JPAC Dalitz plot analysis of $\eta \to 3\pi$

Igor Danilkin

ECT* workshop: Precision tests of fundamental physics with light mesons

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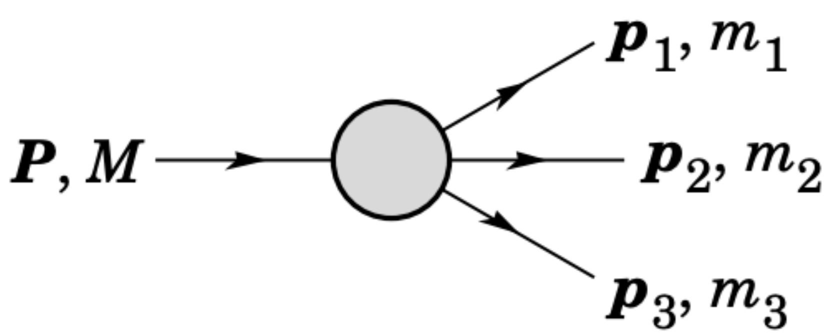
Motivation for Dalitz plot analyses



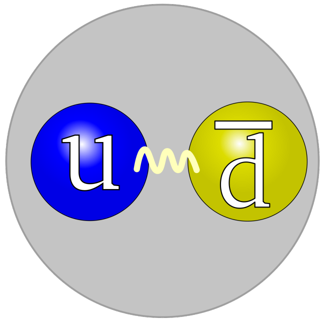
- **Past**: discovery of several prominent meson resonances
  Now (using high precision data): discovery and interpretation of the new **exotic states**

- There is a need to understand better hadronic contributions for **low-energy precision tests** of the Standard Model (e.g. $(g-2)_\mu$ …)

- High-precision data requires the **advancement** of the **theoretical tools**

- Isospin violating decay: sensitive to **quark mass difference** ($\eta \to 3\pi$)

$$\mathscr{L}_{IB} = -\frac{m_u - m_d}{2}(\bar{u}u - \bar{d}d)$$



Theoretical tools

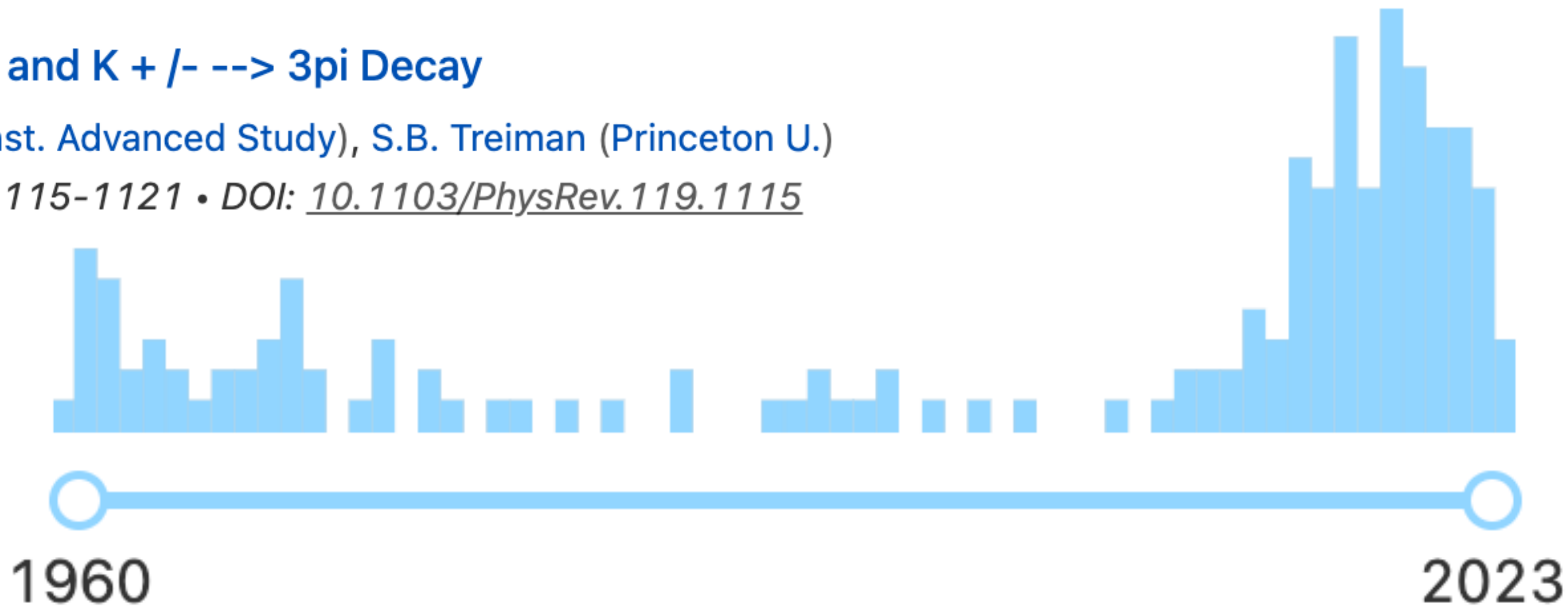
resonances
pw-expansion
analyticity
unitarity
helicity
spin
dalitz-plot
reconstruction-theorem
pasquier-inversion
dispersion-relation
kinematic-constraints
qcd
subtractions
omnes
khuri-treiman

Khuri-Treiman formalism: Citation Summary

Pion-Pion Scattering and K + /- --> 3pi Decay

N.N. Khuri (Princeton, Inst. Advanced Study), S.B. Treiman (Princeton U.)

Phys.Rev. 119 (1960) 1115-1121 • *DOI: 10.1103/PhysRev.119.1115*



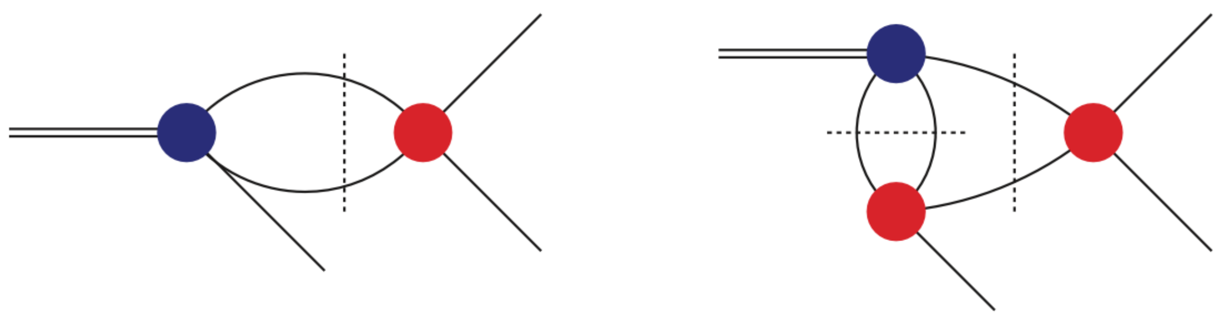


fig from [Gan et al. 2020]

- developed for $K \to 3\pi$
- most prominent applications:

  $\eta \to 3\pi$ [Kambor et al. 1995], [Anisovich et al. 1996], [Descotes-Genon et al. 2014], [JPAC 2015,2017], [Colangelo 2016], [Albaladejo et al. 2017], $\eta' \to \pi\pi\eta$ [Isken et al. 2017]

  $\omega/\phi \to 3\pi/\pi\gamma$ [Niecknig et al. 2012], [JPAC 2014, 2020], [Dax et al. 2018]

  $J/\psi \to 3\pi/\pi\gamma$ [Kubis et al. 2014], [JPAC 2023]

  $e^+e^- \to 3\pi/\pi\gamma$ (HLbL $a_\mu^{\pi^0\text{–pole}}$, HVP $a_\mu^{3\pi}$, $a_\mu^{\pi^0\gamma}$) [Hoferichter et al. 2018, 2019], [Hoid 2020]

  $D \to K\pi\pi$ [Niecknig et al. 2015], [Kou et al. 2022]

  $J^{PC} \to 3\pi$ [JPAC 2019], [Stamen et al. 2022]

p.w. expansion

- p.w. decomposition

$$F(s,t,u) = \sum_{J=0}^{\infty} (2J+1)\, P_J(\cos\theta_s)\, f_J(s)$$

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free from kinematic constraints

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- Symmetrized p.w. decomposition (reconstruction theorem)
(analyticity exact up to NNLO [Stern et al. 1993, Knecht et al. 1995])

$$F(s,t,u) = \sum_{J=0}^{J_{max}} (2J+1)\, P_J(\cos\theta_s)\, (p\, q)^J F_J(s) + \sum_{J=0}^{J_{max}} \ldots (s \to t) + \sum_{J=0}^{J_{max}} \ldots (s \to u)$$

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(for $J_{max} = 0$)

$$\tilde{f}_0(s) = F_0(s) + \underbrace{\int_{-1}^{1} \frac{d\cos\theta_s}{2} \left(F_0(t) + F_0(u)\right)} \equiv F_0(s) + \hat{F}_0(s)$$

right-hand cut *left-hand cut*

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right-hand cut *left-hand cut*

- Truncation J_{max} only neglects the discontinuities in partial waves with $J > J_{max}$

p.w. expansion

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- Functions F_J^{Isospin} will be constrained by analyticity & unitarity

$$F_J^I(s) = \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{disc}\, F_J^I(s')}{s'-s}$$

many more in the coupled-channel analysis [Albaladejo et al. 2017]

$$\eta \to 3\pi : \quad F_0^0, F_0^2, F_1^1 \quad (J_{max} = 1)$$

$$\eta' \to \pi\pi\eta : \quad F_0^0, F_0^1, F_1^1 \quad (J_{max} = 1)$$

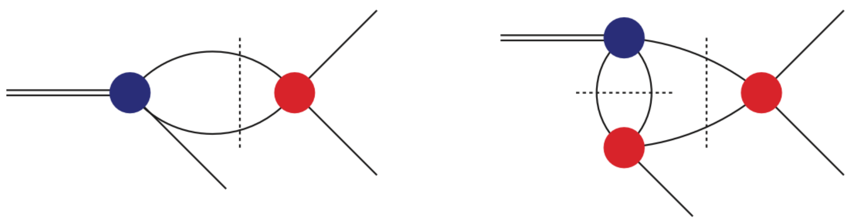
$$D^+ \to \bar{K}\pi\pi^+ : \quad F_0^2, F_0^{1/2}, F_0^{3/2}, F_1^1, F_1^{1/2}, F_1^{3/2} \quad (J_{max} = 1)$$

$$\ldots$$

$$\omega/\phi \to 3\pi : \quad F_1^1 \quad (J_{max} = 1)$$

predictive power (discuss later)

Unitarity & Analyticity

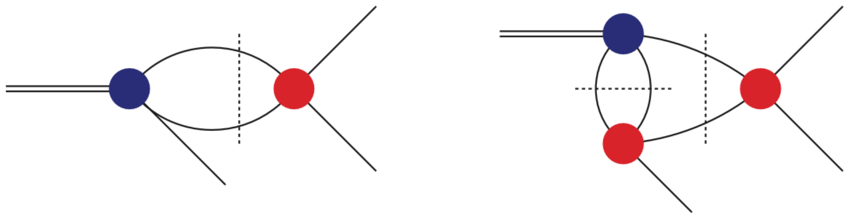


$\pi\pi \to \pi\pi$ amplitude

$$\text{disc}\, F_J^I(s) = t_J^{I*}(s)\rho(s)\left(F_J^I(s) + \hat{F}_J^I(s)\right)$$

$$\hat{F}(s) \equiv \int_{t_-(s)}^{t_+(s)} \frac{dt}{k(s)} \left(F(t) + F(u)\right)$$

Unitarity & Analyticity



ππ → ππ amplitude

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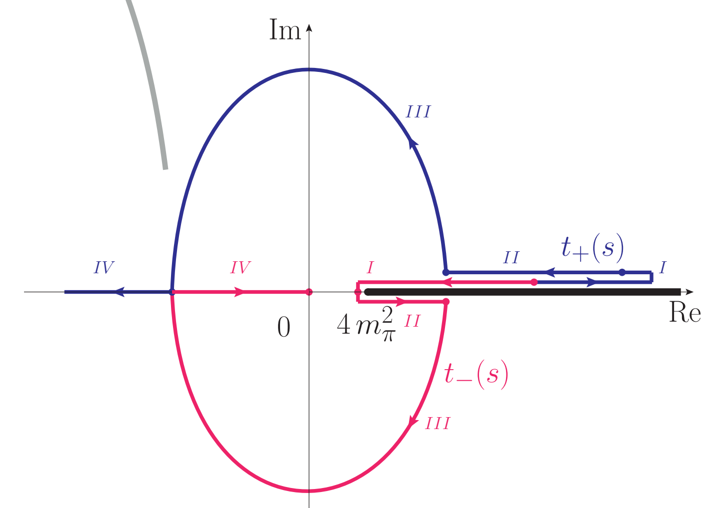
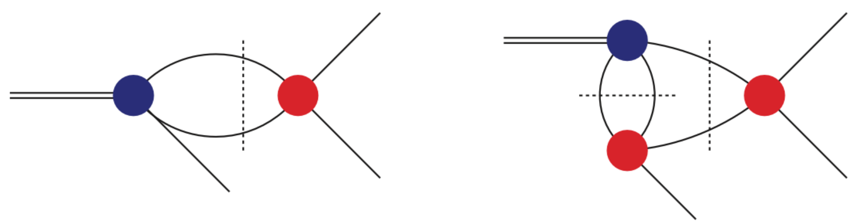


$M^2 \to M^2 + i\epsilon$

Unitarity & Analyticity



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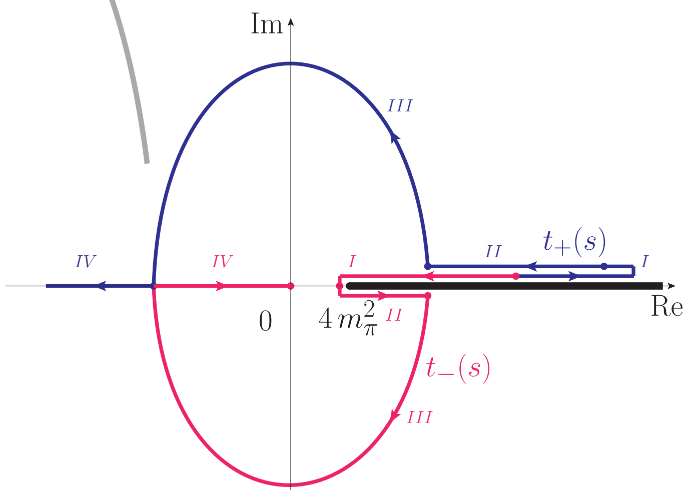


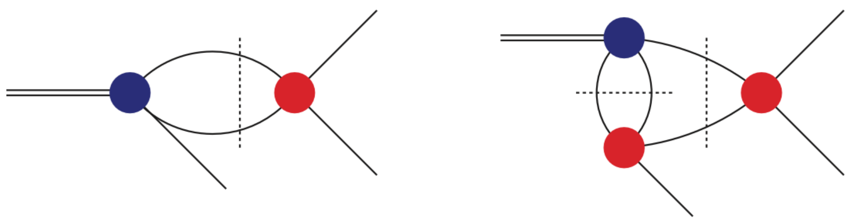
- Conventional solution in terms of the Omnès function

fit parameters

$$F_J^I(s) = \Omega_J^I(s)\left(P_{n-1}(s) + \frac{s^n}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{(s')^n}\frac{\sin\delta_J^I(s')\,\hat{F}_J^I(s')}{|\Omega_J^I(s')|\,(s'-s)}\right)$$

$$\Omega_J^I(s) = \exp\left(\frac{s}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\frac{\delta_J^I(s')}{s'-s}\right)$$

Unitarity & Analyticity



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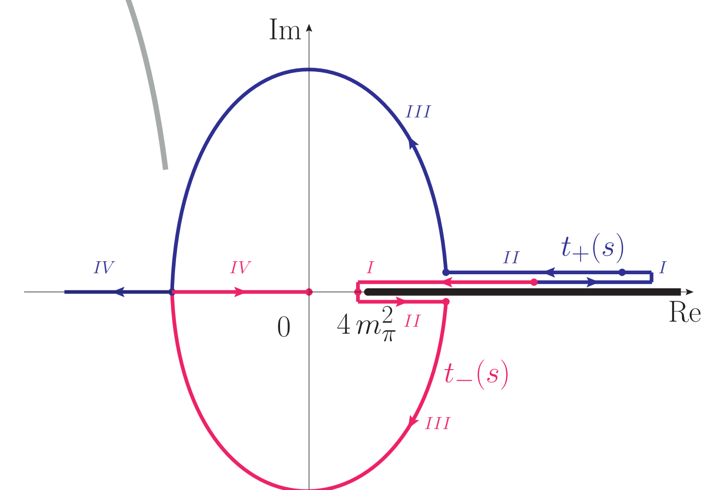

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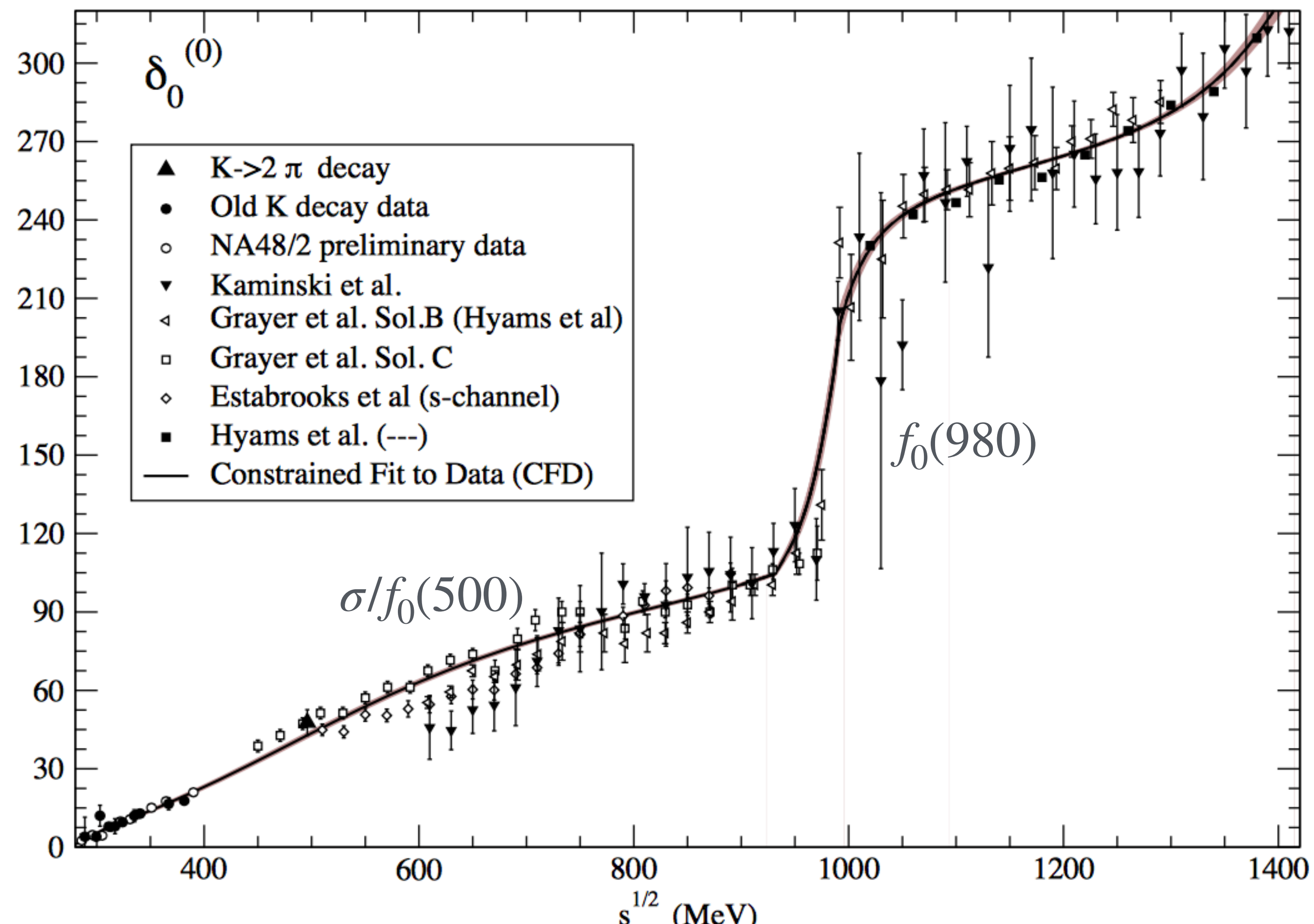
- **Caveats**: one needs to assume high energy behaviour of the phase shifts in the elastic approximation (typically $\delta^I_J(s \to \infty) = n_{res}\,\pi$), therefore many subtractions introduced which are in general complex parameters **(less predictive models**)

S-wave, I=0

- For many decays (e.g. $\eta \to 3\pi, \eta' \to \pi\pi\eta, \ldots$) the **most important contribution** comes from S-wave, $I = 0$ ($\sigma/f_0(500), f_0(980), \ldots$) or $I = 1/2$ ($\kappa/K_0^*(700), \ldots$)

$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right)$$



[Garcia-Martin et al. (2011)]

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- In an effective single-channel problem it is not clear what to do with the $f_0(980)$ resonance …

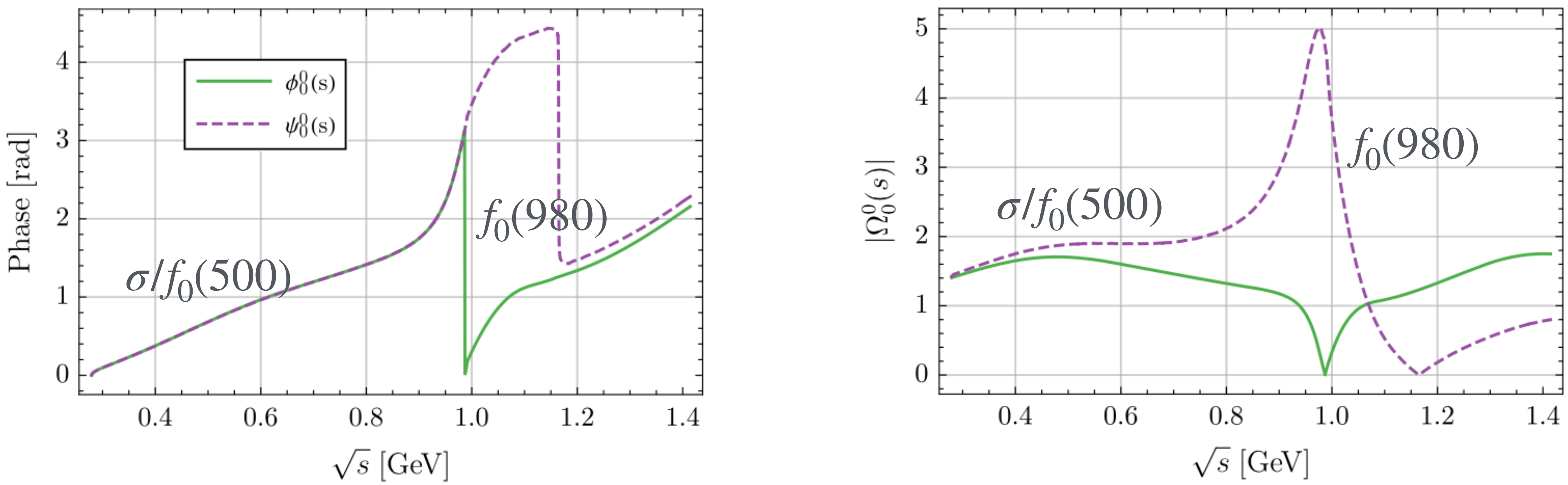

fig from [Colangelo et al. 2016]

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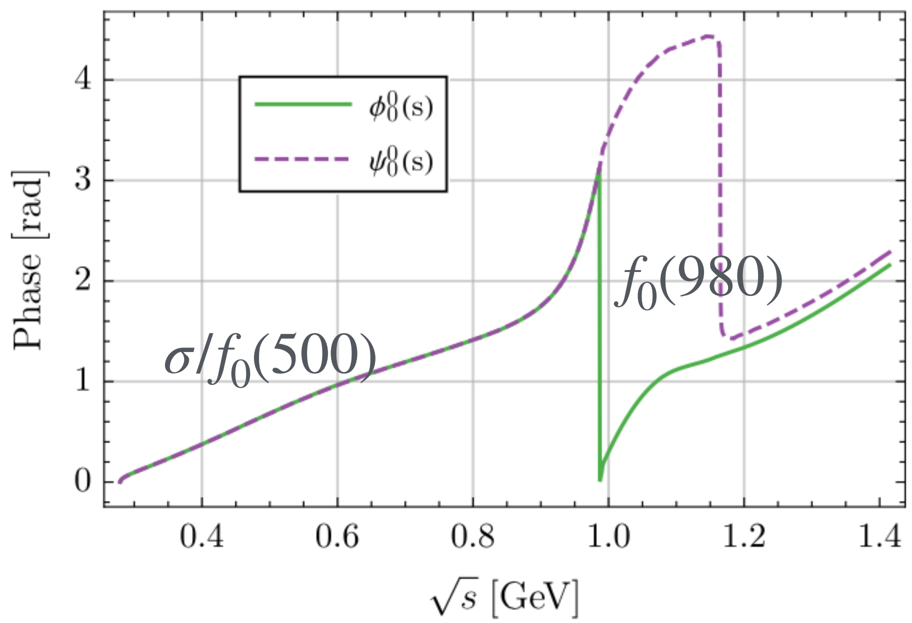


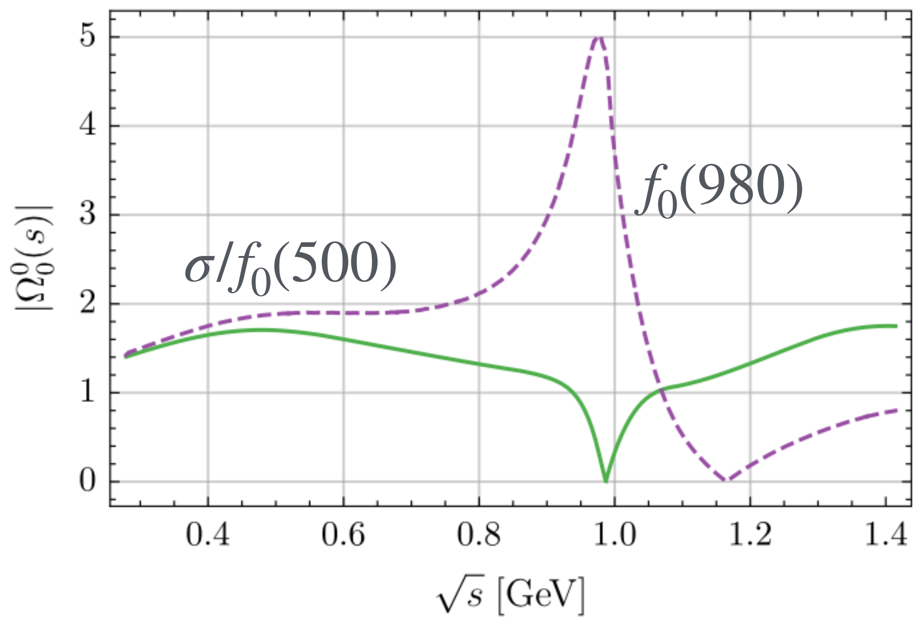
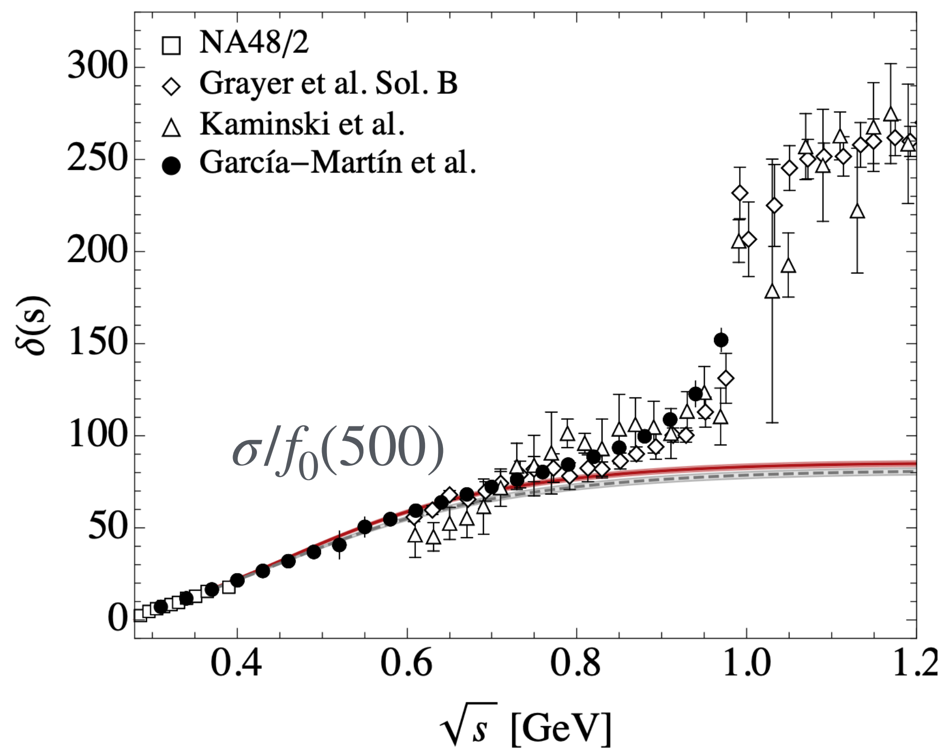


fig from [Colangelo et al. 2016]

- **Possible way out**: use a phase shift from one-channel **p.w. dispersion relations**

Omnès function {ππ}

$\pi\pi \to \pi\pi$



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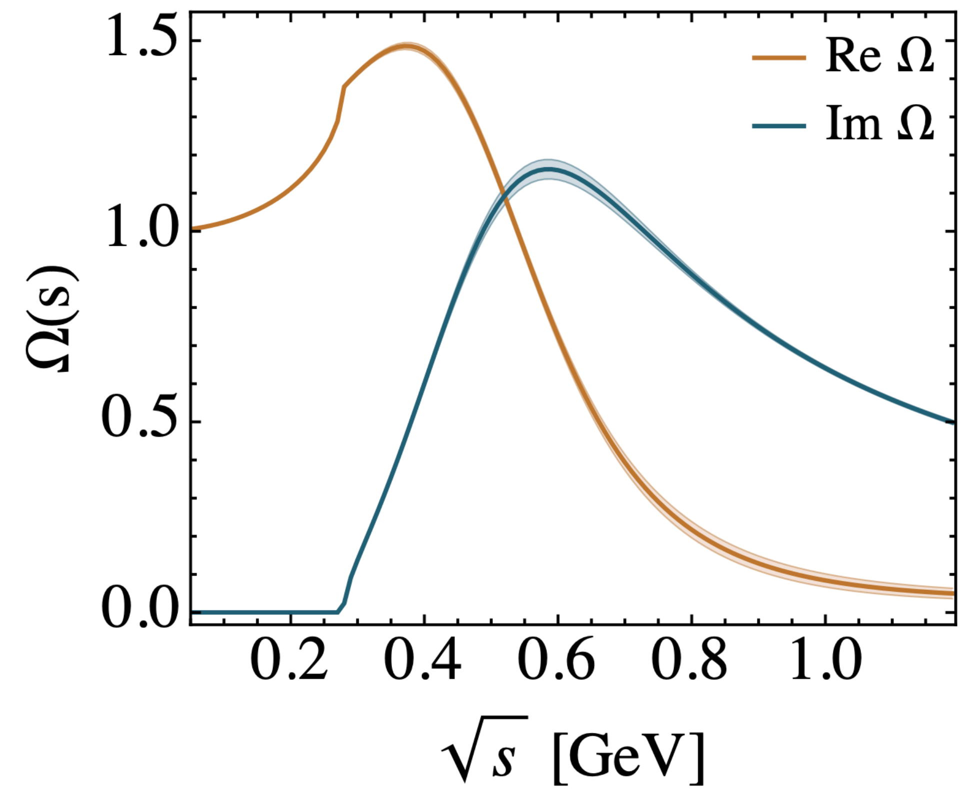

p.w. dispersion relation

$$t(s) = \underbrace{t(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\text{Im}\, t(s')}{s'-s}}_{\approx \sum_{n=0}^{max} C_n\, \xi^n(s)} + \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t(s')\,\rho(s')\,t^*(s')}{s'-s} = D^{-1}(s)\, N(s)$$

[I.D, Deineka, Vanderhaeghen (2021)]

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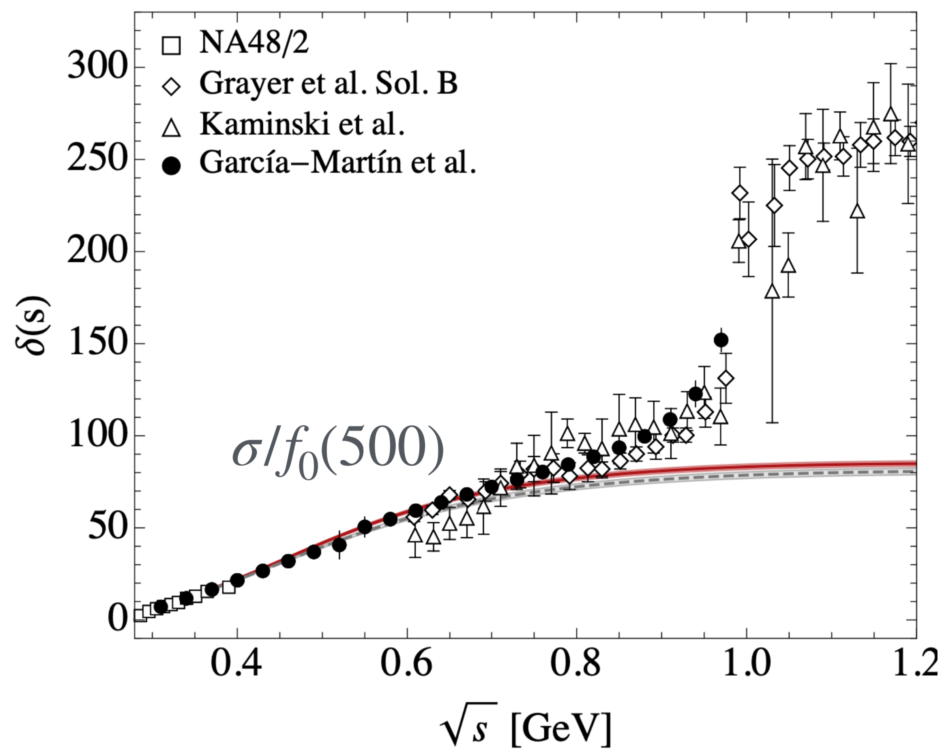

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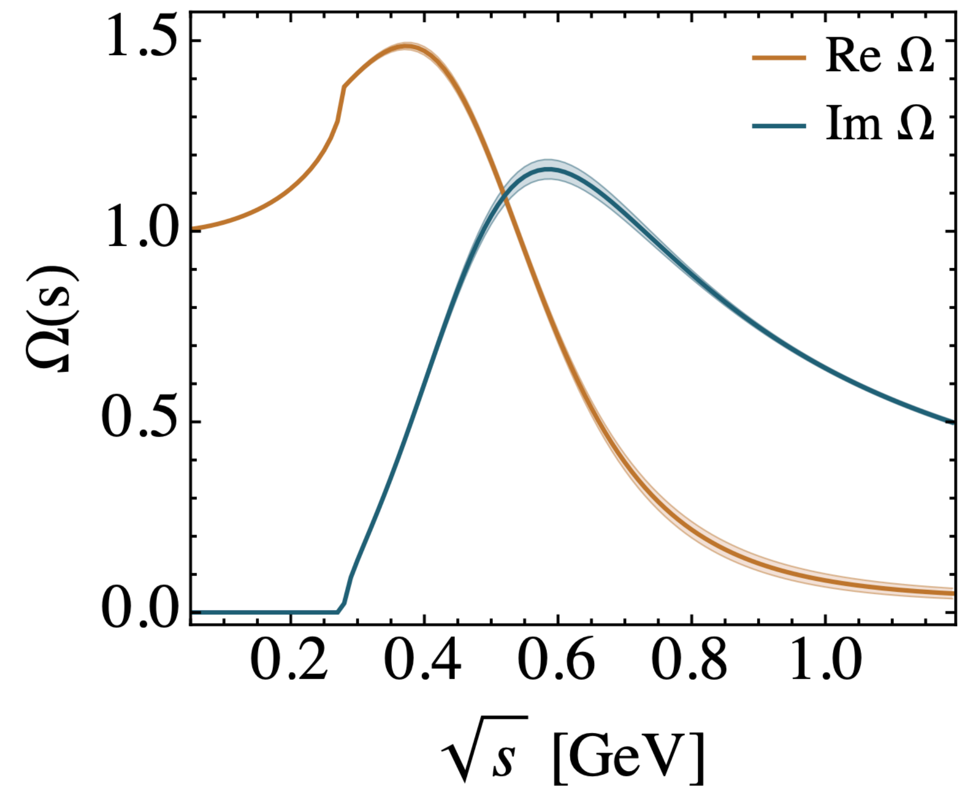

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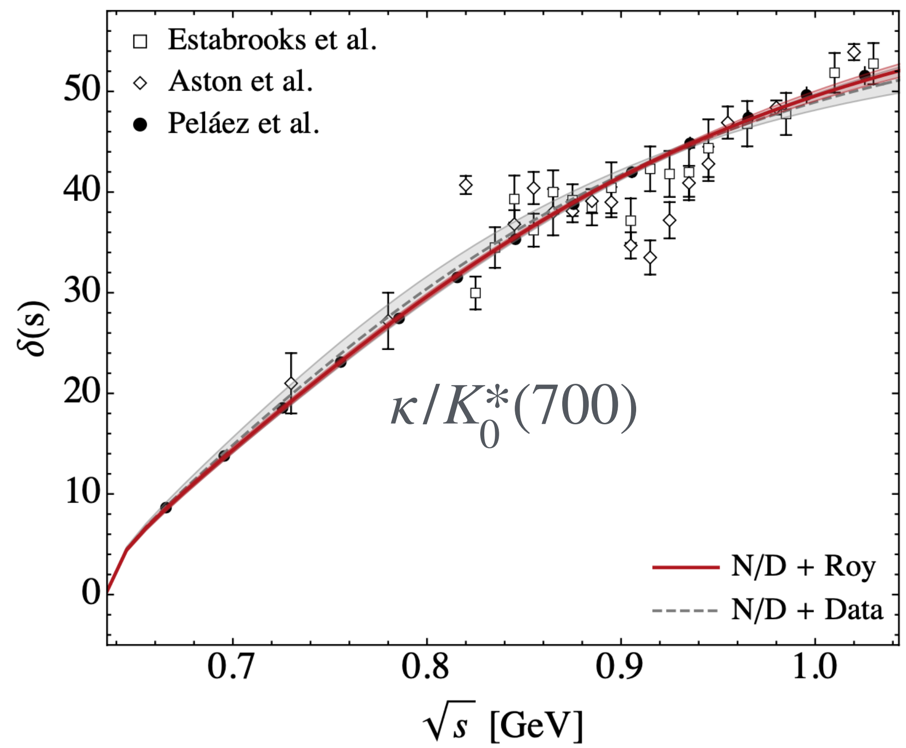
[I.D, Deineka, Vanderhaeghen (2021)]

- Input: experimental data/Roy analysis + threshold parameters NNLO + Adler zero NLO
- The result for $\delta(s)$ is very **similar** to p.w. dispersion relation for $t^{-1}(s)$ (see mIAM [Gomez Nicola et al. 2008] and DIA [I.D, Biloshytskyi, Ren, Vanderhaeghen (2023)])

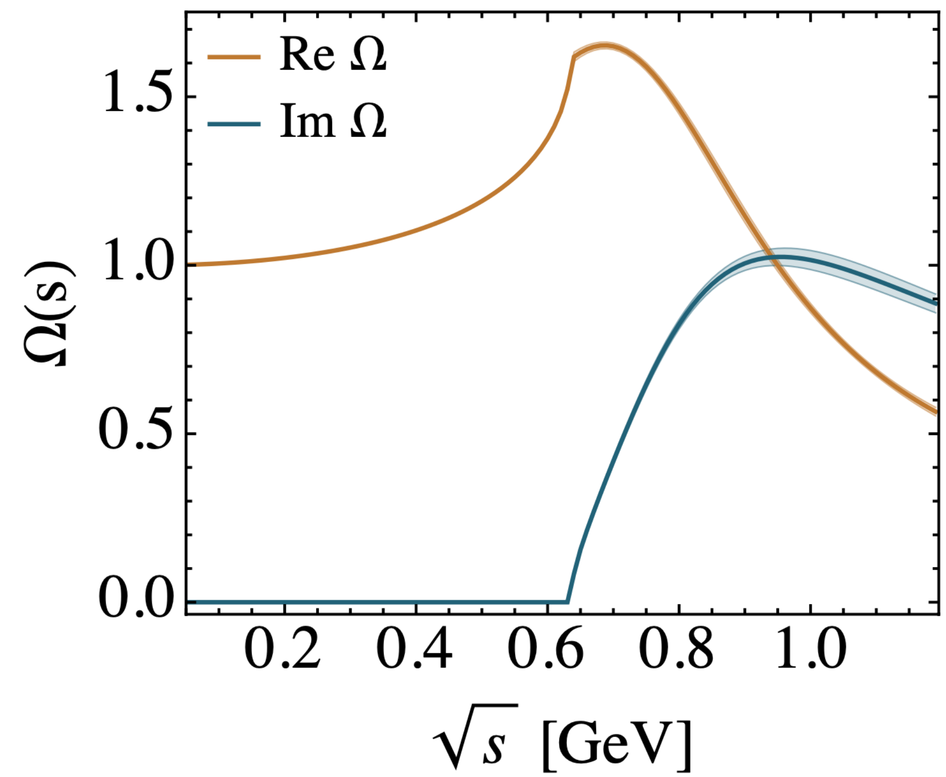
applied to HLbL $a_{\mu,\,J=0}^{\pi\pi,\,\pi\text{-pole LHC}}$
[Colangelo et al. 2017]

Omnès function {πK}

$\pi K \to \pi K$



$\pi K \to \pi K$



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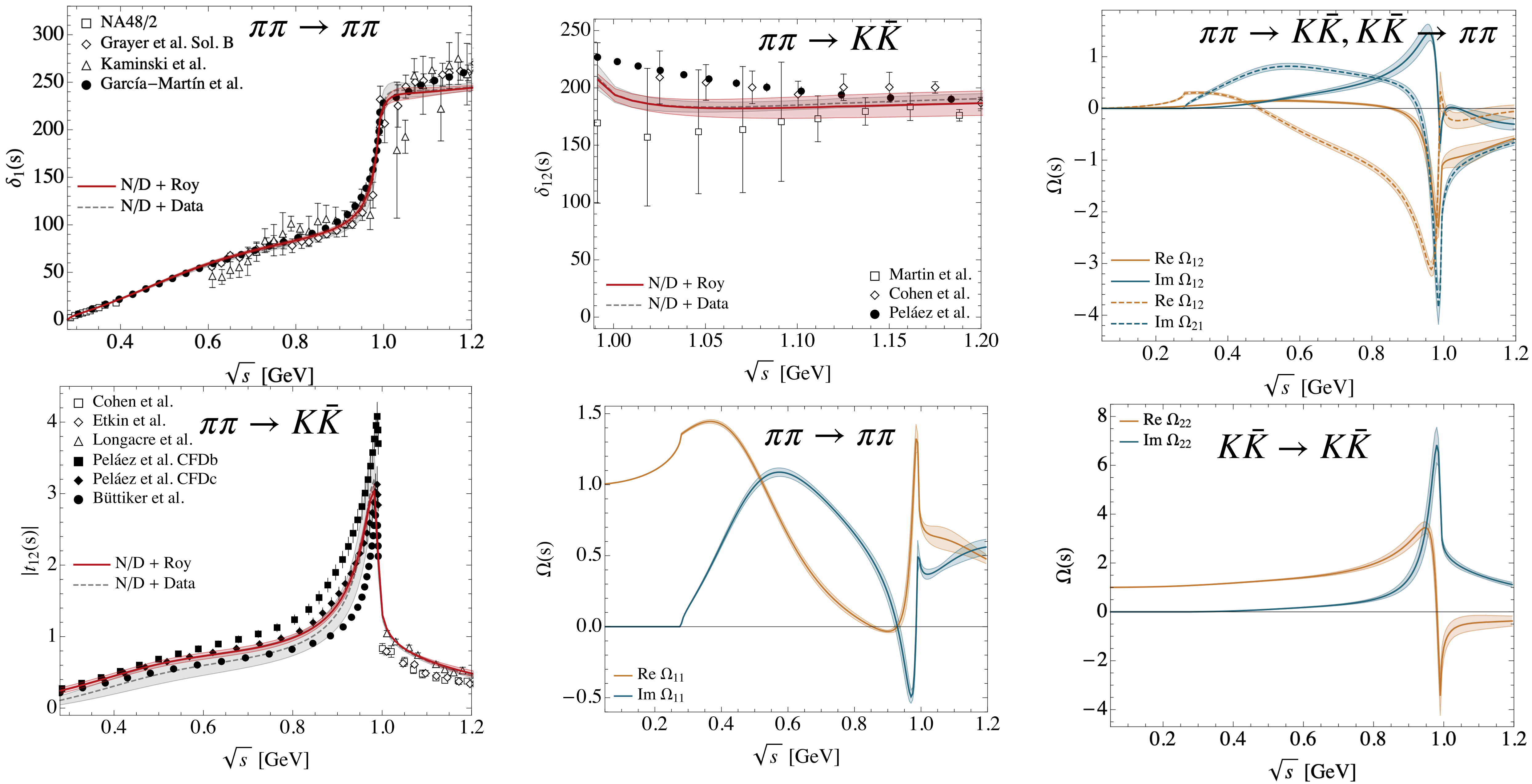
Omnès matrix {ππ, KK}

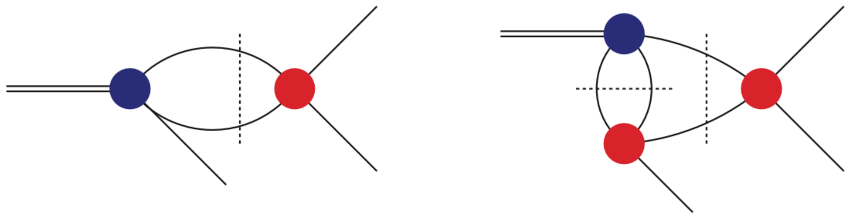


$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'}\frac{\text{Im}\, t_{ab}(s')}{s'-s}}_{\approx \sum_{n=0}^{max} C_{ab,n}\,(\xi_{ab}(s))^n} + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s} = \sum_c D^{-1}_{ac}\, N_{cb}(s)$$

[I.D, Deineka, Vanderhaeghen (2021)]

$$\Omega_{ab}(s) = D^{-1}_{ab}(s)$$

Solution strategies



$\pi\pi \to \pi\pi$ *amplitude*

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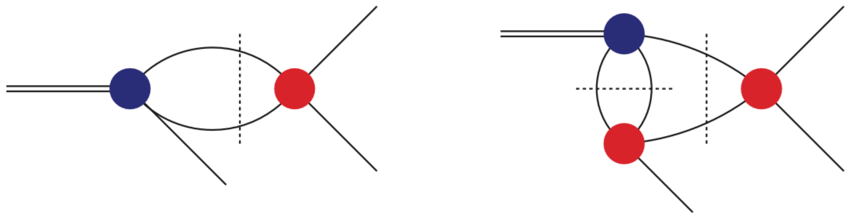
Solution strategies



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- Standard way: $F(s) = \Omega(s)\,G(s)$ solution by iteration

$$F(s) = \Omega(s)\left(P_{n-1}(s) + \frac{s^n}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{(s')^n}\frac{\sin\delta_J^I(s')\,\hat{F}(s')}{|\,\Omega_J^I(s')\,|\,(s'-s)}\right)$$

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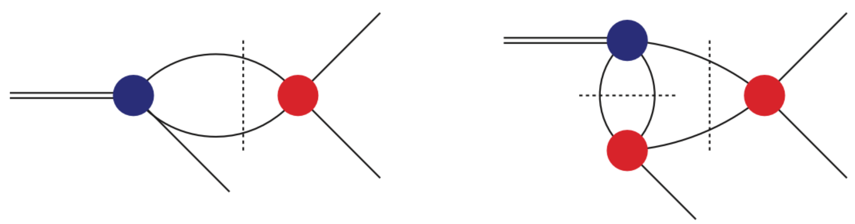

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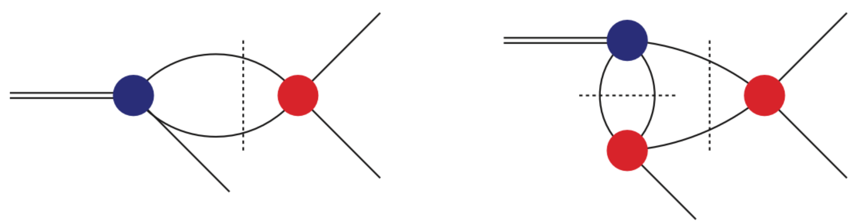
- Write integral eq. for the inhomogeneities (solution by matrix inversion)

$$\hat{F}(s) = A(s) + \int_{s_{th}}^{\infty} ds'\,\hat{F}(s')\,K(s,s')$$

[Niecknig et al. 2015]

Solution strategies



$\pi\pi \to \pi\pi$ amplitude

$$\text{disc}\, F(s) = t^*(s)\rho(s)\,(F(s) + \hat{F}(s))$$

$$\hat{F}(s) \equiv \int_{t_-(s)}^{t_+(s)} \frac{dt}{k(s)}\,(F(t) + F(u))$$

- Standard way: $F(s) = \Omega(s)\, G(s)$ solution by iteration

$$F(s) = \Omega(s)\left(P_{n-1}(s) + \frac{s^n}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{(s')^n}\, \frac{\sin \delta_J^I(s')\, \hat{F}(s')}{|\,\Omega_J^I(s')\,|\,(s'-s)} \right)$$

- Write integral eq. for the inhomogeneities (solution by matrix inversion)

$$\hat{F}(s) = A(s) + \int_{s_{th}}^{\infty} ds'\, \hat{F}(s')\, K(s,s')$$

[Niecknig et al. 2015]

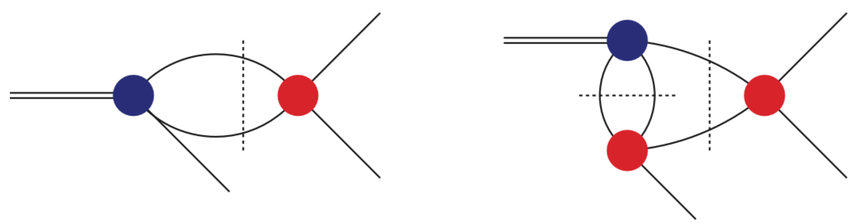
- Pasquier inversion [Pasquier at al. 1968] (solution by matrix inversion)

$$\int_{s_{th}}^{\infty} ds' \ldots \int_{t_-(s')}^{t_+(s')} dt \ldots = \int_{-\infty}^{(M-m)^2} dt \ldots \int_{C'(t)} ds' \ldots$$

$$F(s) = \Omega(s)\left(P_{n-1}(s) + \int_{-\infty}^{(M-m)^2} dt\, F(t)\, \tilde{K}(s,t) \right)$$

[JPAC, Peng et al. 2015]

Solution strategies



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[Niecknig et al. 2015]

- Pasquier inversion [Pasquier at al. 1968] (solution by matrix inversion) $F(s) = t(s)\,g(s)$

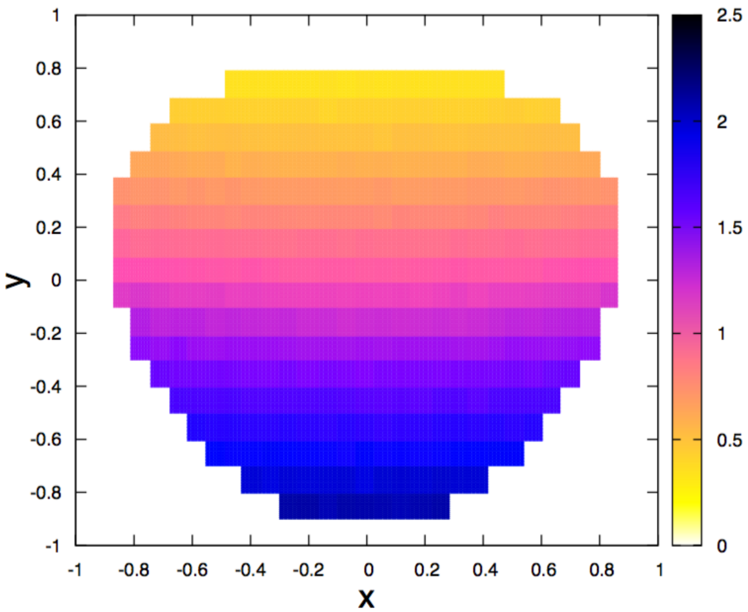
$$F(s) \approx t(s)\left(a + \int_0^{(M-m)^2} dt\,F(t)\,\tilde{K}_g(s,t)\right)$$

universal

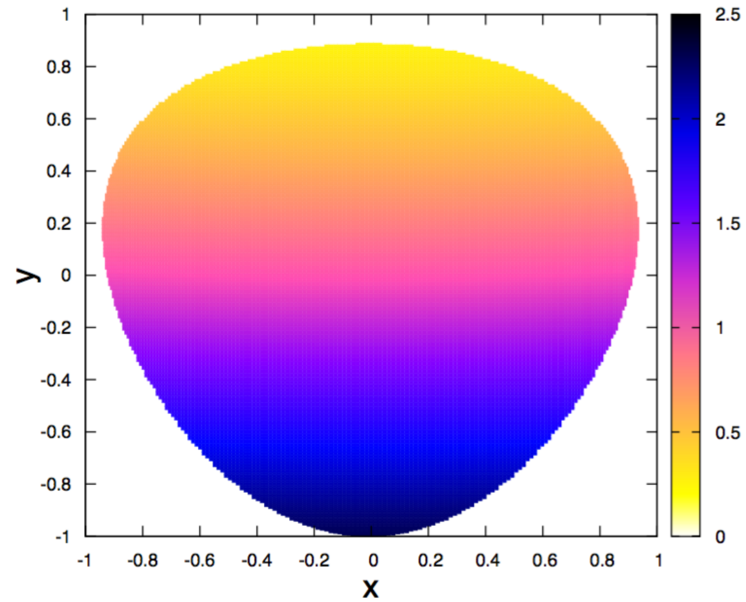
[JPAC, Peng et al. 2015, 2017]

Results: $\eta \to 3\pi$

KLOE-2



result of the fit



Data	F_J^I	χ^2/d.o.f 2b	χ^2/d.o.f 3b
WASA	$F_0^0,\ F_1^1$	1.45	0.96
	$F_0^0,\ F_0^2,\ F_1^1$	0.94	0.90
KLOE-2	$F_0^0,\ F_1^1$	10.4	2.61
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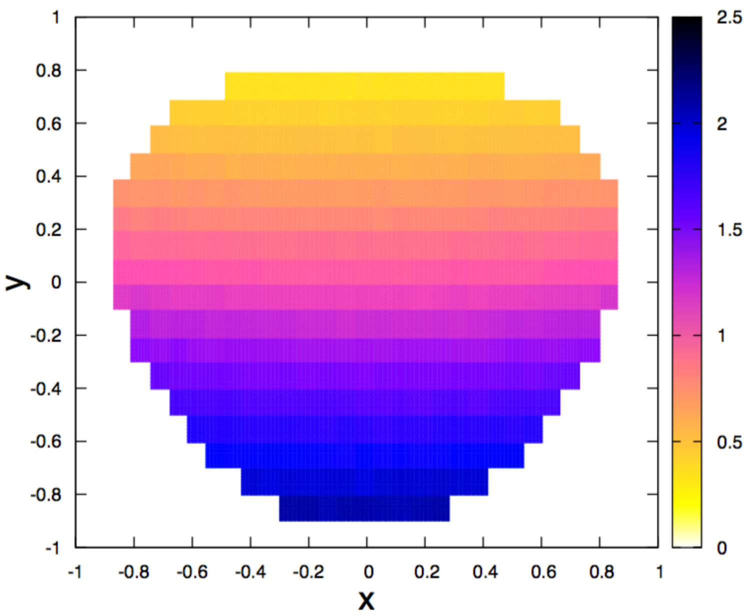

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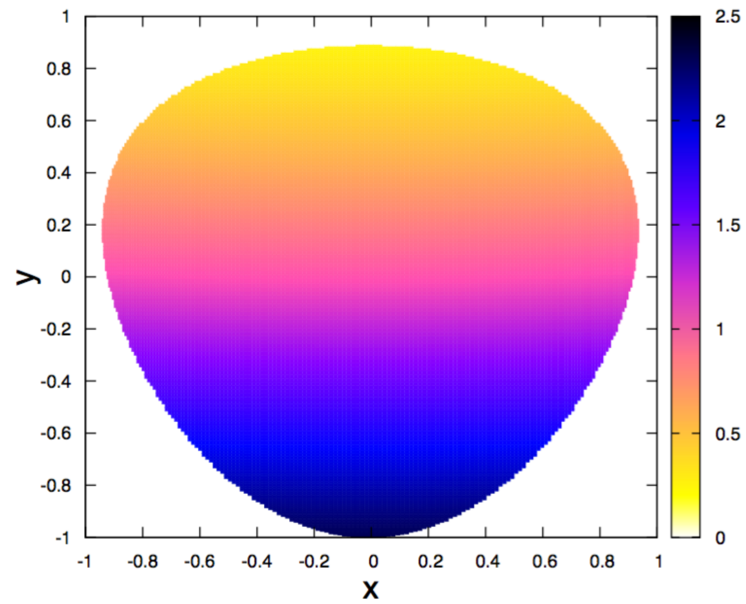

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- Dalitz plot parameter for $|F(s,t,u)_{\eta\to 3\pi^0}|^2 \propto 1 + 2\alpha z + \ldots$

$$\alpha = -0.025 \pm 0.004$$
$$\alpha^{\text{PDG}} = -0.0288 \pm 0.0012$$

Results: $\eta \to 3\pi$

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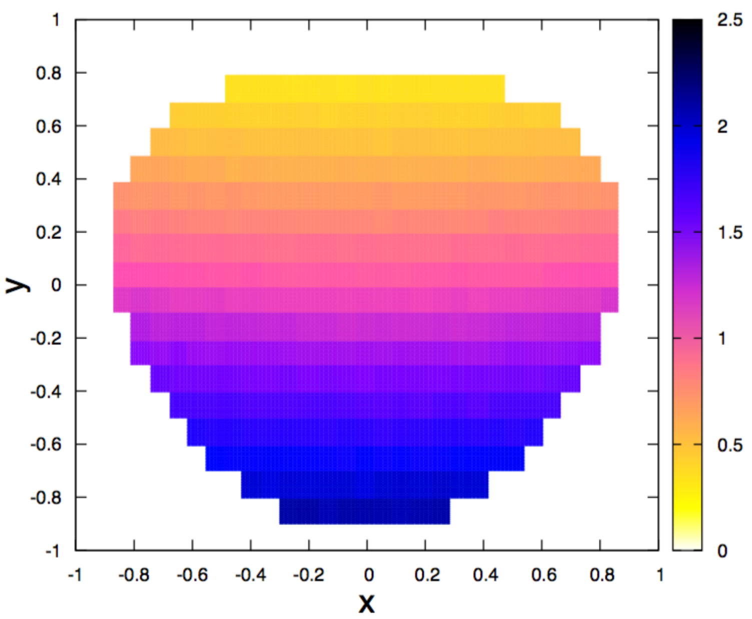
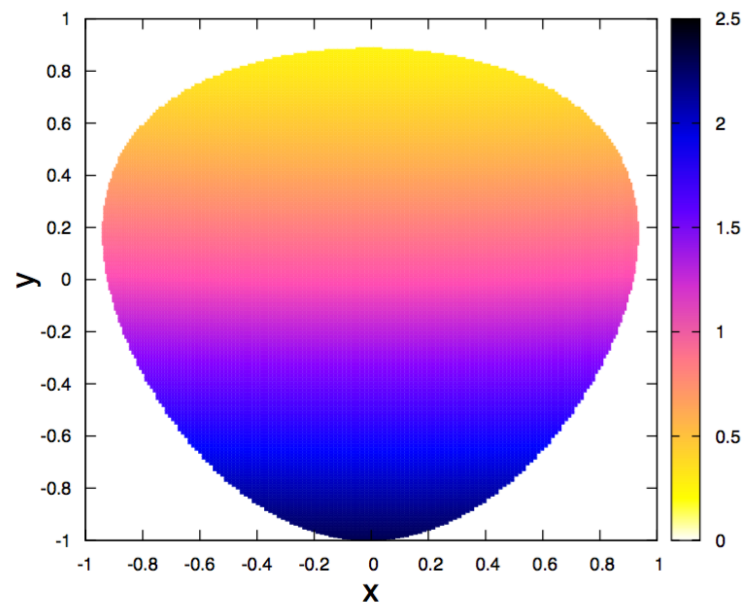


result of the fit



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- Matching to ChPT around **Adler zero** for $F_0^0(s)$ allowed us to calculate quark mass double ratio

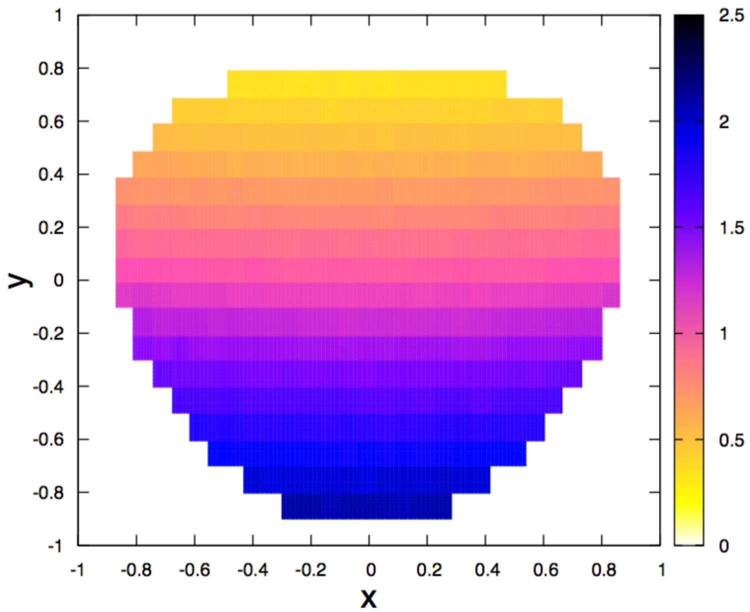
$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

- Different dispersive estimations are consistent between each other and latest FLAG numbers

Theory	Q
JPAC, Peng et al. 2017	21.6(1.1)
Albaladejo et al. 2017	21.5(1.0)
Colangelo et al. 2018	22.1(0.7)
FLAG ($N_f = 2+1+1$), 2016	22.2(1.6)
FLAG ($N_f = 2+1+1$), 2021	22.5(0.5)

Importance of the 3-body effects

KLOE-2



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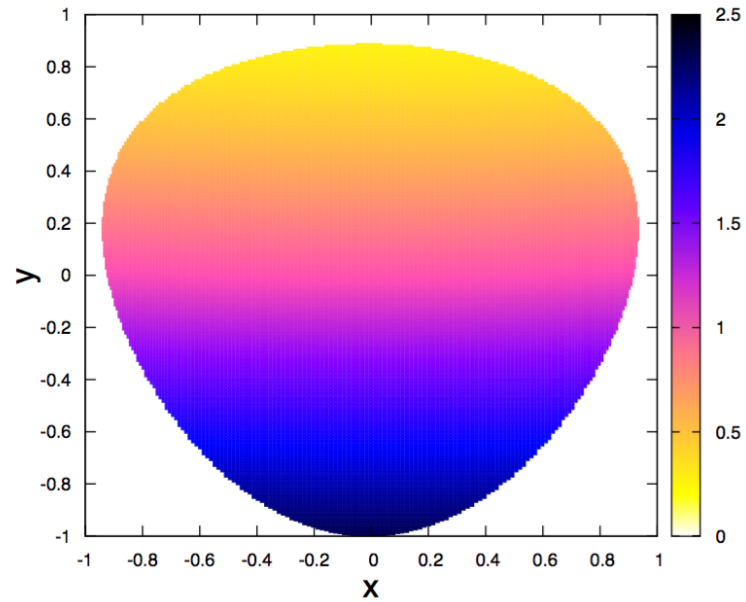

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[JPAC, Peng et al. 2015, 2017]

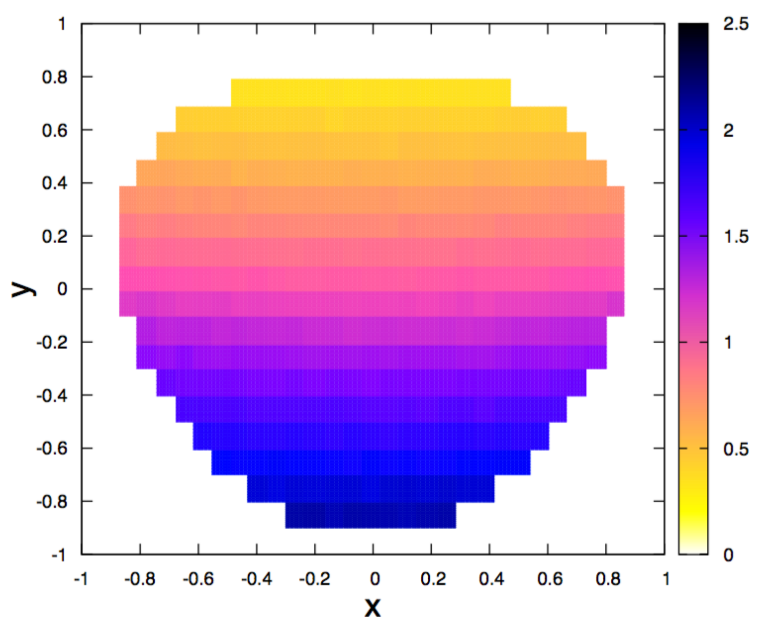
- 3b effects stabilise the fits:
  it is especially important when many amplitudes contribute

$$D^+ \to \bar{K}\pi\pi^+ : \quad F_0^2, F_0^{1/2}, F_0^{3/2}, F_1^1, F_1^{1/2}, F_1^{3/2} \quad (J_{max} = 1)$$

[Niecknig et al. 2015]

Importance of the 3-body effects

KLOE-2



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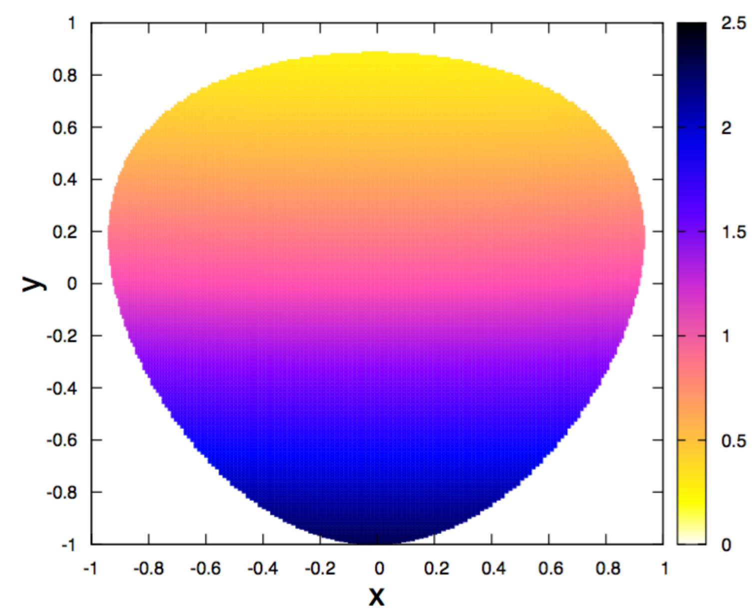

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- Ideal case is $V \to 3\pi$: F_1^1 $(J_{max} = 1)$ [Niecknig et al. 2012], [JPAC 2014, 2020]
  $\phi \to 3\pi$: KLOE data (but not CMD-2 data) favours 3b effects
  $\omega \to 3\pi$: BESIII disfavour 3b effects
  $J/\psi \to 3\pi$: more complicated $(J_{max} > 1)$

see Albaladejo's talk

p.w. expansion (particles with spin)

suffer from kinematic constraints

- For $j^{PC} \to 3\pi$

$$H_\lambda = H^{\mu\nu}\,\epsilon_{\mu\nu}(\lambda) = \left(\sum_i L^{\mu\nu}\,F_i(s,t,u)\right)\epsilon_{\mu\nu}(\lambda) = \sum_{J=0}^{\infty}(2J+1)\,d^J_{\lambda 0}(\theta_s)\,f_{J\lambda}(s)$$

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- The way to pin down kinematic constraints is to express $F_i(s,t,u)$ in terms of $f_{J\lambda}(s)$ (or *vice versa*) (**numerical applications** for $j^{PC} \to 3\pi$ that decays predominantly to $\pi\rho$) [Stamen et al. 2022] Reconstruction-like theorem for $F_i(s,t,u)$ derived using fixed-s, t, u dispersion relations

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- Another way to pin down kinematic constraints is to employ crossing matrices between H_λ in different scattering kinematics [JPAC, Albaladejo et al. 2019] and consequently work with singularity free p.w. helicity amplitudes and generalised isobar decomposition

$$H_\lambda = \sum_{J=0}^{J_{max}}(2J+1)\,d^J_{\lambda 0}(\theta_s)\,\tilde{f}_{J\lambda}(s) + \sum_{\lambda'} d^j_{\lambda'\lambda}(\omega_t)\sum_{J=0}^{J_{max}}(\,..\,s\to t) + \sum_{\lambda'} d^j_{\lambda'\lambda}(\omega_u)\sum_{J=0}^{J_{max}}(\,..\,s\to u)$$

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- For $1^{--}(\omega, \phi, \ldots) \to 3\pi$ everything is easy (there is only one helicity/invariant amplitude)

$$F(s,t,u) \overset{J_{max}=1}{=} F(s) + F(t) + F(u)$$

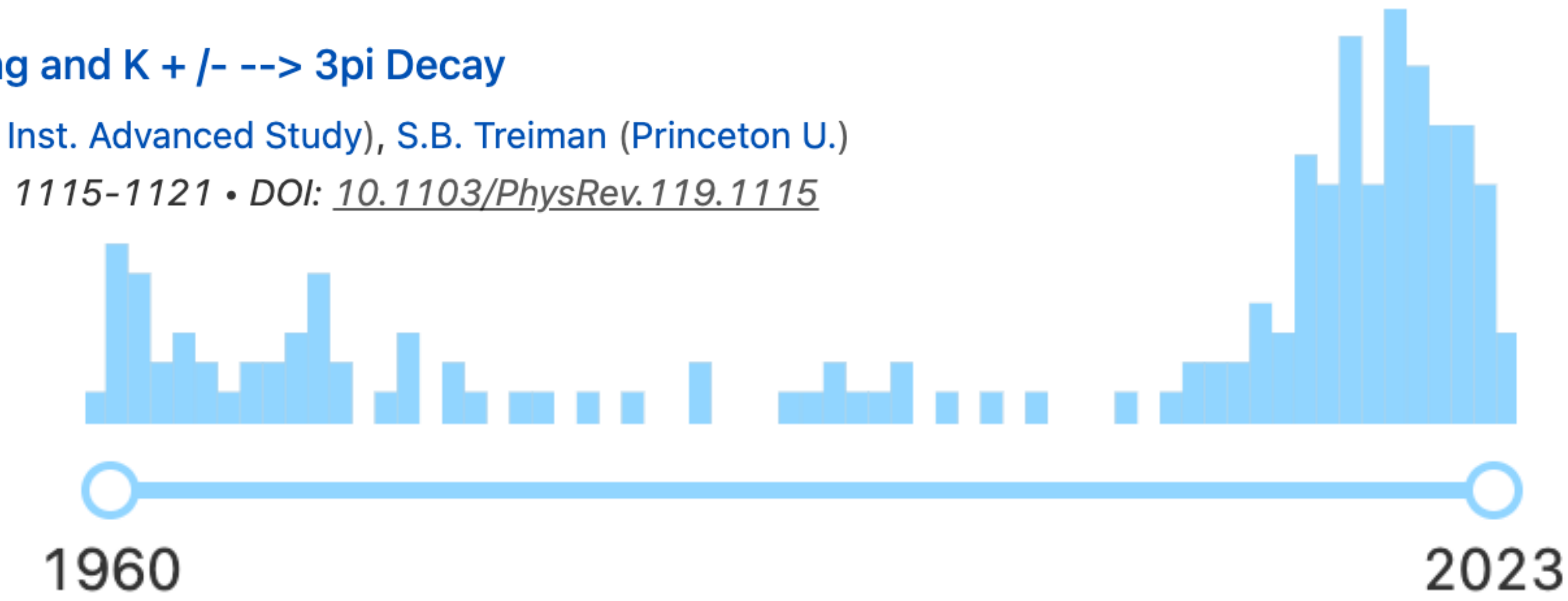
… derivation much more involved for $2^{++}(a_2(1320),..) \to 3\pi$

Conclusion

Pion-Pion Scattering and K + /- --> 3pi Decay

N.N. Khuri (Princeton, Inst. Advanced Study), S.B. Treiman (Princeton U.)

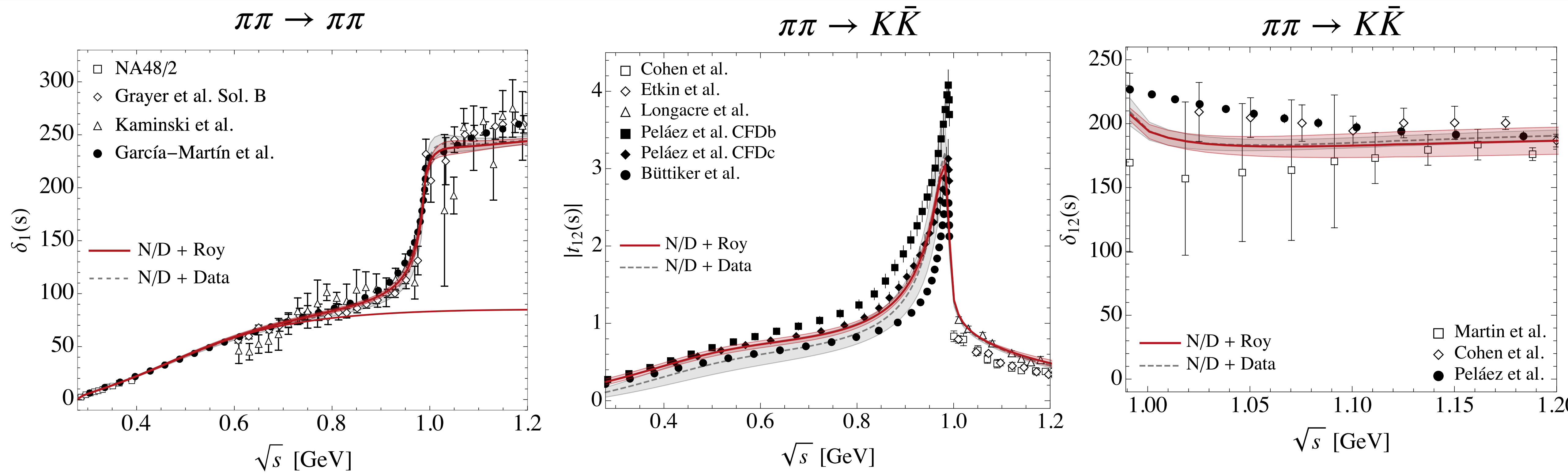
Phys.Rev. 119 (1960) 1115-1121 • DOI: 10.1103/PhysRev.119.1115



stay tuned …

Spares

Coupled-channel analysis {ππ, KK}



Input: experimental data/Roy analysis + threshold parameters NNLO (a, b) + Adler zero NLO

	Our results		Roy-like analyses	
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i\,256(9)^{+5}_{-8}$	$\pi\pi: 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K}: 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i\,275(15)$	$\pi\pi: 3.45^{+0.25}_{-0.29}$ $K\bar{K}: -$
fit to Exp	$454(12)^{+6}_{-7} - i\,262(12)^{+8}_{-12}$			
$f_0(980)$	$993(2)^{+2}_{-1} - i\,21(3)^{+2}_{-4}$	$\pi\pi: 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K}: 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i\,25^{+11}_{-6}$	$\pi\pi: 2.3(2)$ $K\bar{K}: -$
fit to Exp	$990(7)^{+2}_{-4} - i\,17(7)^{+4}_{-1}$			

[Caprini et al. (2006)]
[Garcia-Martin et al. (2011)]
[Moussallam (2011)]

- Omnes function fulfils the unitarity relation on the right-hand cut and analytic everywhere else
- For the case of no bound states or CDD poles: $\Omega_{ab}(s) = D^{-1}_{ab}(s)$

p.w. expansion

- p.w. decomposition

$$F(s,t,u) = \sum_{J=0}^{\infty} (2J+1)\, P_J(\cos\theta_s) f_J(s) = \sum_{J=0}^{\infty} (2J+1)\, P_J(\cos\theta_s)\,(p\,q)^J \tilde{f}_J(s)$$

free from kinematic constraints

- Symmetrized p.w. decomposition (reconstruction theorem)
(analyticity exact up to NNLO [Stern et al. 1993, Knecht et al. 1995])

$$F(s,t,u) = \sum_{J=0}^{J_{max}} (2J+1)\, P_J(\cos\theta_s)\,(p\,q)^J F_J(s) + \sum_{J=0}^{J_{max}} \ldots (s \to t) + \sum_{J=0}^{J_{max}} \ldots (s \to u)$$

$$\tilde{f}_J(s) = F_J(s) + \int_{-1}^{1} \frac{dz_s}{2} \frac{P_J(z_s)}{(p\,q)^J} \sum_{J'=0}^{J_{max}} (2J'+1)\Big(P_{J'}(z_t)(p\,q)^{J'} F_{J'}(t) + (t \to u)\Big) \equiv F_J(s) + \hat{F}_J(s)$$

right-hand cut *left-hand cut*

- Truncation J_{max} only neglects the discontinuities in partial waves with $J > J_{max}$