Trapped-ion Quantum Computing For Collective Neutrino Oscillations

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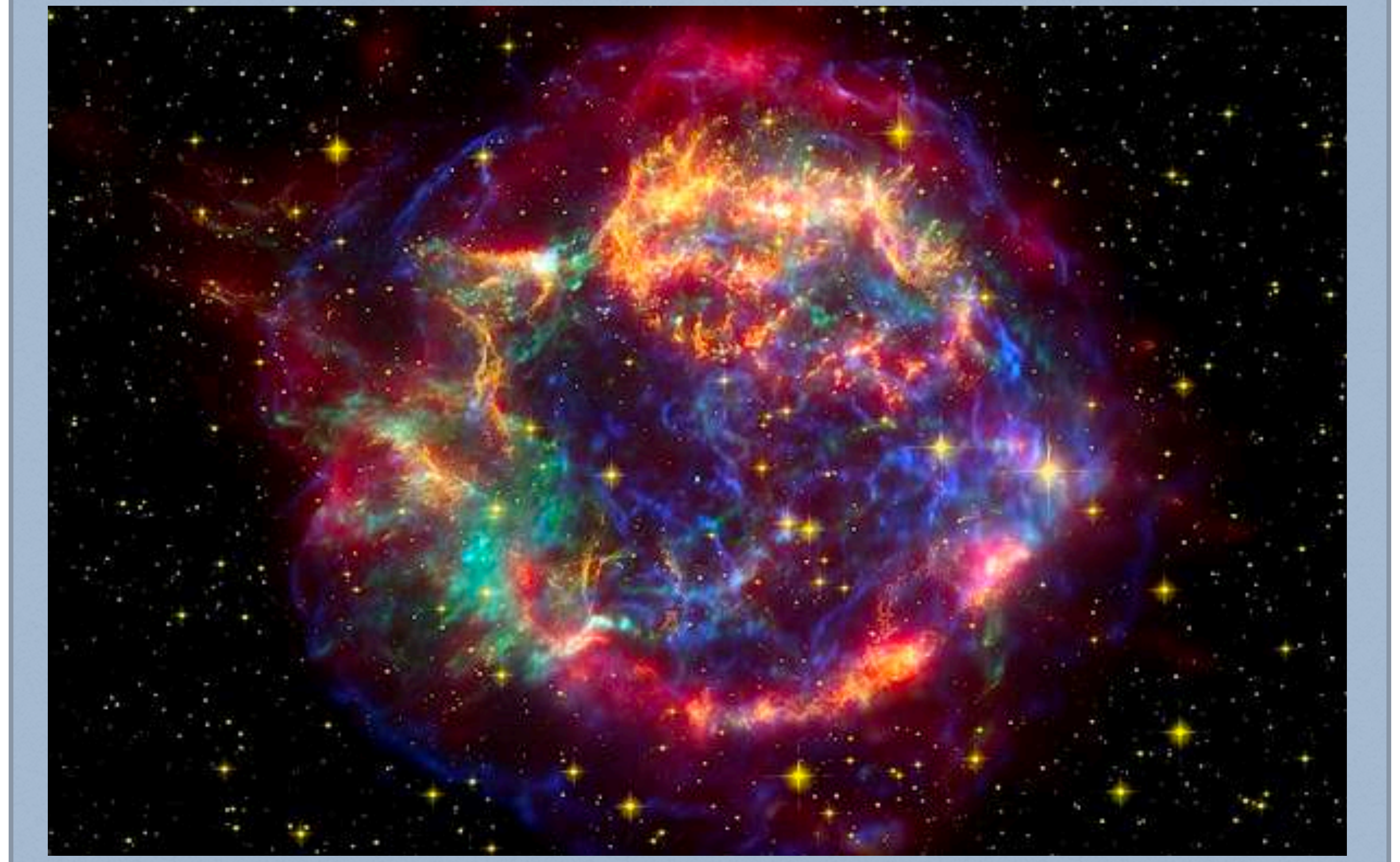
Nuclear and Particle Physics on a Quantum Computer: Where do we stand now?

*Trento, 6 June 2023 - **ECT* workshop***

Outline

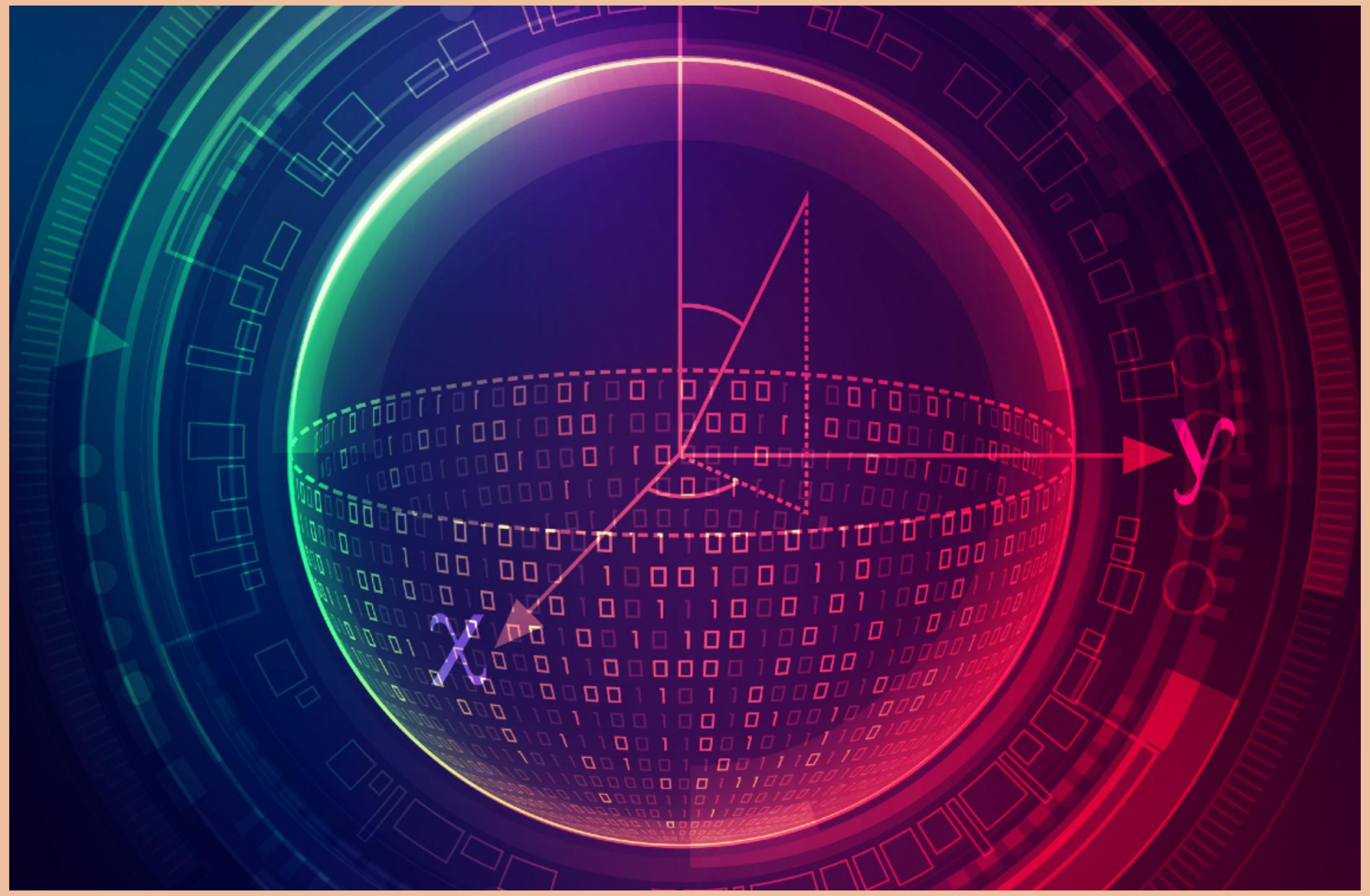
Introduction

- Motivation
- Physical description of the many-neutrino system in high density environment



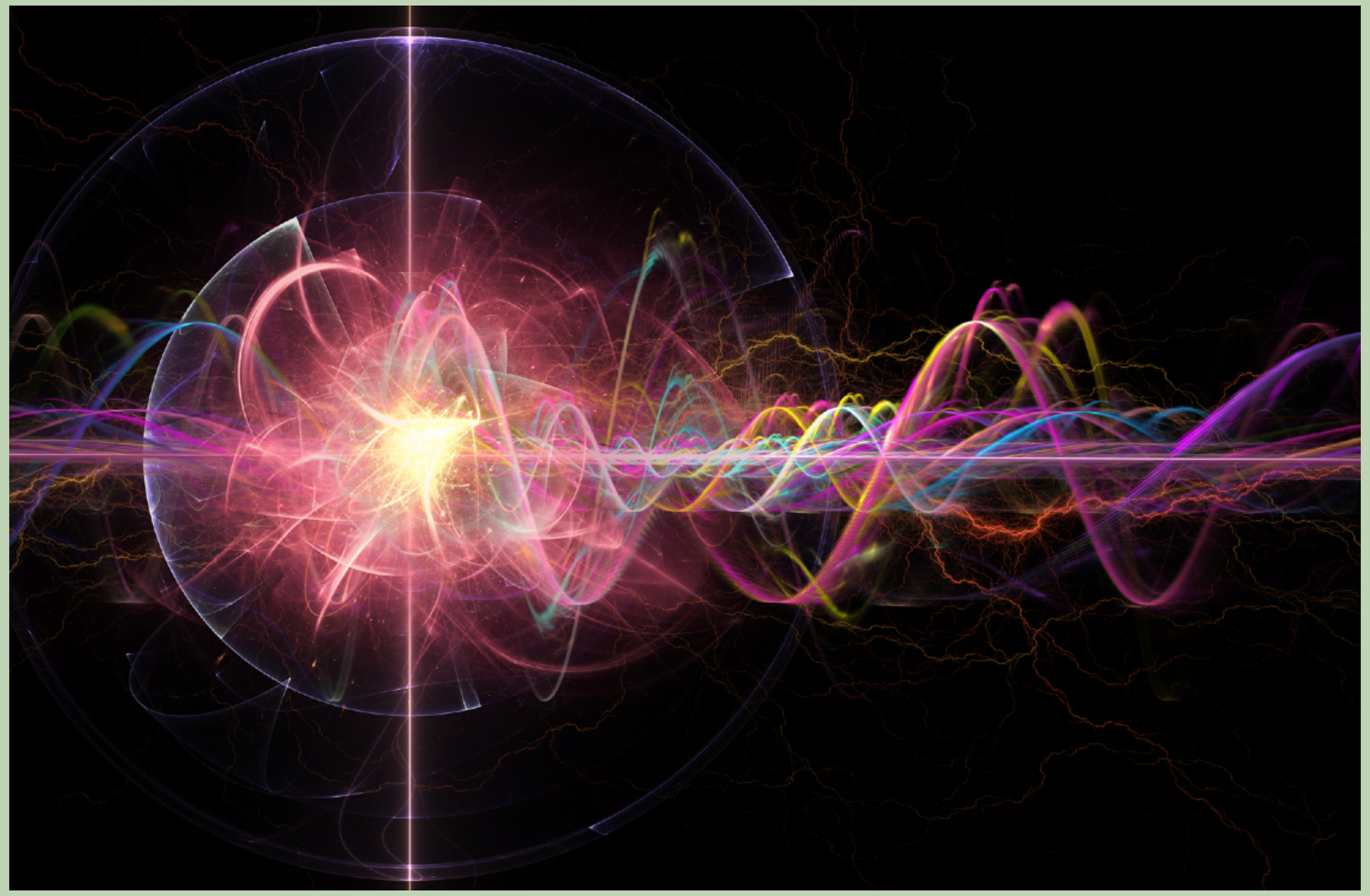
QC simulation

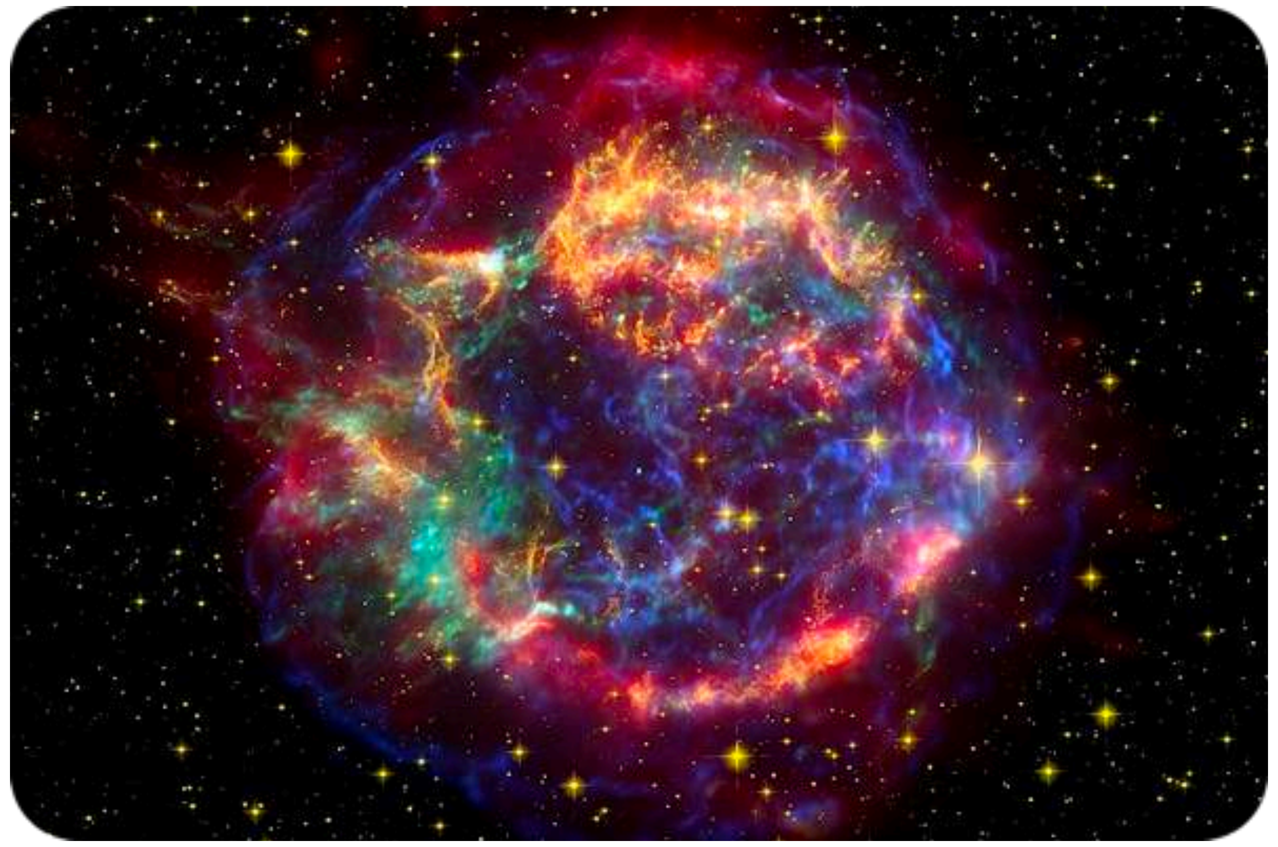
- Hamiltonian simulation
- The quantum algorithm implementation



Results

- Data from the real trapped-ion quantum machine: Quantinuum System Model

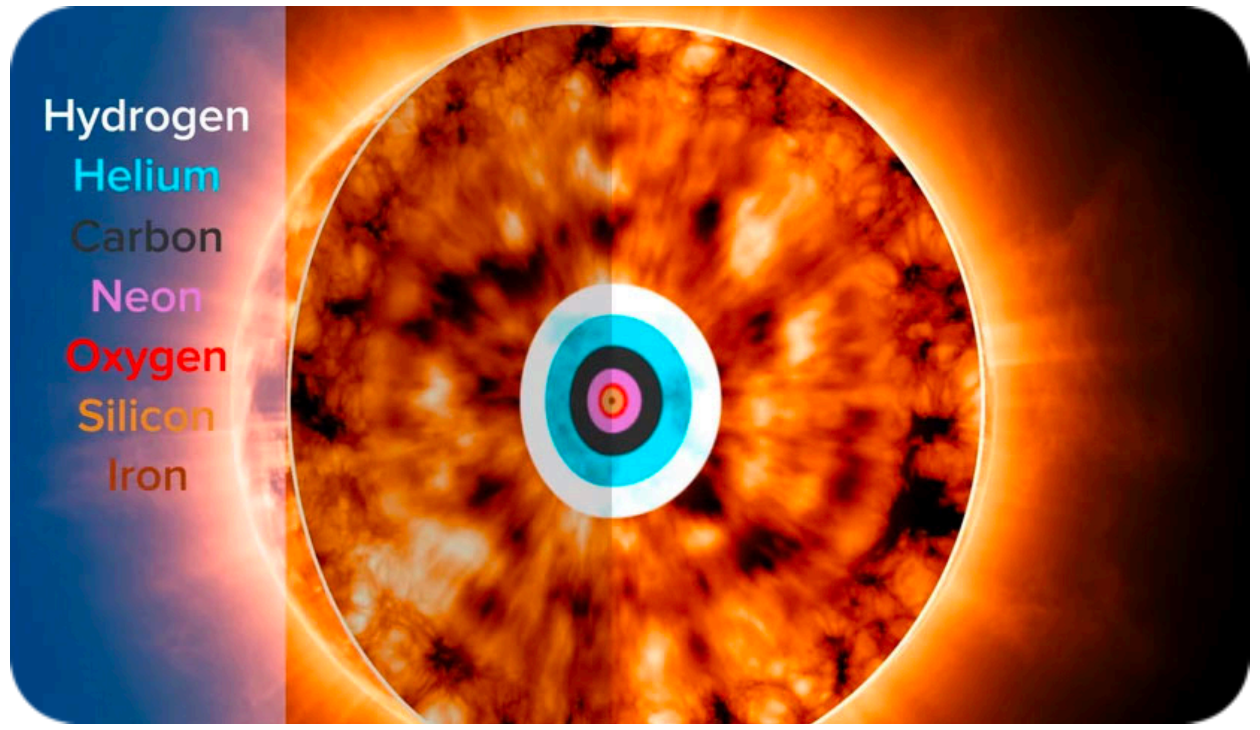




Core-collapse supernovae

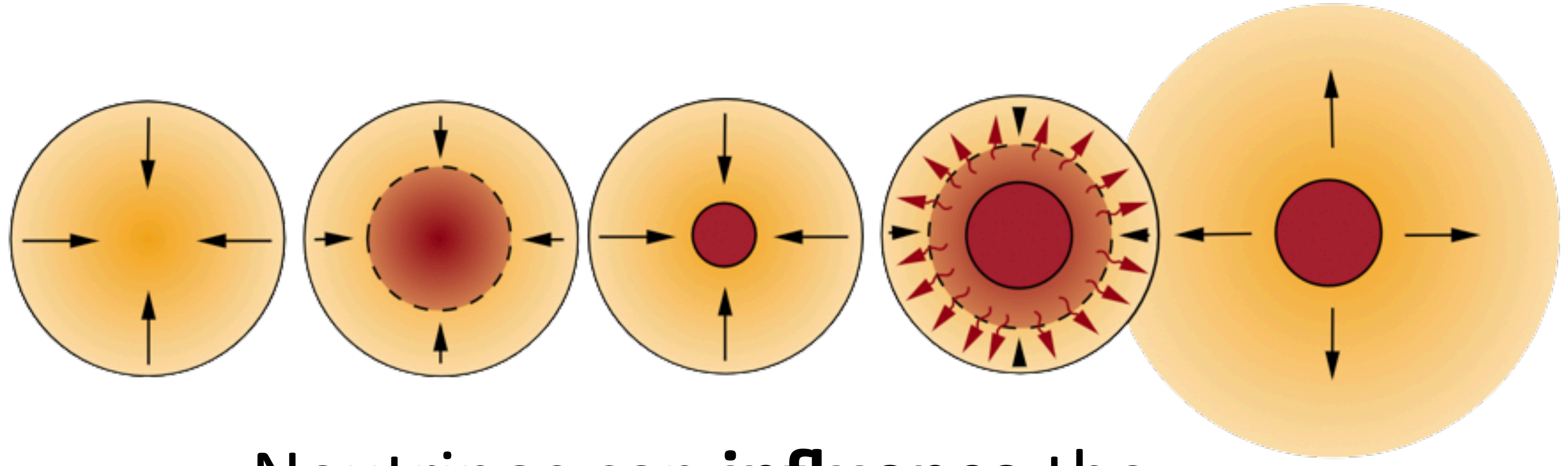
Neutrinos are **messengers of information** of physics under extreme conditions

Massive star mergers



Nucleosynthesis and in general weak interaction is **flavor-dependent**

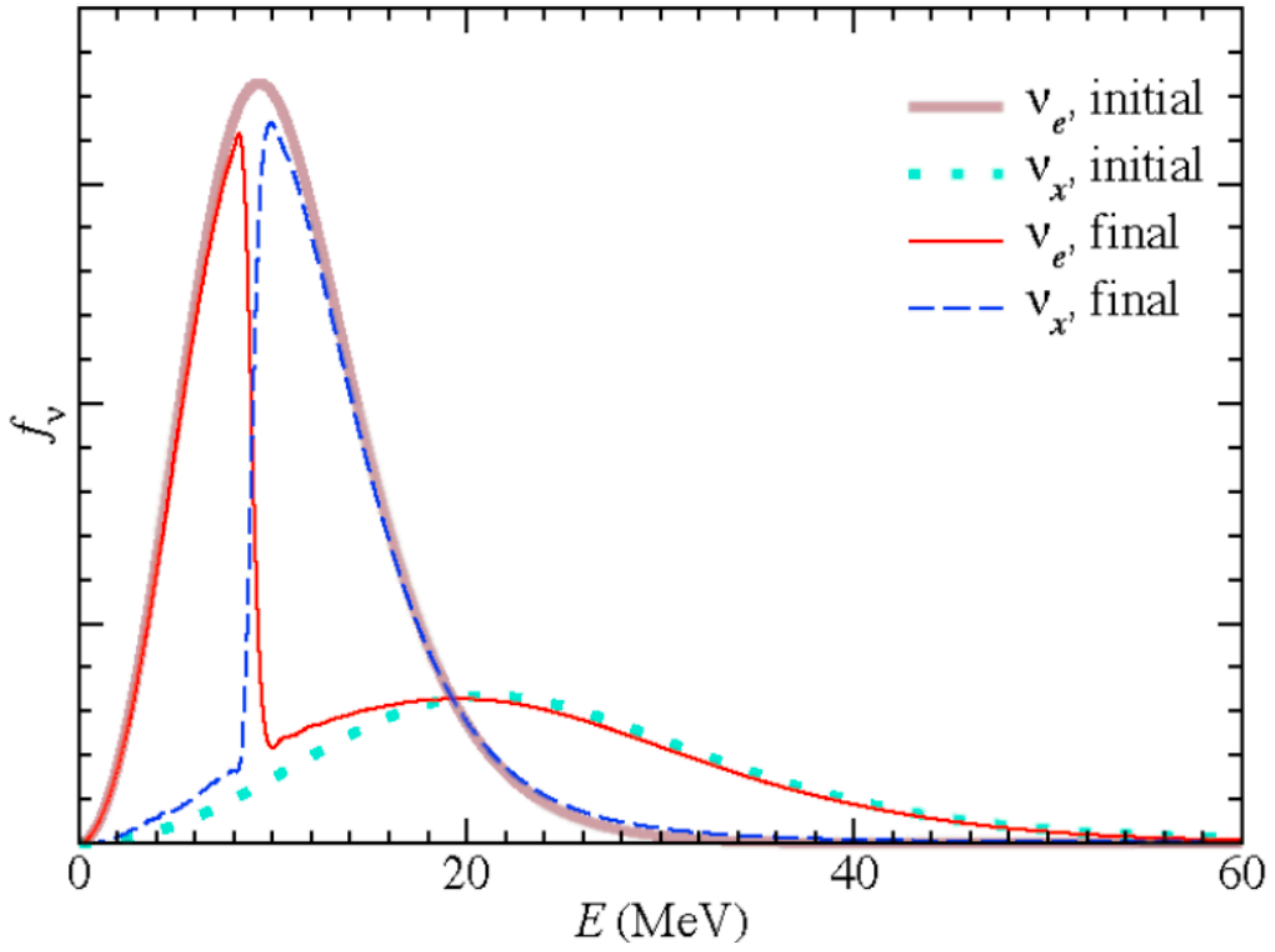
$$\nu_e + n \longleftrightarrow p + e^-$$
$$n \longleftrightarrow p + e^- + \bar{\nu}_e$$
$$\bar{\nu}_e + p \longleftrightarrow n + e^+$$

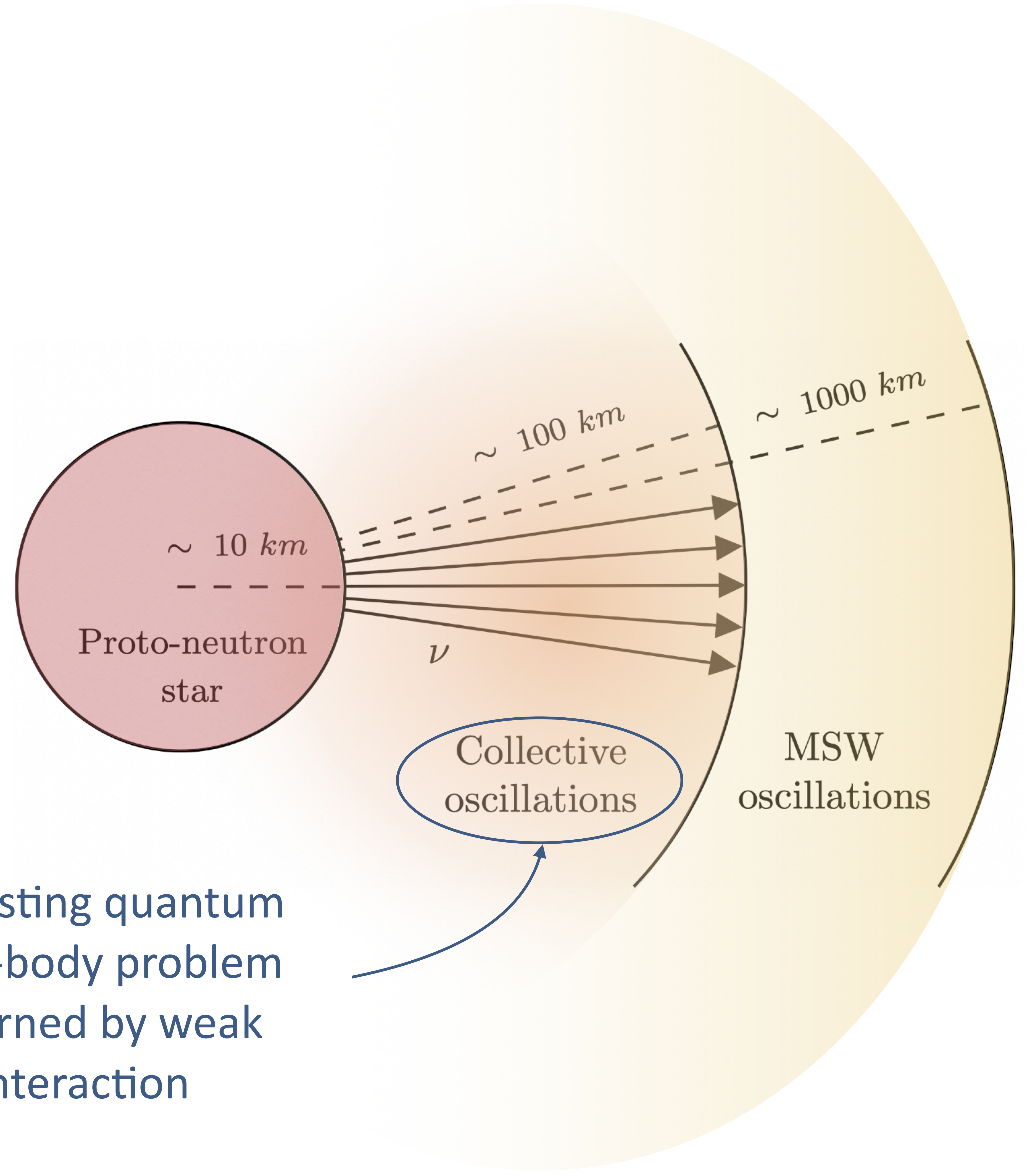


Neutrinos can **influence** the supernovae **explosion**

Spectral splits can happen at some distance from the emission sphere

Duan et. al (2006)





Interesting quantum many-body problem governed by weak interaction

- Massive stars $M \geq 8\, M_{\odot}$ explode releasing a huge amount of energy and neutrinos $\sim 10^{58}$
- Flavor Hamiltonian of many-neutrino system

$$H = H_{vac} + H_{\nu e} + H_{\nu\nu}$$

Vacuum: Mass eigenstates $\neq$ flavor eigenstates

MSW: Scattering with matter

$\nu\nu$-interaction: Forward scattering

Two-flavor Hamiltonian (SU(2) model): the flavor state of a neutrino is a **flavor isospin** $|\nu\rangle = \alpha|\nu_e\rangle + \beta|\nu_x\rangle$

$$H = H_{vac} + H_{\nu\nu}$$

1-body term

$$H_{vac} = \Delta \sum_{i=1}^{N} \vec{b}\cdot\vec{\sigma}_i$$

$$\Delta = \frac{\delta m^2}{4E}$$

$$\vec{b} = (\sin(2\theta_\nu), 0, -\cos(2\theta_\nu))$$

$$\vec{\sigma} = (X, Y, Z)$$

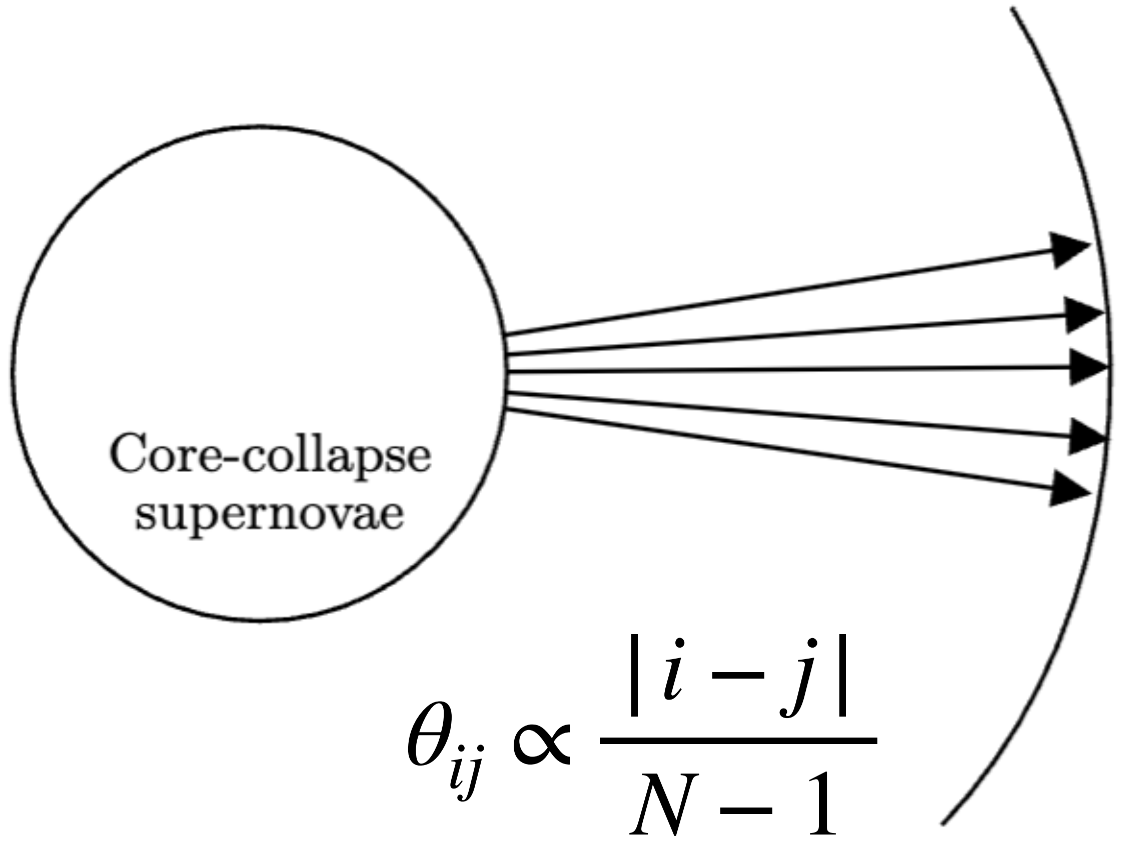
2-body term

$$H_{\nu\nu} = \frac{\mu}{N}\sum_{i<j}^{N} J_{ij}\vec{\sigma}_i\cdot\vec{\sigma}_j$$

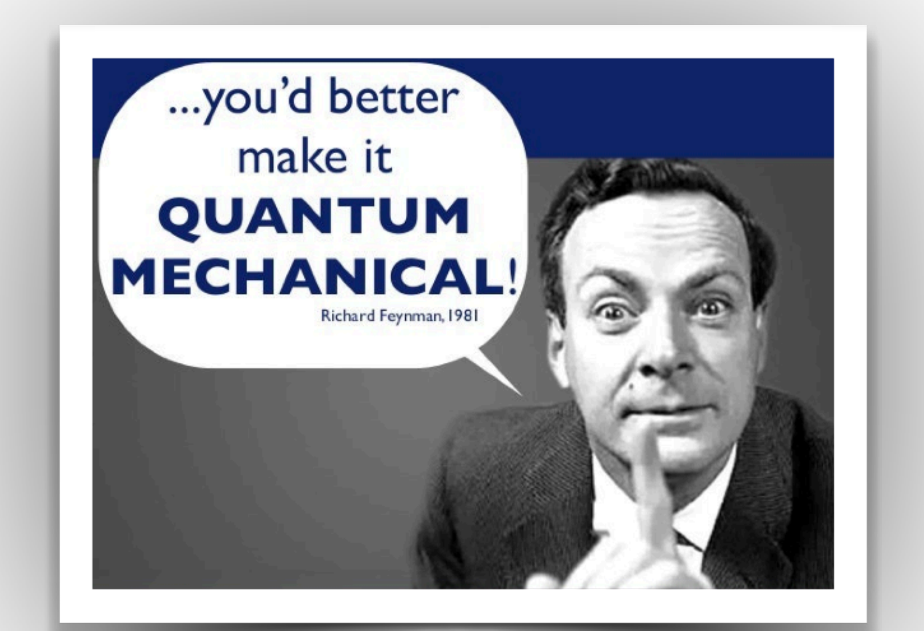
$$\mu = \sqrt{2}G_F n_\nu$$

$$J_{ij} = 1 - \cos(\theta_{ij})$$

$$\cos(\theta_{ij}) = \frac{\vec{p}_i\cdot\vec{p}_j}{\|\vec{p}_i\|\|\vec{p}_j\|}$$



$$\theta_{ij} \propto \frac{|i-j|}{N-1}$$



! Simulating the full dynamics is difficult using classical resource

V. Amitrano et. al. Phys. Rev. D 107, 023007 (2023)

We want to simulate the flavor evolution

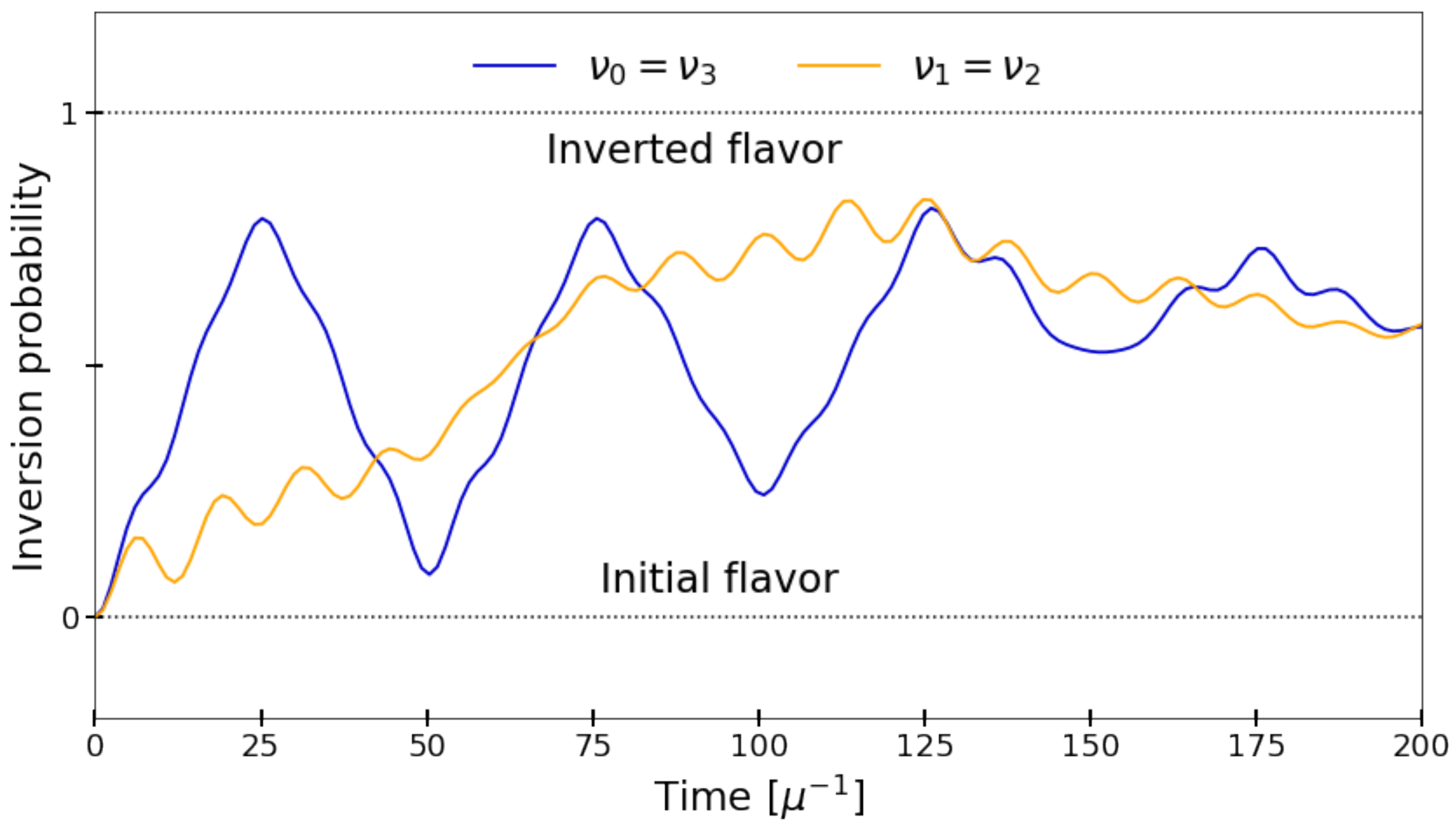
- Initial state $|\Psi_0\rangle = |\nu_e\rangle^{\otimes N/2} \otimes |\nu_x\rangle^{\otimes N/2}$
- Evolved state $|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$
- $\langle\nu_e|Z|\nu_e\rangle = 1$ and $\langle\nu_x|Z|\nu_x\rangle = -1$
- Measure the probability to be in the inverted flavor as a function of time

$$P_{inv}^{(i)}(t) = \frac{|\langle Z_i(0)\rangle - \langle Z_i(t)\rangle|}{2}$$

- Note the symmetry under particle exchange
  - Symmetric Hamiltonian
  - Anti-symmetric initial state
    - $\nu_k \longleftrightarrow \nu_{N-1-k}$

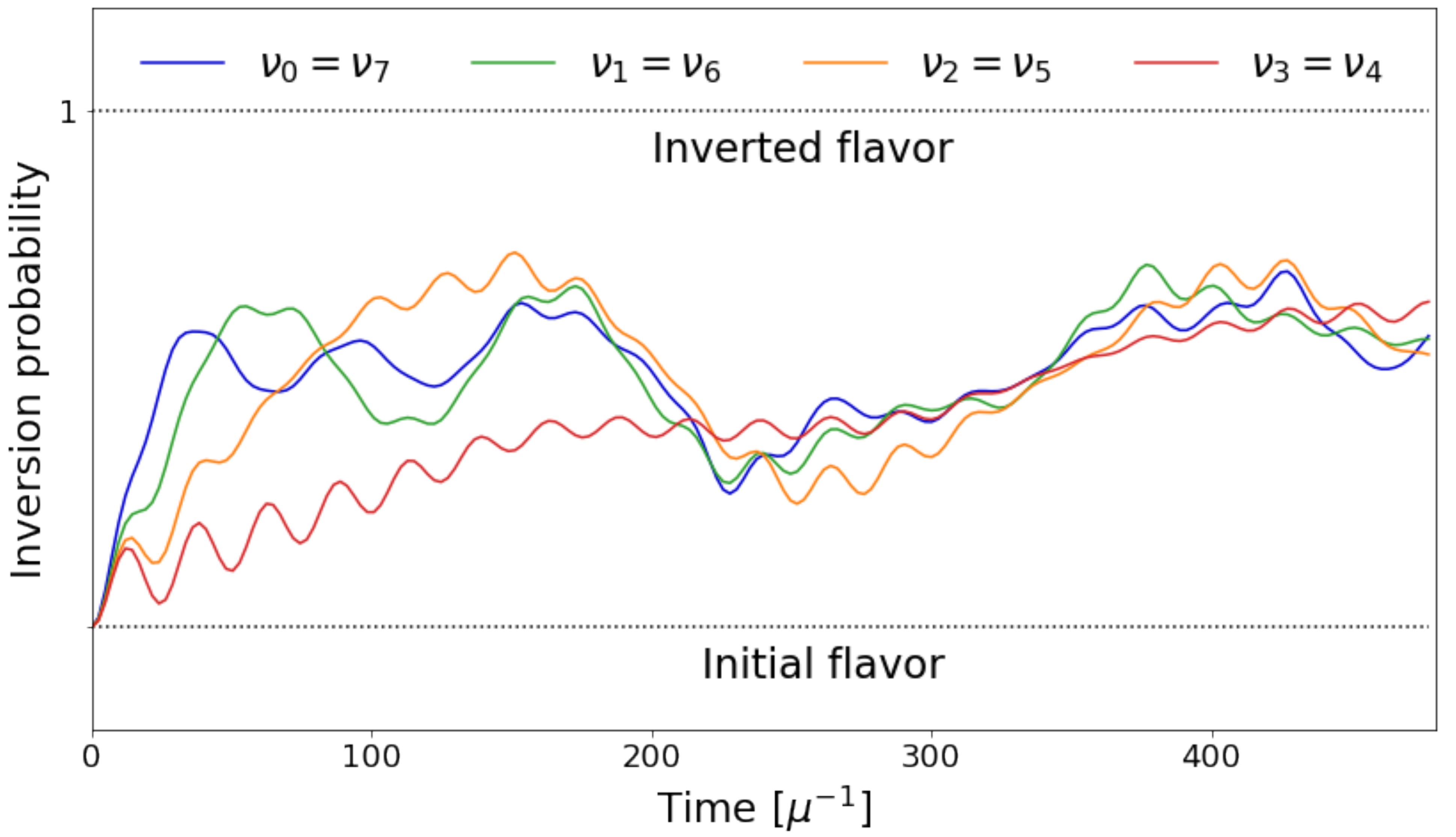
$N = 4$

$2^4 \times 2^4$ matrix multiplications



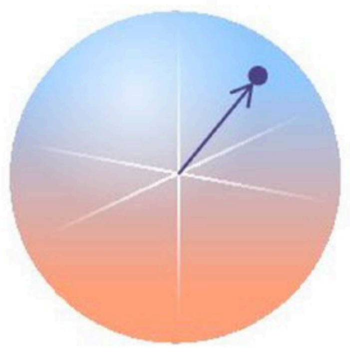
$N = 8$

$2^8 \times 2^8$ matrix multiplications



1° ingredient:
Encoding map

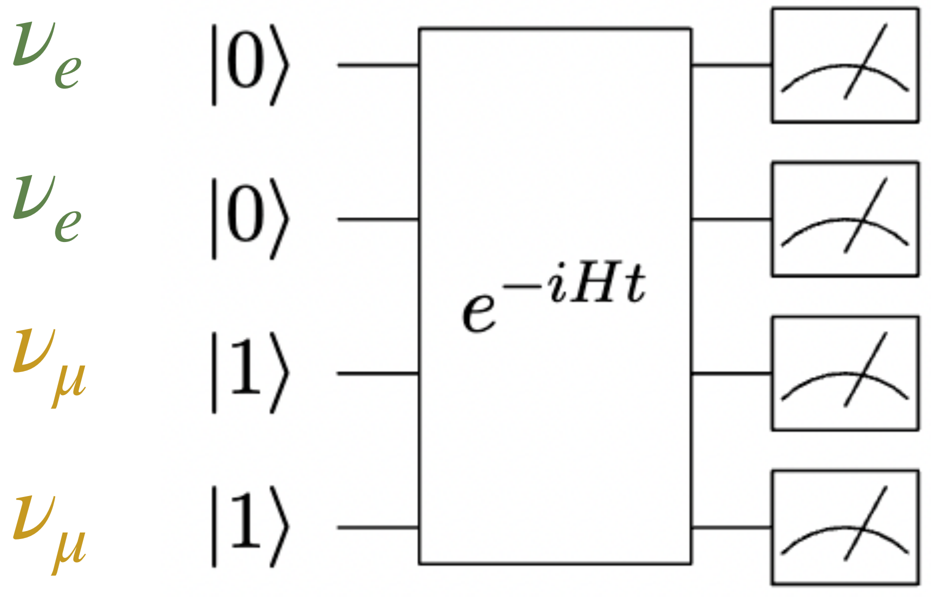
$|\nu_e\rangle \mapsto |0\rangle$



$|\nu_x\rangle \mapsto |1\rangle$

- Two-flavor approximation $|\nu\rangle = \alpha|\nu_e\rangle + \beta|\nu_x\rangle$
- Qubit state $|\nu\rangle = \alpha|0\rangle + \beta|1\rangle$
- N neutrinos encoded into N qubits

2° ingredient:
Unitary implementation

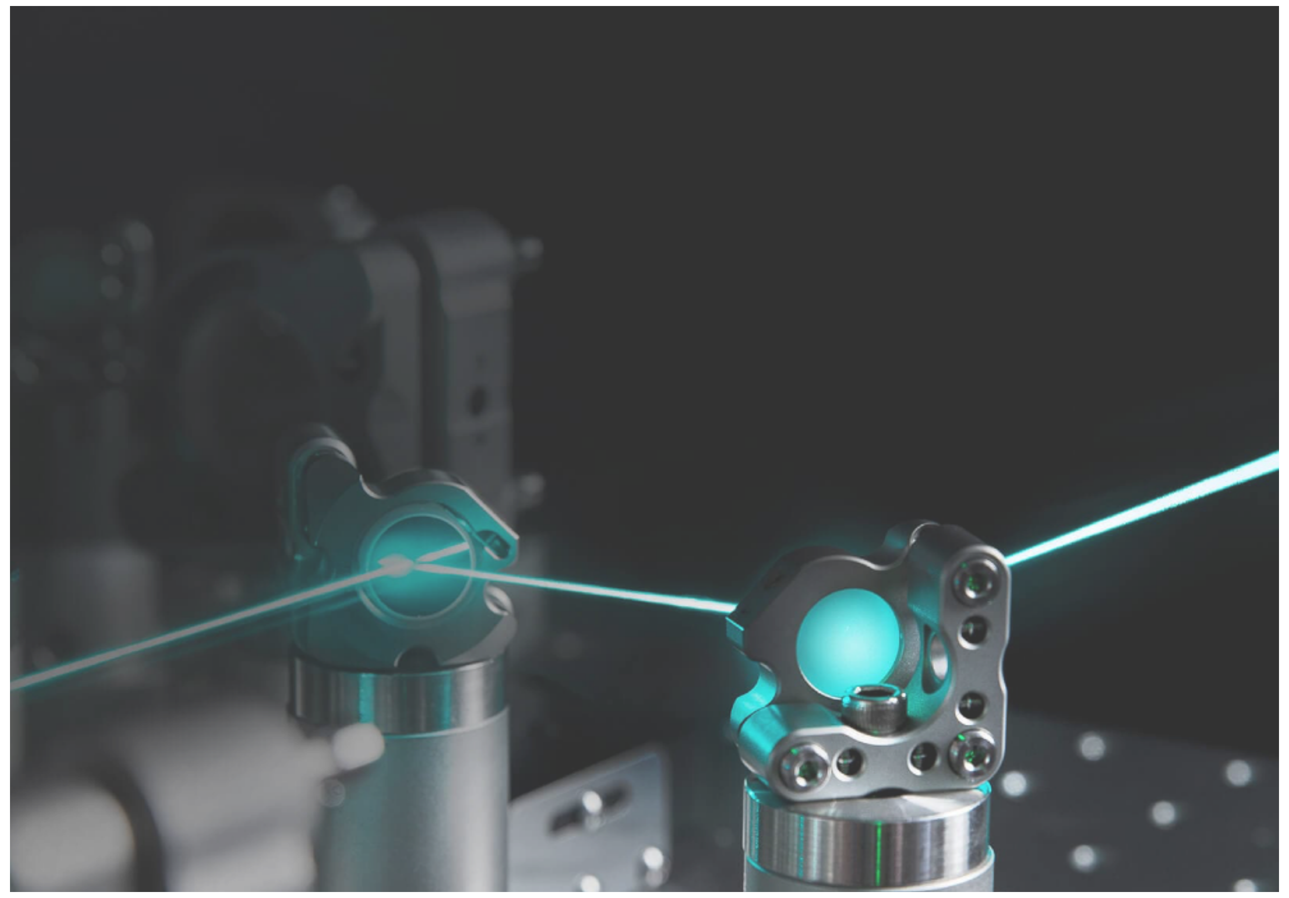


- Implement the propagator $U(t) = e^{-iHt}$ generated by the Hamiltonian

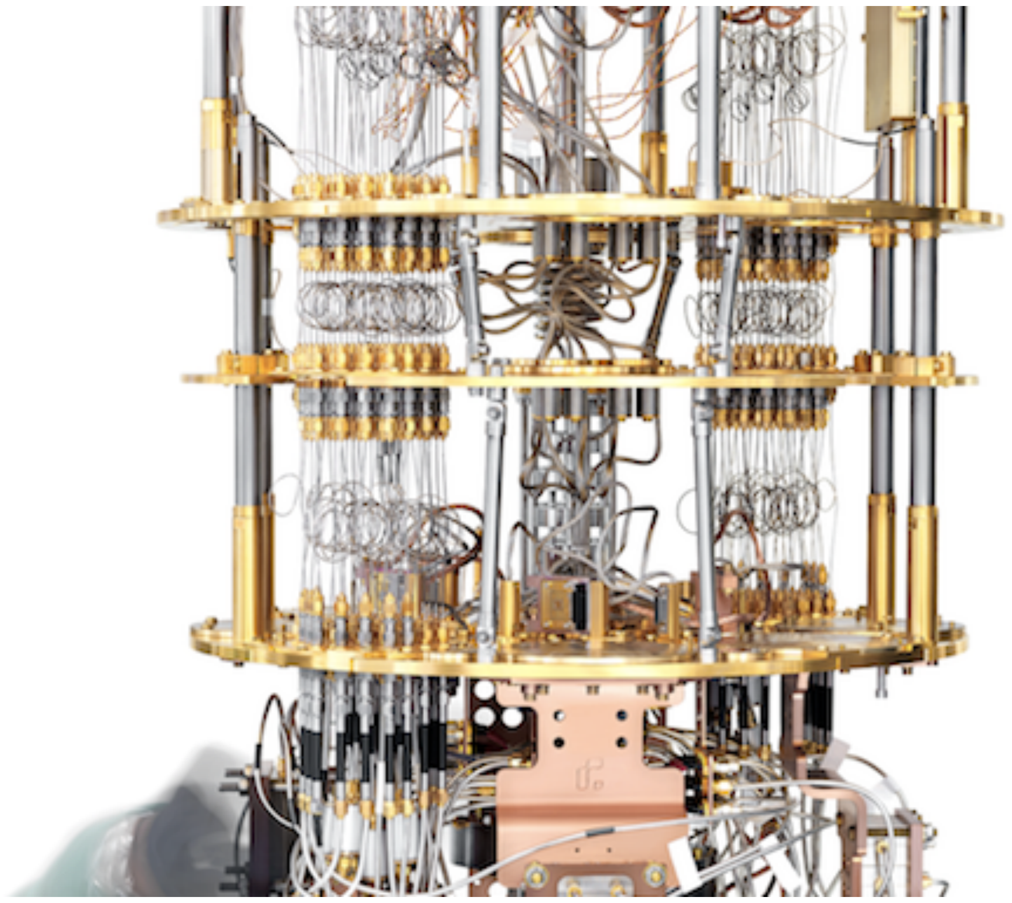
$$H = \sum_i \vec{b} \cdot \vec{\sigma}_i + \sum_{i<j} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j$$

- **Quantum gate decomposition** procedure to obtain a quantum circuit
- Exponential number of operations in general… we need to optimize it!
- All-to-all interactions are difficult with reduced connectivity

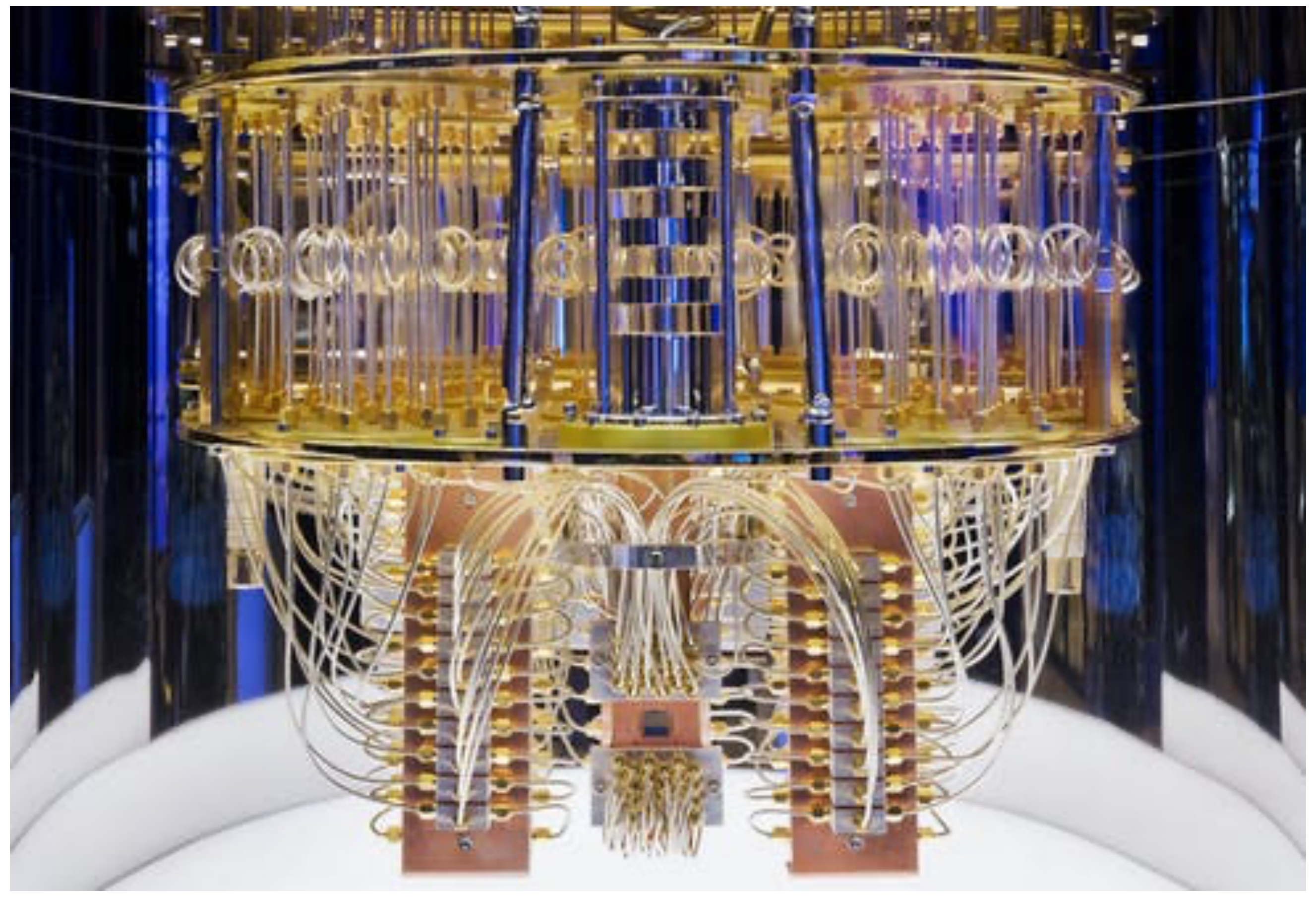
- Different **qubit**
  - Superconductive circuit
  - Trapped ions
- Different universal **gate set**
  - Circuit optimization
  - More control on what we are running
- Different qubit **connectivity**
  - Linear
  - All - to - all
  - Etc…



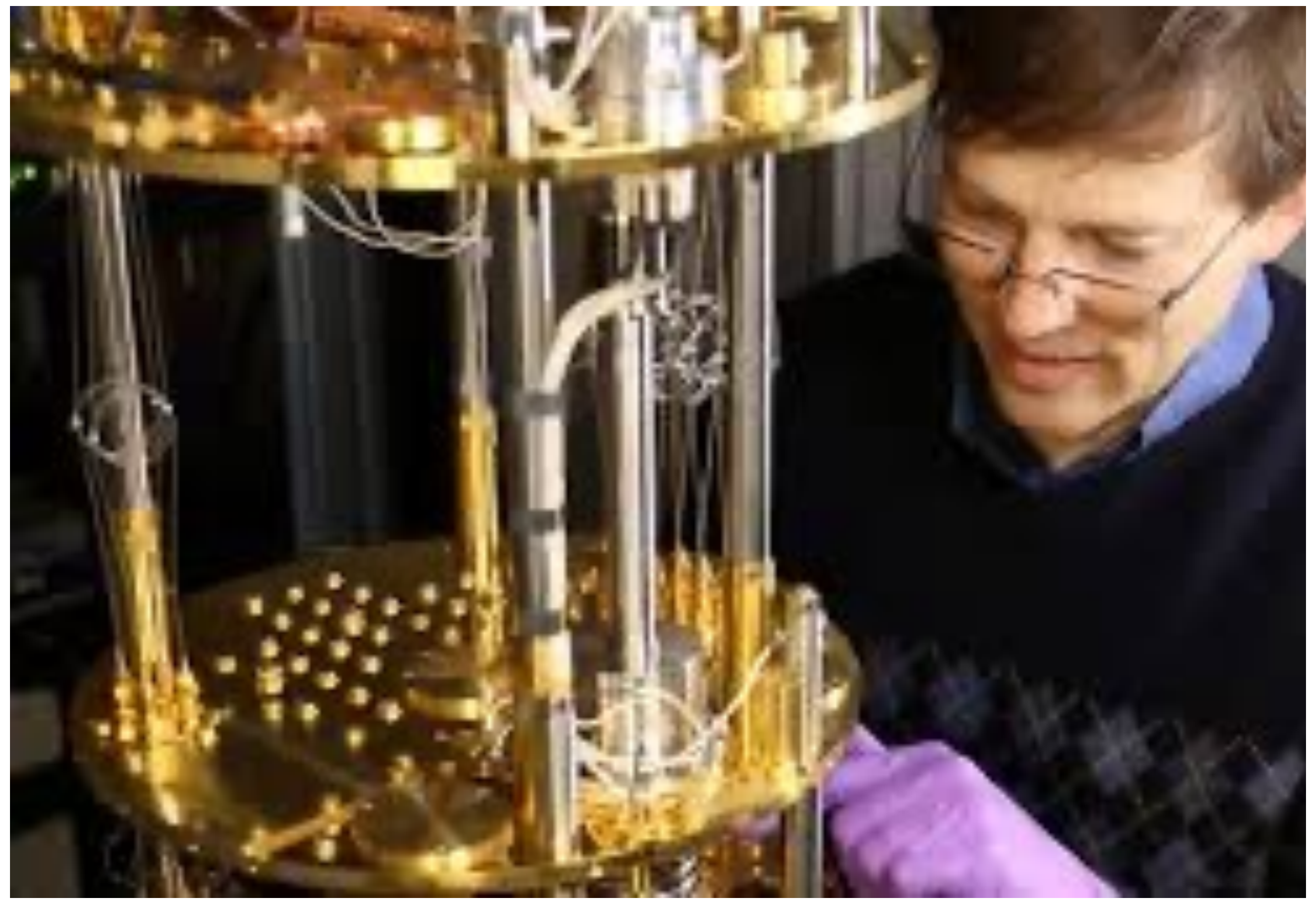
Honeywell Quantum



Rigetti Quantum



IBM Quantum

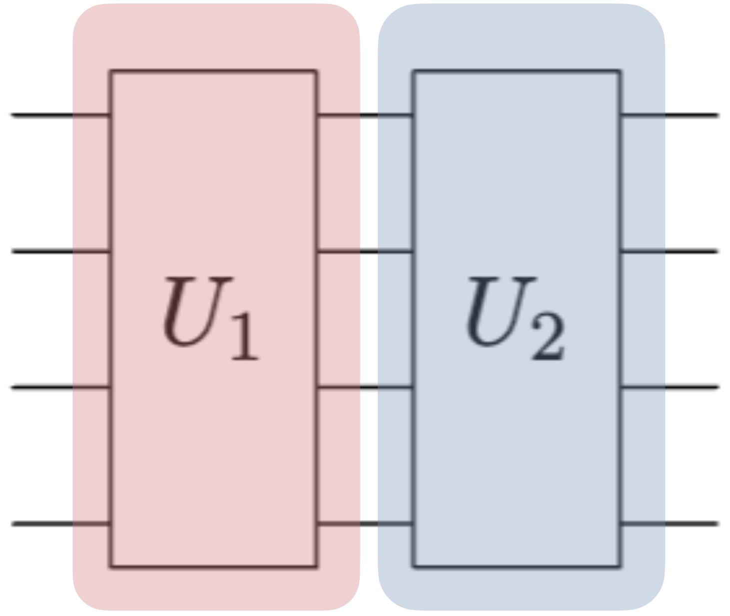


LLNL testbed

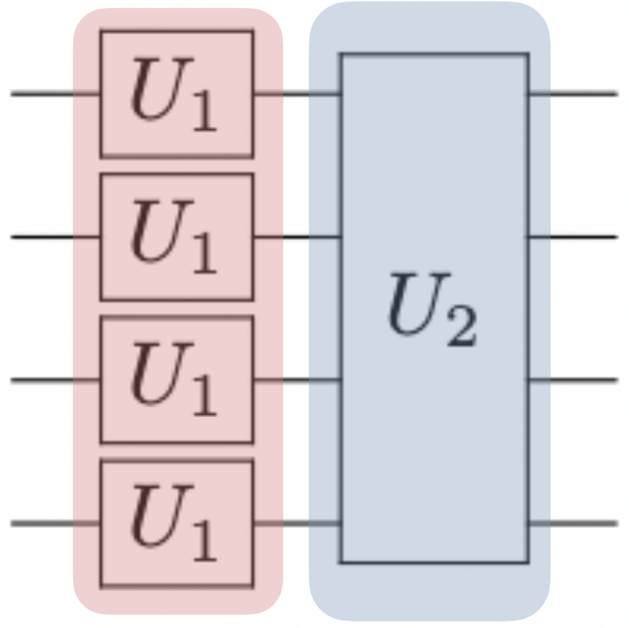
Trapped ions are perfect for the collective neutrino problem

The Unitary Implementation: gate decomposition

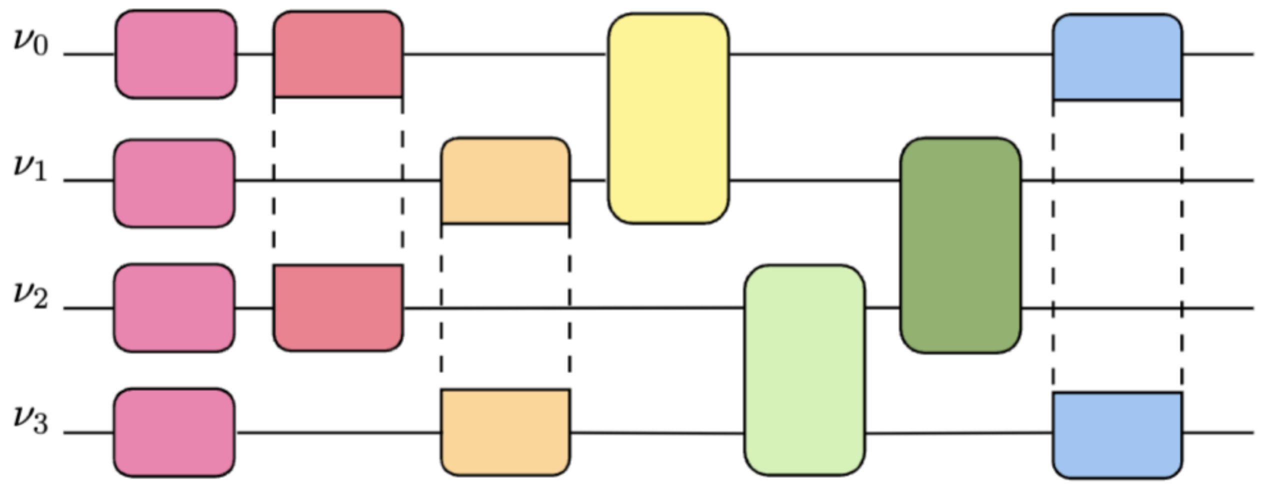
$$U(dt) = U_2(dt)U_1(dt)$$



$$U_1(dt) = \bigotimes_i u_i(t)$$



$$U_2(dt) = \prod_{i<j} u_{ij}(t)$$



The total hamiltonian is

$U(dt) = e^{-i(H_{vac}+H_{\nu\nu})dt}$ and we can split 1-body and 2-body parts without error because $\left[H_{vac}, H_{\nu\nu}\right] = 0$

The 1-body part is simple $U_1(dt) = e^{-iH_{vac}dt} = e^{-i\sum_i h_i dt}$ where $h_i = \vec{b} \cdot \vec{\sigma}_i$ and $[h_i, h_j] = 0$ so we have exactly

$$U_1(dt) = \prod_i e^{-ih_i dt} = \prod_i u_i(dt)$$

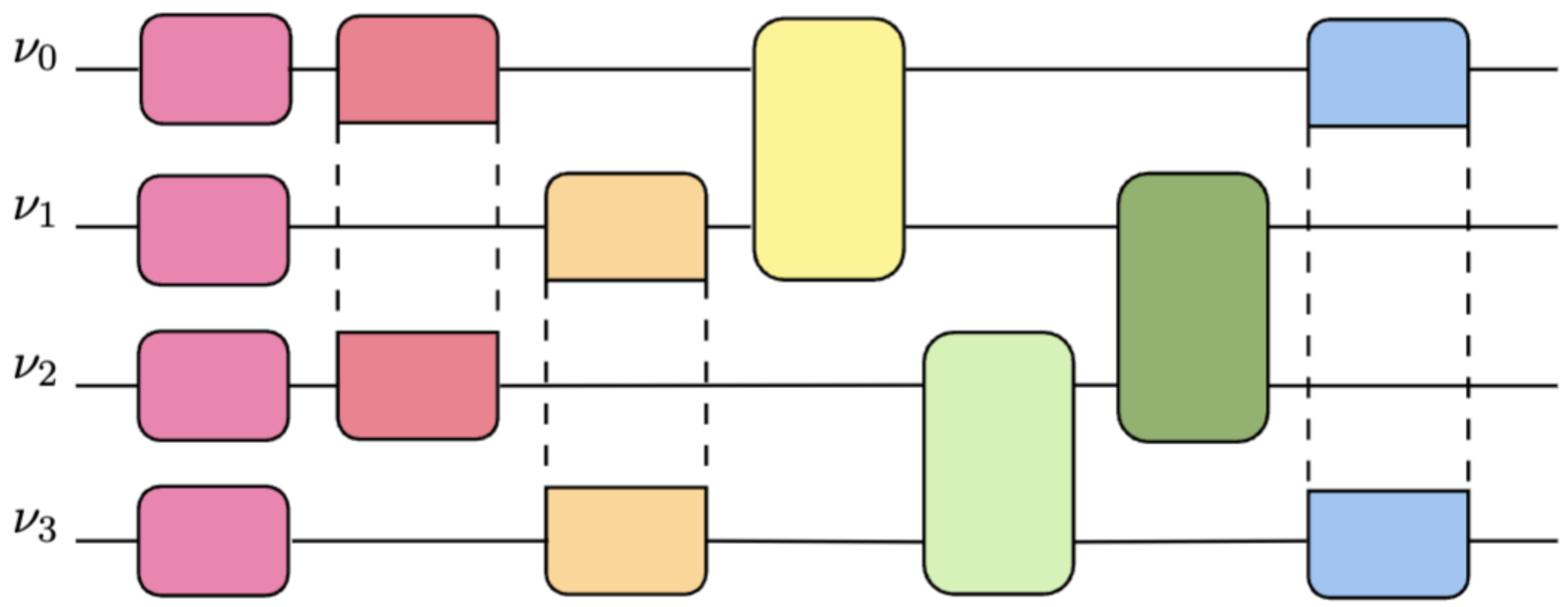
The 2-body part is more tricky $U_2(t) = e^{-iH_{\nu\nu}t} = e^{-i\sum_{i<j} h_{ij}t}$ where $h_{ij} = J_{ij}\vec{\sigma}_i \cdot \vec{\sigma}_j$ and $[h_{ij}, h_{ik}] \neq 0.$

We approximate it in pairs

$$U_2(t) \approx \prod_{i<j} e^{-ih_{ij}t} = \prod_{i<j} u_{ij}(t)$$

with an error $\sim \mathcal{O}(t^2)$

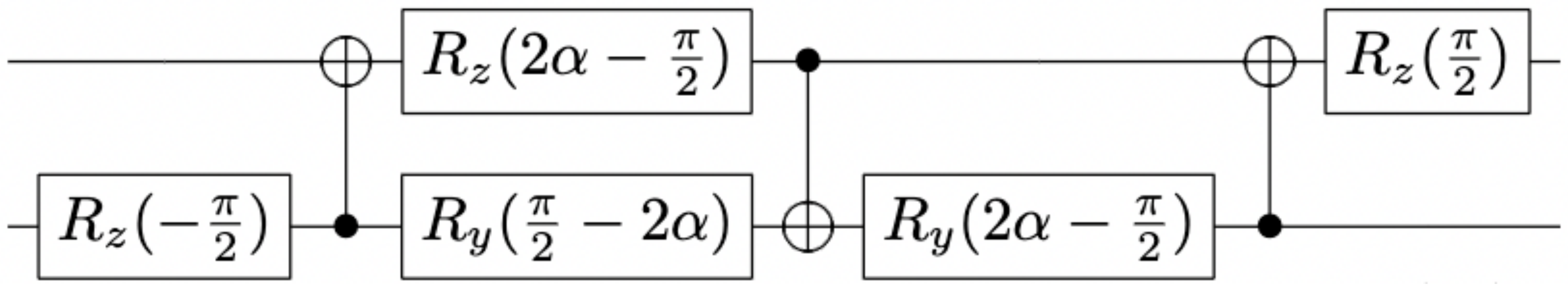
Optimal Circuit for full connectivity



Each pair propagator is simple:

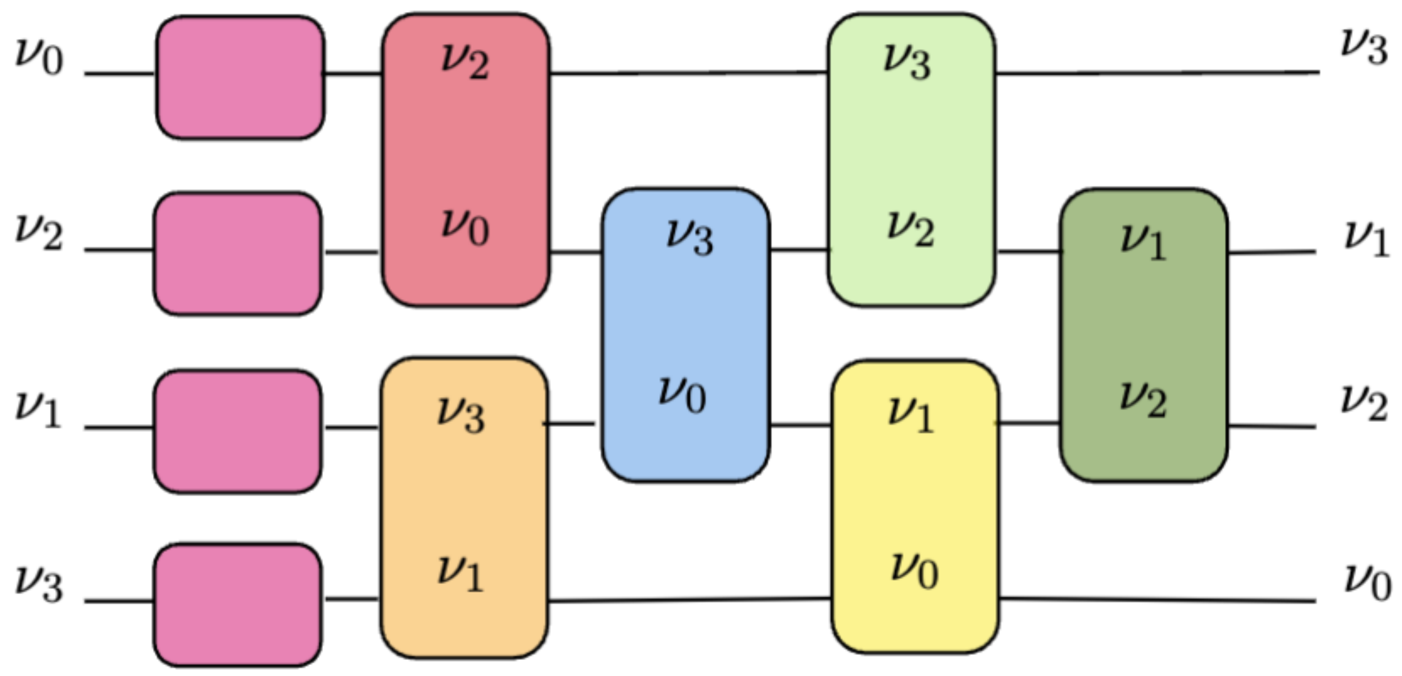
$$u_{ij}(t) = e^{-iJ_{ij}(X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j)t}$$

and has the following optimal CNOT-based circuit where$\alpha = -\, dtJ_{ij}$



F. Vatan and C. Williams (2004)

Swap network for linear connectivity

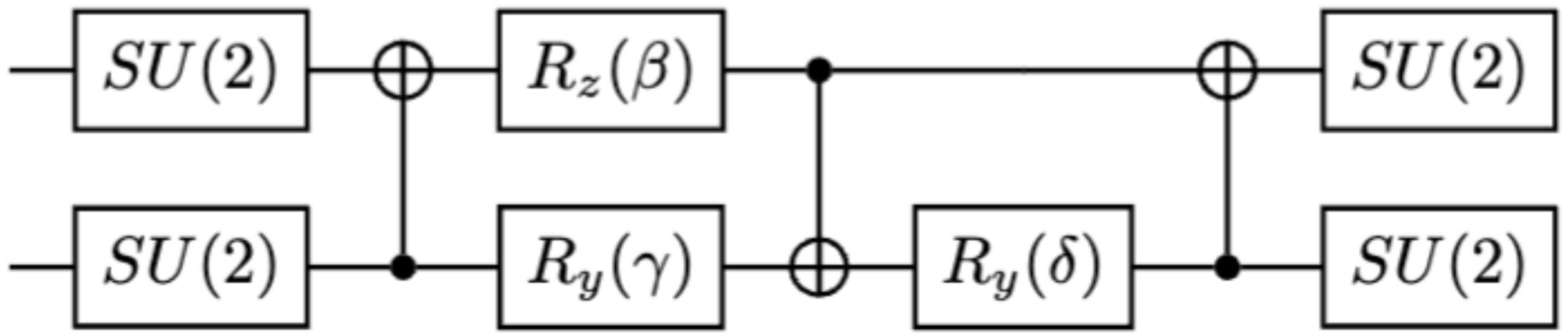


Hall, A. Roggero et. al (2021)

Each pair propagator contains also a SWAP operation:

$$w_{ij}(t) = \text{SWAP}_{ij}\; u_{ij}(t)$$

And it requires more single qubit rotations



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Optimal Circuit for full connectivity

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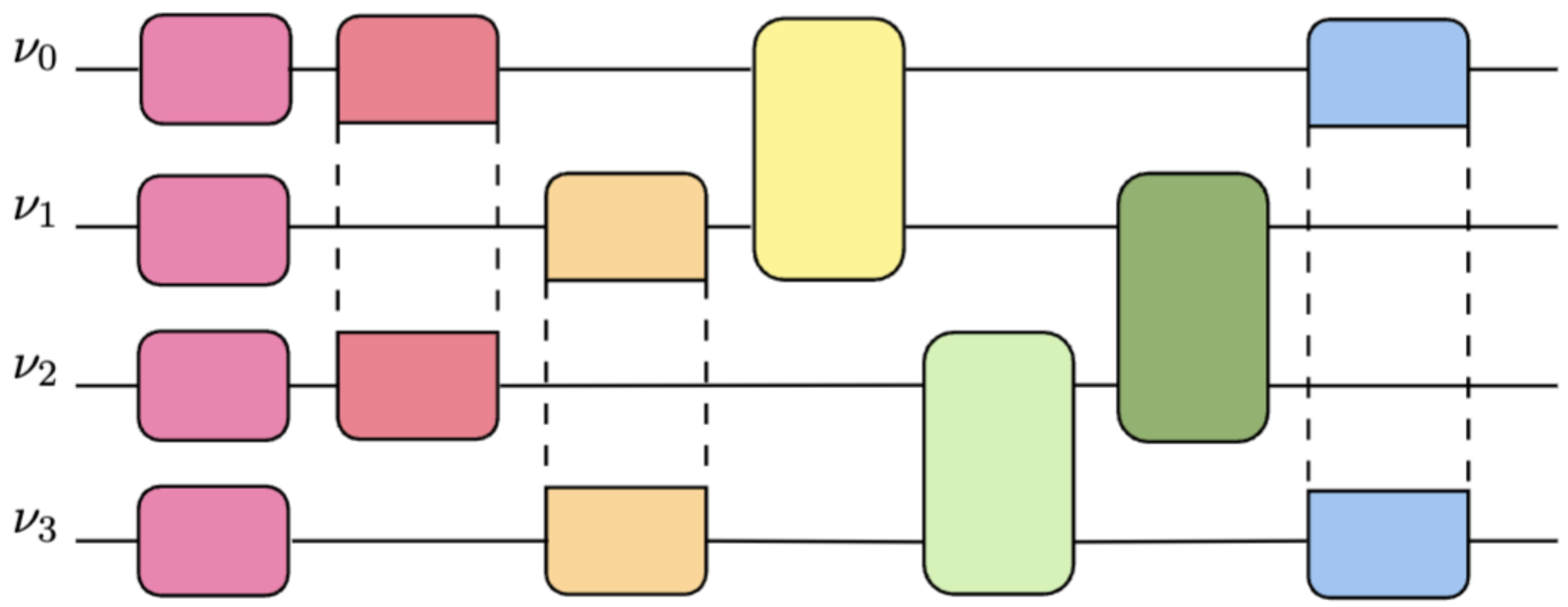
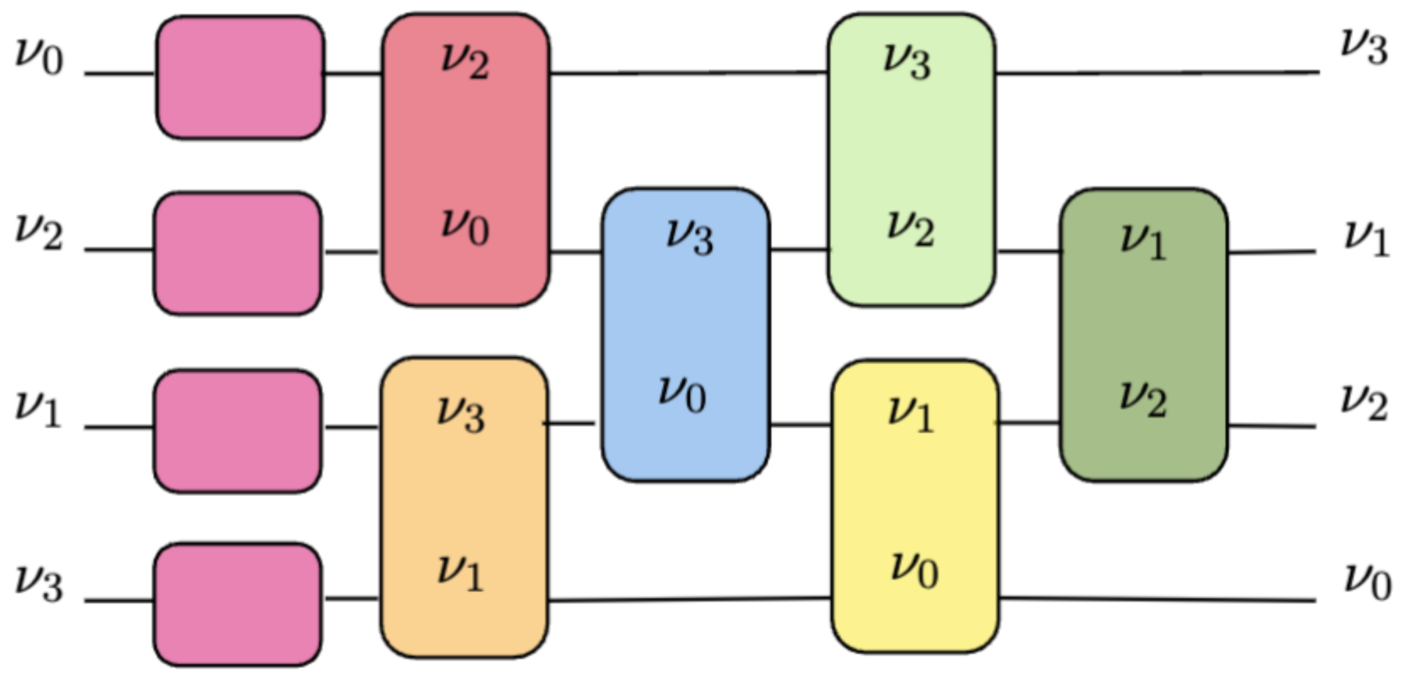
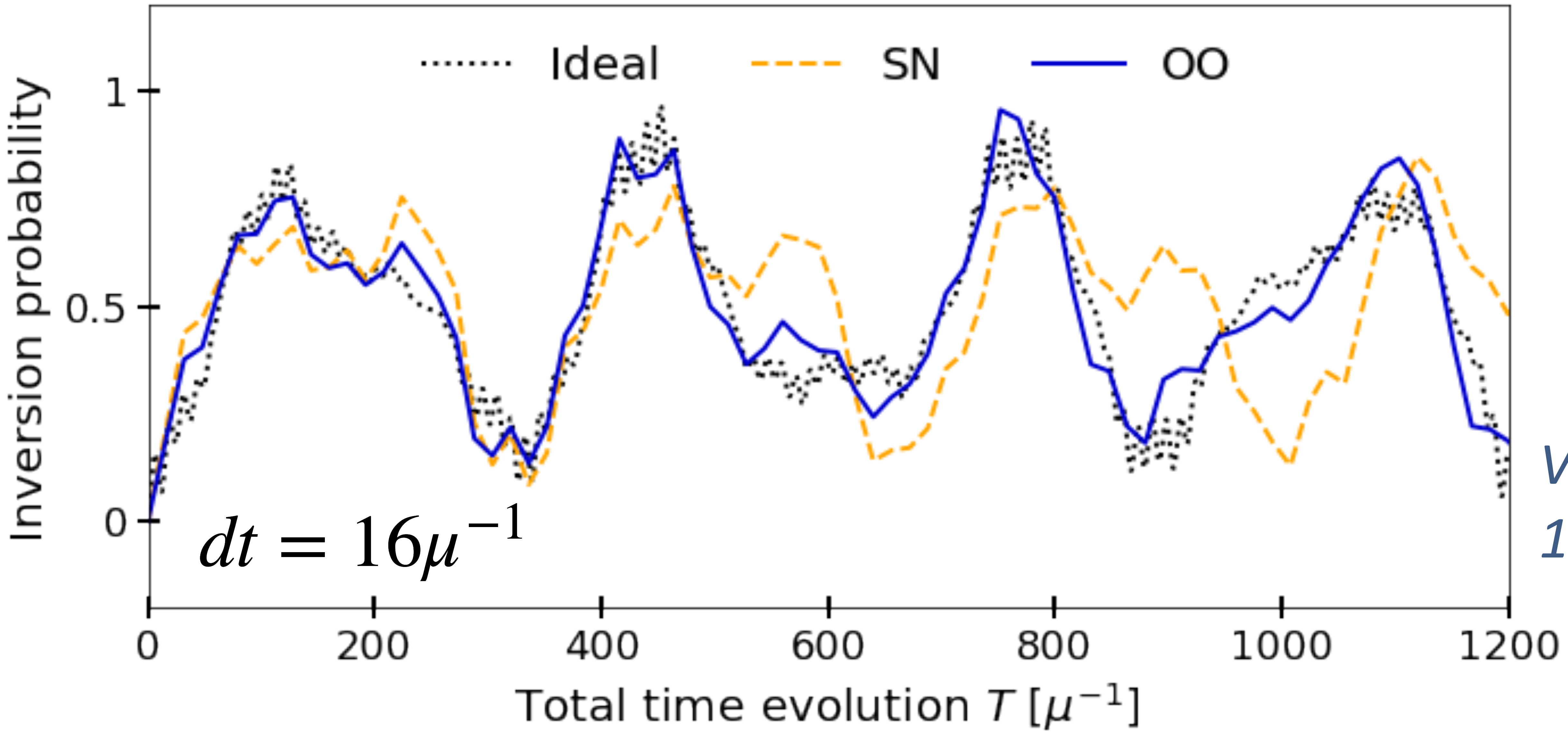


Full freedom in the pair ordering



Hall, A. Roggero et. al (2021)

Full freedom in the initial encoding, but the pair ordering is fixed. Any order can be obtain at the price of adding additional SWAP gates.



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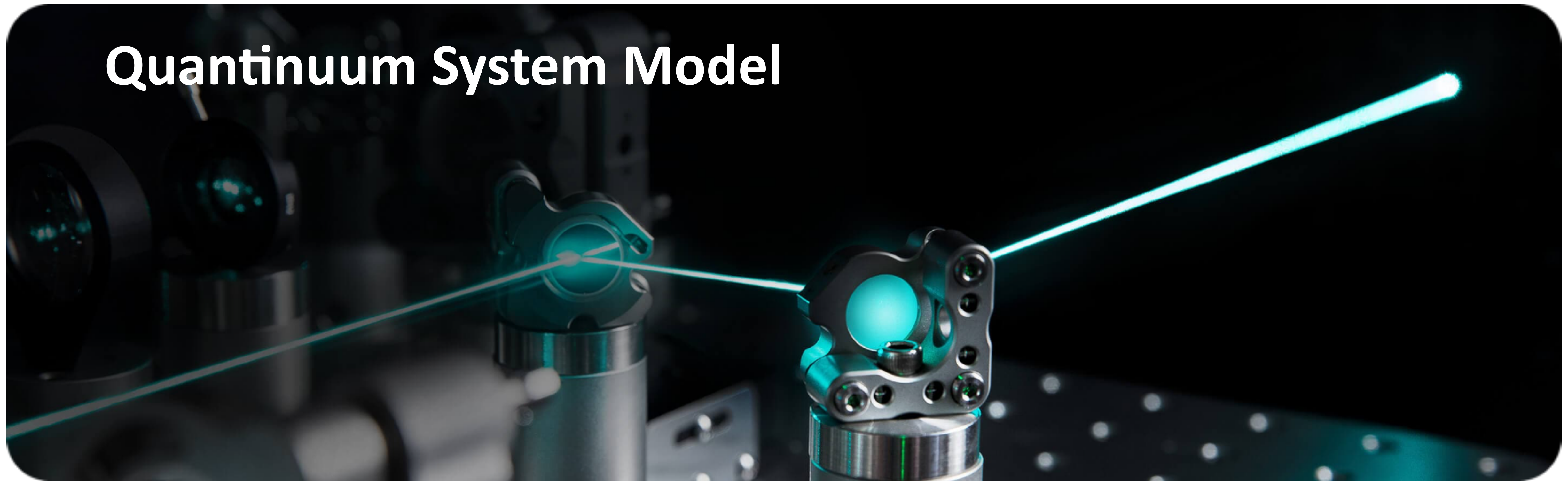
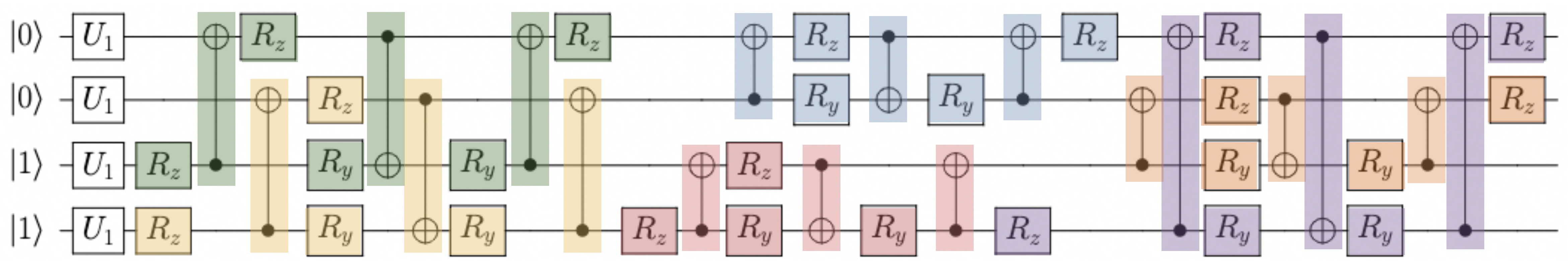
Quantinuum System Model (QSM) H1-2

- Trapped-ion device
- Full-connected qubits
- High fidelity: $\varepsilon_q \sim 10^{-4}$ and $\varepsilon_{qq} \sim 10^{-3}$

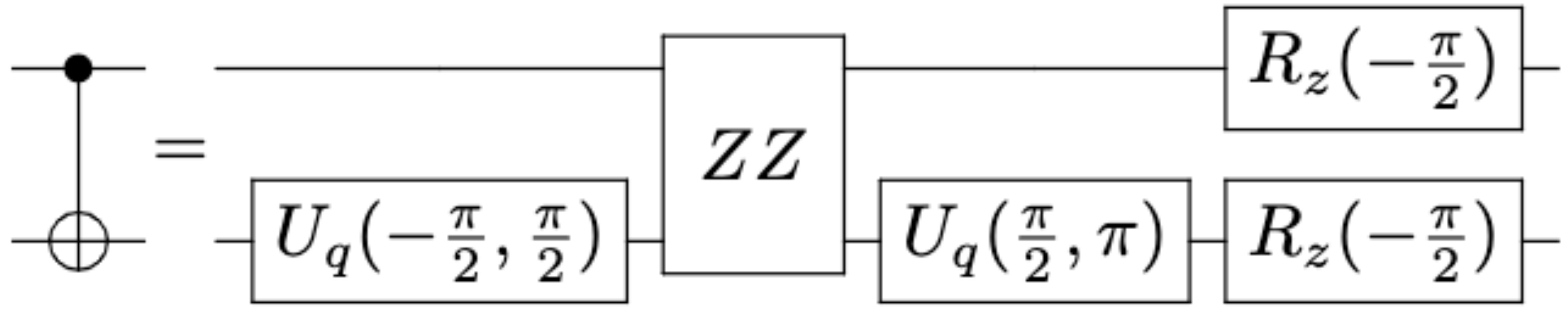
Machine aware compilation:

- Qubit topology
- Quantum gate set
  - $R_z(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}$
  - $U_q(\theta, \varphi) = \begin{pmatrix} \cos\theta/2 & -ie^{-i\varphi}\sin\theta/2 \\ -ie^{i\varphi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}$
  - $ZZ = e^{-i\frac{\pi}{4}Z\otimes Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

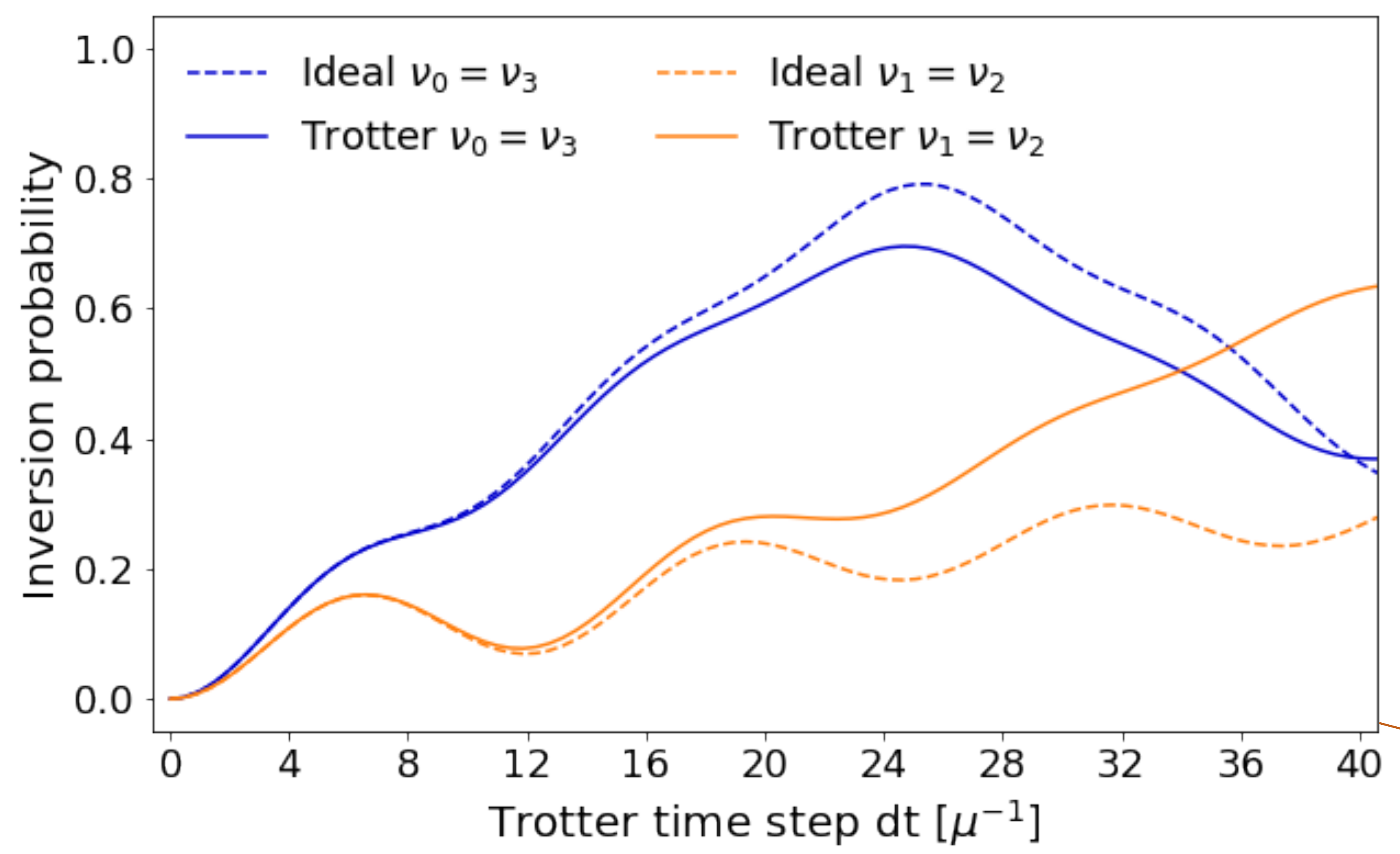


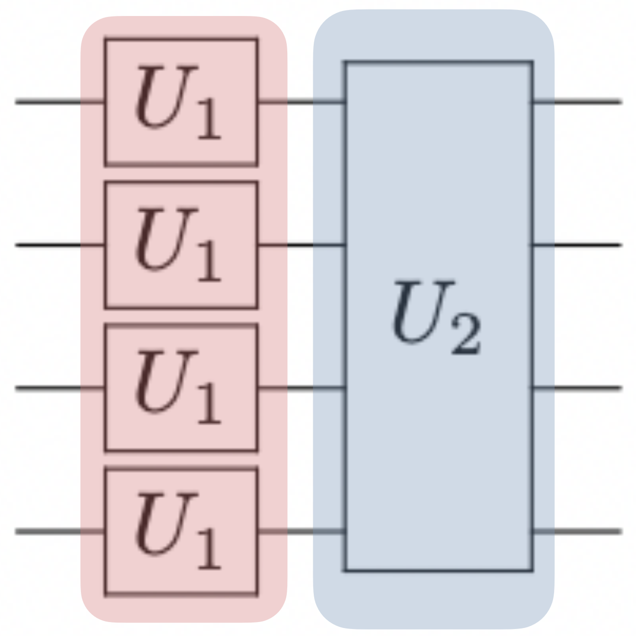


ZZ-based CNOT gate

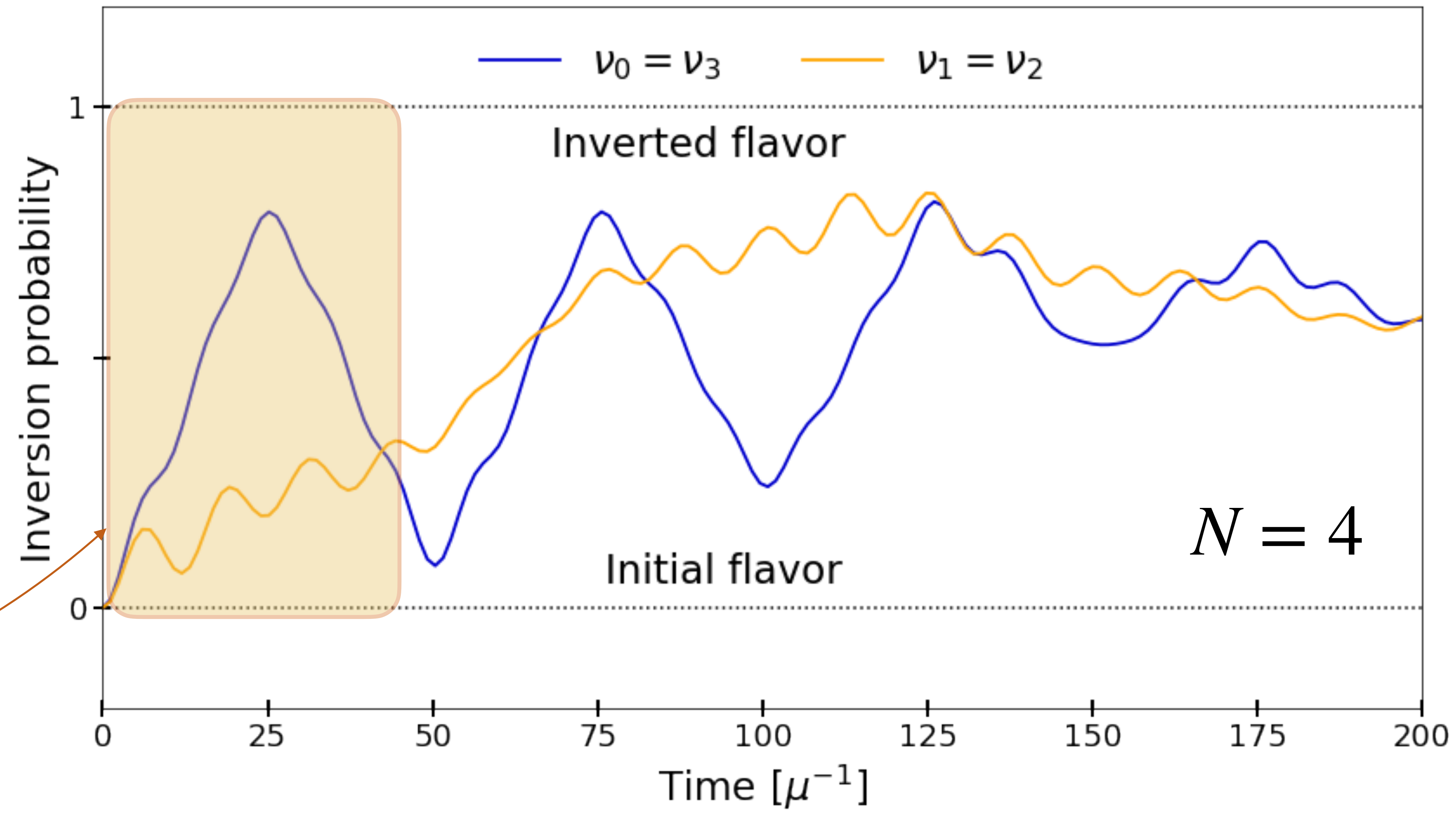


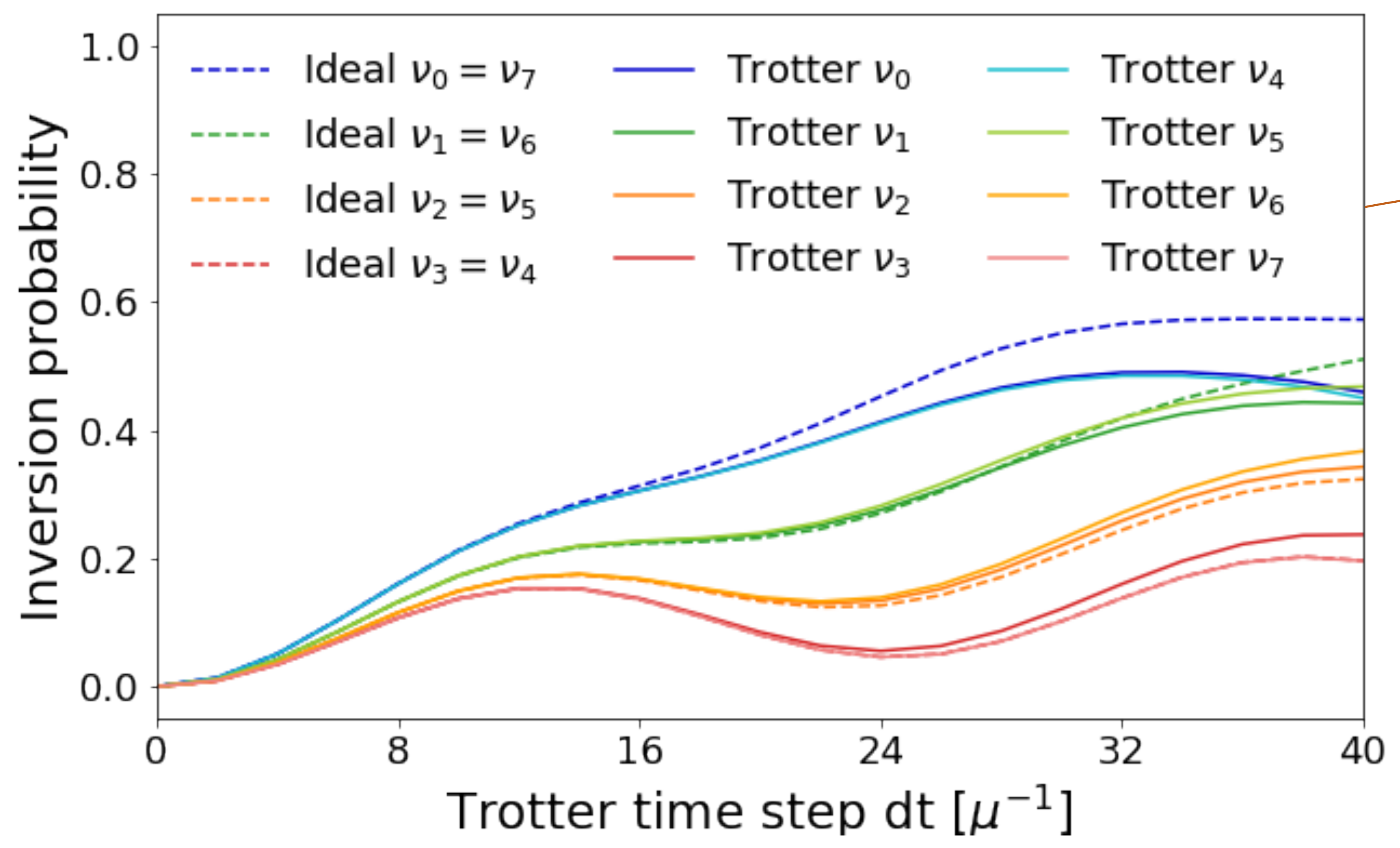
RESULTS: SINGLE TROTTER STEP

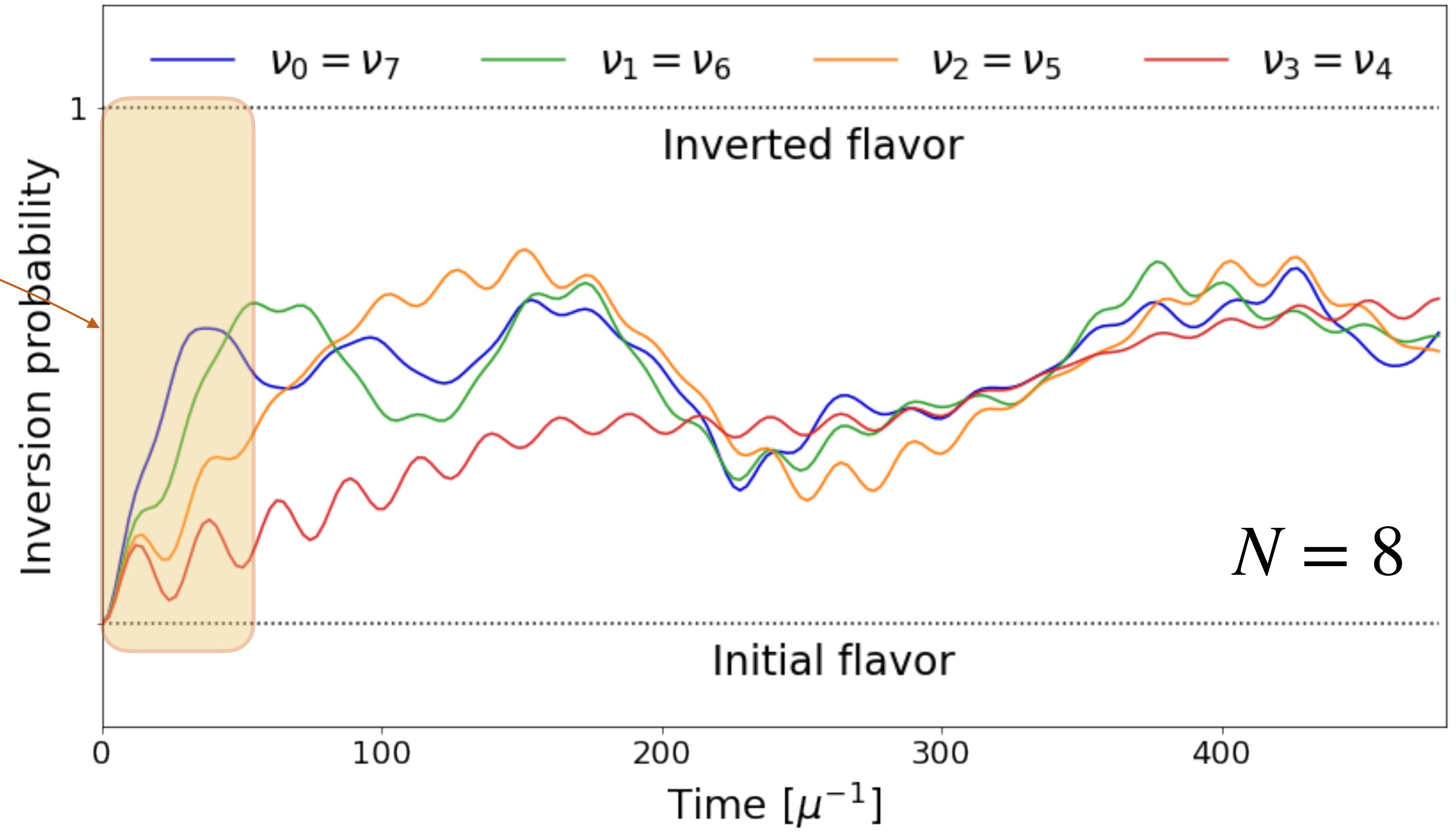




$\varepsilon \sim \mathcal{O}(dt^2)$







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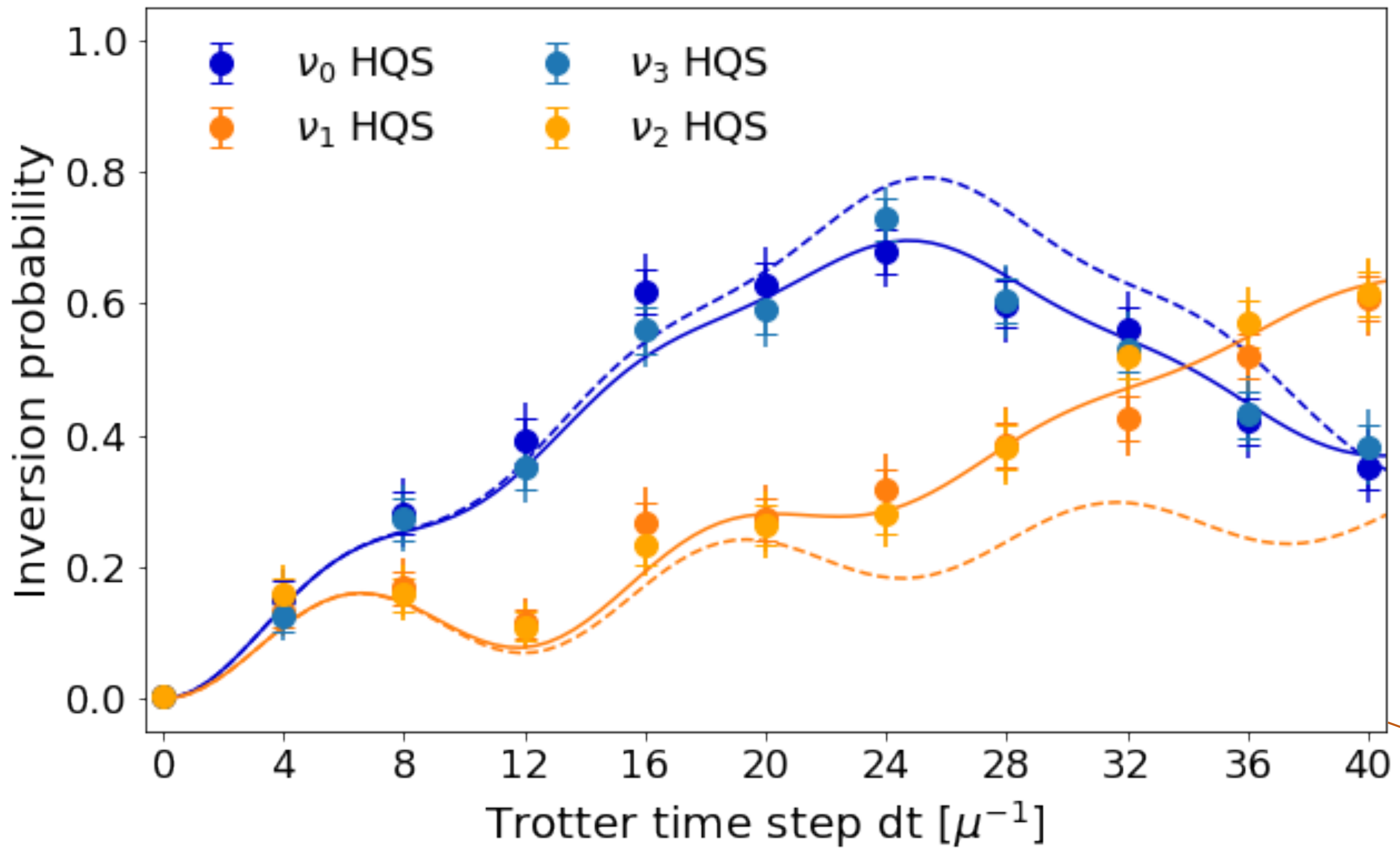

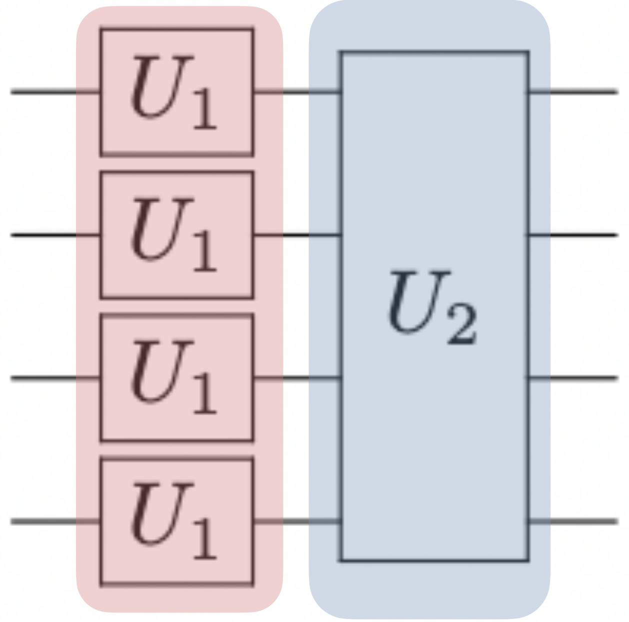

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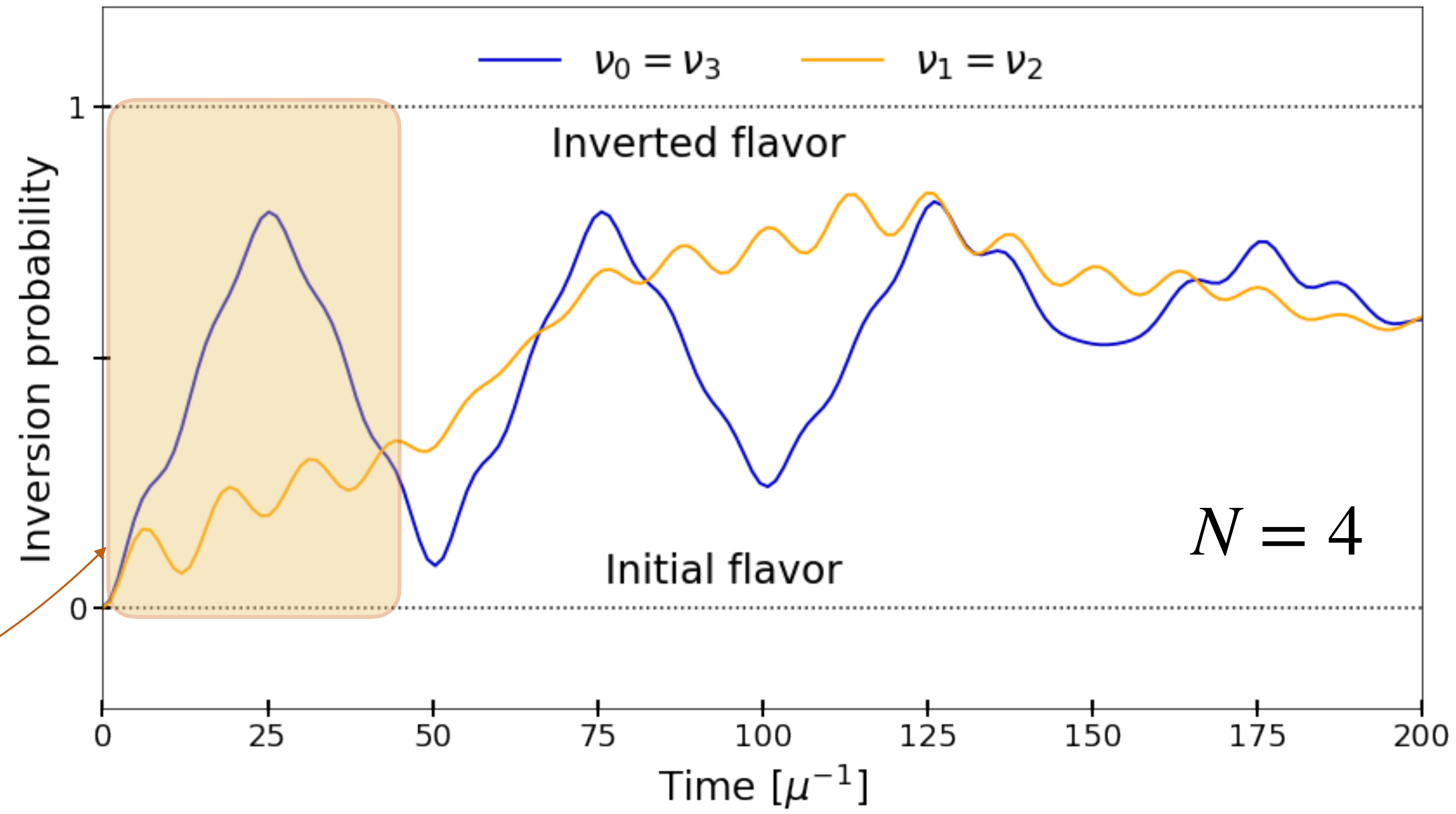

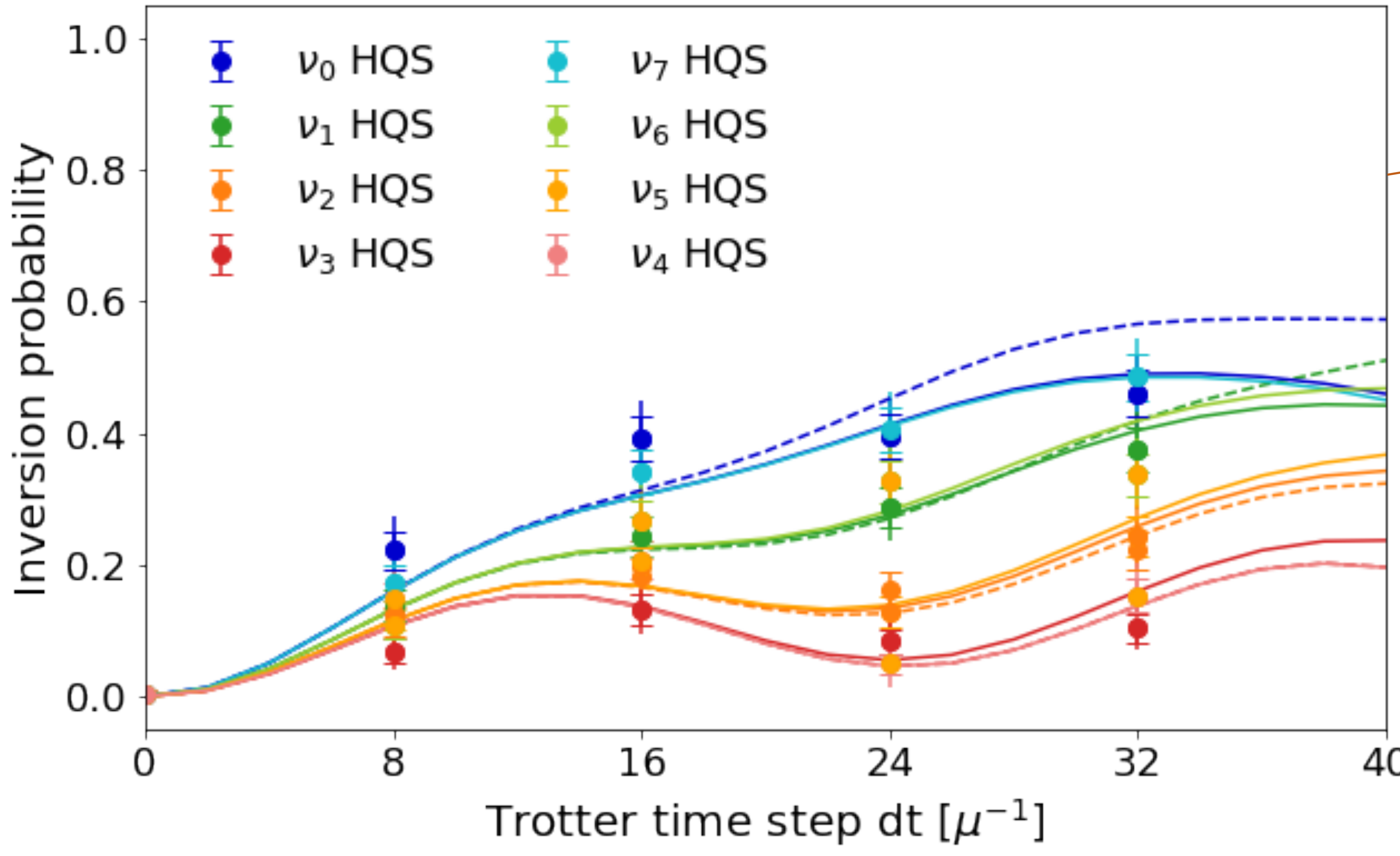

error bar = 68 %

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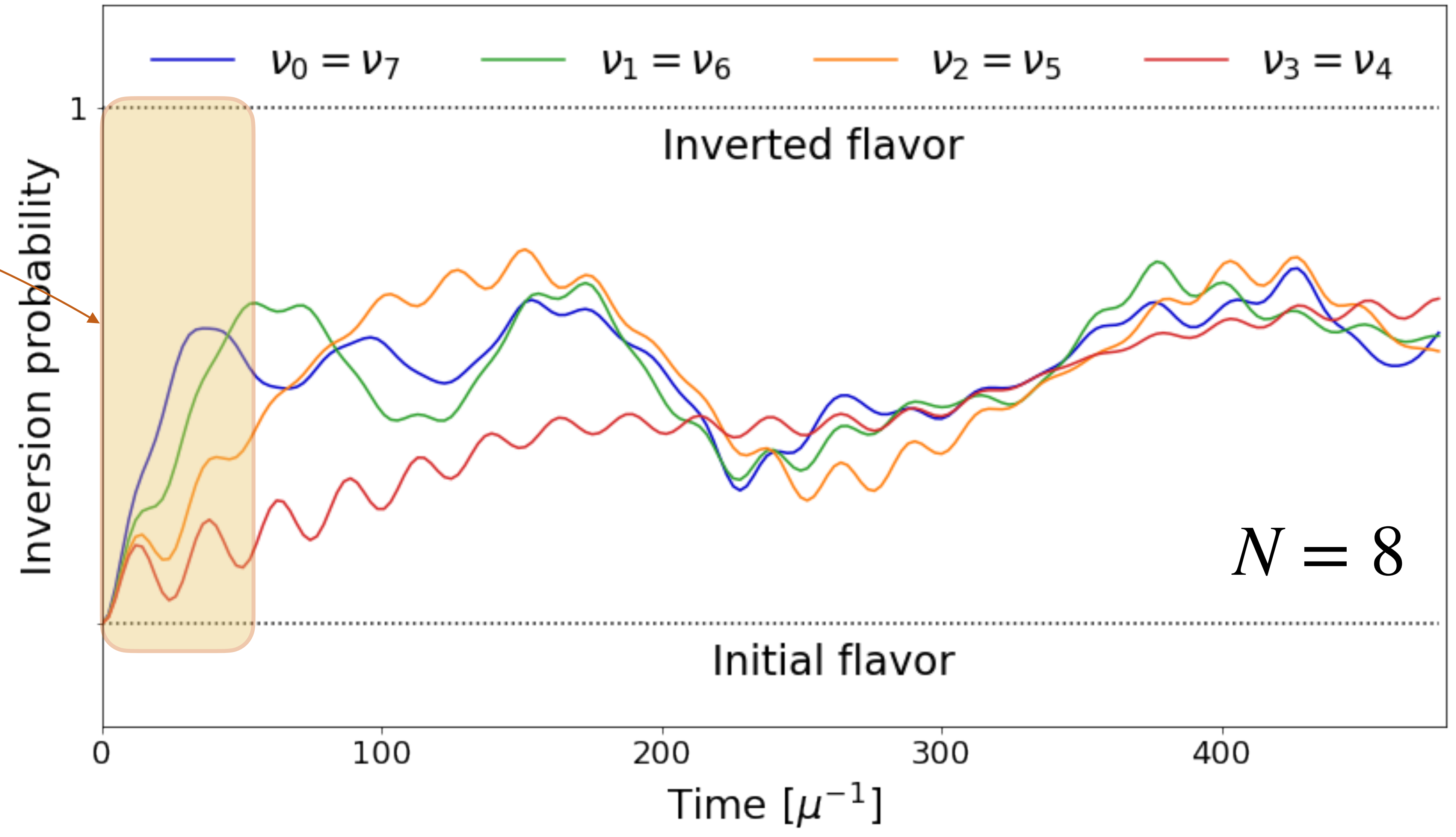

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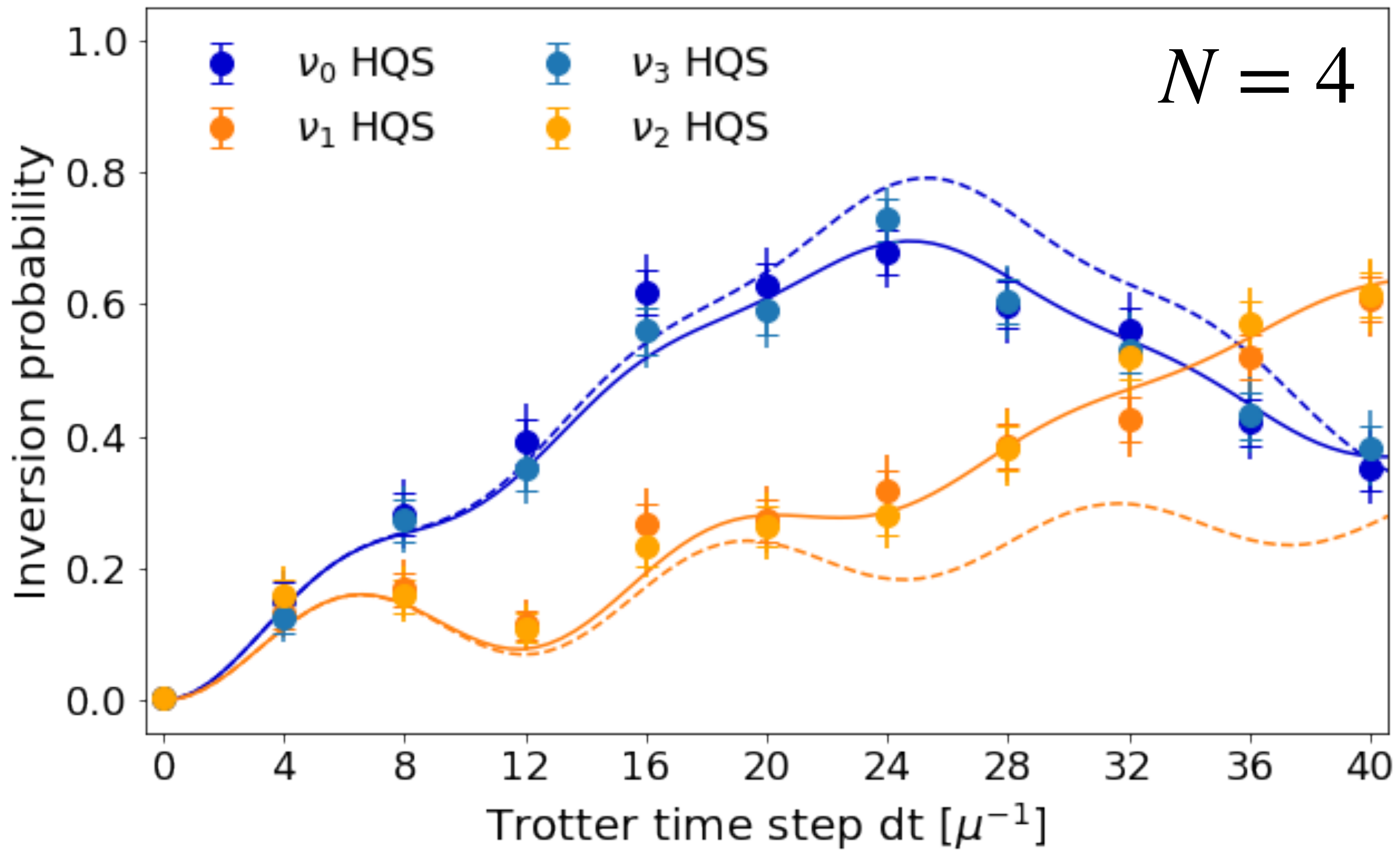
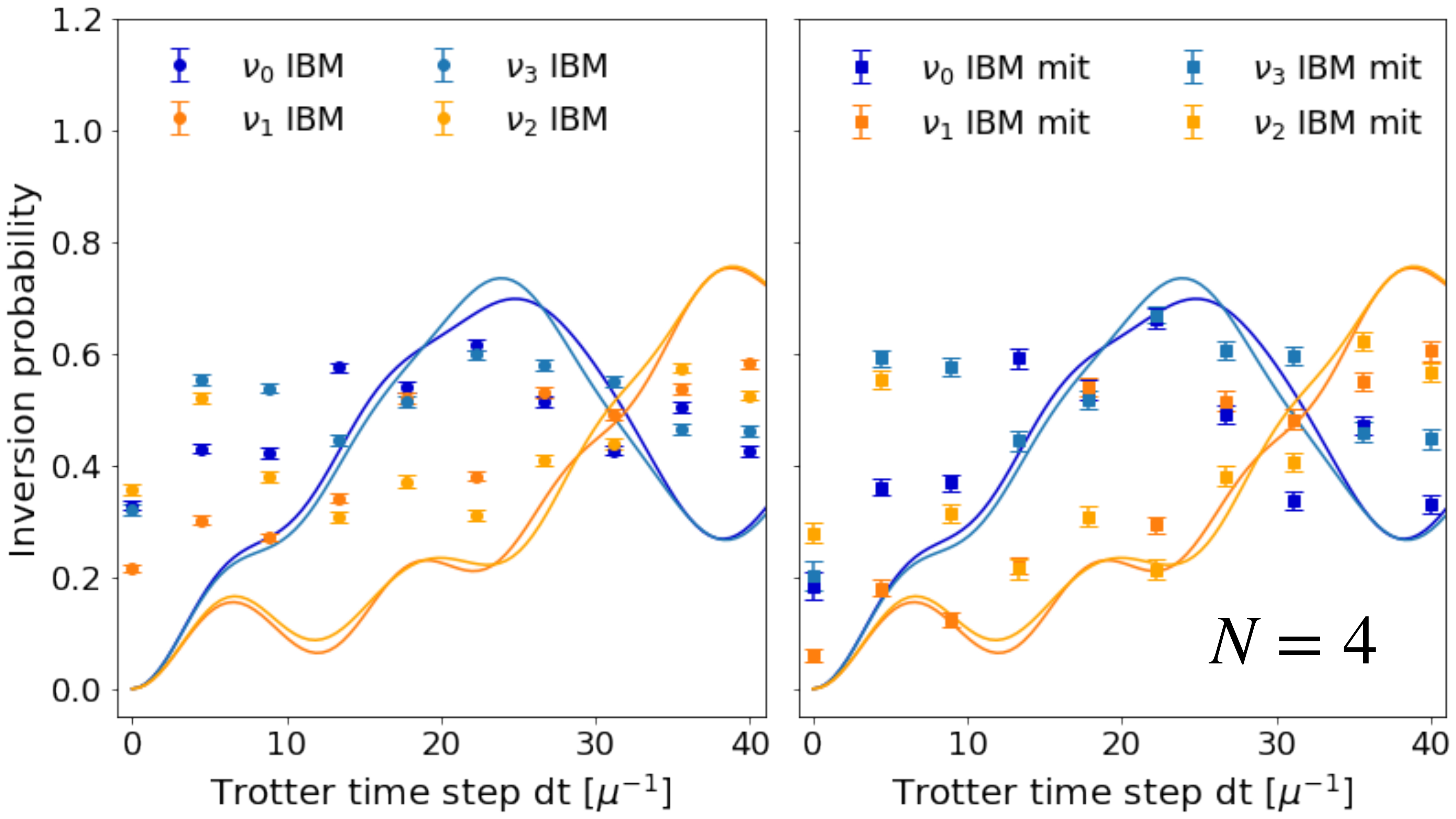
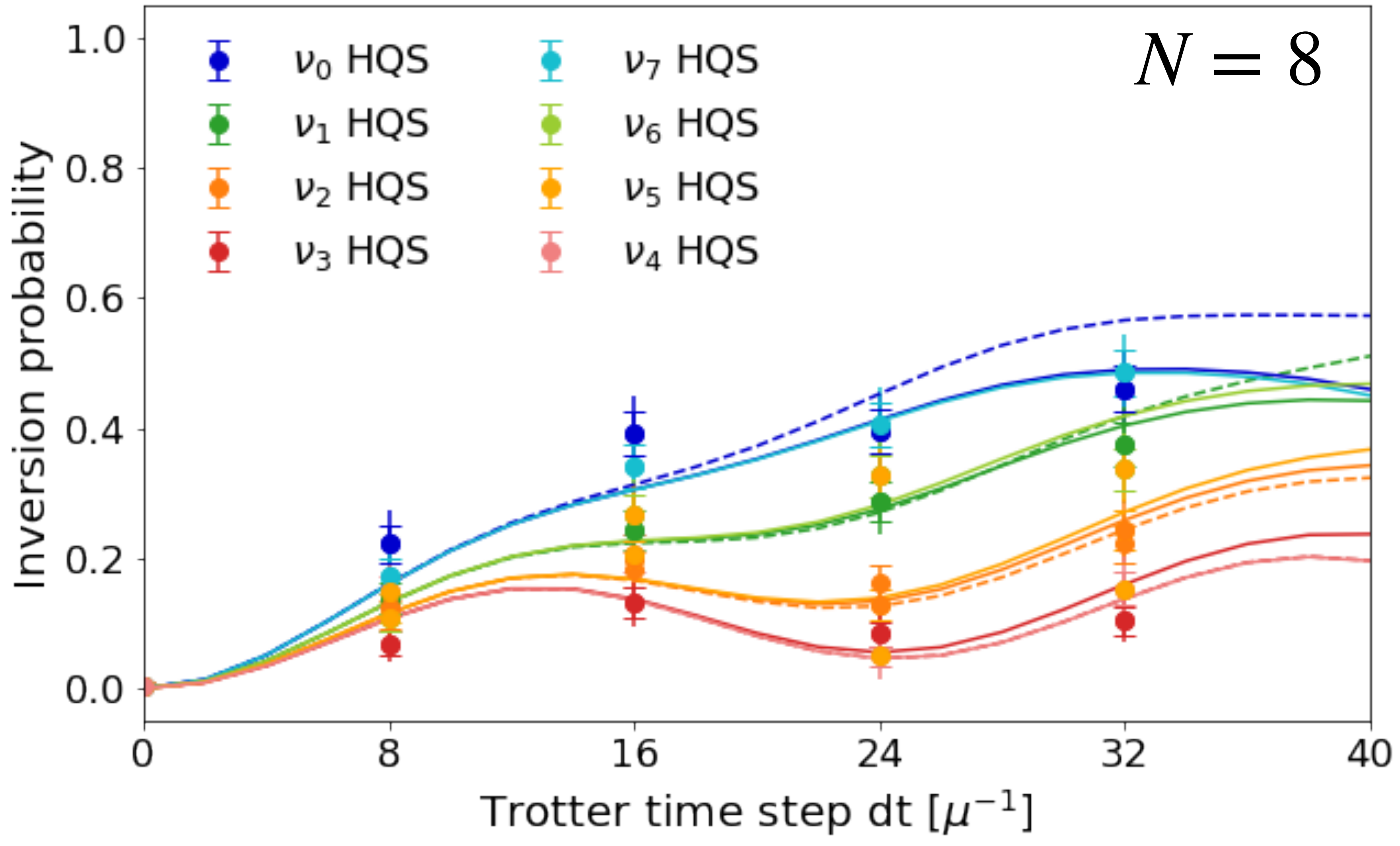




B. Hall et. al. Phys. Rev. D 104, 063009 (2021)



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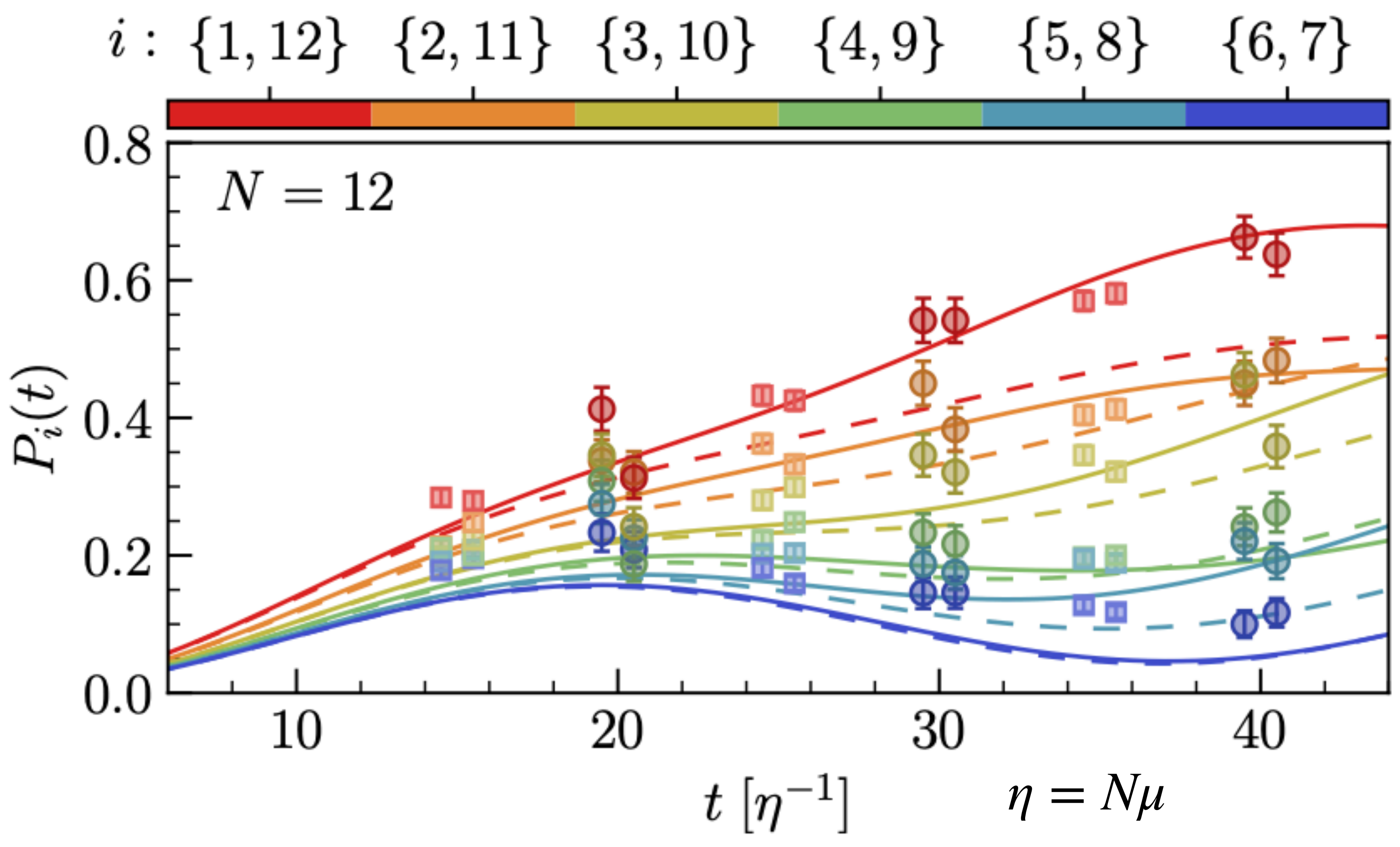
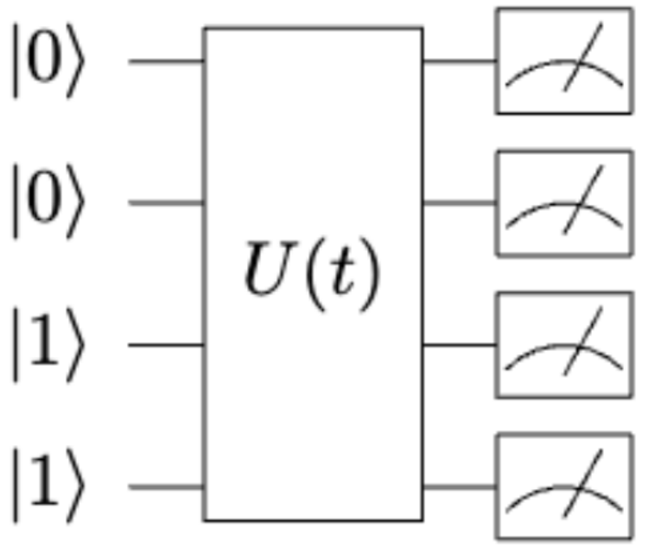
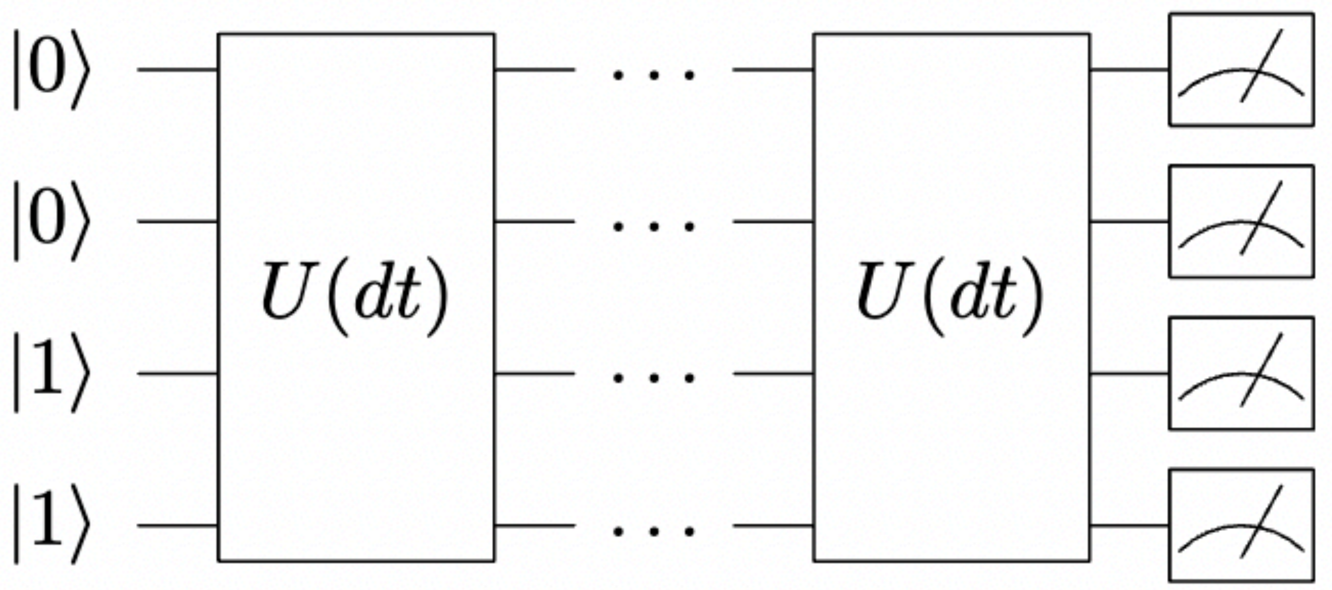


M. Illa and M. J. Savage (2022) [arXiv:2210.08656](https://arxiv.org/abs/2210.08656)

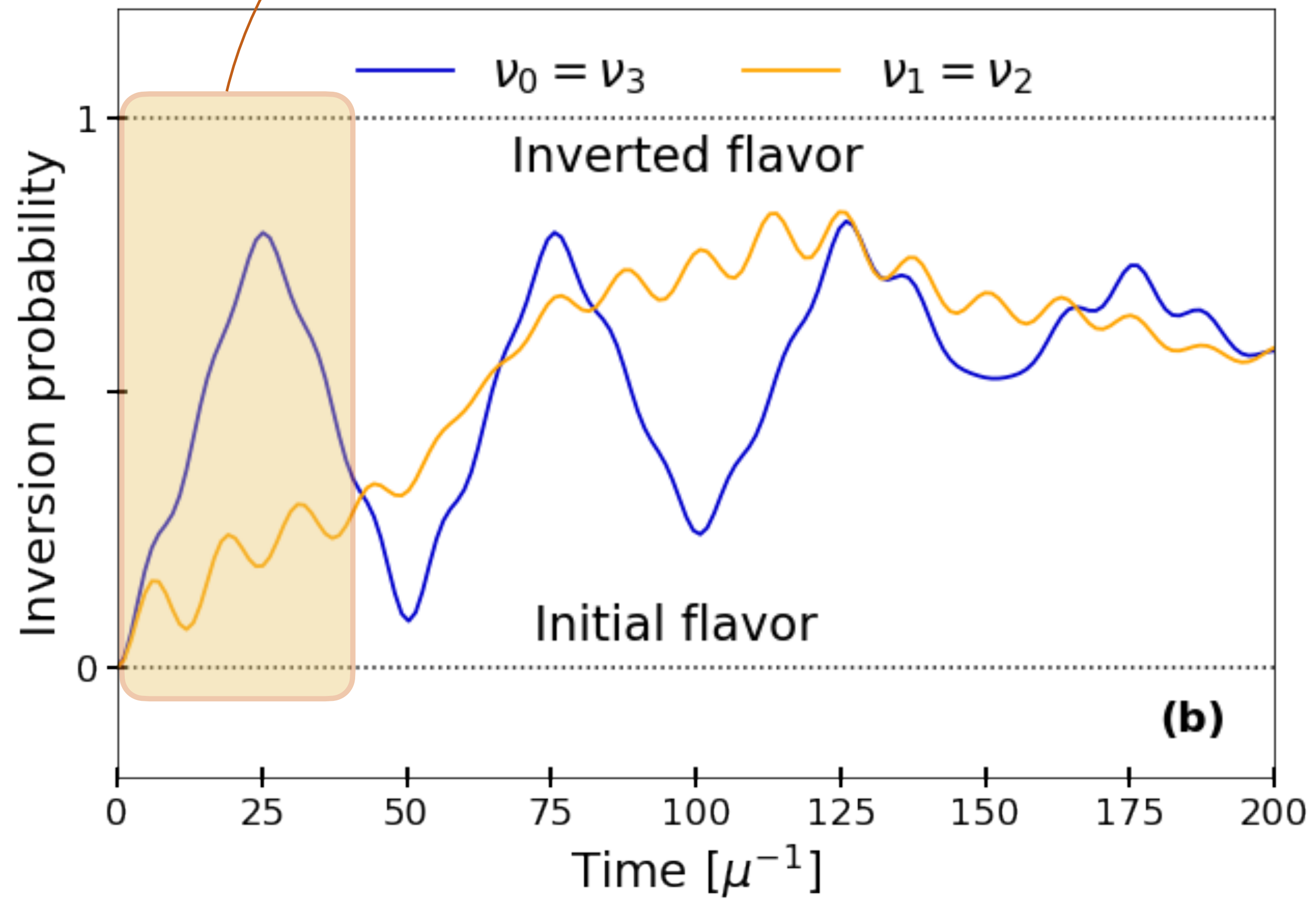


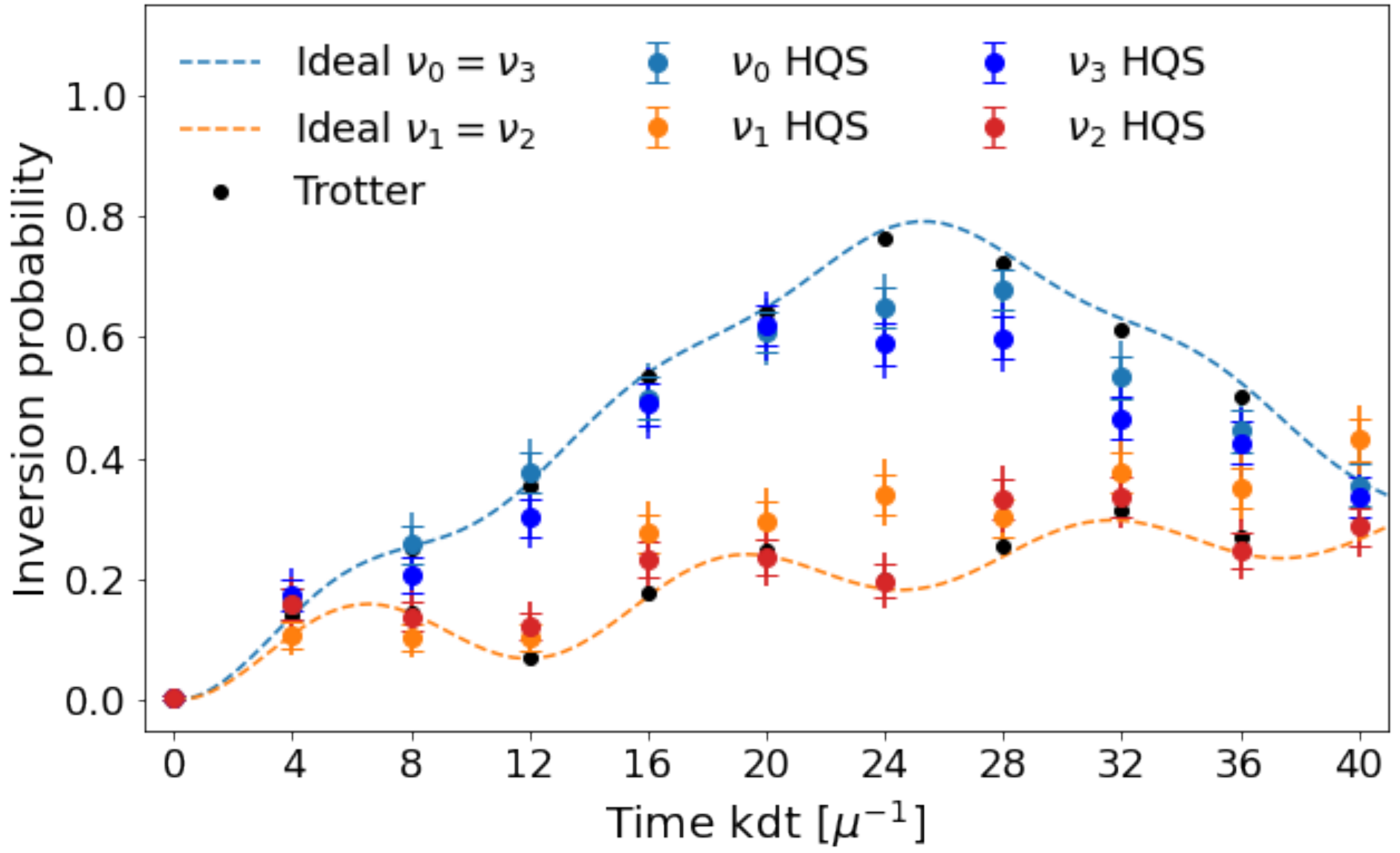


$\varepsilon \sim \mathcal{O}(k^2 dt^2)$
$t = kdt$

$\varepsilon \sim \mathcal{O}(kdt^2)$

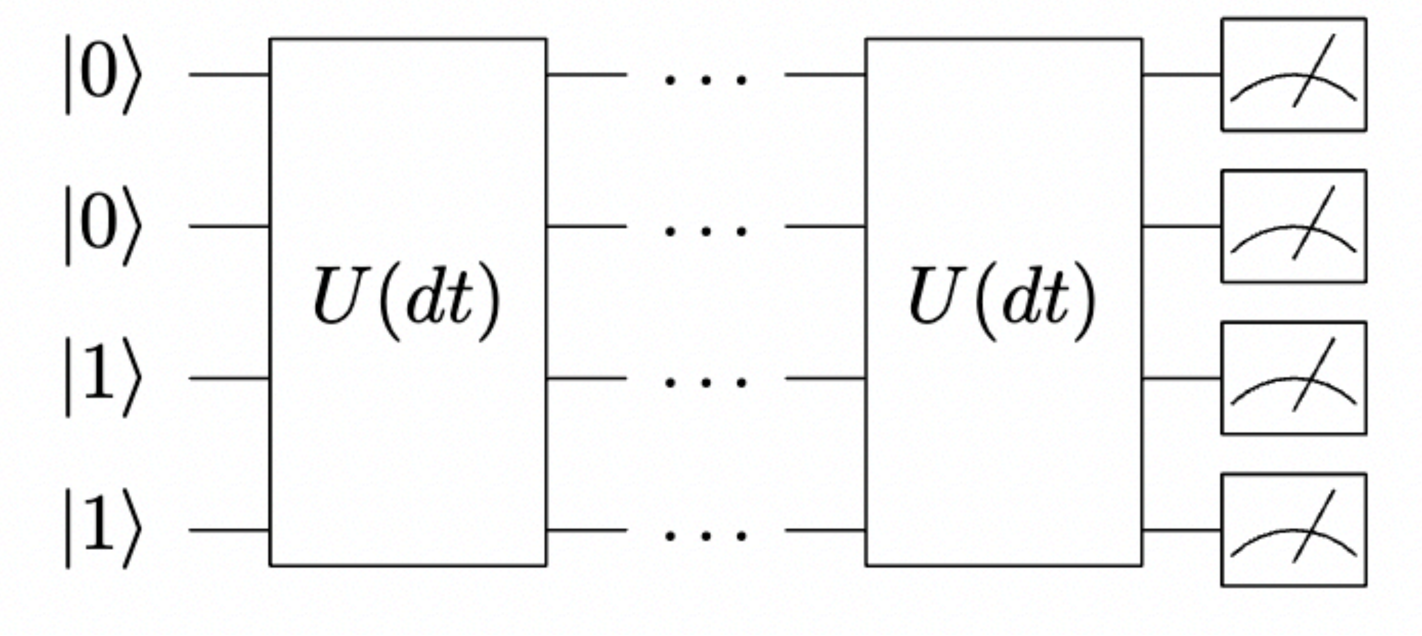
- Short time-step $dt = 4\mu^{-1}$
- Ideal ≈ trotterized evolution
- Very long quantum circuits (noise)

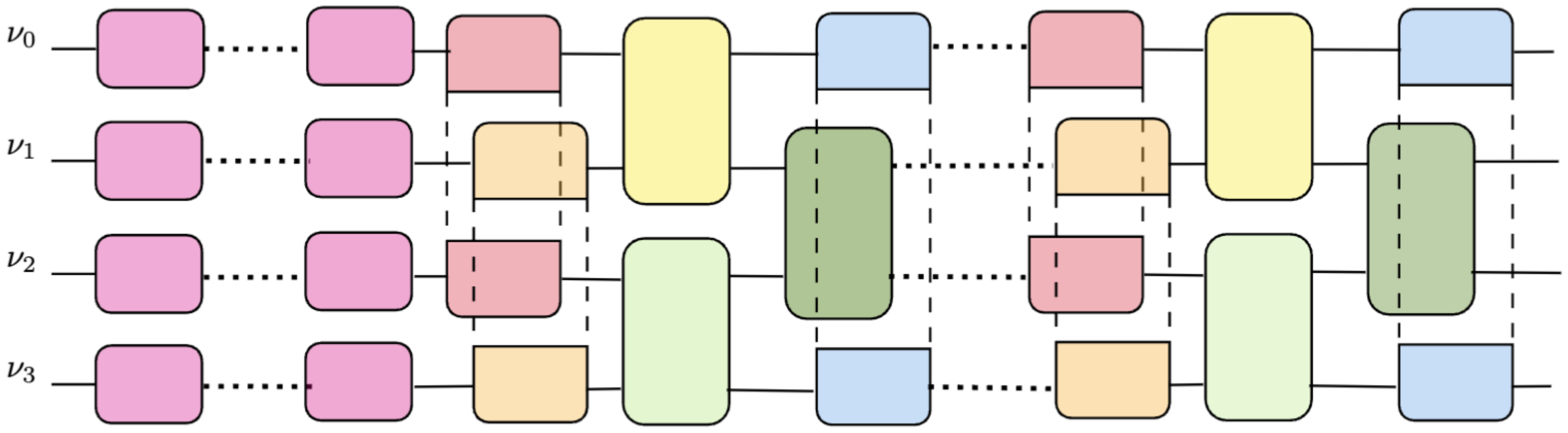


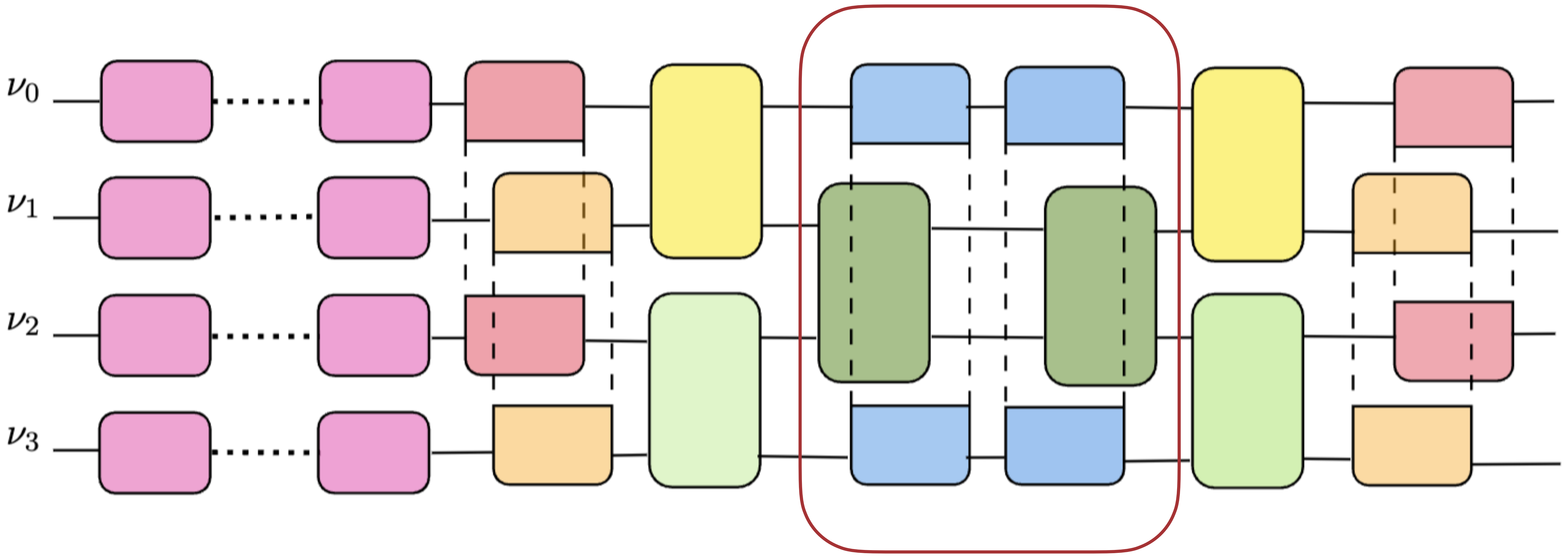


Steps	1	2	3	4	5	6	7	8	9	10
# ZZ	18	36	54	72	90	108	126	144	162	180
# SU(2)	36	68	100	132	164	196	228	260	292	324

Inverse order for the Trotter decomposition





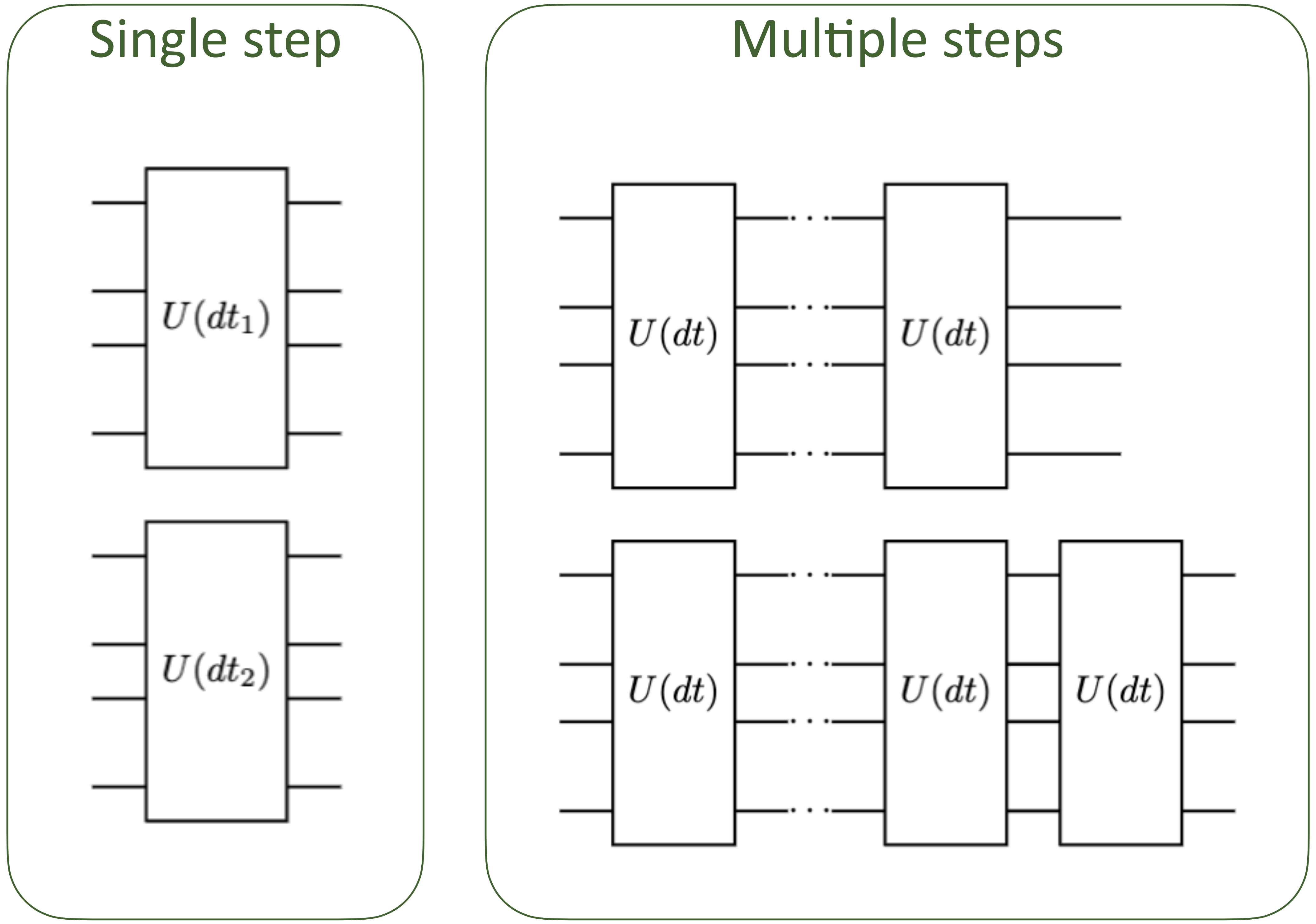


- Alternate steps with inverted order
- Is effectively a second order Trotter decomposition
- For r steps it decreases the number of two-qubit operation of $6r$

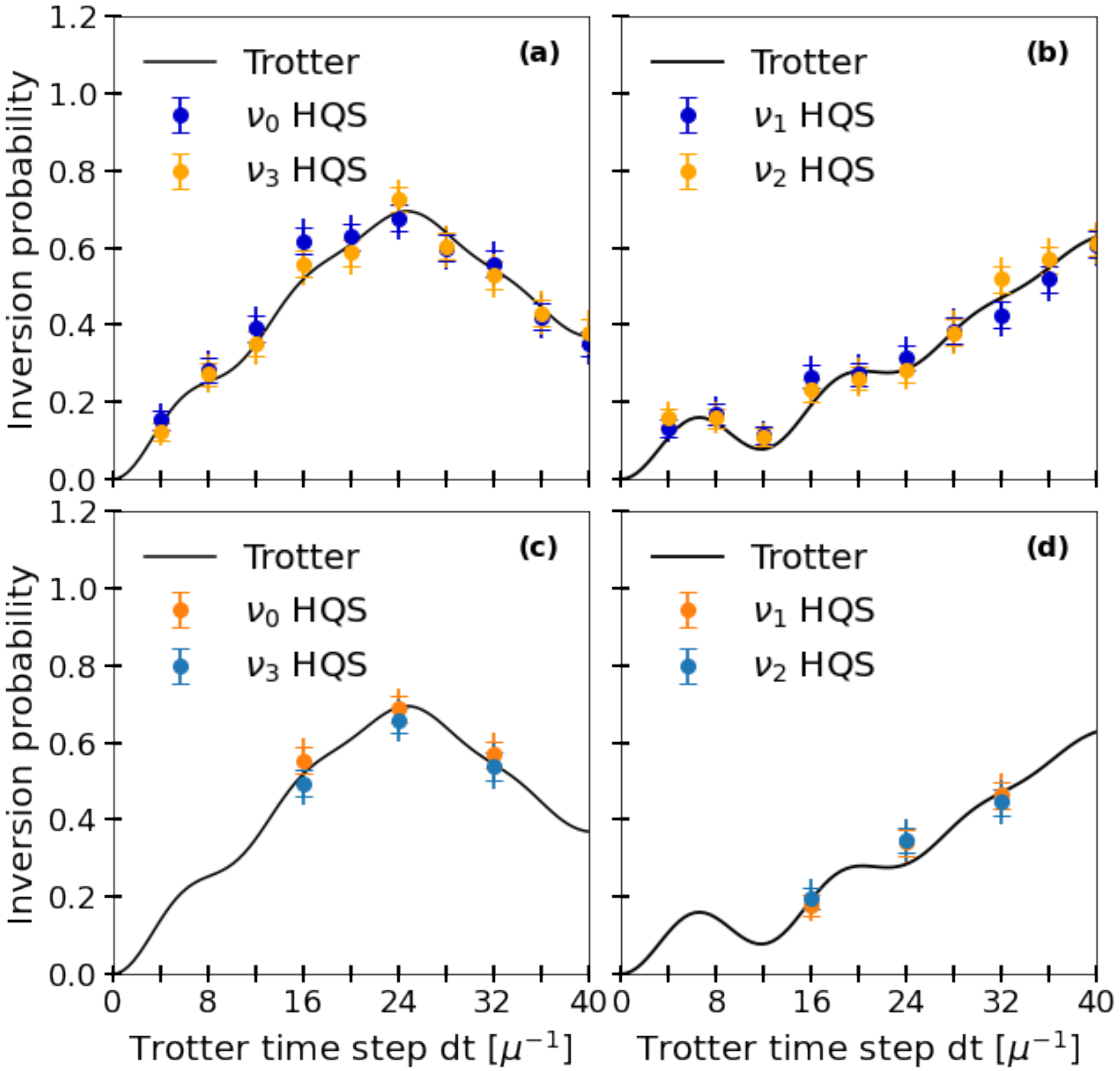
$3\frac{N}{2}r$ in general

Cross talk between disconnected registers

- All results were obtain running two parallel quantum circuit



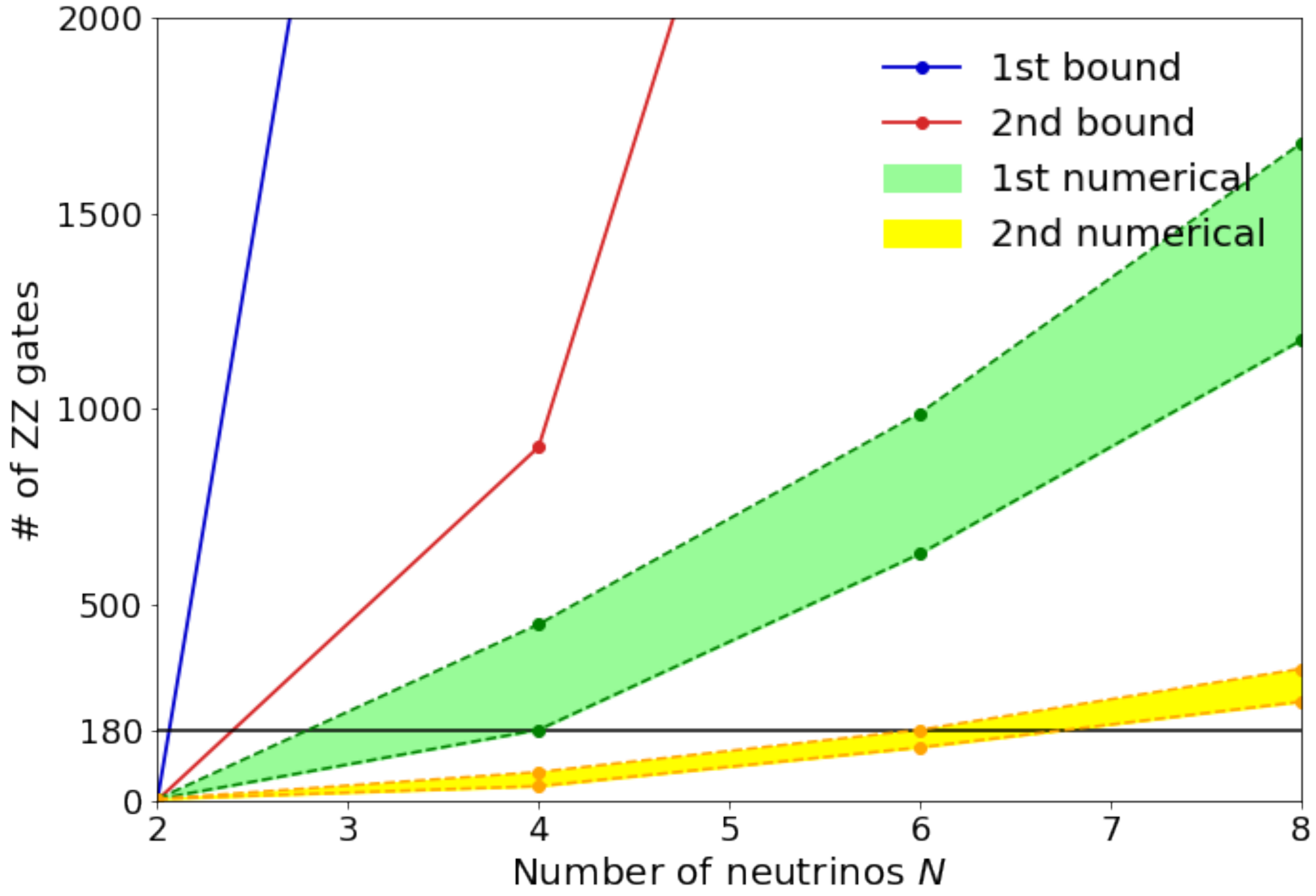
- We run 3 parallel circuits to check the crass talk effect
- The results are compatible with the previous one:
  - Cross talk is negligible



We are interested in systems in which we fix $n_\nu = N/V$ and we look at the scaling with N

Complexity as the number of 2-qubit gates to evolve the system up to T keeping the error $< \epsilon$

- First order Trotter $\mathscr{C}_1 \leq \mathcal{O}\left(\frac{T^2\mu^2N^3}{\epsilon}\right)$
- Second order Trotter $\mathscr{C}_2 \leq \mathcal{O}\left(\frac{(T\mu)^{3/2}}{\sqrt{\epsilon}}N^{5/2}\right)$
- Higher order Trotter $\sim N^{2+\delta}$
- Qubitization $\mathscr{C}_Q \leq \mathcal{O}\left(T\mu N^3 + N^2 \log\left(\frac{1}{\varepsilon}\right)\right)$



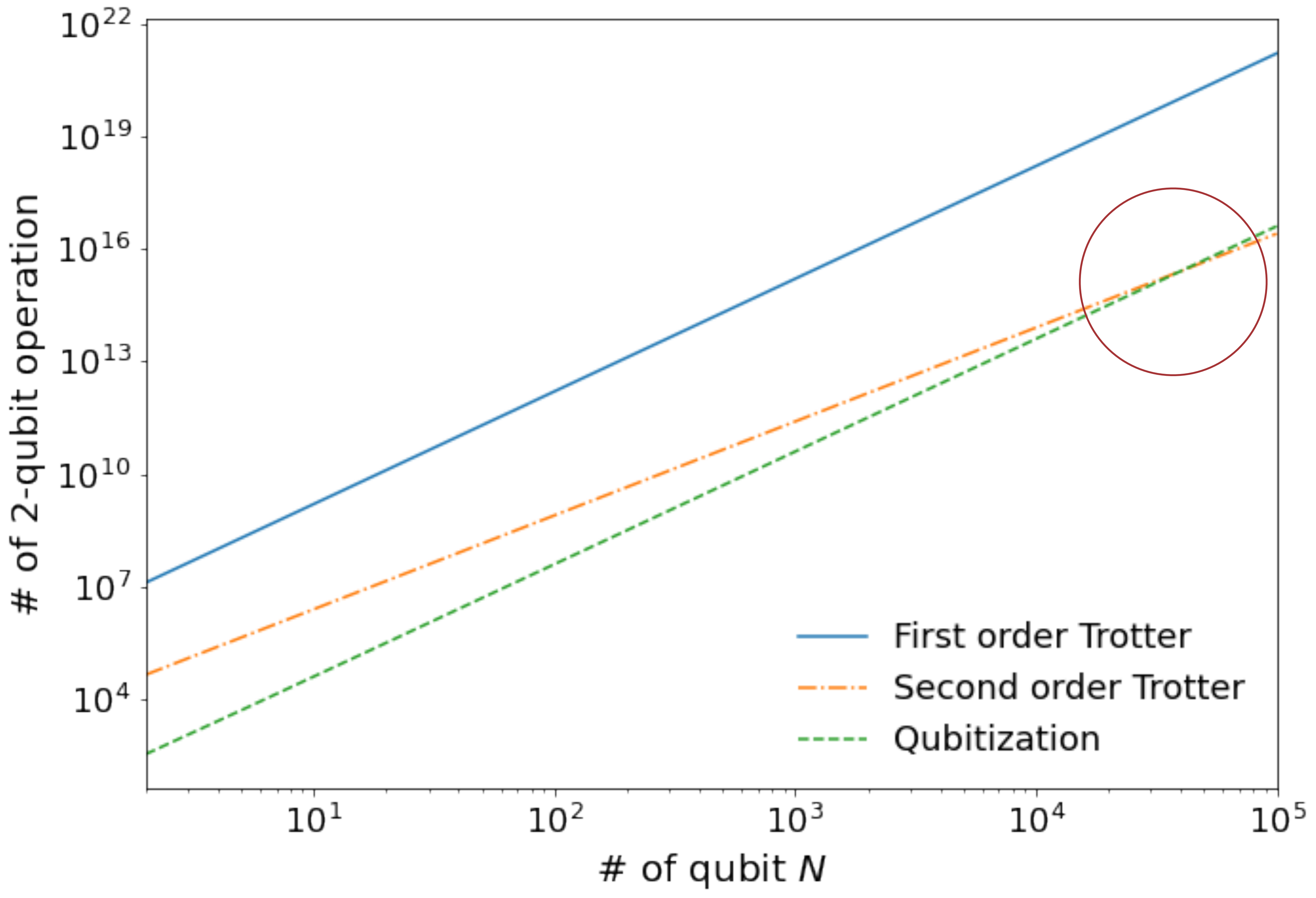
Real cost estimated by calculating the number of steps such that we evolve up to $T = 40\mu^{-1}$ with an error ≤ 0.15

$$\varepsilon(dt) = \|U_{approx}(dt) - U_{exact}(dt)\|_\infty$$

$$\varepsilon(t) \leq r\varepsilon(dt)$$

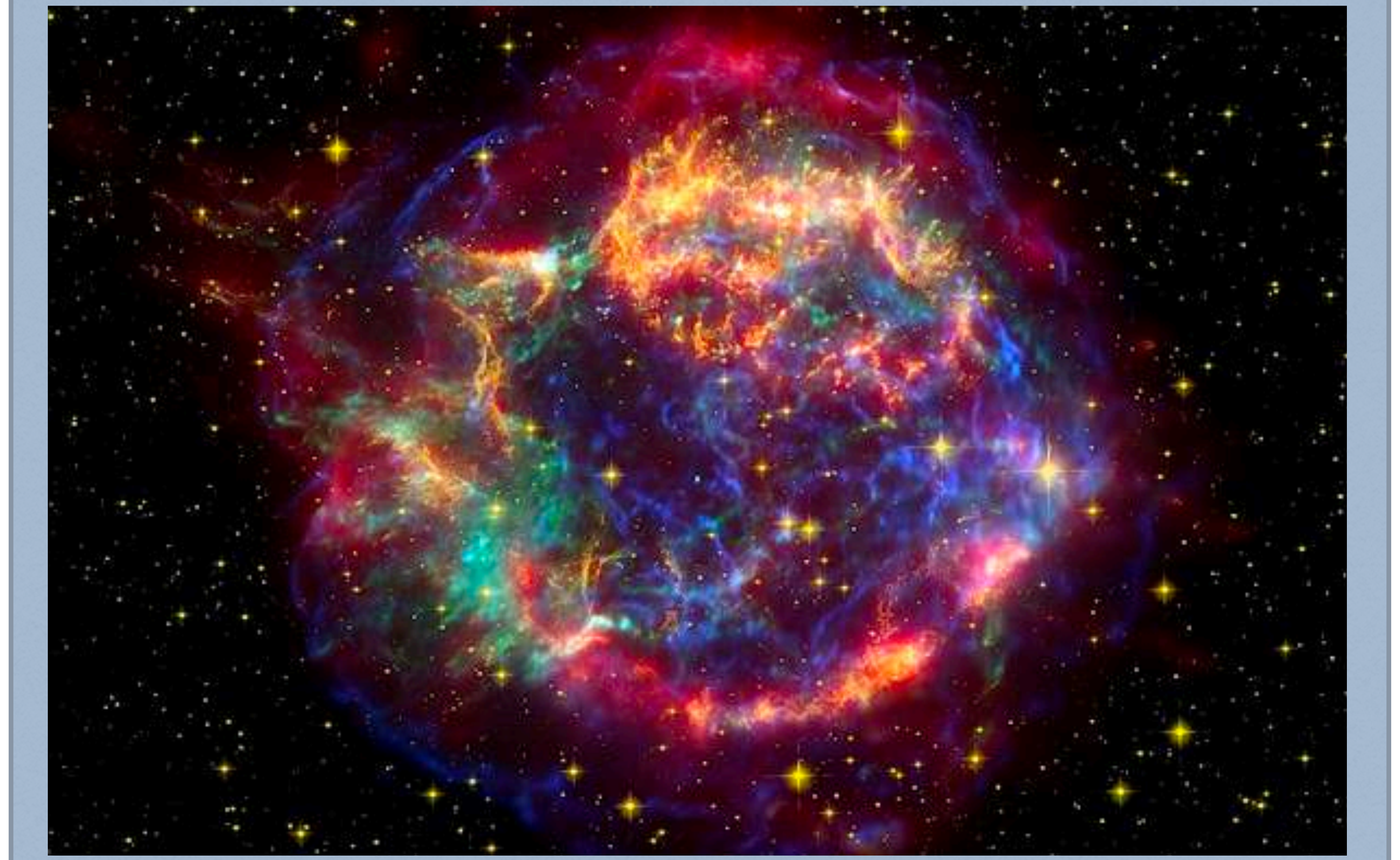
Decomposition type	Single-step error	Number of steps	Circuit complexity
First order Trotter	$\mathcal{O}(dt^2\mu^2 N)$	$\mathcal{O}\left(\frac{T^2\mu^2}{\epsilon}N\right)$	$\mathcal{O}\left(\frac{T^2\mu^2}{\epsilon}N^3\right)$
Second order Trotter	$\mathcal{O}(dt^3\mu^3 N)$	$\mathcal{O}\left(\frac{T^{3/2}\mu^{3/2}}{\sqrt{\epsilon}}\sqrt{N}\right)$	$\mathcal{O}\left(\frac{T^{3/2}\mu^{3/2}}{\sqrt{\epsilon}}N^{5/2}\right)$
Qubitization	-	$\mathcal{O}\left(T\mu N + \log(1/\epsilon)\right)$	$\mathcal{O}\left(T\mu N^3 + \log(1/\epsilon)N^2\right)$

- Qubitization work well for large time T and small error ε
- Trotter method wins for fixed time and error

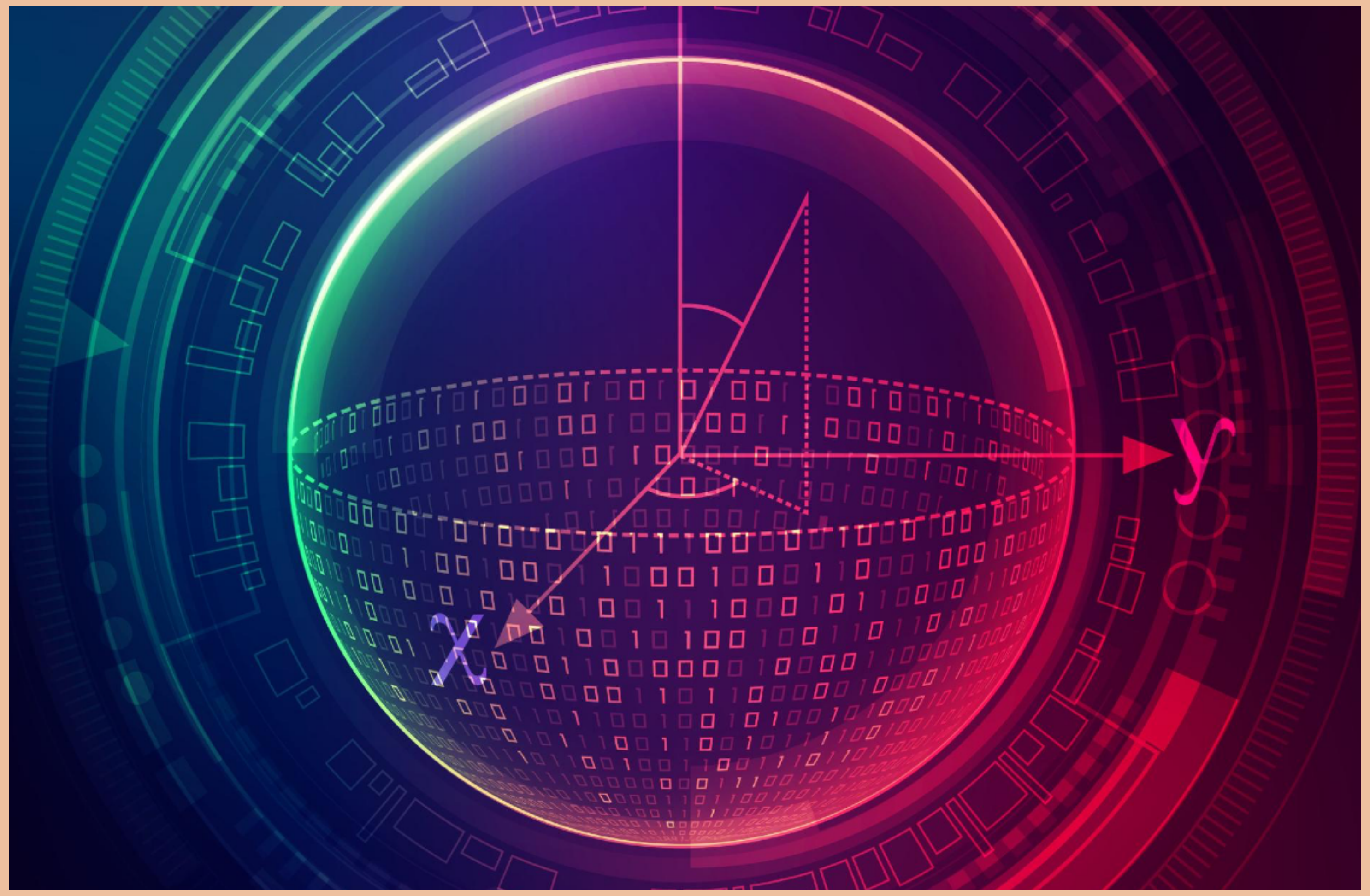


Conclusions

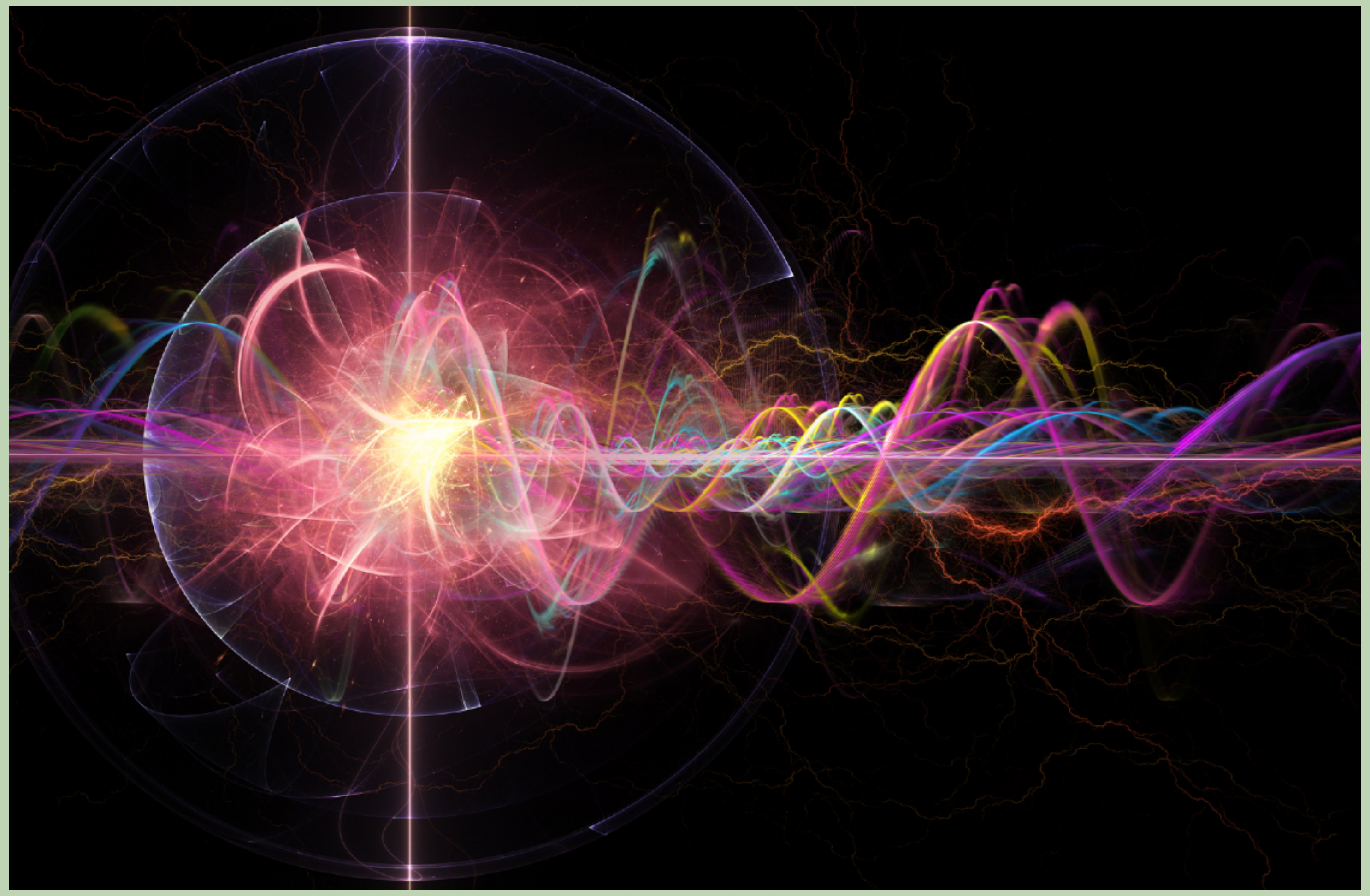
- Flavor dynamics is crucial to describe many effects in core-collapse supernovae
- Collective neutrino oscillations make the problem non linear and interesting to test quantum computing



- QC necessary for full dynamics simulation
- The gate decomposition must be machine aware and circuit optimization is crucial
- Full qubit connectivity allows for more freedom in gate decomposition



- Results are very promising
- We can increase the number of simulated neutrinos
- The complexity of the algorithm scales polynomially with the number of neutrinos



Thank you for your attention

Valentina Amitrano
Francesco Pederiva
Alessandro Roggero

Nuclear and Particle Physics on a Quantum Computer: Where do we stand now?

Supplementary material

- Vacuum mixing (1-body term)

$$H_{vac} = \Delta \sum_{i=1}^{N} \vec{b} \cdot \vec{\sigma}_i = \frac{\delta m^2}{4E} \sum_{i=1}^{N} \left(\sin(2\theta_\nu) X_i - \cos(2\theta_\nu) Z_i \right)$$

- $\nu\nu$ - interaction (2-body term)

$$H_{\nu\nu} = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{\mu}{N} \sum_{i<j}^{N} J_{ij} \left(X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j \right)$$

! $H_{\nu\nu}$ **is an all-to-all interaction that makes the problem non-linear**

The model:

- $\theta_\nu = 0.195$ mixing angle
- Monochromatic flux $E_i = E \; \forall i$
- $\vec{b} = \frac{\delta m^2}{4E} (\sin(2\theta_\nu), 0, -\cos(2\theta_\nu))$
- $\Delta = \frac{\delta m^2}{4E}$
- $J_{ij} = 1 - \cos(\theta_{ij})$
- $\theta_{ij} = \arccos(0.9) \frac{|i-j|}{N-1}$
- Energy scale $\mu = \sqrt{2} G_F n_\nu$
- $X_2 = I \otimes I \otimes X \otimes I$
- $X_0 \otimes X_2 = X \otimes I \otimes X \otimes I$

Two-flavor Hamiltonian

- Mass basis $\{\nu_1, \nu_2\}$ and flavor basis $\{\nu_e, \nu_x\}$
- Creation and annihilation operators

$$\begin{pmatrix} a_1^{(\dagger)} \\ a_2^{(\dagger)} \end{pmatrix} = \begin{pmatrix} \cos(\theta_\nu) & -\sin(\theta_\nu) \\ \sin(\theta_\nu) & \cos(\theta_\nu) \end{pmatrix} \begin{pmatrix} a_e^{(\dagger)} \\ a_x^{(\dagger)} \end{pmatrix}$$

- On the mass basis:

$$H_{vac} = E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2$$

- On the flavor one:

$$H_{vac} = \frac{\delta m^2}{4E} \sin(2\theta_\nu)(a_e^\dagger a_x + a_x^\dagger a_e) + \frac{\delta m^2}{4E} \cos(2\theta_\nu)(a_x^\dagger a_x - a_e^\dagger a_e)$$

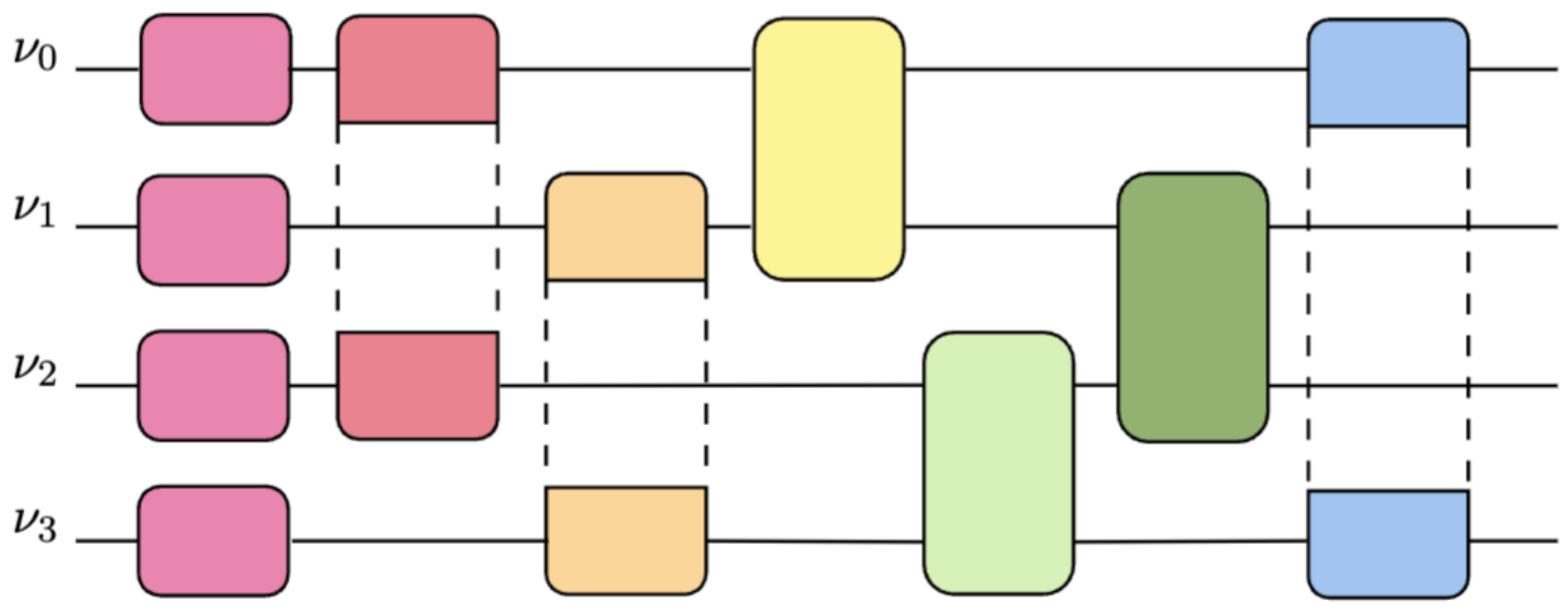
- Mapping:

$$\sigma_z = a_e^\dagger a_e - a_x^\dagger a_x \text{ and } \sigma_x = a_e^\dagger a_x + a_x^\dagger a_e$$

- We have

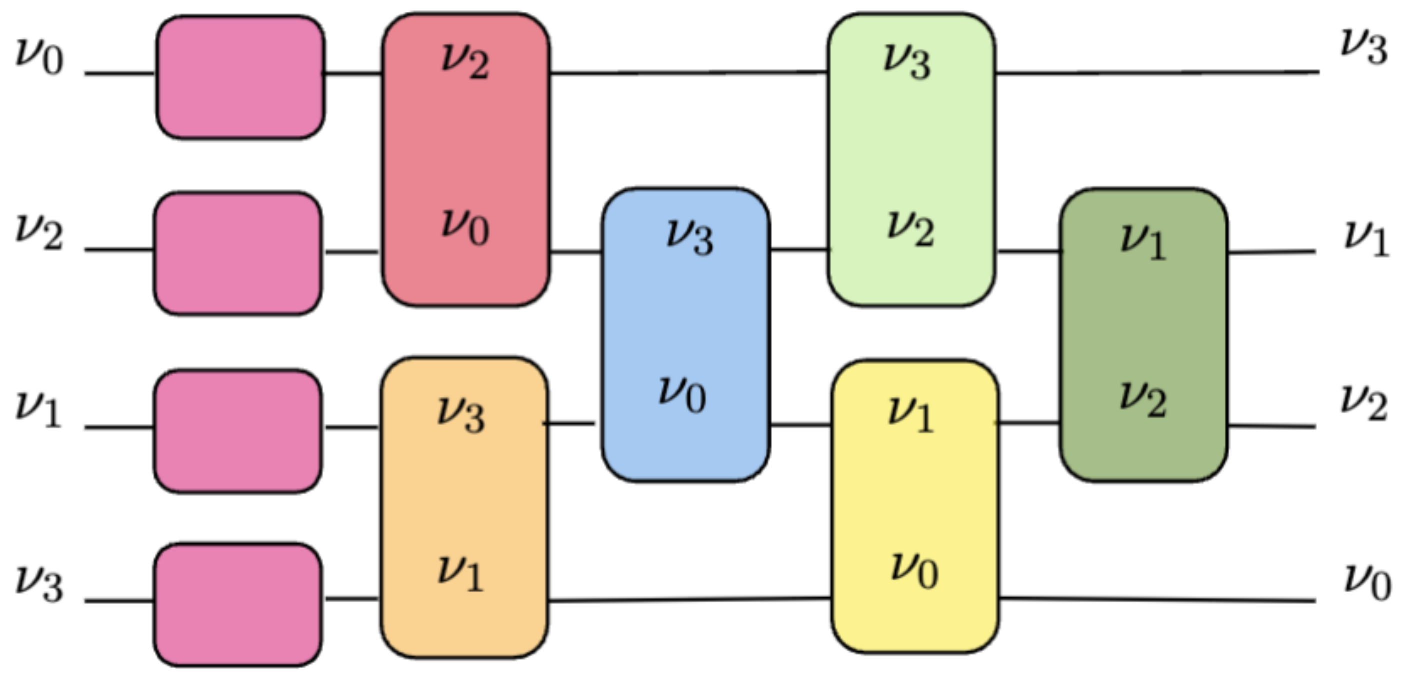
$$H_{vac} = \frac{\delta m^2}{4E}(\sin(2\theta_\nu)X - \cos(2\theta_\nu)Z)$$

Optimal Circuit for full connectivity



Full freedom in the pair ordering

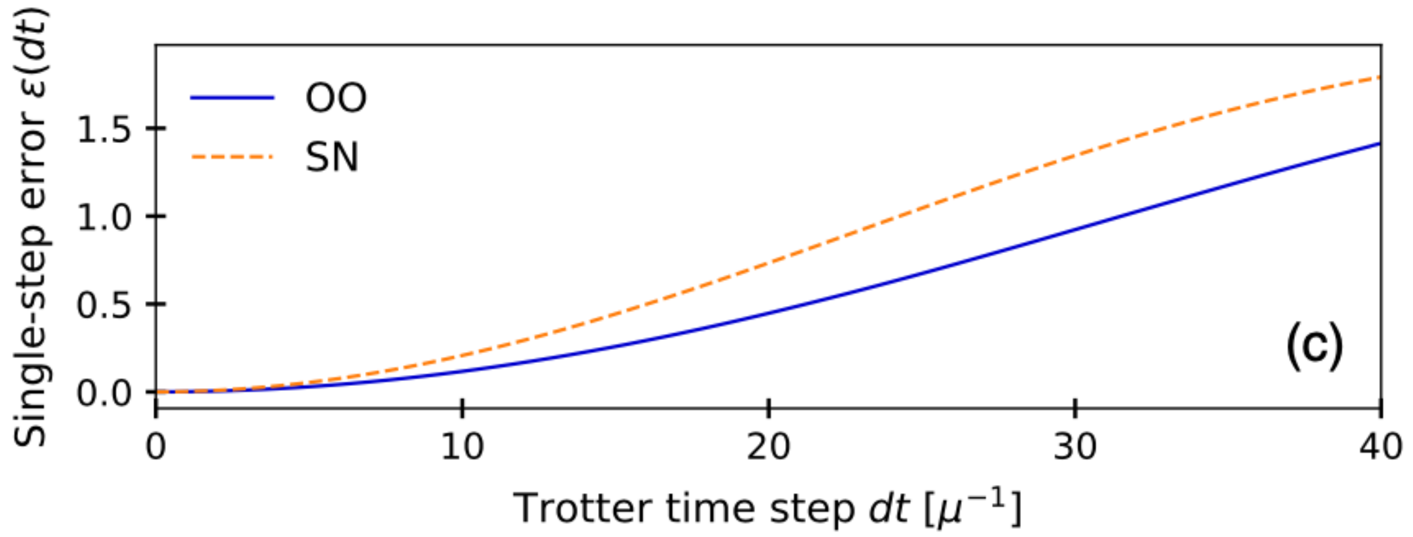
Swap network for linear connectivity

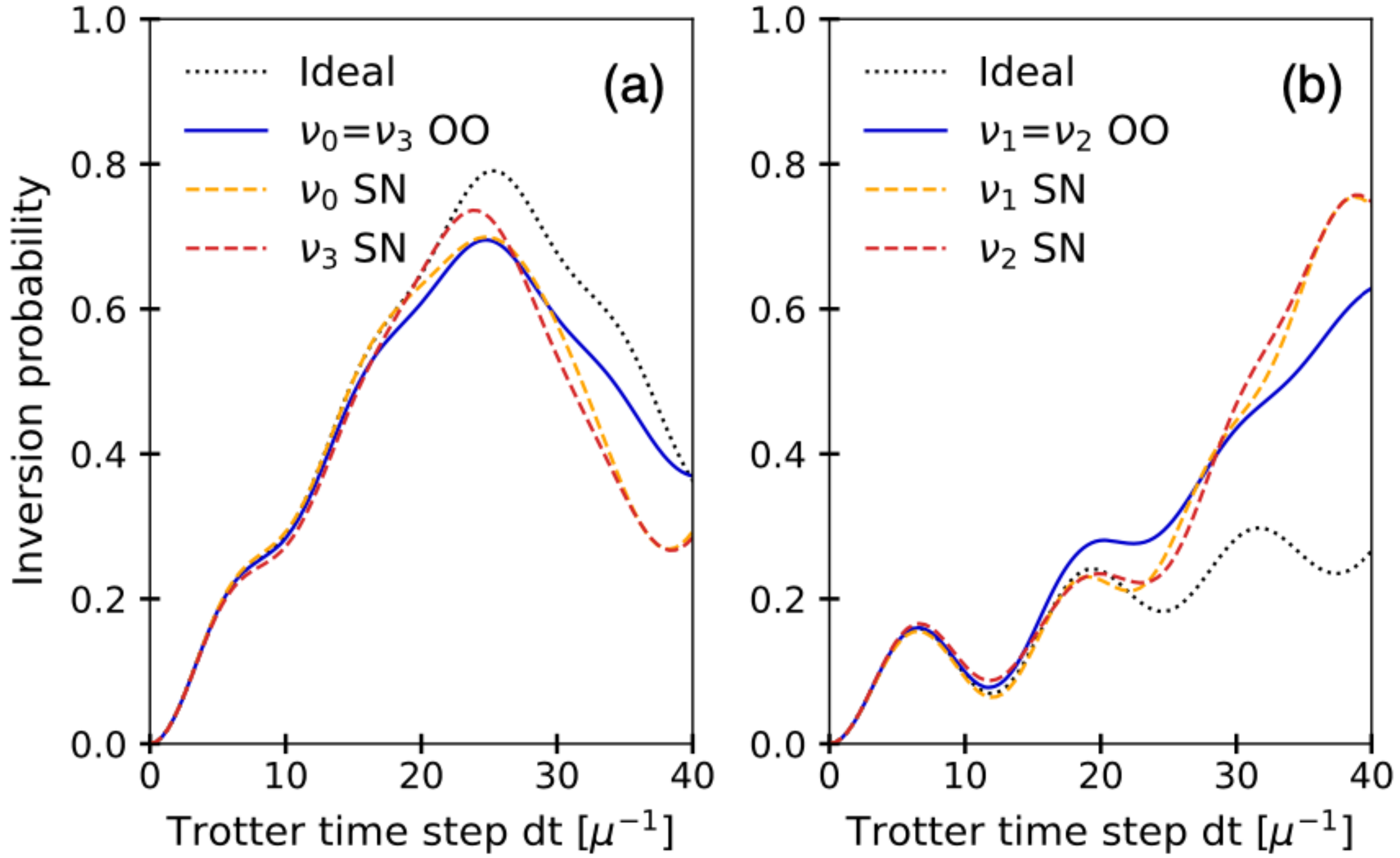


Hall, A. Roggero et. al (2021)

Full freedom in the initial encoding, but the pair ordering is fixed. Any order can be obtain at the price of adding additional SWAP gates.

V. Amitrano et. al. Phys. Rev. D 107, 023007 (2023)



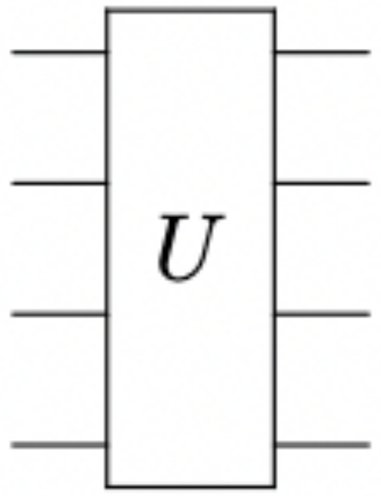


Single vs multiple evolution steps

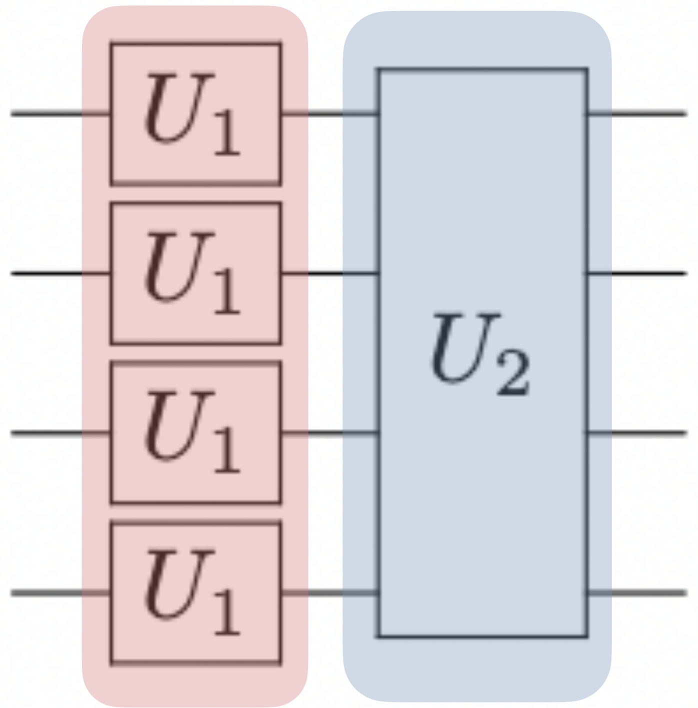
Single Trotter step

$$U(dt) = e^{-iHdt}$$

$$H = H_{vac} + H_{\nu\nu}$$

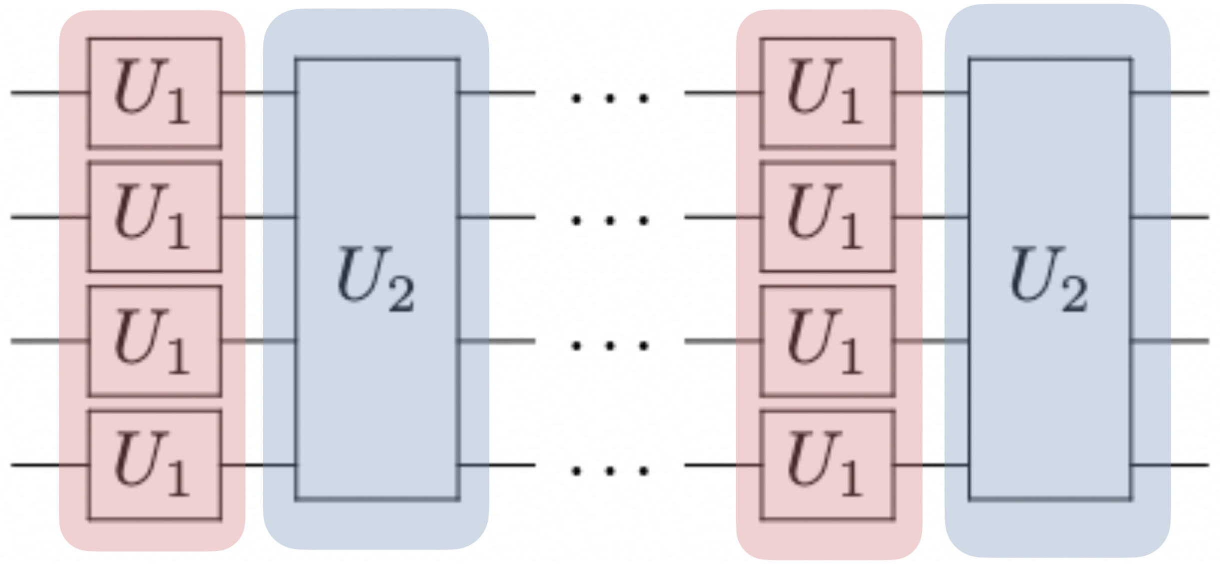


$$U(dt) = U_2(dt)U_1(dt)$$

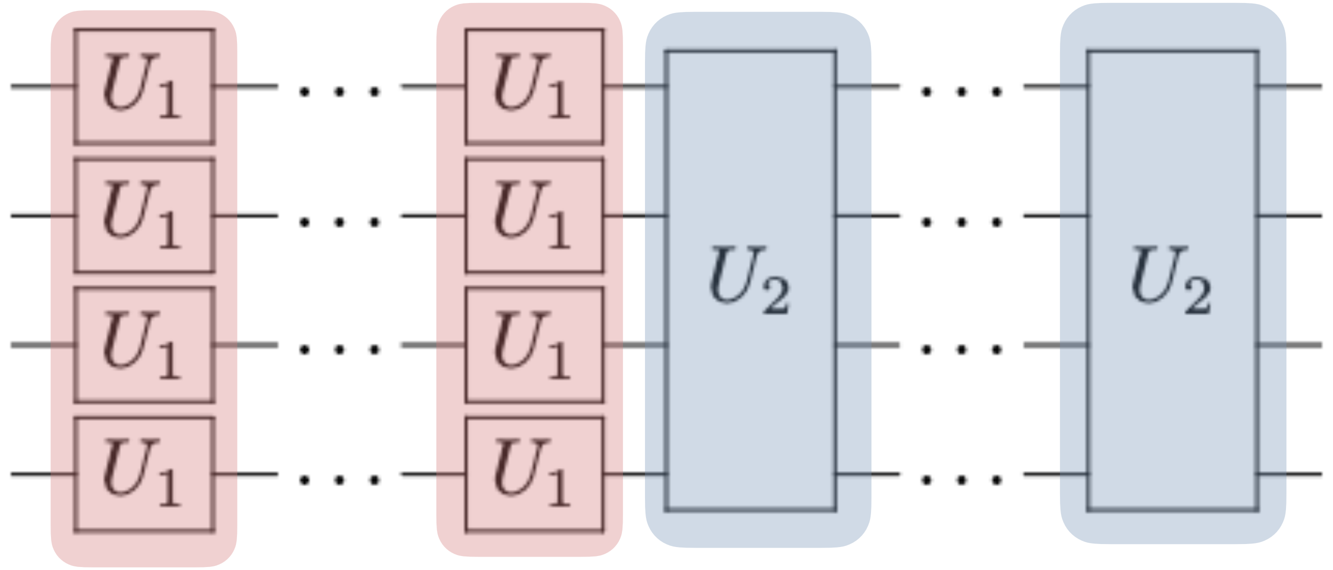


Evolution using multiple Trotter steps

$$U(T) = \prod_{k=1}^{r} U_2\left(\frac{T}{r}\right) U_1\left(\frac{T}{r}\right)$$



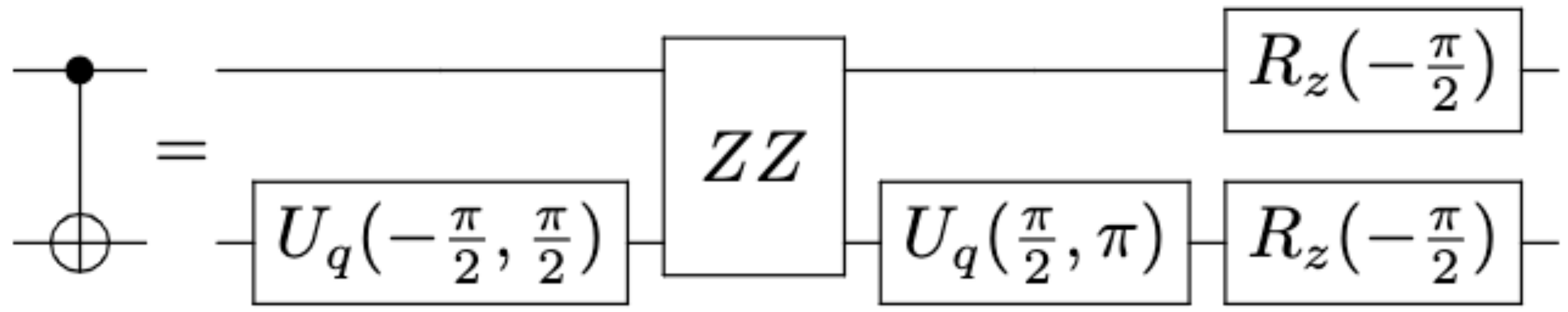
$$U(T) = \prod_{k=1}^{r} U_2\left(\frac{T}{r}\right) \prod_{k=1}^{r} U_1\left(\frac{T}{r}\right)$$



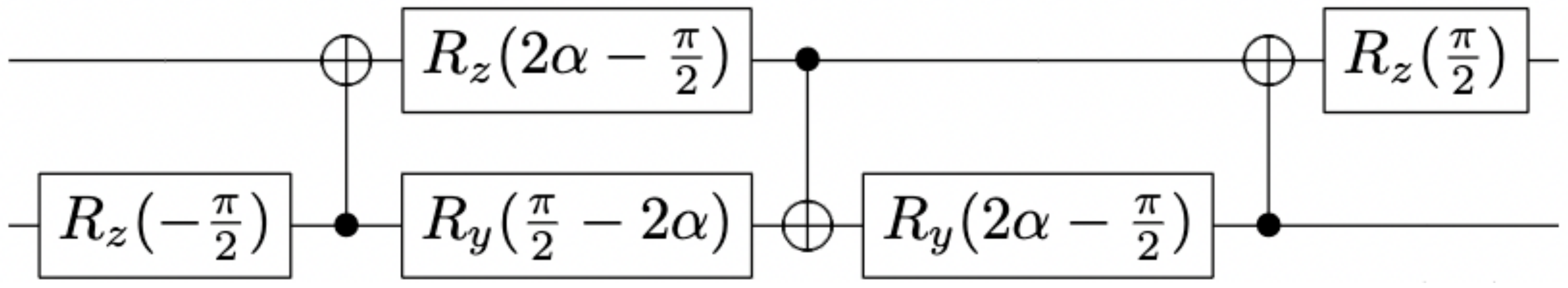
Quantinuum native gate set

- $R_z(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}$
- $U_q(\theta, \varphi) = \begin{pmatrix} \cos\theta/2 & -ie^{-i\varphi}\sin\theta/2 \\ -ie^{i\varphi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}$
- $ZZ = e^{-i\frac{\pi}{4}Z\otimes Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

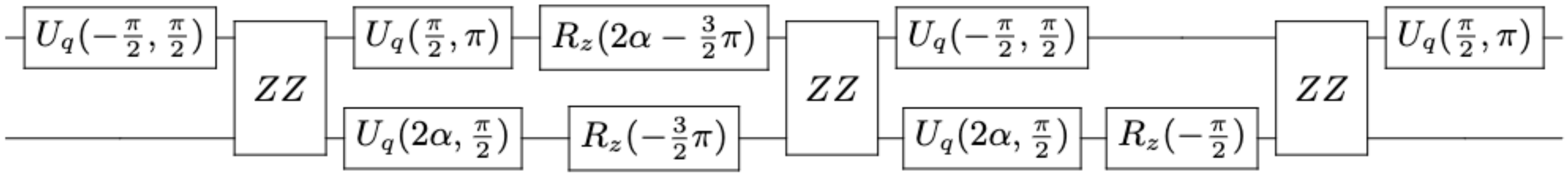
ZZ-based CNOT gate



CNOT-based u_{ij}



ZZ-based u_{ij}



First order

$$U_2(dt) \approx \mathscr{L}_1(dt) = \prod_{K=1}^{\Gamma} e^{-ih_{ij}dt}$$

$$\varepsilon(dt) \leq \frac{dt^2}{2} \sum_{K=1}^{\Gamma} \left\| \sum_{L=K+1}^{\Gamma} [h_K, h_L] \right\|$$

$$\varepsilon(dt) \leq 12 dt^2 \mu^2 \frac{\Theta^2}{N^2} \binom{N}{3} = \mathcal{O}\left(dt^2 \mu^2 N\right)$$

$$\varepsilon(T) \leq r\varepsilon(dt) \quad T = rdt$$

$$r \leq 12 \frac{T^2 \mu^2 \Theta^2}{\epsilon N^2} \binom{N}{3} = \mathcal{O}\left(\frac{T^2 \mu^2 N}{\epsilon}\right)$$

$$\mathscr{C} \leq \binom{N}{2} r$$

$$\mathscr{C}_1 = \mathcal{O}\left(\frac{T^2 \mu^2 N^3}{\epsilon}\right)$$

Second order

$$U(dt) \approx \mathscr{L}_2(dt) = \mathscr{L}_1\left(\frac{dt}{2}\right)\mathscr{L}_1^\dagger\left(-\frac{dt}{2}\right)$$

$$\varepsilon(dt) \leq \frac{dt^3}{12}\sum_K^\Gamma \| \sum_{L>K}^\Gamma \sum_{M>K}^\Gamma [h_L,[h_M,h_K]] \| + \frac{dt^3}{24}\sum_K^\Gamma \| \sum_{L>K}^\Gamma [h_K,[h_K,h_L]] \|$$

$$\varepsilon(dt) \leq dt^3 \frac{\mu^3\Theta^3}{N^3}\left[20\binom{N}{3} + 56\binom{N}{4}\right] = \mathcal{O}\left(dt^3\mu^3 N\right)$$

$$\varepsilon(T) \leq r\varepsilon(dt) \quad T = rdt$$

$$r \leq \frac{(T\mu\Theta)^{3/2}}{\sqrt{\epsilon}N^{3/2}}\sqrt{20\binom{N}{3} + 56\binom{N}{4}} = \mathcal{O}\left(\frac{T^{3/2}\mu^{3/2}\sqrt{N}}{\sqrt{\epsilon}}\right)$$

$$\mathscr{C} \leq \left(2\binom{N}{2} - \frac{N}{2}\right)r$$

$$\mathscr{C}_2 \leq \left(2\binom{N}{2} - \frac{N}{2}\right)r = \mathcal{O}\left(\frac{(T\mu)^{3/2}}{\sqrt{\epsilon}}N^{5/2}\right)$$

- Number of repetitions $M = 200$
- Bayesian approach
  - Probability distribution of obtaining m times the output $|q\rangle$:

$$\mathscr{P}_b(m|p) = \binom{M}{m} p^m (1-p)^{M-m}$$

Likelihood distribution:

Binomial distribution

  - Bayes theorem:

$$\mathscr{P}(p|m) = \frac{\mathscr{P}(m|p)\mathscr{P}(p)}{\mathscr{P}(m)}$$

A priori distribution: Beta distribution

$$\mathscr{B}(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

Posterior distribution

- Prior conjugate

$$\mathscr{B}(\alpha',\beta') = = \frac{\mathscr{P}_b(m|p)\mathscr{B}(\alpha,\beta)}{\int dq \mathscr{P}_b(m|p)\mathscr{B}(\alpha,\beta)}$$ where $\alpha' = \alpha + m$ and $\beta' = \beta + M - m$

- $\alpha = 1$ and $\beta = 1$. We used $\mathscr{B}(\alpha',\beta')$ as posterior distribution and look for:
  - $\mathscr{P}(p_{min} < p < p_{max}) = 0.68$