)\,\mathrm{MeV}$: pion decay constant P. Zyla et al., 2020

Backup

Discontinuity equation:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i(\mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2))\theta(s-4M_\pi^2)\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

- $\mathcal{F}(s,q^2)$: right-hand cut
- $\hat{\mathcal{F}}(s,q^2)$: left-hand cut; angular averages of $\mathcal{F}(t,q^2)$ & $\mathcal{F}(u,q^2)$

Backup

A once-subtracted dispersive solution to the discontinuity equation:

Hoferichter et al., 2014

$$\mathcal{F}(s,q^2) = a(q^2)\Omega(s)\Big\{1+\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\mathsf{d}s'\,\frac{\hat{\mathcal{F}}(s',q^2)\sin\delta_1^1(s')}{s'(s'-s)|\Omega(s')|}\Big\}$$

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\mathsf{d}s'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

is the Omnès function

Omnès, 1958

Backup

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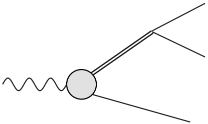
Omnès, 1958

$\hat{\mathcal{F}}(s,q^2)$ absent:

$\mathcal{F}(s,q^2) =$



$\hat{\mathcal{F}}(s,q^2)$ present:

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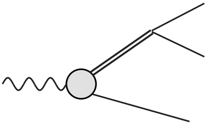
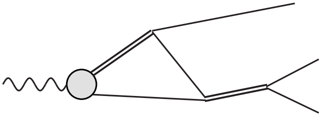


$+$



$+ \cdots$

- Incorporated crossed-channel interactions

Backup

$a(q^2)$ fit to different $e^+e^- \to 3\pi$ cross-section data with parameterization:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im}\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\,\Gamma_V(q^2)}, \qquad V = \omega, \phi, \omega', \omega''$$

$$C_n(q^2) = \sum_{i=1}^{n} c_i \big(z(q^2)^i - z(0)^i\big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

Backup

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$$C_n(q^2) = \sum_{i=1}^{n} c_i \big(z(q^2)^i - z(0)^i\big), \qquad z(q^2) = \frac{\sqrt{s_{\rm inel} - s_1} - \sqrt{s_{\rm inel} - q^2}}{\sqrt{s_{\rm inel} - s_1} + \sqrt{s_{\rm inel} - q^2}}$$

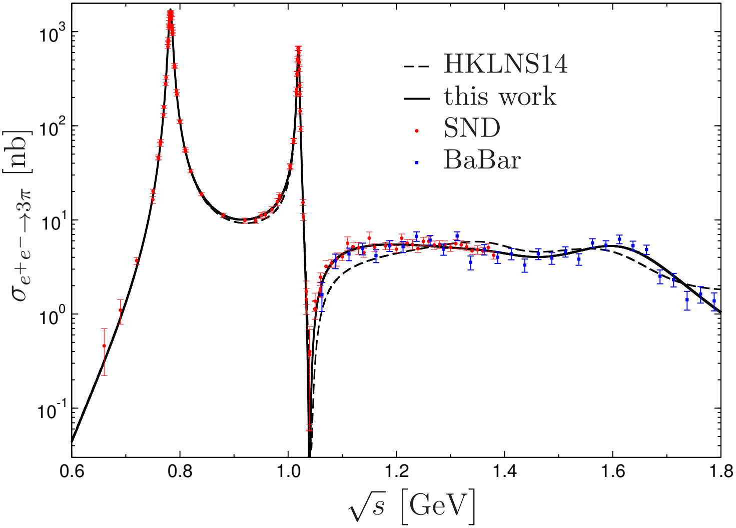
- S-wave cusp eliminated
- Exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\rm thr}}^{\infty} \mathrm{d}s' \frac{\mathrm{Im}\,a(s')}{s'}$$

- Asymptotic behavior of $C_n(q^2)$ controlled

Backup

6 (7) parameters c_ω, c_ϕ, $c_{\omega'}$, $c_{\omega''}$, c_1, c_2 & (c_3) fit to $e^+e^- \to 3\pi$ data:



- Substantially improved above the ϕ peak

Backup

Rewrite the asymptotic form into a double-spectral representation:

$$F^{\text{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \mathrm{d}x\mathrm{d}y\,\frac{\rho^{\text{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)},$$

$$\rho^{\text{asym}}(x,y) = -2\pi^2 F_\pi xy\delta''(x-y)$$

Decomposition of the pion-transition form factor:

$$\begin{aligned}F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) &= \frac{1}{\pi^2}\int_0^{s_{\text{m}}}\mathrm{d}x\int_0^{s_{\text{m}}}\mathrm{d}y\,\frac{\rho^{\text{disp}}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\text{m}}}^\infty\mathrm{d}x\int_{s_{\text{m}}}^\infty\mathrm{d}y\,\frac{\rho^{\text{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)}\\ &+ \frac{1}{\pi^2}\int_0^{s_{\text{m}}}\mathrm{d}x\int_{s_{\text{m}}}^\infty\mathrm{d}y\,\frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\text{m}}}^\infty\mathrm{d}x\int_0^{s_{\text{m}}}\mathrm{d}y\,\frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}\end{aligned}$$

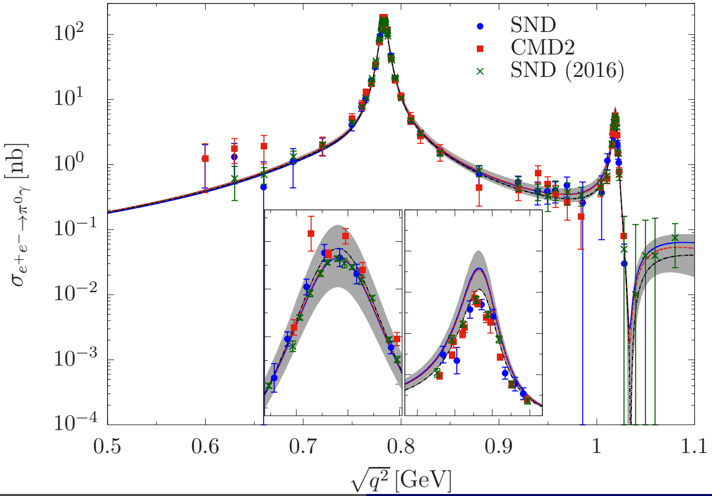
- s_{m}: continuum threshold
- $\rho(x,y)$ not known rigorously, $\rho^{\text{asym}}(x,y)$ applied in mixed regions vanishes

  $\Rightarrow$ All constraints can be fulfilled discarding mixed regions

Backup

Our predictions of the time-like form factor in $e^+e^- \to \pi^0\gamma$:

- Entirely based on the dispersive framework
- Input quantities $F_{\pi\gamma\gamma}$, $F_{3\pi}$, $\delta_1^1(s)$ and $e^+e^- \to 3\pi$ data



Backup

- Large relative discrepancy between direct data-integration works

$a_\mu^{\pi^0\gamma} \times 10^{11}$		
Davier et al., 2019	Keshavarzi et al., 2019	Hoid et al. 2020
$44.1(1.0)_{\leq 1.8\,\text{GeV}}$	$45.8(1.0)_{\leq 1.937\,\text{GeV}}$	$43.8(6)_{\leq 1.35\,\text{GeV}}$

Even lower value $a_\mu^{\pi^0\gamma}|_{\leq 2.0\,\mathrm{GeV}} = 40.0(1.6) \times 10^{-11}$! Jegerlehner, 2017

- Other independent analyses necessary

$\Rightarrow$ Dispersive global fit function fulfilling analyticity, unitarity & QCD constraints

Backup

Experiment $\mathrm{Re}\,\mathcal{A}(q^2)\big|_{\mathsf{KTeV}} = 11.89^{+0.94}_{-1.02}$
Matching to ChPT:

$$\mathrm{Re}\,\mathcal{A}(q^2)|_{\mathsf{ChPT}} = \frac{\mathrm{Li}_2\big[-\frac{1-\sigma_e(q^2)}{1+\sigma_e(q^2)}\big] + \frac{1}{4}\log^2\big[\frac{1-\sigma_e(q^2)}{1+\sigma_e(q^2)}\big] + \frac{\pi^2}{12}}{\sigma_e(q^2)} + 3\log\frac{m_e}{\mu} - \frac{5}{2} + \chi^{(\mathrm{r})}(\mu)$$

Low-energy constant $\chi^{(\mathrm{r})}(\mu = 0.77\,\mathrm{GeV}) = 2.69(10)$

Backup

This form factor $\tilde{F}(-Q^2)$ defines a single-variable function that is closely related to the input required for a space-like evaluation of the loop integral