Light pseudoscalar transition form factors from lattice QCD and the muon $g-2$

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Based on 2305.04570 [hep-lat]

Precision tests of fundamental physics with light mesons
Trento - June 14, 2023

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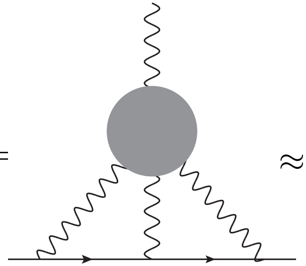
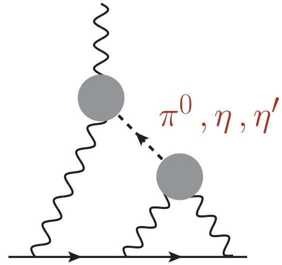
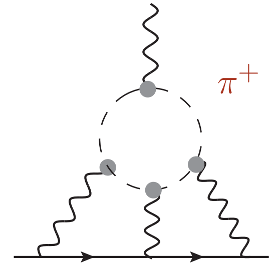
CPT

Motivation : hadronic light-by-light scattering in the muon $g-2$

$a_\mu^{\mathrm{HLbL}} =$  $\approx$ 

$+$ 

$+ \quad \dots$

Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
HLbL (data-driven) '20	92 ± 19
LO HVP	6931 ± 40
HLbL (lattice) Mainz '22	109.6 ± 15.9
HLbL (lattice) RBC '23	124.7 ± 15.2

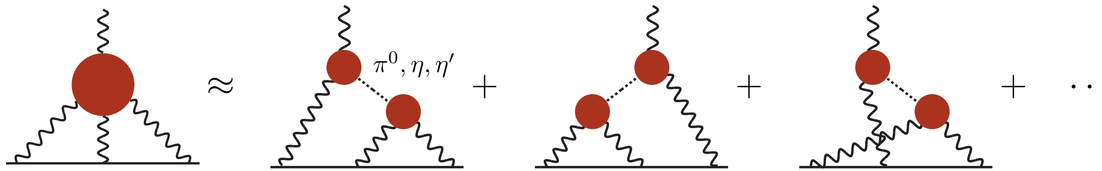
- results from the 2020 White Paper
- target precision : <10%

Two approaches on the lattice :

- direct lattice calculation

  [Mainz '22] & [RBC/UKQCD '23]

- π^0, η, η' : accessible on the lattice



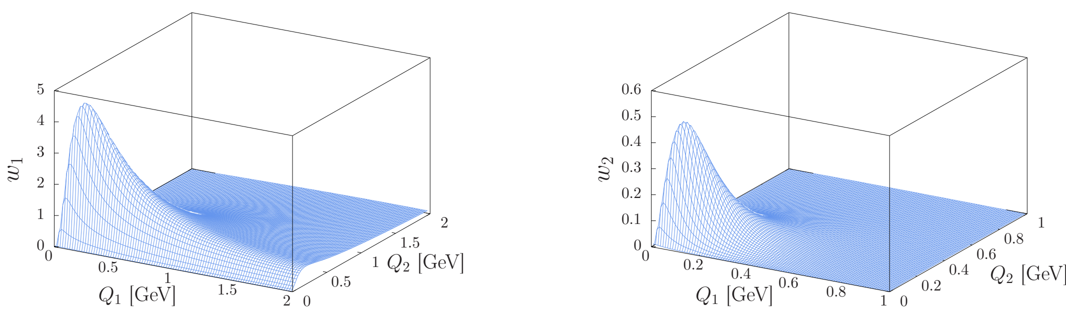
[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) \; + $$
$$w_2(Q_1,Q_2,\tau)\, \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\, \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

→ Product of one single-virtual and one double-virtual transition form factors

→ $w_{1,2}(Q_1,Q_2,\tau)$ are model-independent weight functions

→ The weight functions are **concentrated at small momenta below 1** GeV



↪ Hadronic input : TFF for arbitrary spacelike virtualities in the momentum range $[0-3]$ GeV^2

- **pion transition form factor**

  → first lattice calculation done in collaboration with the Mainz group [Phys.Rev.D 100 (2019) 3]

$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.7 \pm 3.6) \times 10^{-11} \quad [6\%]$$

$$\Gamma(\pi^0 \to \gamma\gamma) = 7.17 \pm 0.49 \ \mathrm{eV}$$

  → new calculation (completely independent)

  → warm-up with staggered quarks before moving to η and η' TFFs

- **η and η' transition form factors**

  → first lattice calculation at the physical point

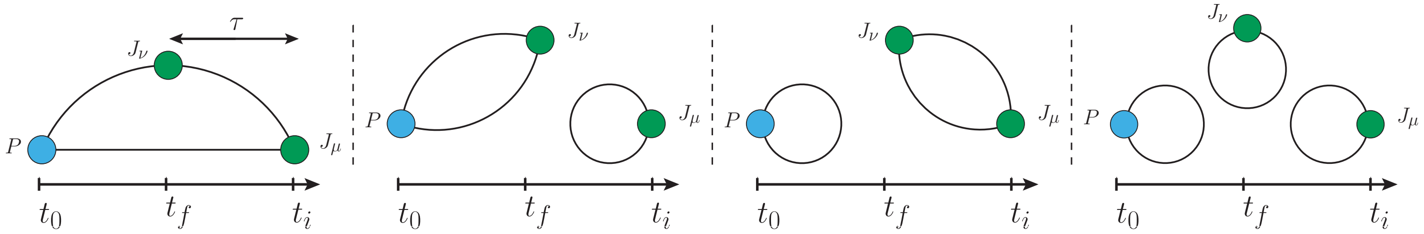
  → $(g-2)_\mu$ contribution from Canterbury approximants [PRD 95, 054026 (2017)] / Dyson-Schwinger equations

$$a_\mu^{\mathrm{HLbL};\eta} \approx 15 \times 10^{-11}$$

$$a_\mu^{\mathrm{HLbL};\eta'} \approx 14 \times 10^{-11}$$

  → much more difficult : $\eta - \eta'$ mixing, noisy quark-disconnected contributions

  → 25-30% on $a_\mu^{\mathrm{HLbL};\eta+\eta'}$ precision would suffice to reach 10%



- **Diagram 1** : contributes to all meson TFFs.

- **Diagram 2** : contributes to all, only disconnected diagram for the pion

  → we work in the isospin limit of QCD ⇒ single pseudoscalar loop vanishes

  → numerically small : O(1%) [Mainz '19]

- **Diagram 3 and 4** : only contribute to the η and η'

  → fourth diagram expected to be small : SU(3)-suppressed vector loops

  → third diagram is large, noisy and enters with an opposite sign : challenging to compute.

Calculation based on Budapest-Marseille-Wuppertal (BMW) gauge ensembles

- Goldstone pion/kaon are tuned to their physical pion/kaon masses

  → $N_f = 2+1+1$ dynamical staggered fermions with four steps of stout smearing

- up to 6 lattice spacings from 0.13 fm down to 0.064 fm

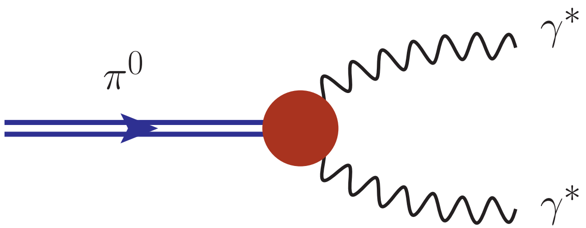
- different physical volumes from 3 to 6 fm

  → pion TFF : based on large 6 fm ensembles only

  → smaller boxes also used for the η and η' to boost statistics

All the details are given in 2305.04570 [hep-lat]

The pion transition form factor



In Minkowski space-time :

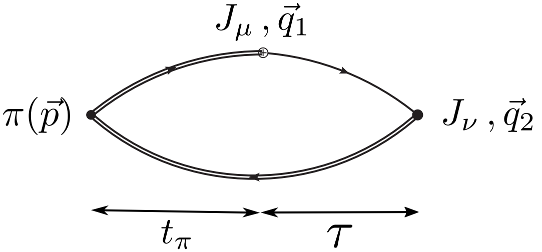
$$M_{\mu\nu}(q_1^2,q_2^2) = i\int \mathrm{d}^4x\, e^{iq_1x}\, \langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2,q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \ldots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M^E_{\mu\nu}(q_1^2,q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1\tau}\int \mathrm{d}^3x\, e^{-i\vec{q}_1\vec{x}}\, \langle 0|T\left\{J_\mu(\vec{x},\tau)J_\nu(\vec{0},0)\right\}|\pi(p)\rangle = \int_{-\infty}^{\infty}\mathrm{d}\tau\, \widetilde{A}^{(\pi^0)}_{\mu\nu}(\tau)\, e^{\omega_1\tau}$$

- Kinematics : $q_1 = (\omega_1, \vec{q}_1)$
- $|\vec{q}_1|^2 = (2\pi/L)^2|\vec{n}|^2$, $|\vec{n}|^2 = 1,2,3,4,5,\ldots$



We can extract $\widetilde{A}^{(\pi^0)}_{\mu\nu}(\tau)$ from an Euclidean three-point correlation function :

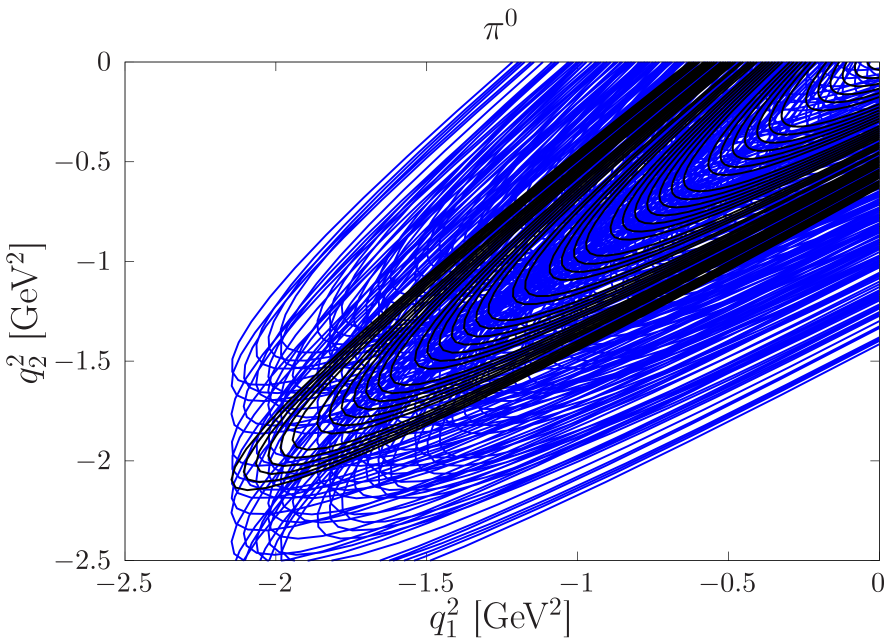
$$C^{(3)}_{\mu\nu}(\tau,t_\pi;\vec{p},\vec{q}_1,\vec{q}_2) = \sum_{\vec{x},\vec{z}} \left\langle T\left\{J_\nu(\vec{0},t_f)J_\mu(\vec{z},t_i)P(\vec{x},t_0)\right\}\right\rangle e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1\vec{z}}$$

Photons virtualities (pion rest frame) :

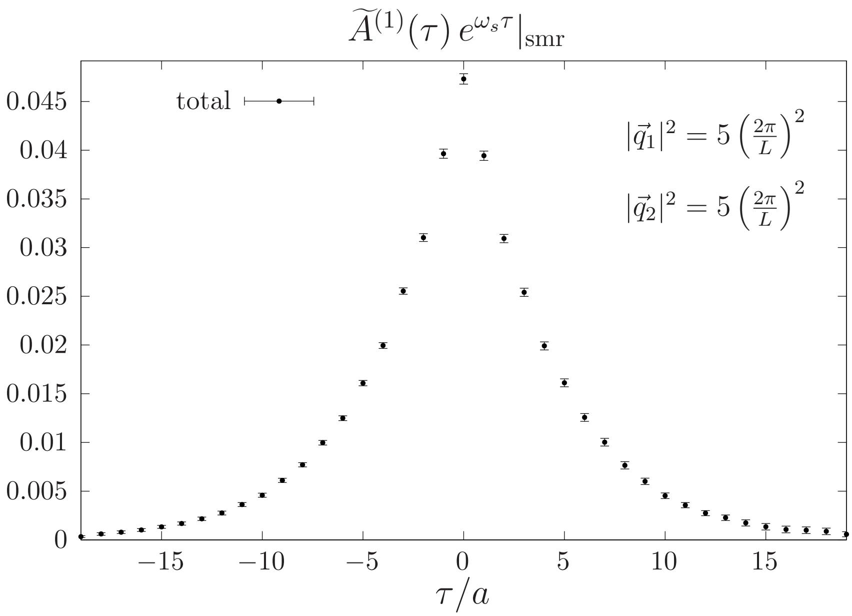
$$q_1^2 = {\omega_1}^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

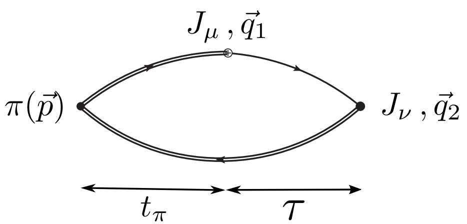
$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter



$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau} \qquad \widetilde{A}_{\mu\nu}^{(\pi)}(\tau) = \lim_{t_P \to +\infty} \frac{2E_\pi}{Z_\pi} e^{E_\pi(t_f - t_0)} C_{\mu\nu}^{(3)}(\tau, t_P)$$





On the lattice :

- Discrete sum over lattice points :

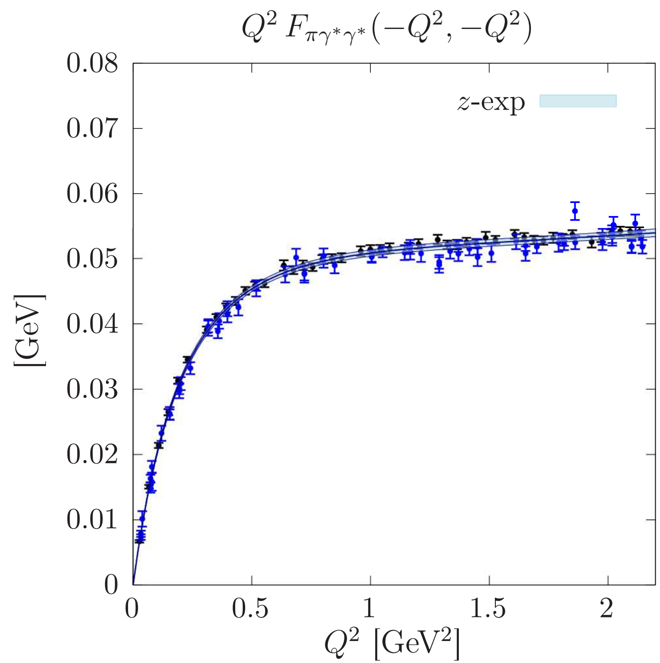
$$\int \mathrm{d}\tau \to a \sum_\tau$$

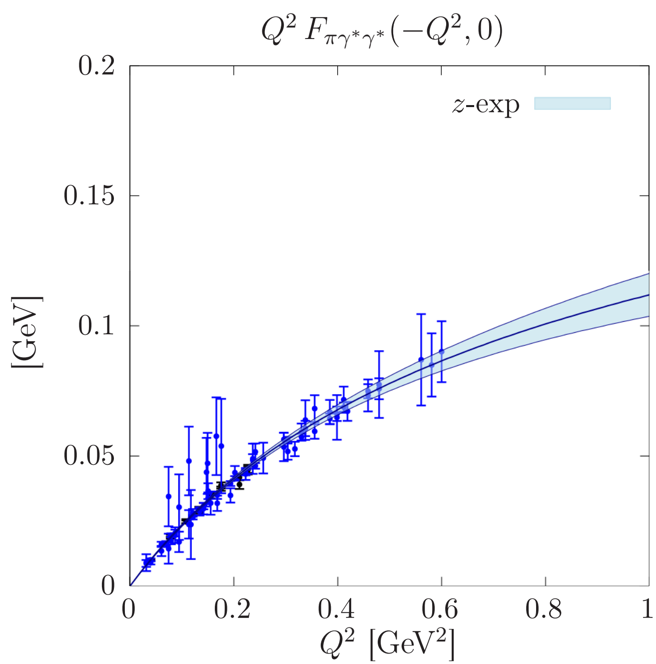
- Finite size of the box :

$$|\tau| \leq \tau_{\mathrm{max}} \neq \infty$$

- The small contribution $\tau > \tau_{\mathrm{max}}$ is evaluated assuming a VMD (or LMD) parametrization

Single ensemble - $a = 0.0787$ fm - physical pion mass





→ Results at our finest lattice spacing

→ Black/blue points : two different pion frames

→ Reminder : lattice data are not restricted to these kinematics

- Model independent double z-expansion for space-like momenta :

$$\mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\, z_1^n\, z_2^n \quad , \qquad z_i = \frac{\sqrt{t_c + Q_i^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_i^2} + \sqrt{t_c - t_0}}$$

$\rightarrow$ $t_c = 4m_\pi^2$

$\rightarrow$ t_0 chosen to reduce the maximum value of $|z_i|$ in the range $[0, Q^2_{\mathrm{max}}]$

$\rightarrow$ cut the sum to finite N

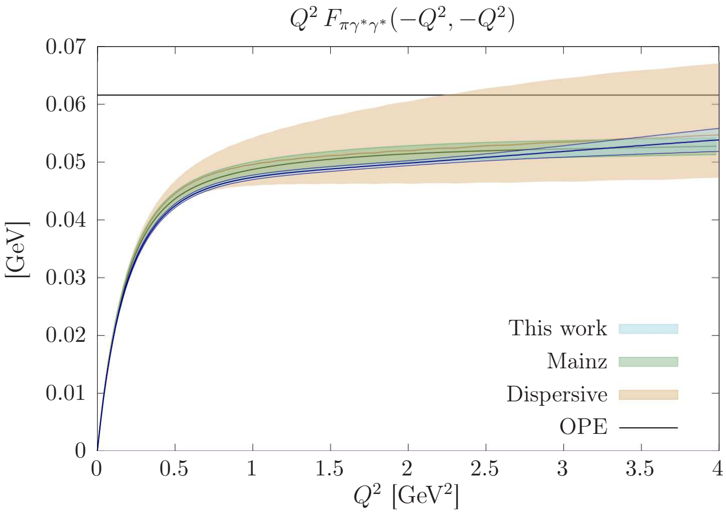
- Analytical + short-distance constraints (Brodsky-Lepage behavior and OPE) :

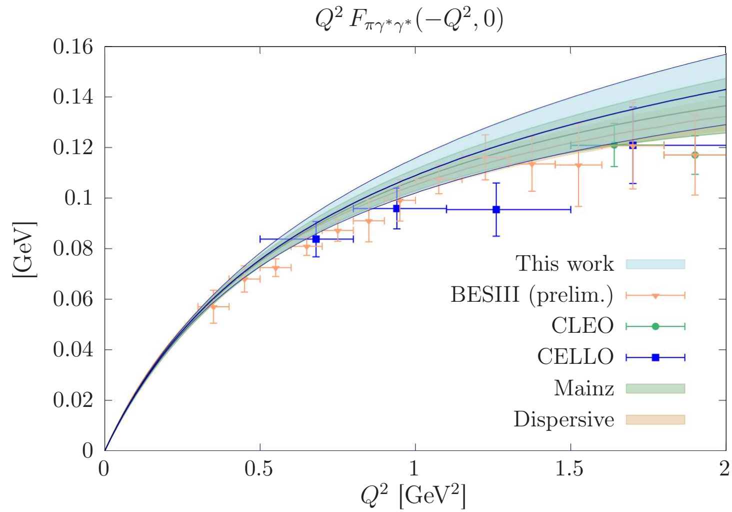
$$P(Q_1^2, Q_2^2)\ \mathcal{F}_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N-1} c_{nm} \left(z_1^n - \frac{(-1)^{N+n}\, n\, z_1^N}{N} \right) \left(z_2^m - \frac{(-1)^{N+m}\, m\, z_2^N}{N} \right)$$

with

$$P(Q_1^2, Q_2^2) = 1 + \frac{Q_1^2 + Q_2^2}{M_V^2} \qquad \Rightarrow \qquad \mathcal{F}_{\pi^0\gamma^*\gamma} \sim \frac{1}{Q^2}$$

- Discretization effects : $\tilde{c}_{mn}(a) = c_{mn} + \gamma_{mn}^{(1)}\, a^2 + \gamma_{mn}^{(2)}\, a^4$





$$\Gamma(\pi^0 \to \gamma\gamma) = 7.11(0.44)_{\text{stat}}(0.21)_{\text{syst}}\ \text{eV}$$
$$a_\mu^{\text{HLbL};\pi^0} = 57.8(1.8)_{\text{stat}}(0.9)_{\text{syst}} \times 10^{-11}$$

→ PrimEx-II : $7.802(52)_{\text{stat}}(105)_{\text{syst}}$ eV

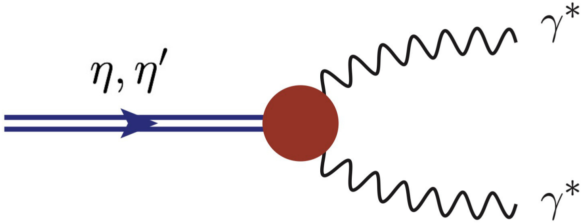
Other lattice estimates :

→ Mainz '19 : $a_\mu^{\text{HLbL};\pi^0} = 59.7(3.6) \times 10^{-11}$

→ ETM (prelim) : $a_\mu^{\text{HLbL};\pi^0} = 55.4(2.1) \times 10^{-11}$ → next talk

Dispersive framework : $a_\mu^{\text{HLbL};\pi^0} = 63.6(2.7) \times 10^{-11}$ [2006.04822] → 1.7 σ larger

The η and η' transition form factors



- In principle, the same techniques work for η

$$C^{(i)}_{\mu\nu}(\tau, t_P) = \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, t_i) J_\nu(\vec{0}, t_f) \mathcal{O}_i(\vec{x}, t_0) \rangle \, e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}$$

with the η_8 and η_0 interpolating operators :

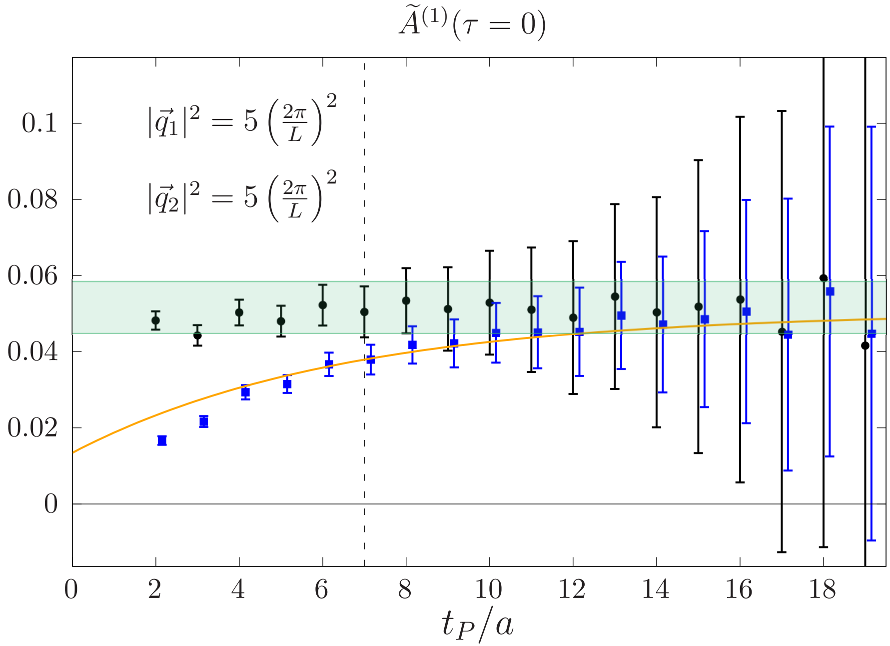
$$\mathcal{O}_8(x) = \frac{1}{\sqrt{6}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right) ,$$
$$\mathcal{O}_0(x) = \frac{1}{\sqrt{3}} \left(\overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right) .$$

For the η, the spectral decomposition reads

$$\begin{aligned} C^{(i)}_{\mu\nu}(\tau, t_P) = & \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_\eta} \langle \eta(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E_\eta (t_0 - t_f)} \\ & + \sum_{\vec{z}} \langle 0 | J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) | \eta'(\vec{p}) \rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_{\eta'}} \langle \eta'(\vec{p}) | \mathcal{O}_i | 0 \rangle e^{E'_\eta (t_0 - t_f)} \\ & + \cdots \end{aligned}$$

$\rightarrow$ η' is just an excited states, its contribution vanishes exponentially with t_P

- $\Delta E = E_{\eta'} - E_\eta \approx 400$ MeV not so large
- Large excited state contribution : large statistical error at large t_P



- does not work for the η'

Solution : Generalized eigenvalue problem to deal with excited states.

Spectral decomposition :

$$C^{(i)}_{\mu\nu}(\tau,t_P) = \sum_{\vec{z}} \langle 0|J_\mu(\vec{z},\tau)J_\nu(\vec{0},0)|\eta(\vec{p})\rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_\eta}\langle \eta(\vec{p})|\mathcal{O}_i|0\rangle e^{E_\eta(t_0-t_f)}$$
$$+\sum_{\vec{z}} \langle 0|J_\mu(\vec{z},\tau)J_\nu(\vec{0},0)|\eta'(\vec{p})\rangle e^{-i\vec{q}_1\cdot\vec{z}} \times \frac{1}{2E_{\eta'}}\langle \eta'(\vec{p})|\mathcal{O}_i|0\rangle e^{E'_\eta(t_0-t_f)}$$

Matrix notation :

$$\begin{pmatrix} C^{(8)}_{\mu\nu} \\ C^{(0)}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} T^{(8)}_\eta & T^{(8)}_{\eta'} \\ T^{(0)}_\eta & T^{(0)}_{\eta'} \end{pmatrix} \begin{pmatrix} \widetilde{A}^{(\eta)}_{\mu\nu} \\ \widetilde{A}^{(\eta')}_{\mu\nu} \end{pmatrix} ,$$

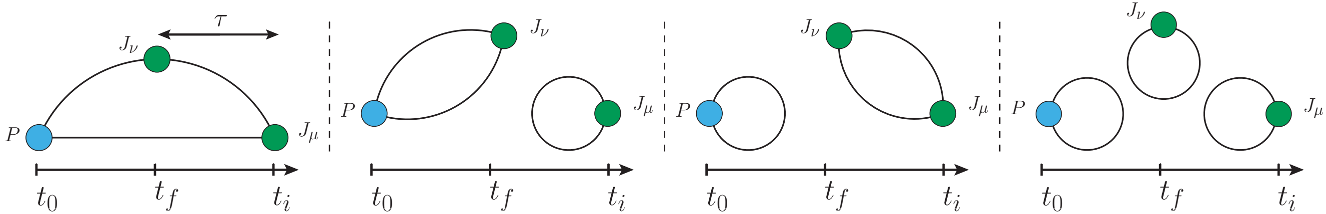
with

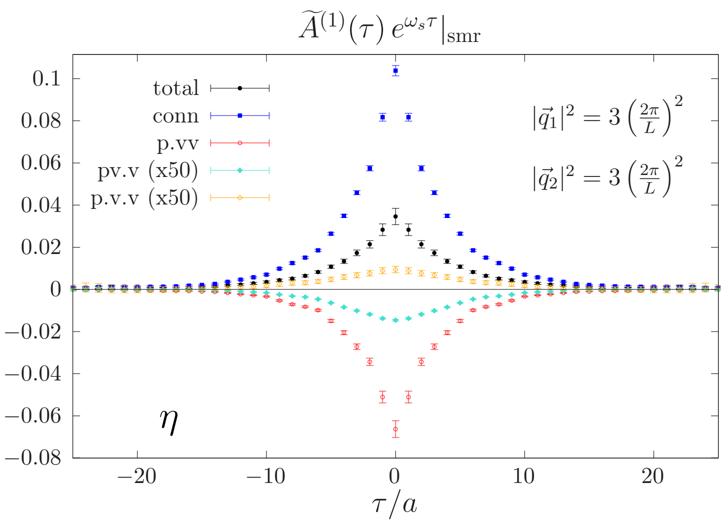
$$T^{(i)}_n = \frac{Z^{(i)}_n}{2E_n} e^{-E_n(t_f-t_0)} , \quad \widetilde{A}^{(n)}_{\mu\nu}(\tau) = \sum_{\vec{z}} \langle 0|J_\mu(\vec{z},\tau)J_\nu(\vec{0},0)|n(\vec{p})\rangle\, e^{-i\vec{q}_1\cdot\vec{z}}$$

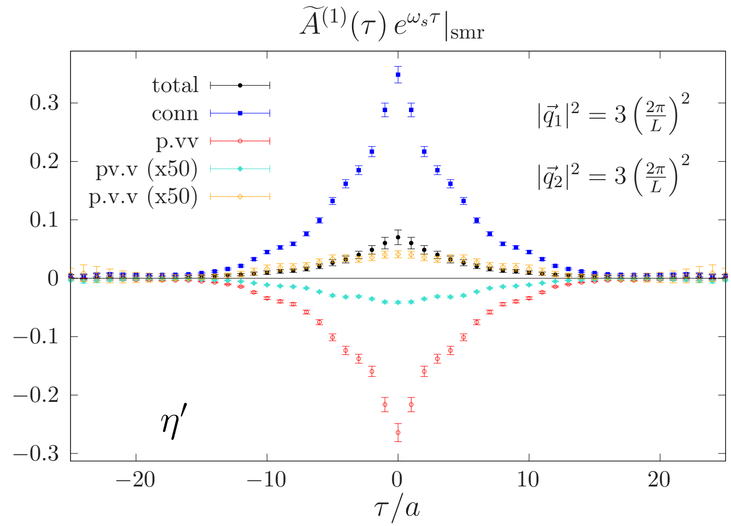
Inverting the system :

$$\widetilde{A}^{(\eta)}_{\mu\nu} = \cos^2\phi_I \, \frac{C^{(8)}_{\mu\nu}}{T^{(8)}_\eta} + \sin^2\phi_I \, \frac{C^{(0)}_{\mu\nu}}{T^{(0)}_\eta}$$
$$\widetilde{A}^{(\eta')}_{\mu\nu} = \sin^2\phi_I \, \frac{C^{(8)}_{\mu\nu}}{T^{(8)}_{\eta'}} + \cos^2\phi_I \, \frac{C^{(0)}_{\mu\nu}}{T^{(0)}_{\eta'}}$$

with $\tan^2\phi_I = -(Z^{(8)}_{\eta'} Z^{(0)}_\eta)/(Z^{(8)}_\eta Z^{(0)}_{\eta'})$ (mixing angle)

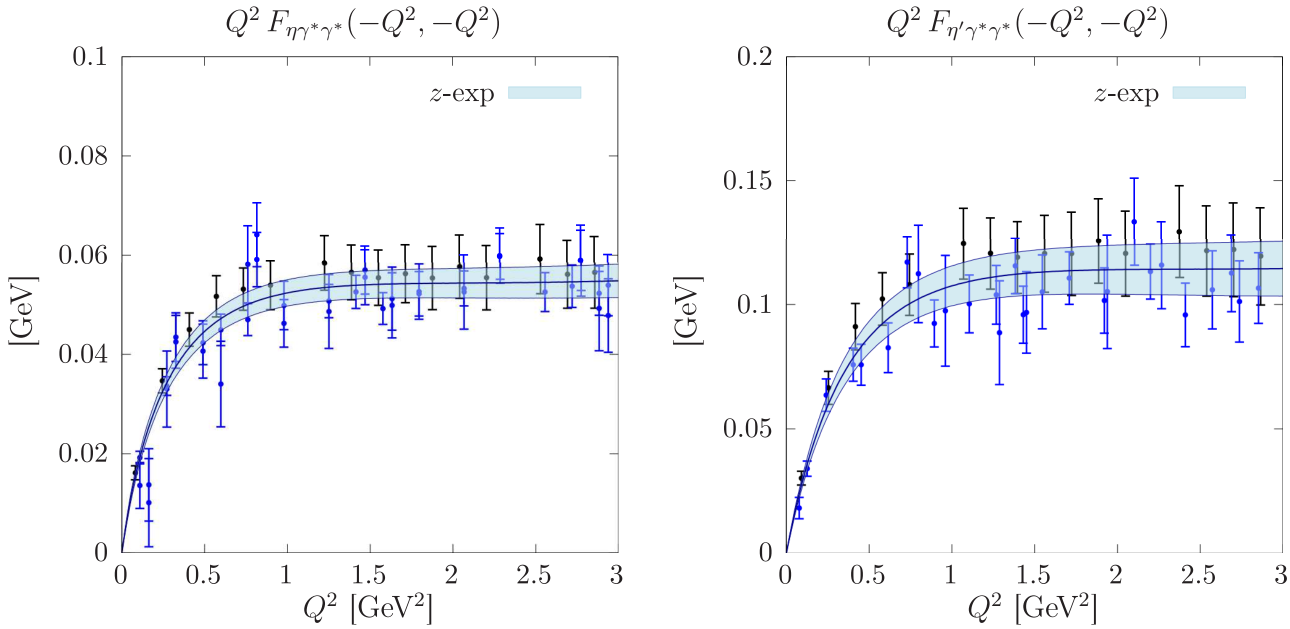






$$\mathcal{F}_{P\gamma\gamma}(q_1^2, q_2^2) \;\propto\; \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau}$$

- Double-virtual TFFs

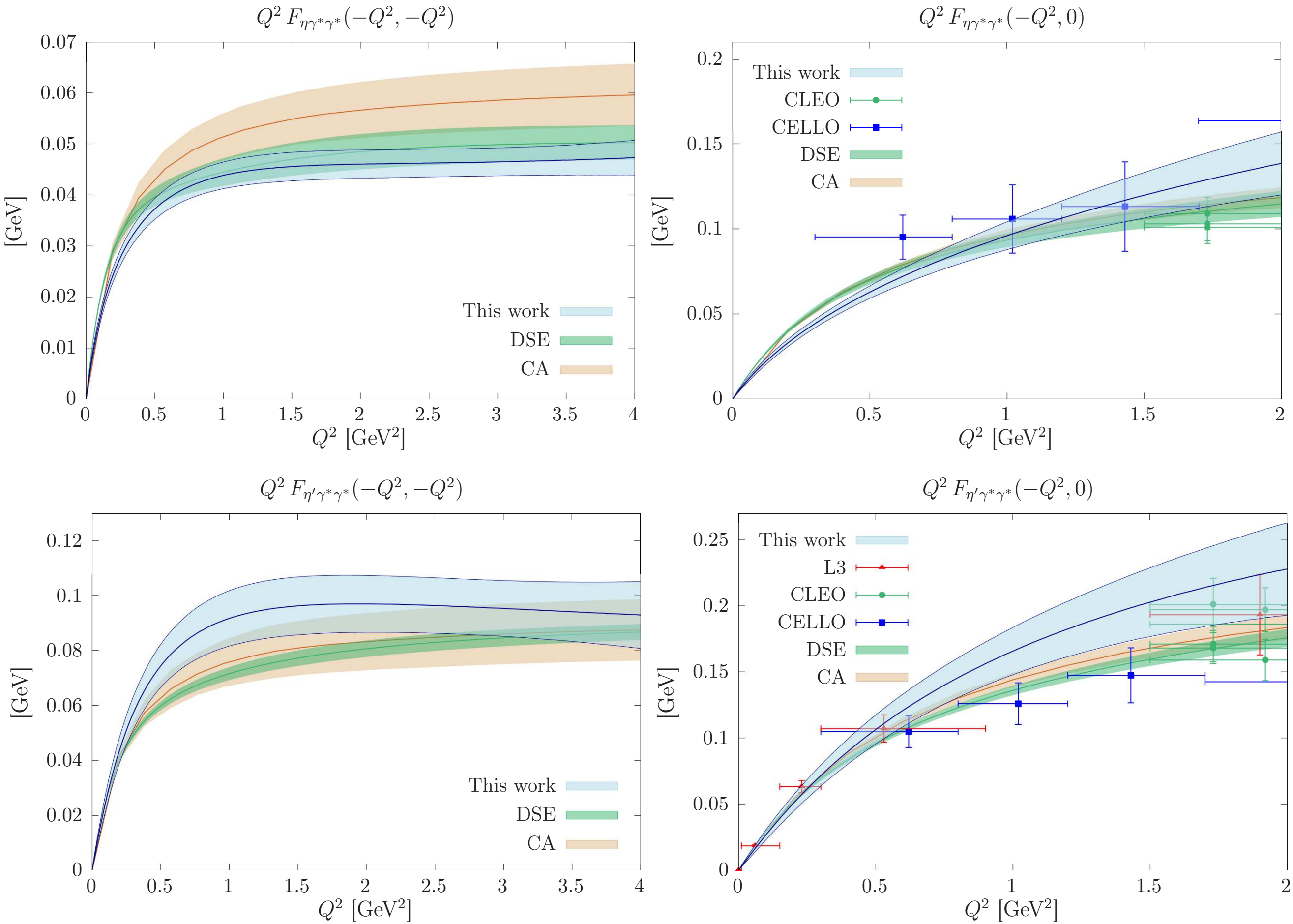


- Next step : continuum extrapolation

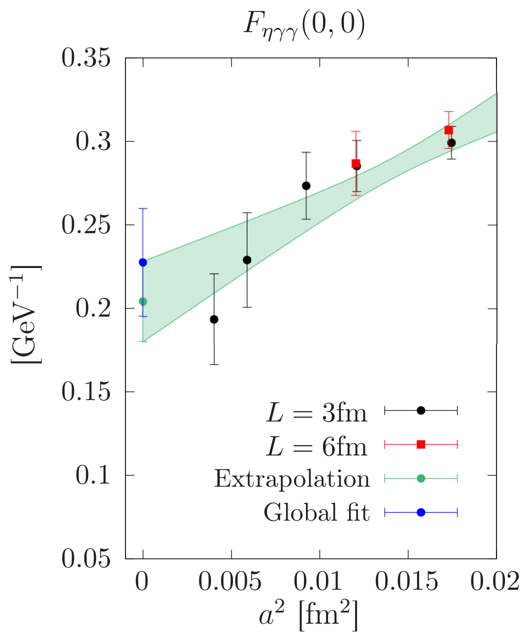
  $\rightarrow$ same strategy as for the pion : z-expansion parametrization

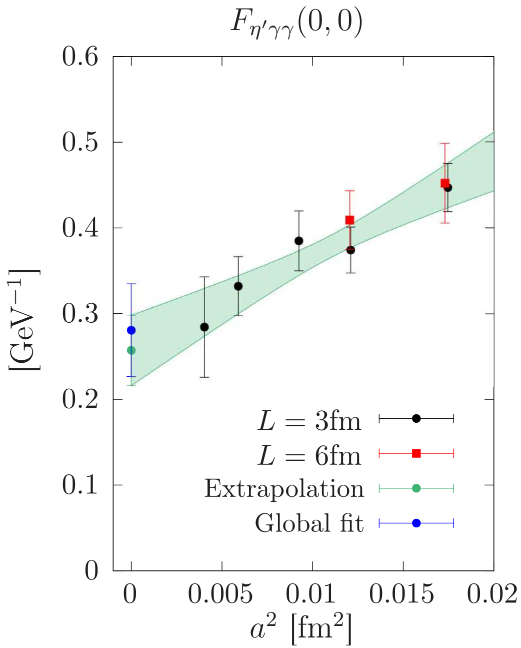
  $\rightarrow$ $\tilde{c}_{mn}(a) = c_{mn} + \gamma_{mn}^{(1)}\, a^2$

Model-independent extrapolation to the physical point



$$\Gamma(P \to \gamma\gamma) = \frac{\pi \alpha^2 m_P^3}{4} \mathcal{F}^2_{P\gamma\gamma}(0,0)$$



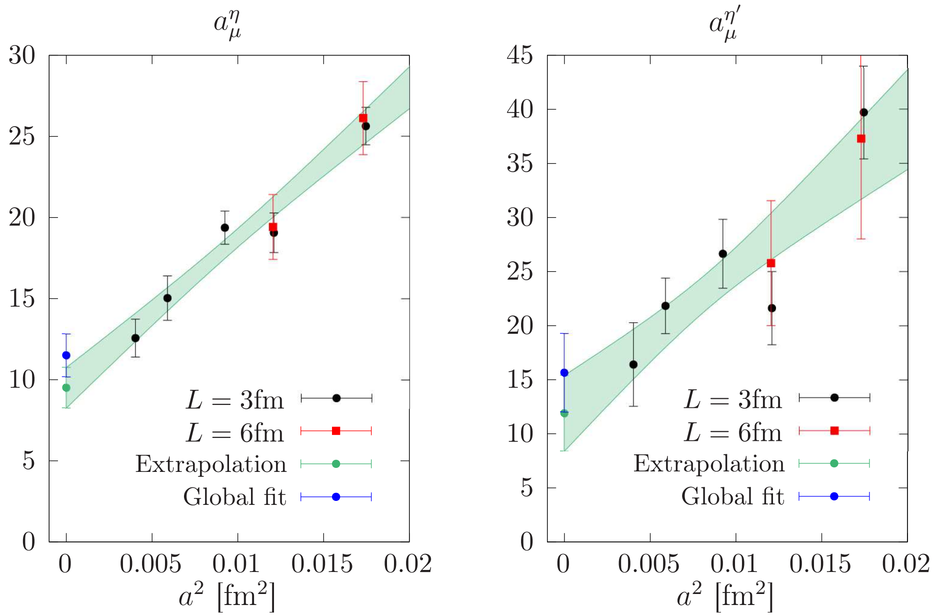


$$\Gamma(\eta \to \gamma\gamma) = 338(94)_{\mathrm{stat}}(35)_{\mathrm{syst}}\ \mathrm{eV}$$
$$\Gamma(\eta' \to \gamma\gamma) = 3.4(1.0)_{\mathrm{stat}}(0.4)_{\mathrm{syst}}\ \mathrm{keV}$$

$\to$ PDG : $\Gamma(\eta \to \gamma\gamma) = 516(18)$ eV

$\to$ PDG : $\Gamma(\eta' \to \gamma\gamma) = 4.28(19)$ keV

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_2^2, 0) \;+$$
$$w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

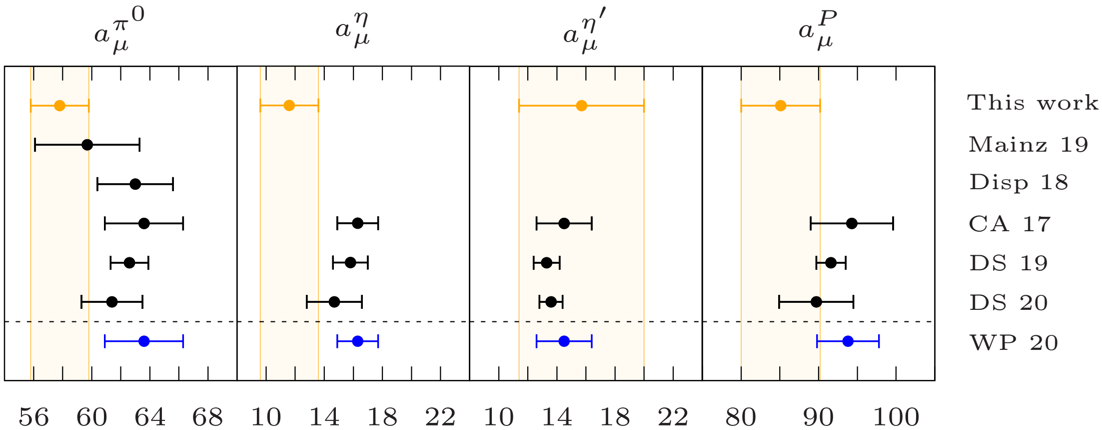


$$a_\mu^{\mathrm{HLbL};\eta} = 11.6(1.6)_{\mathrm{stat}}(0.5)_{\mathrm{syst}}(1.1)_{\mathrm{FSE}} \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL};\eta'} = 15.7(3.9)_{\mathrm{stat}}(1.1)_{\mathrm{syst}}(1.3)_{\mathrm{FSE}} \times 10^{-11}$$

Canterbury approximants [PRD 95, 054026 (2017)]

$\to a_\mu^{\mathrm{HLbL};\eta} = 16.3(1.4) \times 10^{-11}$

$\to a_\mu^{\mathrm{HLbL};\eta'} = 14.5(1.9) \times 10^{-11}$



- Our final estimate

$$a_\mu^{\mathrm{HLbL;ps-poles}} = (85.1 \pm 4.7_{\mathrm{stat}} \pm 2.3) \times 10^{-11} \,.$$

- Pion transition form factor

  $\rightarrow$ good agreement with Mainz'19 and with exp. data for $\mathcal{F}_{\pi^0\gamma\gamma}(Q^2, 0)$

  $\rightarrow$ calculation of $\mathcal{F}_{\pi^0\gamma\gamma}(0, 0)$ might help to reduce the error (+ comparison with PrimEx)

- η-η' transition form factors

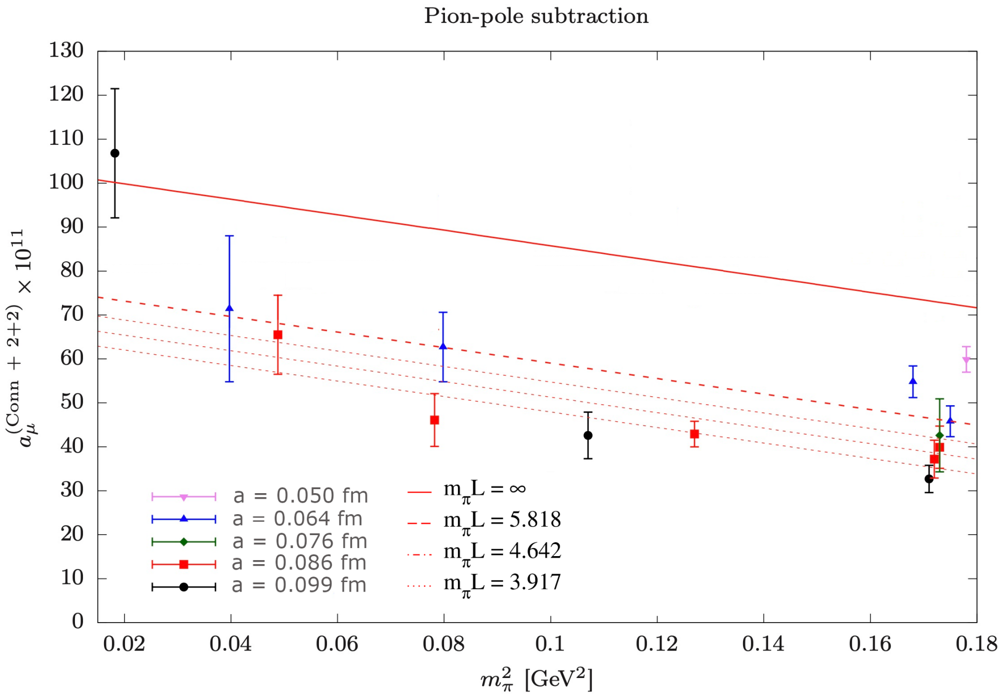
  $\rightarrow$ first ab-initio calculation - error dominated by statistics

  $\rightarrow$ some tensions for the η TFF at very low virtualities

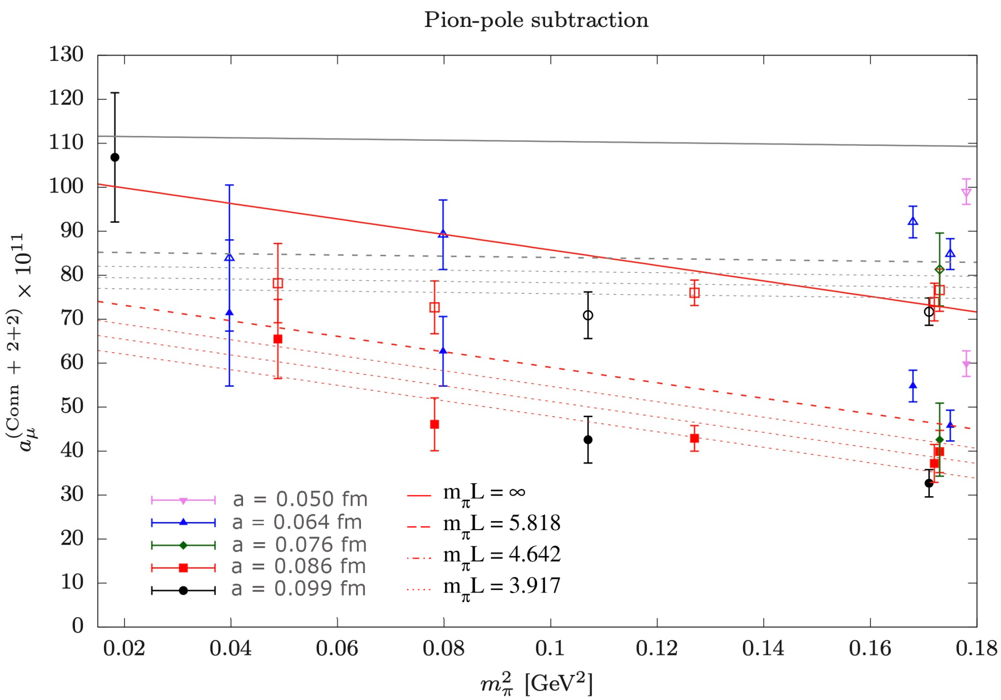
  $\rightarrow$ additional large-volume ensembles at fine lattice spacing would certainly help.

Table – Relative contribution of the pseudoscalar-pole contributions $a_\mu^{\mathrm{hlbl,P}}$ as a function of the momentum cutoff Q_{cut}

Q_{cut} [GeV]	π^0	η	η'
0.25	54%	31%	20%
0.50	71%	50%	38%
0.75	80%	61%	51%
1.00	84%	68%	60%
1.50	89%	76%	71%
2.00	92%	81%	78%
3.00	94%	86%	85%
5.00	97%	91%	92%



- Results obtained with the Mainz group [Eur.Phys.J.C 81 (2021) 7, 651]
- Statistical precision deteriorates rapidly at low pion masses
- Dashed lines : finite-volume correction



- Open symbols : $a_\mu^{\mathrm{hlbl,cor}}(a, m_\pi) = a_\mu^{\mathrm{hlbl,data}}(a, m_\pi) + \left(a_\mu^{\pi^0,\mathrm{phys}(a,m_\pi)} - a_\mu^{\pi^0}(a, m_\pi) \right)$
- Correction term : from a dedicated lattice QCD calculation [Phys.Rev.D 100 (2019) 3]

Significantly improve the chiral extrapolation !. Allow to improve on the continuum extrap. at heavier pion masses.