UNIVERSITÄT GRAZ

Towards TMDs with contour deformations

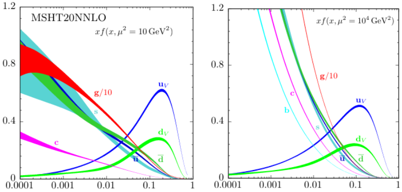
Eduardo Ferreira *with G. Eichmann, A. Stadler*

From first-principles QCD to experiments workshop, ECT*

Hadrons on the Light Front

Goal: Use DSE/BSE to study hadrons on the light front, $x^+ = 0$.

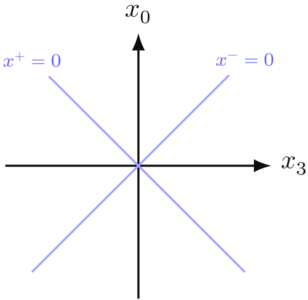
- Natural frame for defining parton distribution functions: PDFs, TMDs, ...

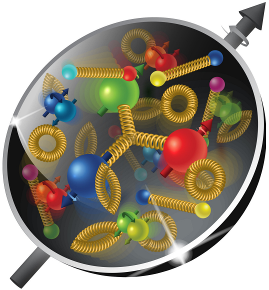


- Future: COMPASS/AMBER @ CERN
  EIC @ Brookhaven National Laboratory.

(AMBER: arXiv:1808.00848)

(EIC: Eur. Phys. J. A 52.9 (2016))

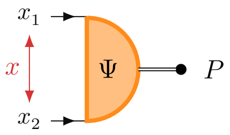




Hadronic quantities

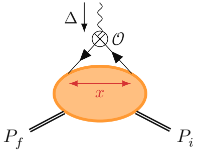
- Bethe-Salpeter Wavefunction

$$\langle 0|\, \mathrm{T}\Phi(x)\Phi(0)\, |P\rangle$$

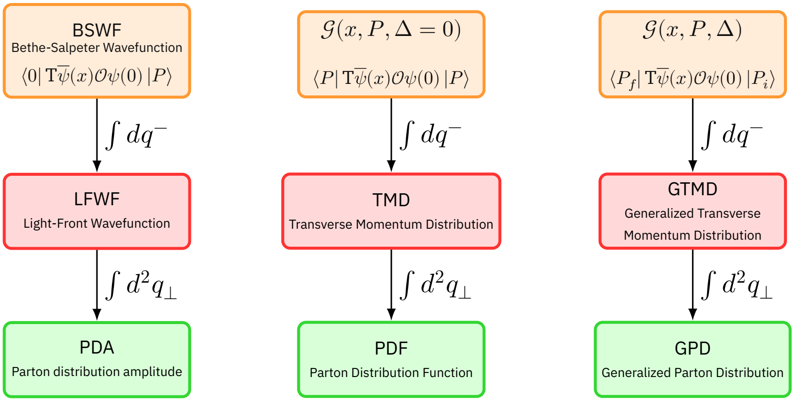


- Generic Correlator

$$\langle P_f|\, \mathrm{T}\Phi(x)\mathcal{O}\Phi(0)\, |P_i\rangle$$



- With $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$, $\vec{x}_\perp = \{x^1, x^2\}$.



(Lorce, Pasquini, Vanderhaeghen; 2011)

1 Light-Front Wavefunction

2 TMDs: Triangle diagram

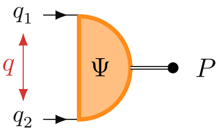
Light-Front Wavefunction

- **Light-Front Wavefunction** (LFWF): Fourier transform of the BSWF at $x = \lambda n + \vec{x}_\perp$, with n along the light front.

$$\Psi_{LF}(q^+, \vec{q}_\perp, P) = \\ = \mathcal{N} \int dq^- \Psi\left(q^-, q^+, \vec{q}_\perp, P\right),$$

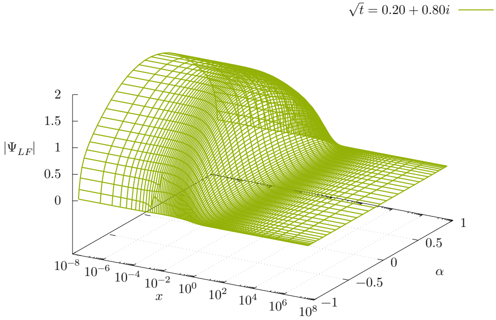
- **Parton Distribution Amplitude** (PDA): Integration of the Ψ_{LF} over $q_\perp$.

$$\phi(\alpha, P) = \int d^2 q_\perp \Psi_{LF}\left(\frac{\alpha}{2} P^+, \vec{q}_\perp, P\right).$$



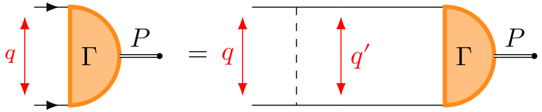
$\xi = \frac{q_1^+}{P^+} = \frac{1+\alpha}{2}$ is the longitudinal momentum fraction

$q = k + \frac{\alpha}{2} P$ is the relative momentum (with $k^+ = 0$).



Scalar toy model

- Scalar model:
  - ϕ of mass m
  - χ of mass μ
- BS amplitude for bound-state of two ϕ:



$$q = k + \frac{\alpha}{2}P \qquad q' = k' + \frac{\alpha}{2}P$$

- Tree-level propagators
- Single χ exchange kernel

(Wick; 1954), (Cutkosky; 1954)

- The BSWF is a function of the kinematic invariants:

$$-M^2 = \frac{P^2}{4m^2} = t \qquad \frac{k^2}{m^2} = x \qquad \omega = \hat{k}\cdot\hat{P}$$

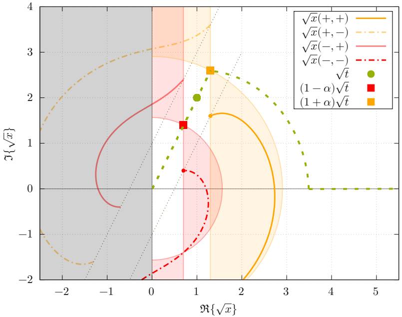
$$\psi(x,\omega,t,\alpha) = \frac{m^4}{(2\pi)^3}\frac{1}{2}\int_0^\infty dx' x' \times \int_{-1}^{1} d\omega' \sqrt{1-\omega'^2}\,\mathbf{G_0}(x',\omega',t,\alpha) \times \int_{-1}^{1} dy\,\mathbf{K}(x,\omega,x',\omega',y)\psi(x',\omega',t,\alpha)$$

Analytic Structure

$$\mathbf{G_0} = \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2}$$

- Poles when $q_{1/2}^2 = -m^2$
- Integration in $\omega \implies$ branch cuts in complex x plane:

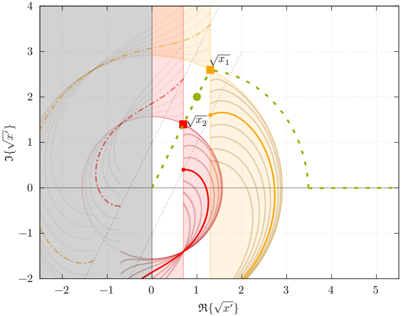
$$\sqrt{x}_{\pm}^{\lambda} = \mp(1 \pm \alpha)\sqrt{t} \left[\omega + i\lambda \sqrt{1 - \omega^2 + \frac{1}{(1 \pm \alpha)^2 t}} \right]$$



$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$

- Poles when $(q - q')^2 = -\mu^2$.
- Branch cuts in complex x' plane:

$$\sqrt{x'} = \sqrt{x} \left(\Omega \pm i \sqrt{1 - \Omega^2 + \frac{\beta^2}{x}} \right)$$



Light-front wavefunction

- The LFWF is defined as:

$$\Psi_{LF}(\alpha, k_\perp, P) = \mathcal{N} \int dq^- \; \Psi(q, P)\Big|_{q^+ = \frac{\alpha}{2} P^+, q_\perp = k_\perp}$$

- In our kinematic variables:

$$q^- = -\frac{2m^2}{P^+}\left(2\sqrt{x}\sqrt{t}\omega + \alpha t\right)$$

α and $x = \frac{k^2}{m^2} = \frac{k_\perp^2}{m^2}$ and t are external variables

- Need the BSWF in $\omega \in (-\infty, \infty)$.
- We use the Schlessinger method for analytic continuation:

$$R(\omega) = \cfrac{f(\omega_1)}{1 + \cfrac{a_1(\omega - \omega_1)}{1 + \cfrac{a_2(\omega - \omega_2)}{1 + \cfrac{a_3(\omega - \omega_3)}{\cdots}}}}$$

(L. Schlessinger, 1968) (Tripolt et al., 2019) (D. Binosi, R-A. Tripolt; 2019)

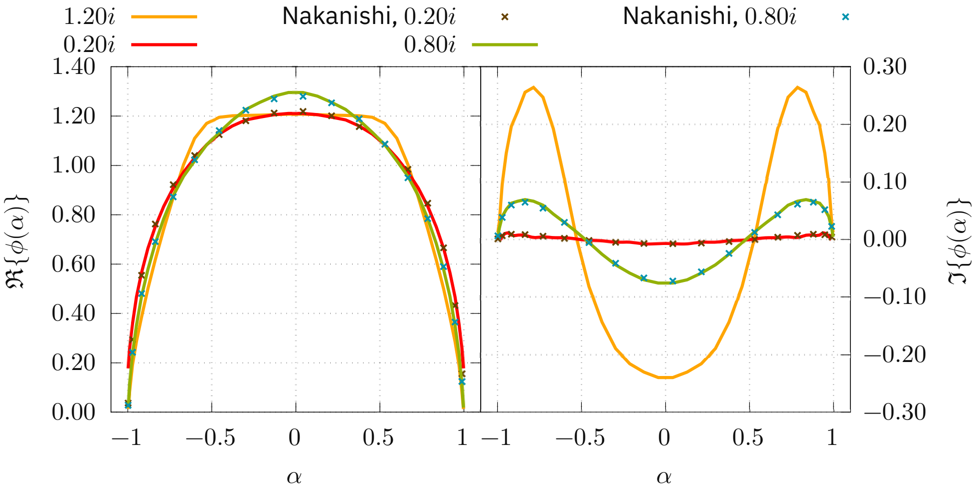
- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$

Definition of the LFWF

$$\Psi_{LF}(\alpha, x, t) = \mathcal{N}\frac{2\sqrt{x}\sqrt{t}}{i\pi}\int_{-\infty}^{\infty} d\omega \, \Psi(x, \omega, t, \alpha)$$

PDA: α dependance

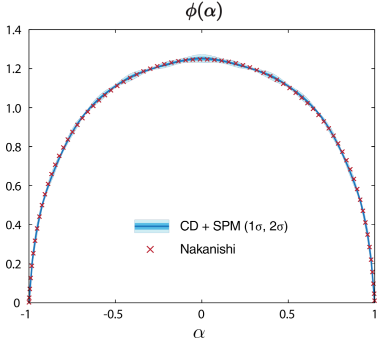
$$\phi(\alpha) = \int_0^\infty dx\, \Psi_{LF}(x, \alpha, t)$$



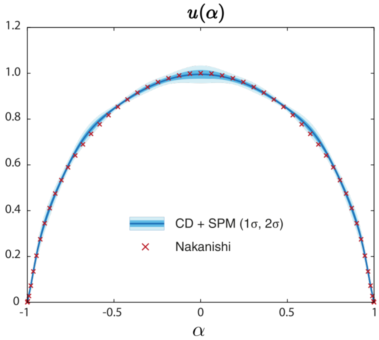
(Frederico, Salmé, Viviani; 2014) (G. Eichmann, EF, A. Stadler; 2022)

PDA: real masses

- Use SPM to get to $\sqrt{t} = \frac{iM}{2m} = bi$, $b \in \mathbb{R}$ (cuts prevent direct evaluation!).
- Expand PDA in Chebyshev-U: $\phi(\alpha) = (1-\alpha^2)\sum \phi_n U_n(\alpha)$.
- Analytic continuation of ϕ_n with SPM.



$$\phi(\alpha) \propto \int dx\, \Psi_{LF}(x,\alpha)$$



$$u(\alpha) \propto \int dx\, |\Psi_{LF}(x,\alpha)|^2$$

(Frederico, Salmé, Viviani; 2014)

- **Input data:** $N \in [10, 50]$ different points starting with $\Re\sqrt{t} = 0.1$, with steps in N of 2.

G. Eichmann, EF, A. Stadler; Phys. Rev. D 105, 034009 (2022)

Unequal masses

- Consider two ϕ of different masses:

$$m_1 = m(1+\varepsilon) \qquad m_2 = m(1-\varepsilon)$$

$$\frac{m_1}{m_2} = \frac{1+\varepsilon}{1-\varepsilon} \qquad 2m = m_1 + m_2$$

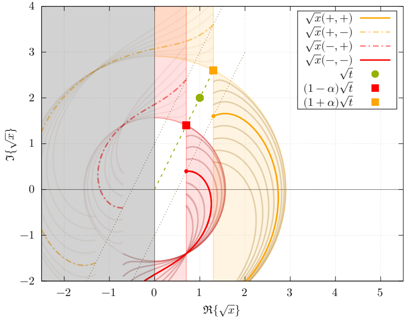
- $\varepsilon \in [-1, 1]$ sets the ratio of the masses
- $\mathbf{G_0}$ is now:

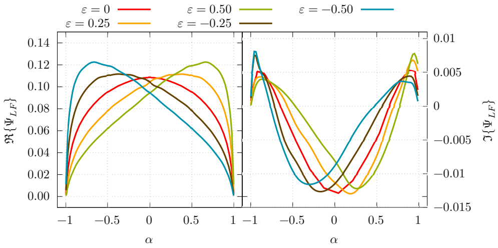
$$\mathbf{G_0} = \frac{1}{q_1^2 + m_1^2}\frac{1}{q_2^2 + m_2^2}$$

- Cuts in x:

$$\sqrt{x}^{\lambda}_{\pm} = \mp(1 \pm \alpha)\sqrt{t} \times \left[\omega + i\lambda\sqrt{1 - \omega^2 + \frac{1}{t}\left(\frac{1 \pm \varepsilon}{1 \pm \alpha}\right)^2}\right]$$

- Integration path still works
- ε adds skewness





Complex conjugate masses

- Also consider complex conjugate mass poles:

$$D_\phi(q,m) = \frac{1}{2}\left(\frac{1}{q^2+m^2} + \frac{1}{q^2+(m^*)^2}\right)$$

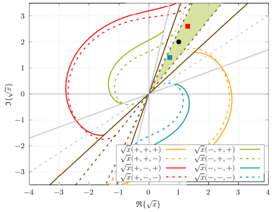
- $m^2 \to m^2(1+i\delta)$, with $m^2, \delta \in \mathbb{R}_+$
- $\mathbf{G_0}$ becomes:

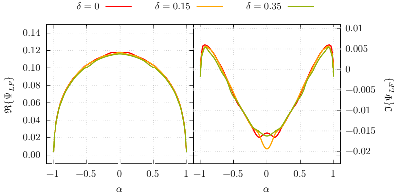
$$\mathbf{G_0} = D_\phi(q_1,m)D_\phi(q_2,m)$$

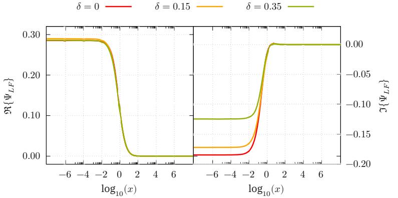
- There are now 8 cuts:

$$\sqrt{x}_\pm^{\{\lambda,\nu\}} = \mp(1\pm\alpha)\sqrt{t} \times \left[\omega + i\lambda\sqrt{1-\omega^2+\frac{1}{t}\frac{1+\nu i\delta}{(1\pm\alpha)^2}}\right]$$

- For $\delta < \delta_{crit}$, contour deformation always possible.





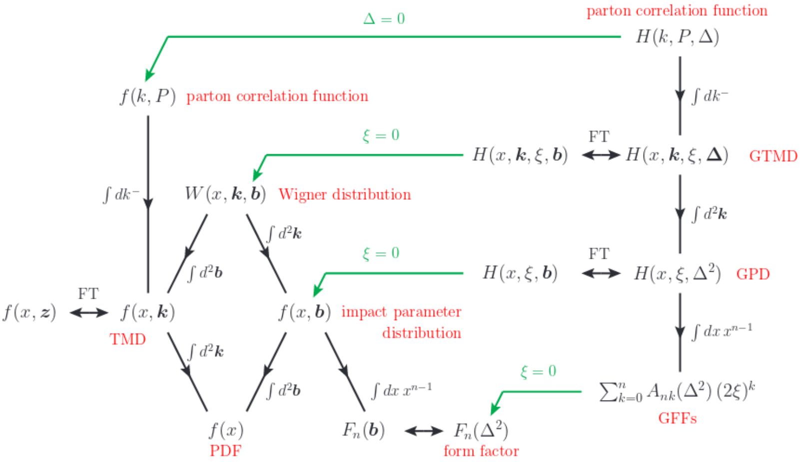


1 Light-Front Wavefunction

2 TMDs: Triangle diagram

Reminder: partonic picture of hadrons

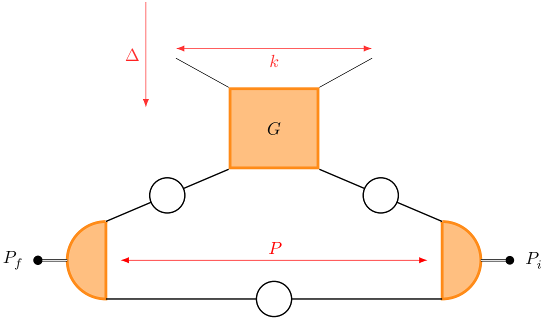
- LF projection of the hadronic correlator uncovers the internal dynamics of the hadron



(Picture: Diehl, 2016); (Diehl, 2003); (Meissner, Goeke, Metz, Schlegel; 2008); (Meissner, Metz, Schlegel; 2009)

Writing the hadronic correlation

- **Main Goal:** Get partonic distribution functions from hadron-hadron correlations via **FUN ctional Methods**



- G is the four-point quark correlation function, calculated with scattering equation.
- The quark propagator is calculated via quark DSE.
- The BSWF is calculated via the meson BSE.

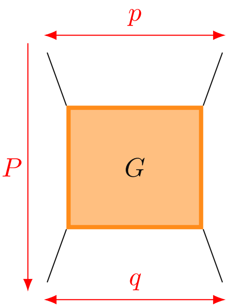
(Mezrag, arXiv:1507.05824); (Diehl, Gousset, 1998); (Tiburzi, Miller, 2003); (Mezrag, Chang, Moutarde, Roberts, Rodríguez-Quintero, Sabatié, Schmidt, 2015); (Cloët, Roberts, 2018), many many others, ...

$$\mathcal{G}^{[\Gamma]}(P,k,\Delta) = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \int \frac{d^4z}{2\pi^4} e^{ik\cdot z} \langle P_f | \overline{\psi}(z)\mathcal{W}\Gamma\psi(0) | P_i\rangle\right]$$

- Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.

First piece: 4-point function

- 4- point function determined from scattering equation:
  $\mathbf{G} = \mathbf{G_0} + \mathbf{G_0 T G_0} \implies \mathbf{T} = \mathbf{K} + \mathbf{K G_0 T} \implies \mathbf{T} = (\mathbb{1} - \mathbf{K G_0})^{-1} \mathbf{K}.$



$$T(t, X, R, Z, Y, Q) = K(X, R, p \cdot q)$$
$$+ \frac{1}{2} \frac{m^4}{2\pi^4} \int_0^\infty dx\, x \int_{-1}^{1} dz \sqrt{1 - z^2} G_0 (x, z, t)$$
$$\times \int_{-1}^{1} dy \int_0^{2\pi} d\Psi K(X, x, k \cdot q) T(t, x, y, z, R, Q)$$

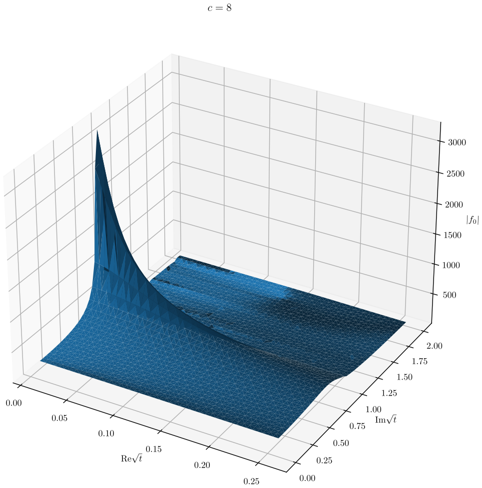
- Fully off-shell: 6 Lorentz invariants
  - 3 radial: X, t, R;
  - 3 angular: Y, Z, Q;

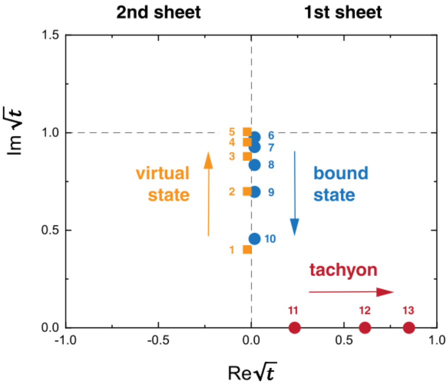
- Same G_0 and K as in the BSE
- Solved numerically as linear system
- Contains *all* dynamics of 2 ϕ particles
  - **Must produce bound state poles dynamically!**

(Eichmann, Duarte, Peña, Stadler; 2019)

4-point function results

- T is made fully on-shell
- Partial-wave expansion shows bound-state pole in the first Riemann sheet



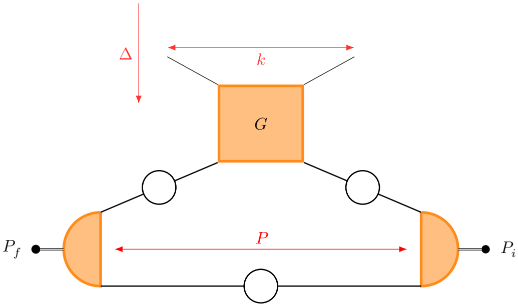


(Eichmann, Duarte, Peña, Stadler; 2019)

- Describes both long-range and short-range qq dynamics.

Second step: Triangle Diagram

- We start by solving a simple model, and gradually build up the complexity of the calculation.
  - Hadrons are on-shell: $P^2 = -M^2$

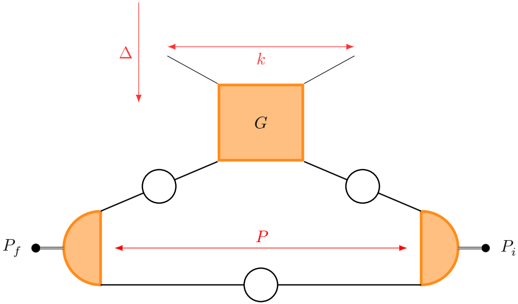


$$\Delta = 2M\sqrt{t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad P = -iM\sqrt{1+t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$k = \frac{1+\alpha}{2}P + M\sqrt{X}\begin{pmatrix}0\\ \sqrt{1-Z^2}\sqrt{1-Y^2}\\ \sqrt{1-Z^2}Y\\ Z\end{pmatrix}$$

Second step: Triangle Diagram

- We start by solving a simple model, and gradually build up the complexity of the calculation.
  - Hadrons are on-shell: $P^2 = -M^2$
  - **Forward limit:** $\Delta \to 0 \implies t \to 0$ – We get the PDFs and TMDs



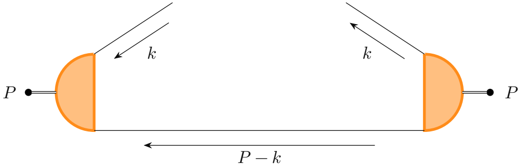
$$\Delta = 0 \qquad P = -iM \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$k = \frac{1+\alpha}{2} + M\sqrt{X} \begin{pmatrix} 0 \\ \sqrt{1-Z^2}\sqrt{1-Y^2} \\ \sqrt{1-Z^2}Y \\ Z \end{pmatrix}$$

$$\omega \implies Y\sqrt{1-Z^2}$$

Second step: Triangle Diagram

- We start by solving a simple model, and gradually build up the complexity of the calculation.
  - Hadrons are on-shell: $P^2 = -M^2$
  - **Forward limit:** $\Delta \to 0 \implies t \to 0$ – We get the PDFs and TMDs



- Using tree-level propagators S
- The amplitudes Γ are calculated with the BSE (as before)
- Two diagrams:
  - Upper line spectating
  - Lower line spectating
- TMD obtained by projecting to the light-front (integration on k^-)

Definition of the TMD

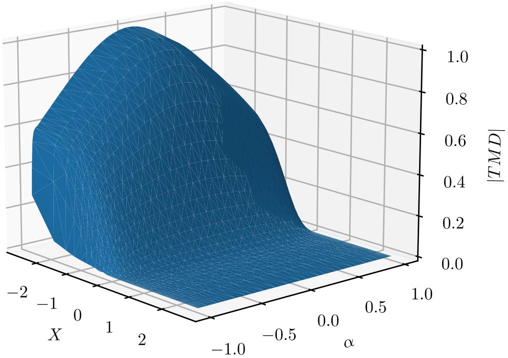
$$\mathrm{TMD}(X,\alpha) \propto -2i\sqrt{X}\int_{-\infty}^{\infty} d\omega\, \mathcal{G}(X,\omega,t,\alpha)$$

G. Eichmann, EF, A. Stadler; Phys. Rev. D 105, 034009 (2022)

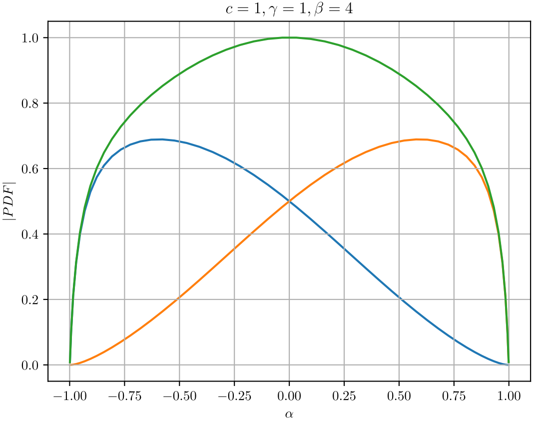
Some results

$$\frac{\mu}{m} = \beta \qquad \frac{m}{M} = \gamma \qquad \frac{g^2}{16\pi^2 m^2} = c$$

$c = 1, \gamma = 1, \beta = 4$



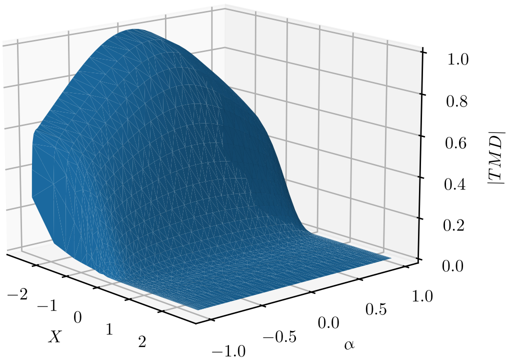
- Lower line spectating (blue)
- Upper line spectating (orange)
- Sum of both diagrams (green)



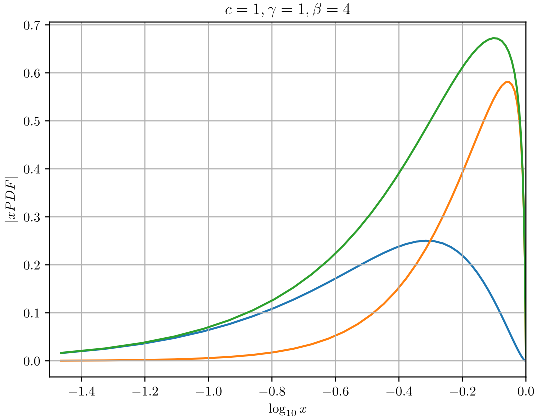
Some results

$$\frac{\mu}{m} = \beta \qquad \frac{m}{M} = \gamma \qquad \frac{g^2}{16\pi^2 m^2} = c$$

$c = 1, \gamma = 1, \beta = 4$

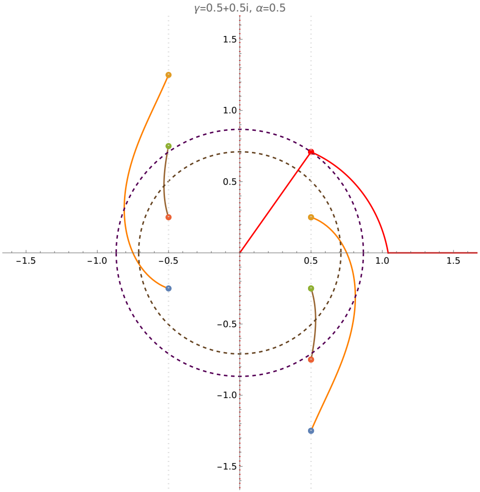


- Lower line spectating (blue)
- Upper line spectating (orange)
- Sum of both diagrams (green)

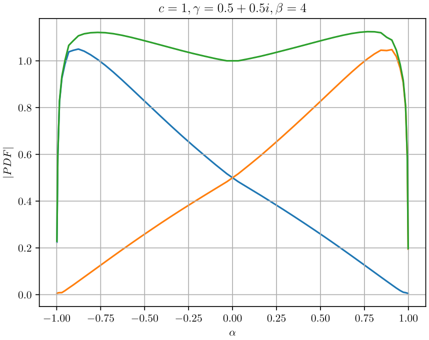


Contour deformations once again

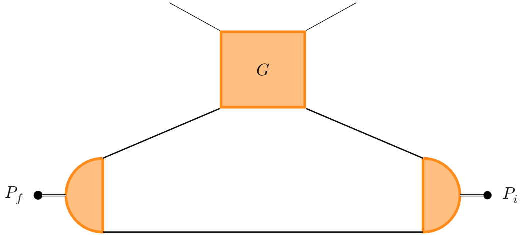
- $\gamma < 1$ implies contour deformations again!
  - Equivalent to the $\Im\sqrt{t} > 1$ region in the BSE: $\frac{i}{2\gamma} = \sqrt{t}$.



- Poles cross the real axis on:
  - "Quark" legs of the triangle diagram
  - Inside the BSE
- Again, purely real masses do not work! Need always some imaginary part.



- **Current step:** close the loop with the previous calculation of the 4-point function.



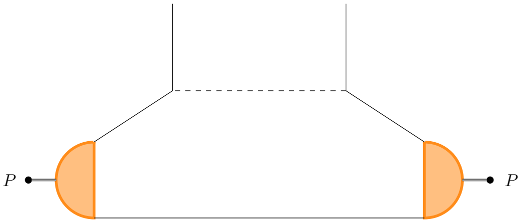
- Need to understand the projection of the complete triangle with the loop.
- G is calculated using a ladder kernel.
- Difficulties when using the one-particle exchange.
- **LF Projection here requires:**

$$\propto \int_{-\infty}^{\infty} d\omega \frac{1}{a\omega + b\sqrt{1-\omega^2} + c}$$

$a, b, c \in \mathbb{C}$

Moving forward *Closing the loop*

- **Current step:** close the loop with the previous calculation of the 4-point function.



- Need to understand the projection of the complete triangle with the loop.
- G is calculated using a ladder kernel.
- Difficulties when using the one-particle exchange.
- **LF Projection here requires:**

$$\propto \int_{-\infty}^{\infty} d\omega \frac{1}{a\omega + b\sqrt{1-\omega^2} + c}$$

$a, b, c \in \mathbb{C}$

BACKUP

Bethe-Salpeter Wavefunction

- The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:

$$\Psi(x, P) = \langle 0 | \, \mathrm{T}\phi(0)\phi(x) \, | P \rangle$$

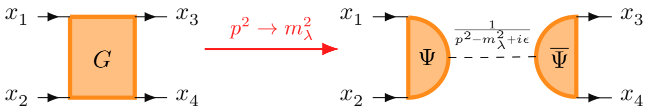
$$\Psi(k, P) = \int d^4 x e^{-ik\cdot x} \Psi(x, P)$$

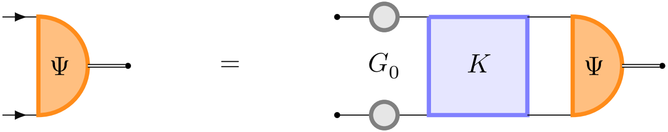
- Determined by the Bethe-Salpeter Equation:

$$\Psi = \mathbf{G_0 K} \Psi$$

$\mathbf{G_0}$ product of the dressed propagators;

$\mathbf{K}$ interaction kernel between the two particles.





Nakanishi Method

- BSWF defined from a smooth weight function $g(x, \alpha)$.

$$\Psi(q, P) = \frac{1}{m^4} \int_0^\infty dx' \int_{-1}^1 d\alpha' \frac{g(x', \alpha')}{\left[\kappa + 1 + x' + (1 - \alpha'^2)t\right]^3}, \qquad \kappa = \frac{1}{m^2}\left(q - \frac{\alpha'}{2}P\right)^2.$$

- Light front quantities obtained from the weight function g, for example LFWF:

$$\Psi_{LF} = \frac{\mathcal{N}}{m^2} \int_0^\infty dx' \frac{g(x', \alpha)}{\left[x' + 1 + x + (1 - \alpha^2)t\right]^2}$$

- The BSE can be rewritten for g:

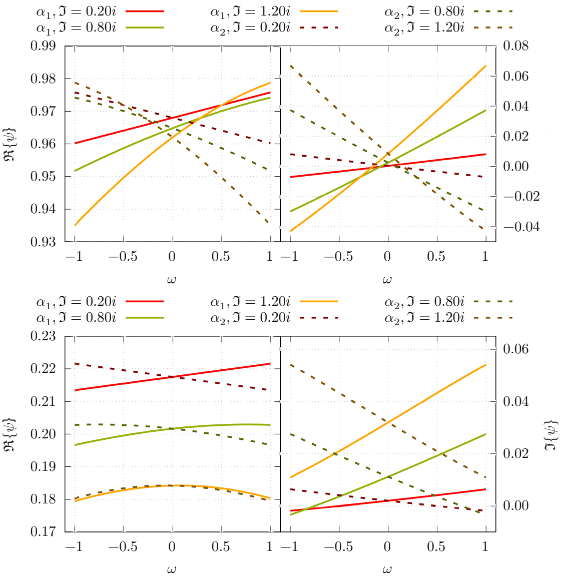
$$\int_0^\infty dx' \frac{g(x', \alpha)}{\left[x' + 1 + x + (1 - \alpha^2)t\right]^2} = c \int_0^\infty dx' \int_{-1}^1 d\alpha' V(x, x', \alpha, \alpha') g(x', \alpha')$$

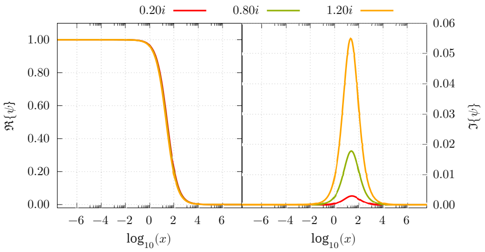
$$V(x, x', \alpha, \alpha') = \frac{K(x, x', \alpha, \alpha') + K(x, x', -\alpha, -\alpha')}{2\left[x + 1 + (1 - \alpha^2)t\right]}$$

$$K(x, x', \alpha, \alpha') = \int_0^1 dv \frac{\theta(\alpha - \alpha')(1 - \alpha)^2}{\left[v(1 - \alpha)(x' + 1 + (1 - \alpha'^2)t) + (1 - v)C\right]^2}$$

$$C = (1 - \alpha')(1 + x + (1 - \alpha^2)t) + (1 - \alpha)\left(\frac{\beta}{v} + x'\right)$$

BSWF Results

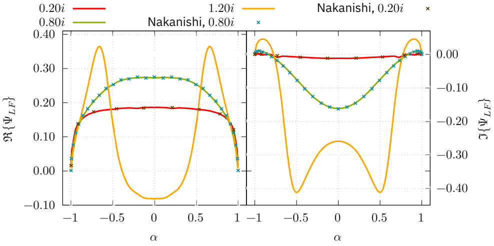


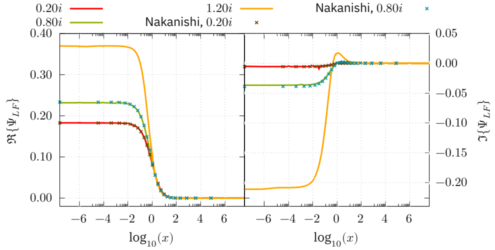


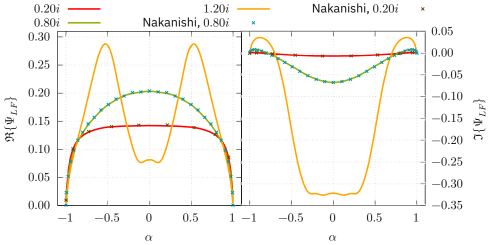
- Small ω and α dependence.
- Symmetry for the combined transformation $\alpha \to -\alpha$ and $\omega \to -\omega$.
- Approximately a monopole:

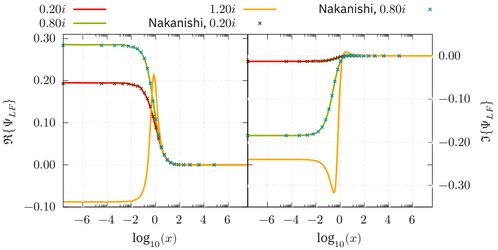
$$\psi \approx \frac{1}{q^2 + \gamma}$$

LFWF: α and x dependance







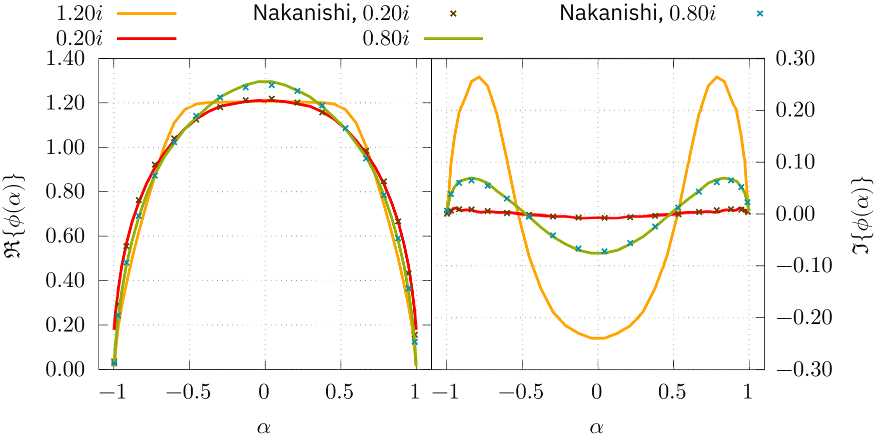


- Symmetric in α.
- Vanishes at $\alpha = \pm 1$.

PDA: α dependance

- In our variables:

$$\phi(\alpha) = \int_0^\infty dx\, \Psi_{LF}(x, \alpha, t)$$



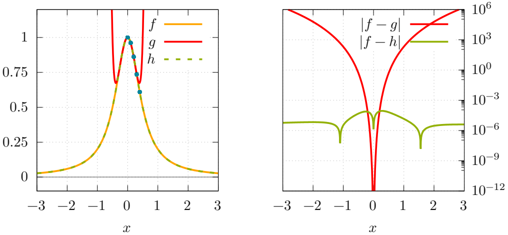
- Symmetric in α.
- Vanishes at $\alpha = \pm 1$.

Schlessinger Point Method

- Numerical analytic continuation method:

$$R(\omega) = \cfrac{f(\omega_1)}{1 + \cfrac{a_1(\omega - \omega_1)}{1 + \cfrac{a_2(\omega - \omega_2)}{1 + \cfrac{a_3(\omega - \omega_3)}{\dots}}}}$$

- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$



- Recurrence relations:

$$R(\omega) = \frac{f(\omega_1)}{1 + \mathcal{Z}_1} = \cfrac{f(\omega_1)}{1 + \cfrac{a_1(\omega - \omega_1)}{1 + \mathcal{Z}_2}} = \dots$$

$$\mathcal{Z}_k = \frac{a_k(\omega - \omega_k)}{1 + \mathcal{Z}_{k+1}} \Leftrightarrow \mathcal{Z}_{k+1} = \frac{a_k(\omega - \omega_k)}{\mathcal{Z}_k} - 1,$$

$$\omega = \omega_k \implies \mathcal{Z}_k = 0$$

$$f(\omega_2) = \frac{f(\omega_1)}{1 + a_1(\omega_2 - \omega_1)},$$

$$\mathcal{Z}_1 = \frac{f(\omega_1)}{f(\omega_2)} - 1 \Leftrightarrow a_1 = \frac{\mathcal{Z}_1}{\omega_2 - \omega_1},$$

Why not do one more iteration?

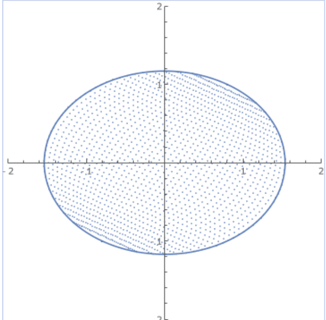
- Do *one more iteration* for a value of $\omega = W \in \mathbb{C}$, with the obtained Ψ

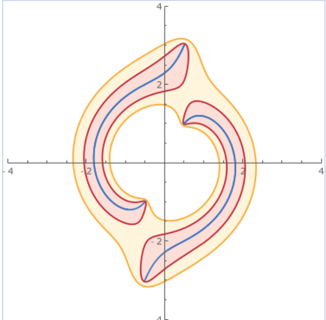
$$\Psi(x, W, t, \alpha) = \mathcal{N} \int_0^\infty dx' \int_{-1}^{1} d\omega' \mathcal{K}(x, x', W, \omega') \Psi(x', \omega', t, \alpha)$$

- **Problem:** Kernel cuts will change
  - For $\omega \in \mathbb{C}$, and $y, \omega' \in [-1, 1]$, Ω turns into a region bounded by the $r(\theta)$ ellipse, with $\omega = a + ib$ and $\sqrt{1-\omega^2} = c + id$:

$$r(\theta) = \sqrt{a^2 + c^2}\sqrt{\cos^2\theta + E^2 \sin^2\theta} \qquad E = \begin{cases} \frac{d^2}{a^2} & \alpha \neq 0 \\ \frac{b^2}{1+b^2} & \alpha = 0 \end{cases}$$

  - Kernel cuts will eventually overlap





Cuts for complex conjugate mass poles

$$\Im\{\sqrt{\tau}\}\Re\{i\sqrt{1+i\delta}\} < \Im\{i\sqrt{1+i\delta}\}\Re\{\sqrt{\tau}\}.$$