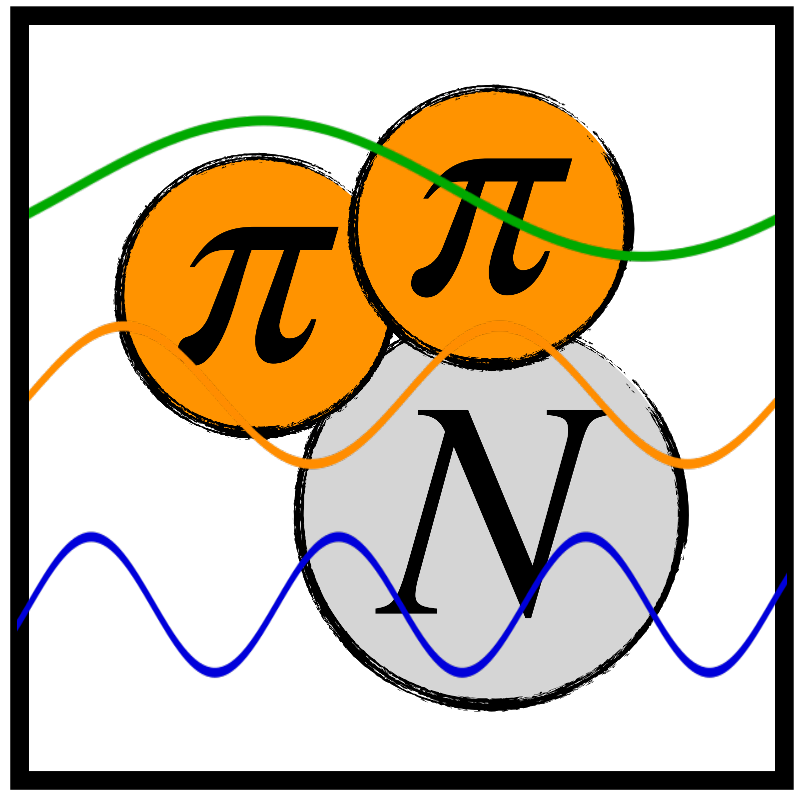
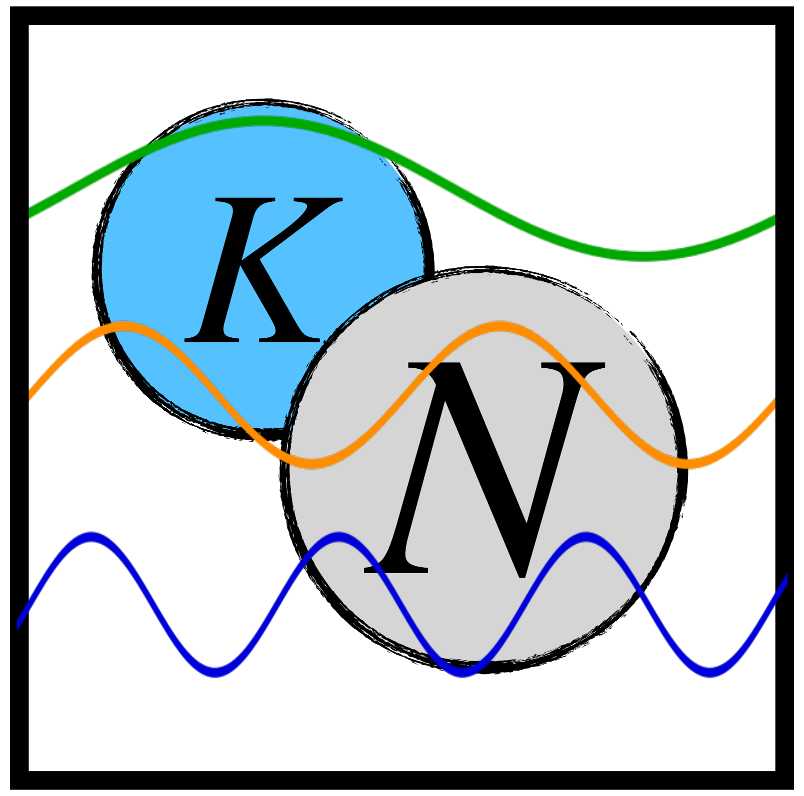
# Hadronic resonances from lattice QCD


Fernando Romero-López
MIT
fernando@mit.edu

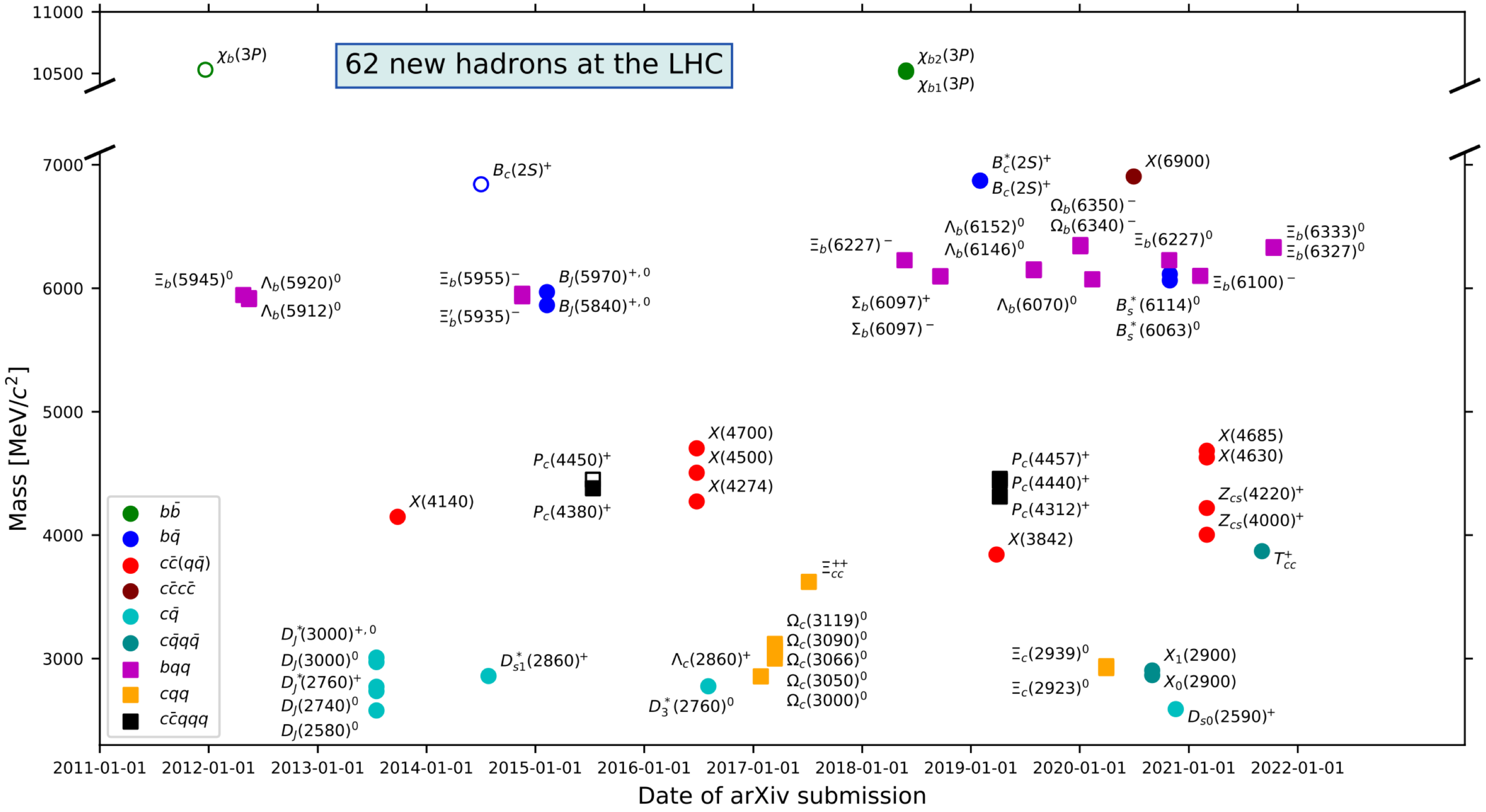




from first-principles QCD to experiment @ ECT*
May 24th



# The Hadron Spectrum

**Lattice QCD offers a systematically improvable approach to investigate the hadron spectrum**




**[I. Danilkin, talk@INT, March 2023]** **(+ Babar, Belle, COMPASS)**

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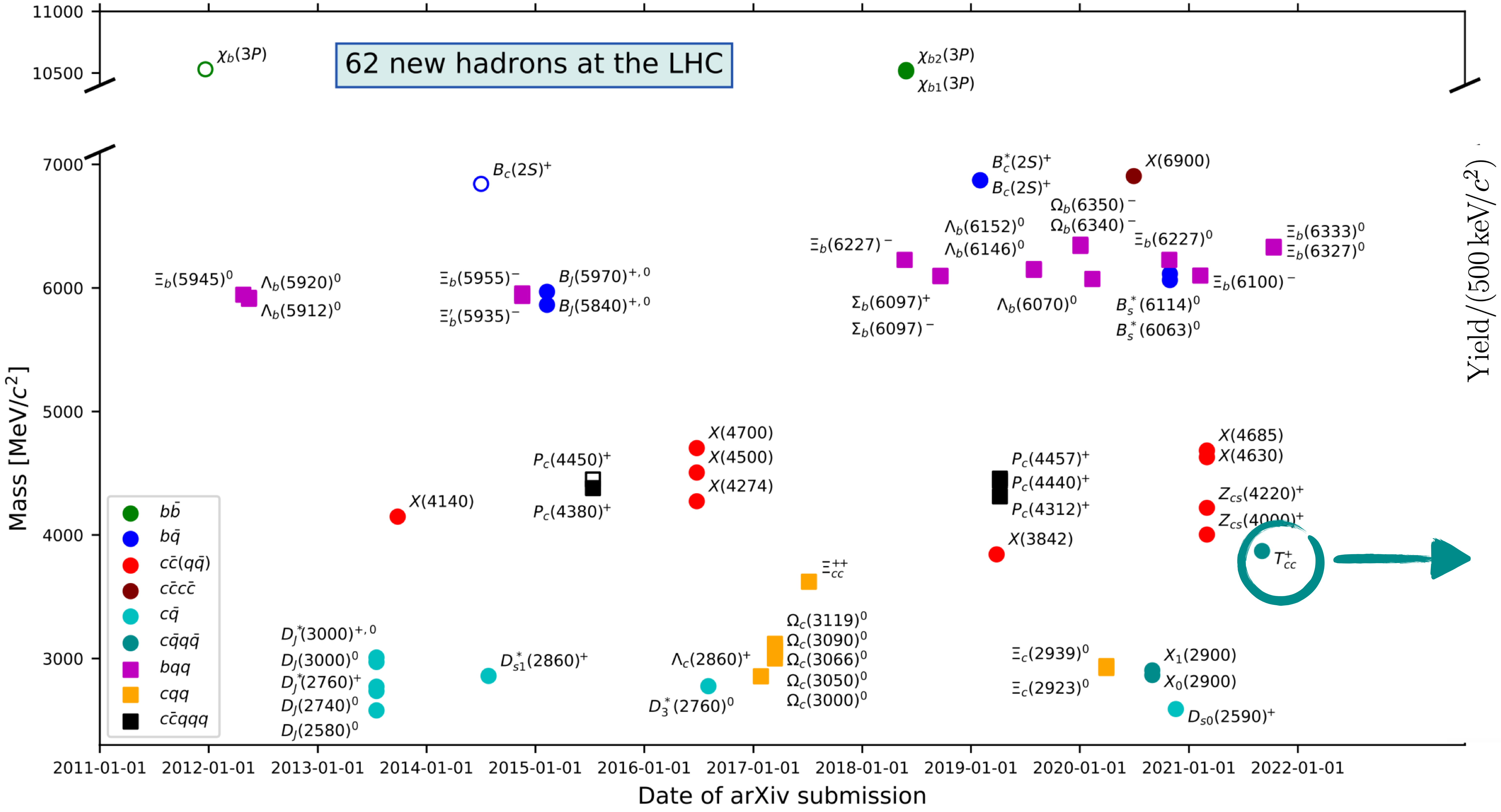


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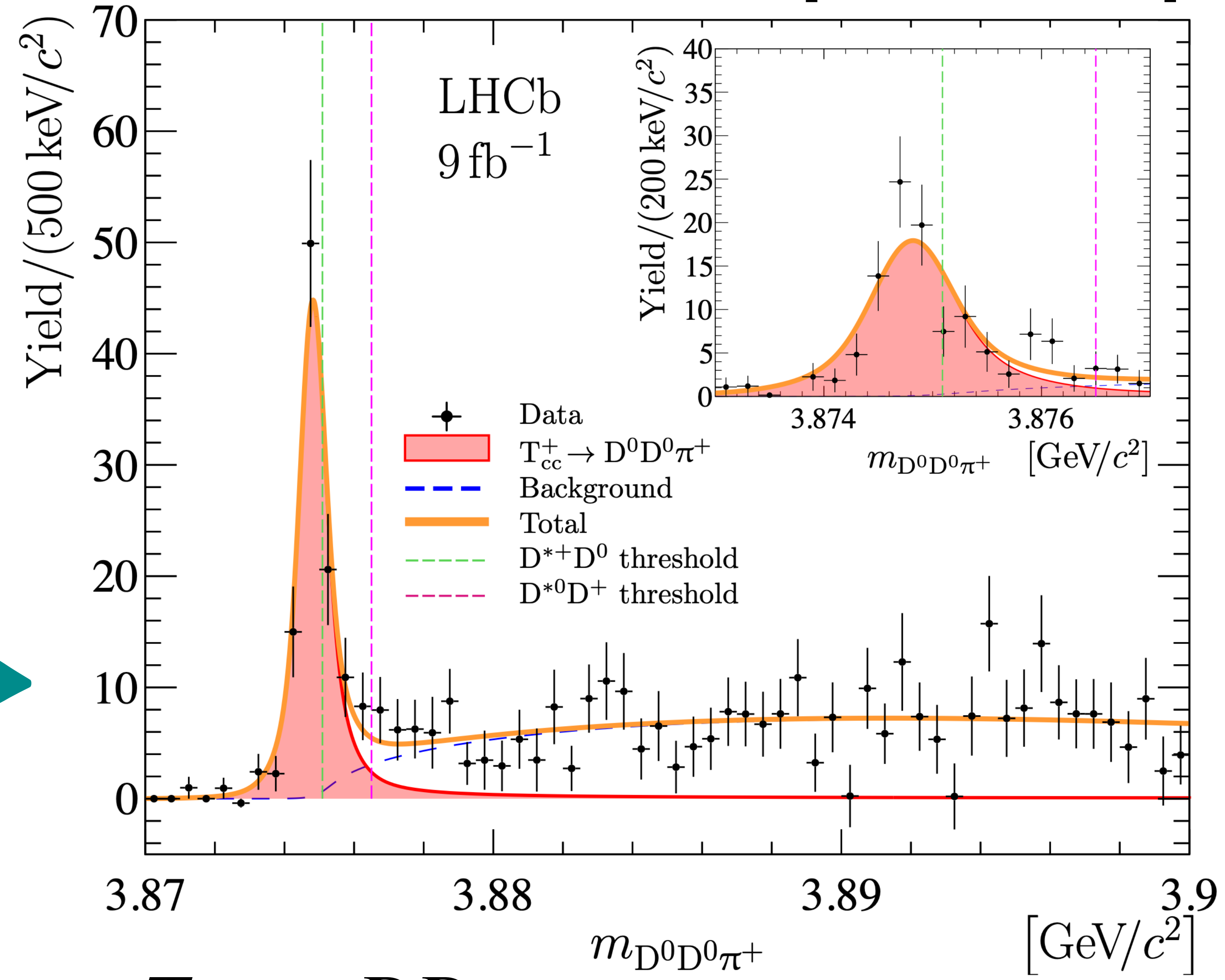
**[arXiv:2109.01038]**




$T_{cc} \to DD\pi$

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# Multi-hadron interactions

Beyond QCD, understanding resonance properties is important for new physics searches

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- **Tests of the Standard Model in meson weak decays**

CP violation in $K \to \pi\pi$ weak decays

$$\left(\varepsilon'/\varepsilon\right)_{\rm exp} = (16.6 \pm 2.3) \times 10^{-4}$$

**[NA48 & KTeV, 2002 & 2009]**

$\sigma$ **resonance**

CP violation in $D^0 \to K^+K^-/\pi^+\pi^-$ decays

$$\Delta a_{CP}^{\rm dir} = (-15.7 \pm 2.9) \times 10^{-4}$$

**[LHCb, 2019]**

$f_0(1710)$ **resonance**

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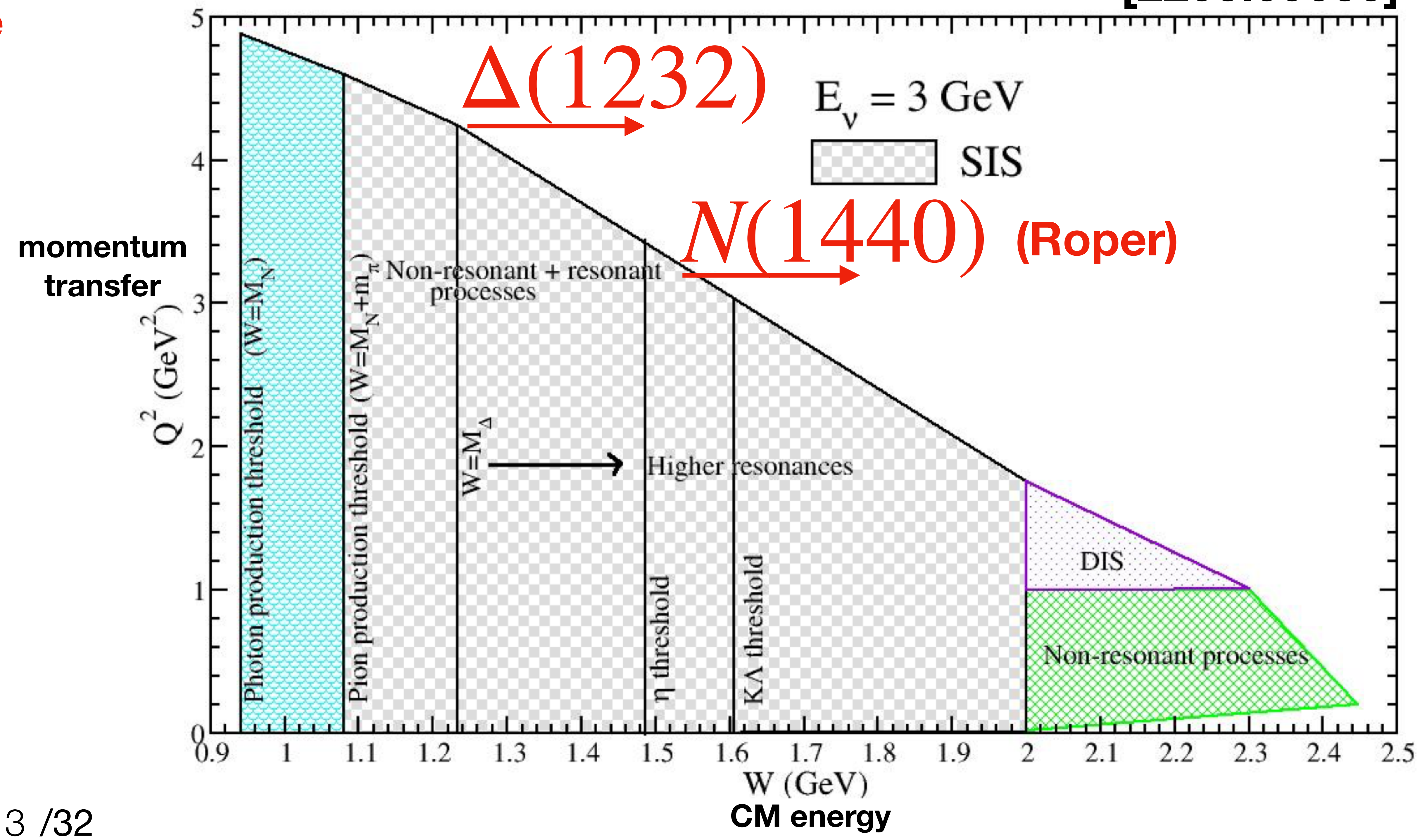
- **Neutrino-nucleus scattering (DUNE, Hyper-K)**

$$\nu N \to \ell N \pi$$

**[2203.09030]**

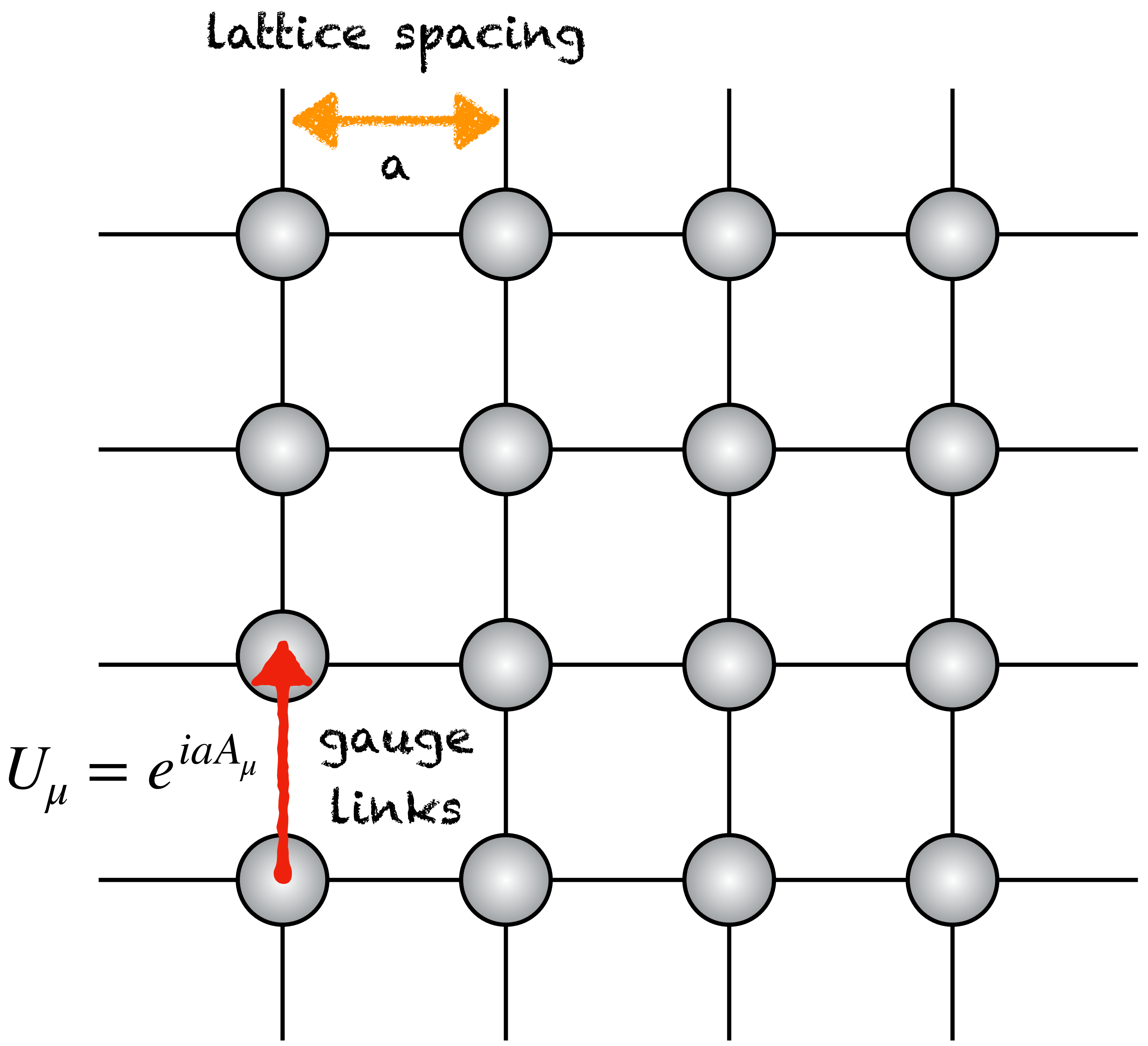


# Lattice QCD

- Lattice QCD is a first-principles numerical approach to the strong interaction

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{\mathcal{Z}}\int D\psi D\bar{\psi} DA\, \mathcal{O}(t)\mathcal{O}(0)e^{-S_E(\psi,\bar{\psi},A_\mu)}$$

Euclidean action



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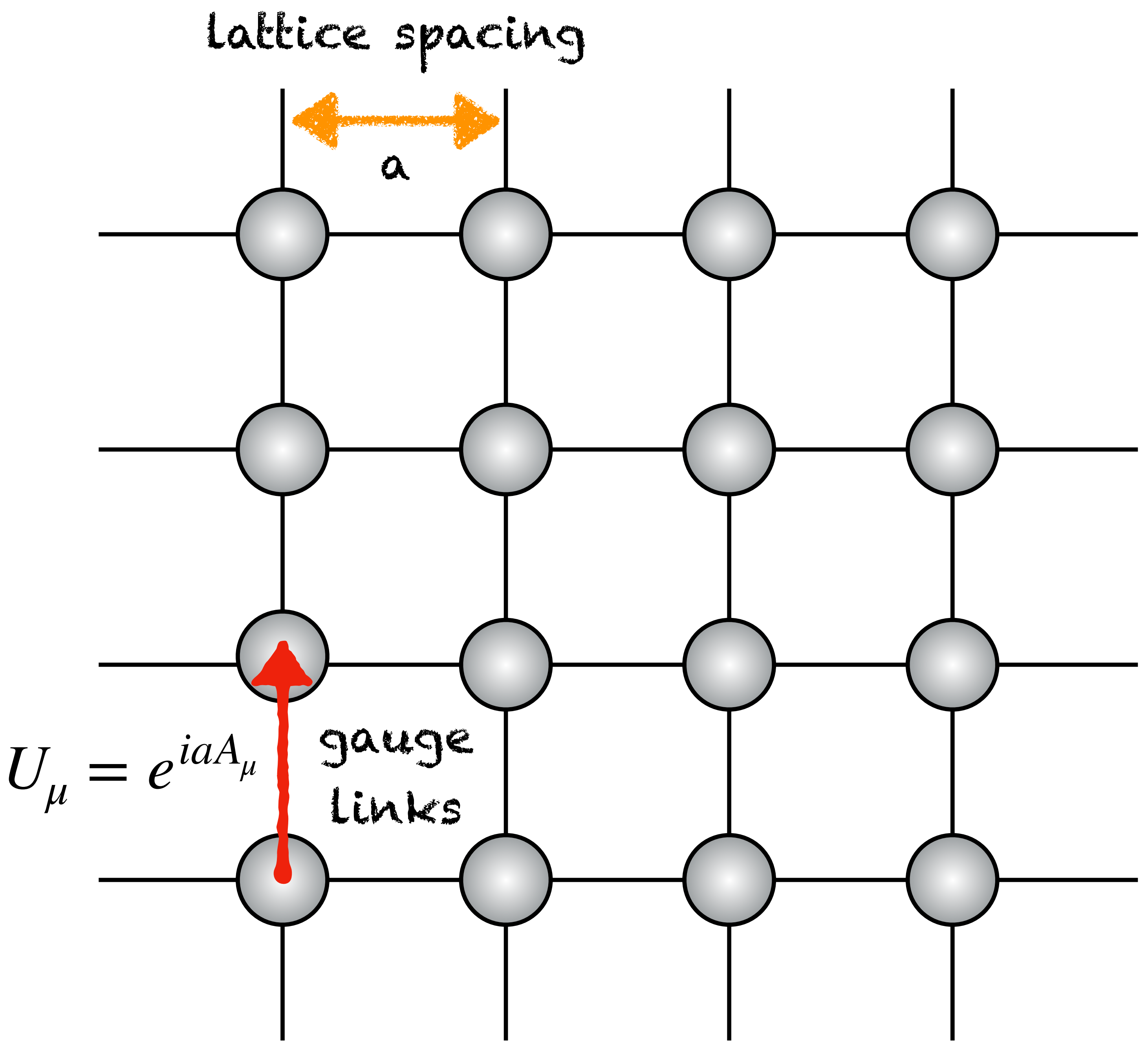


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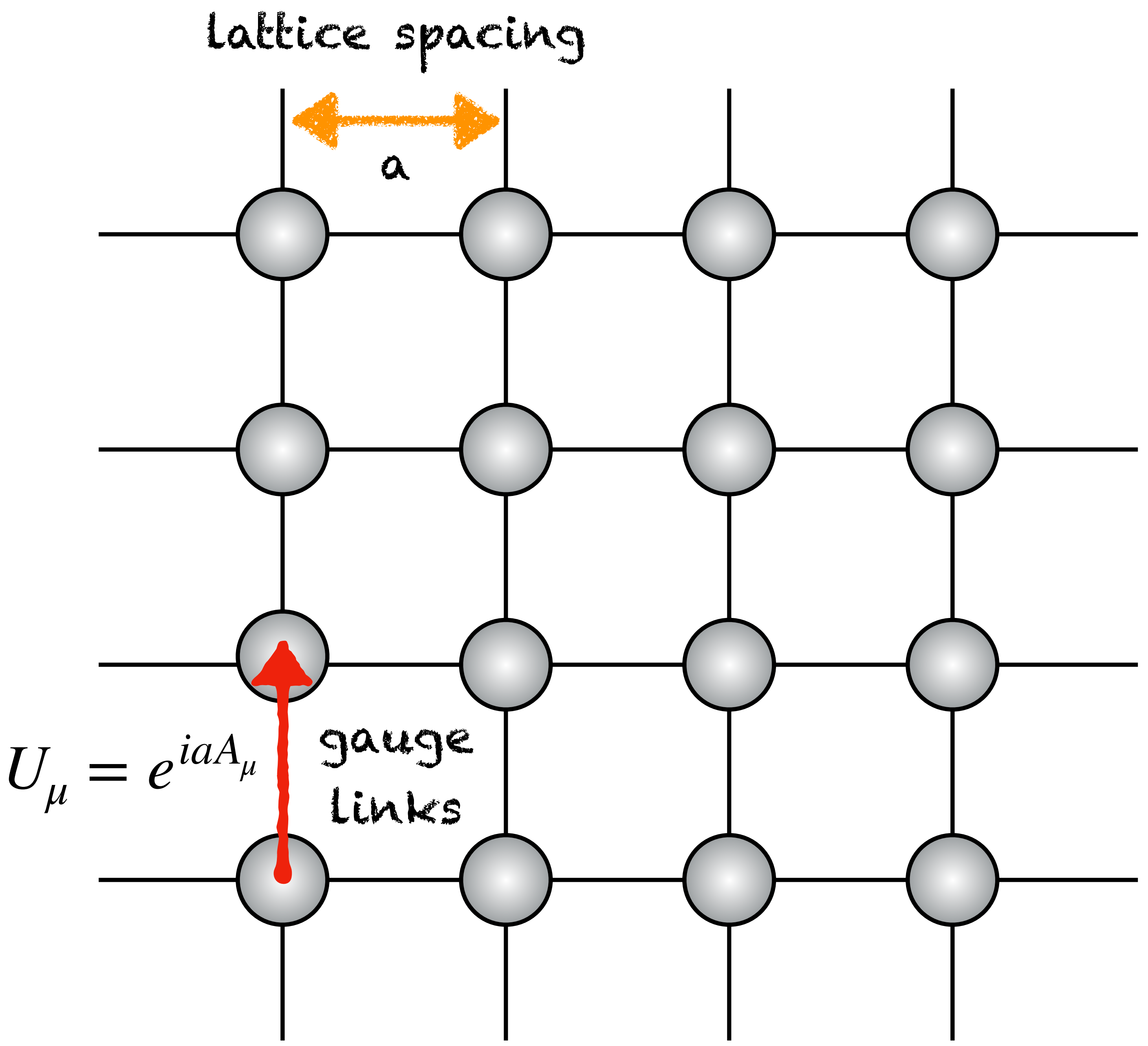


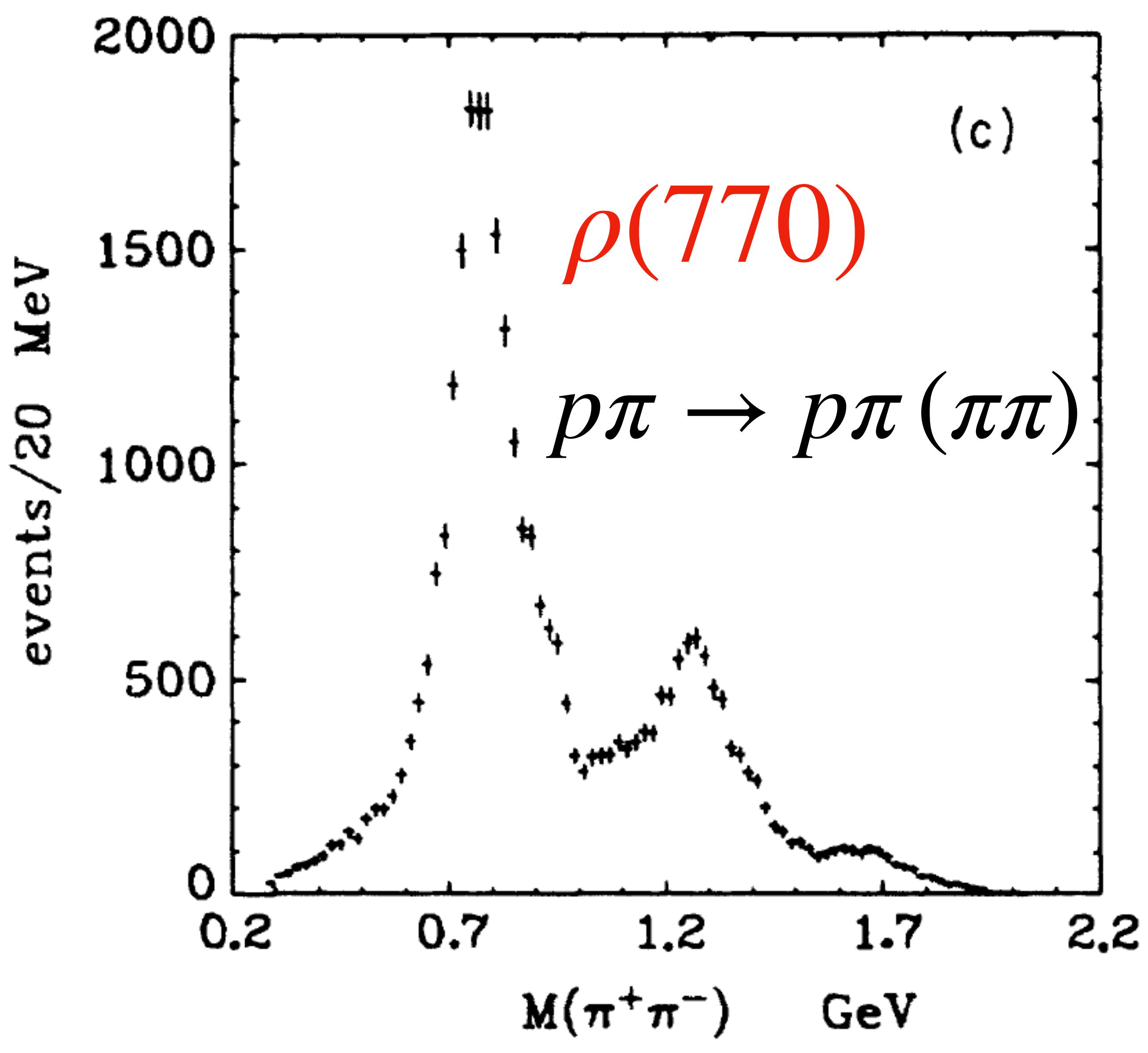

Can we obtain resonance properties from Euclidean correlation functions?

→ Yes, but not that simple!

# Infinite vs finite volume

## Experiments

- Asymptotic states
- Direct access to scattering amplitudes




[Protopopescu et al, PRD7 1973]

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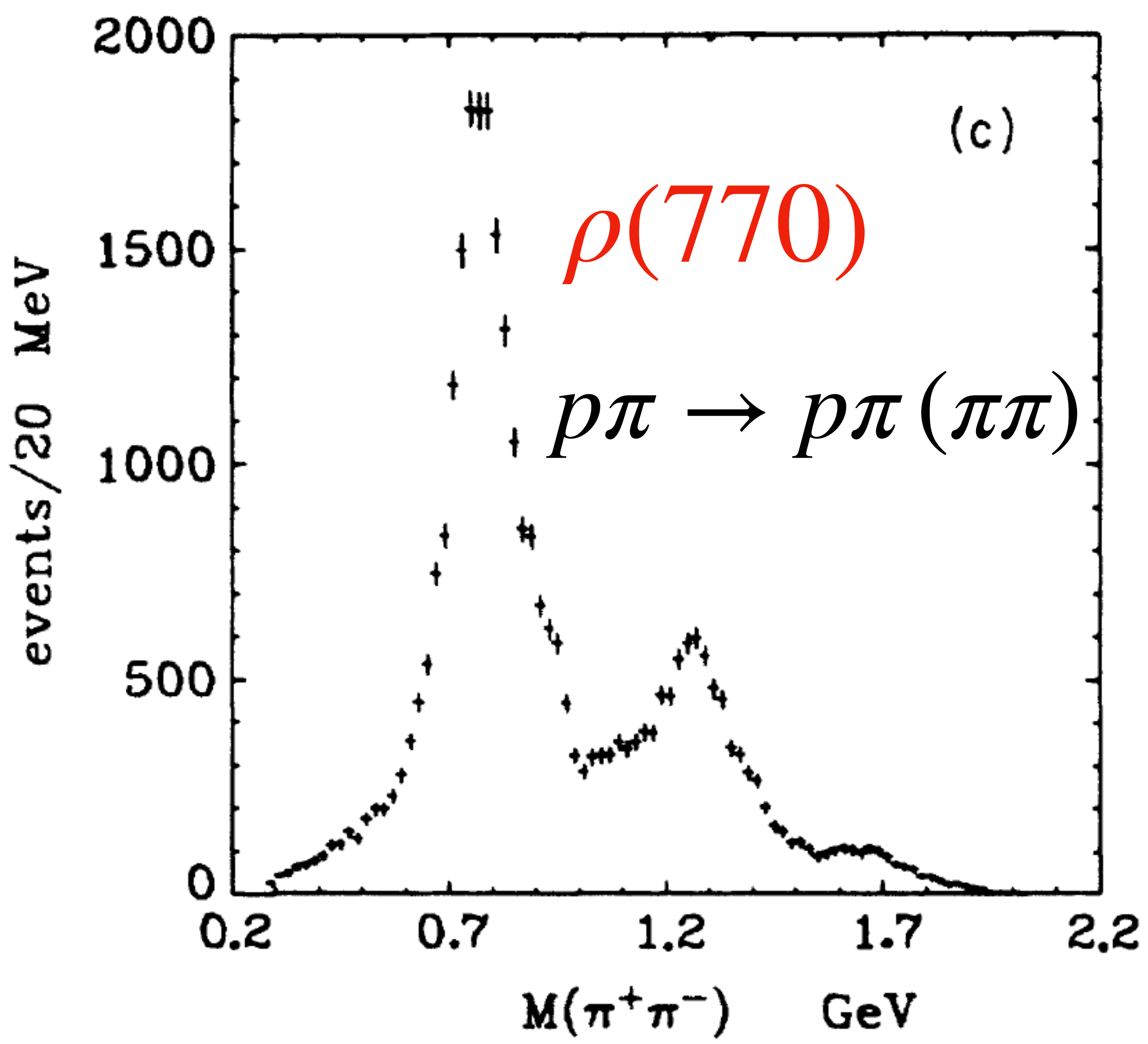


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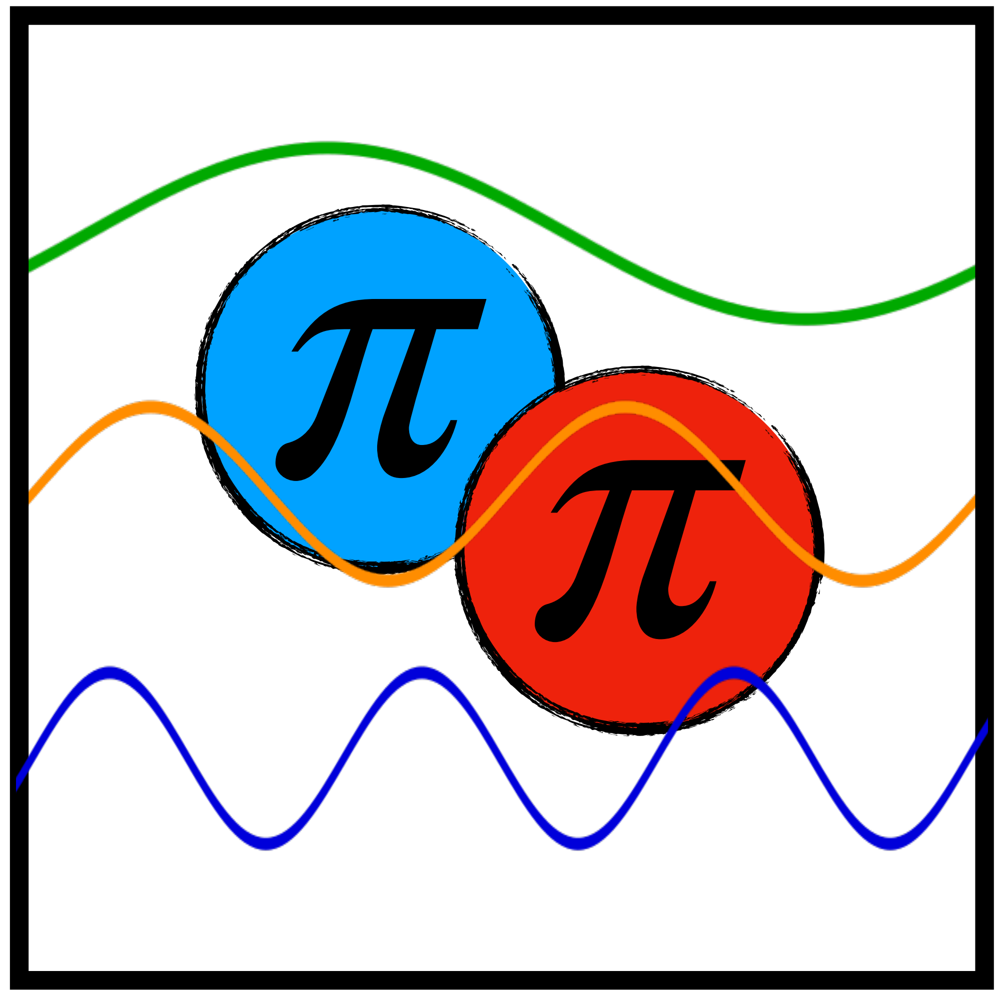
## Lattice QCD

- Euclidean time
- Stationary states in a box



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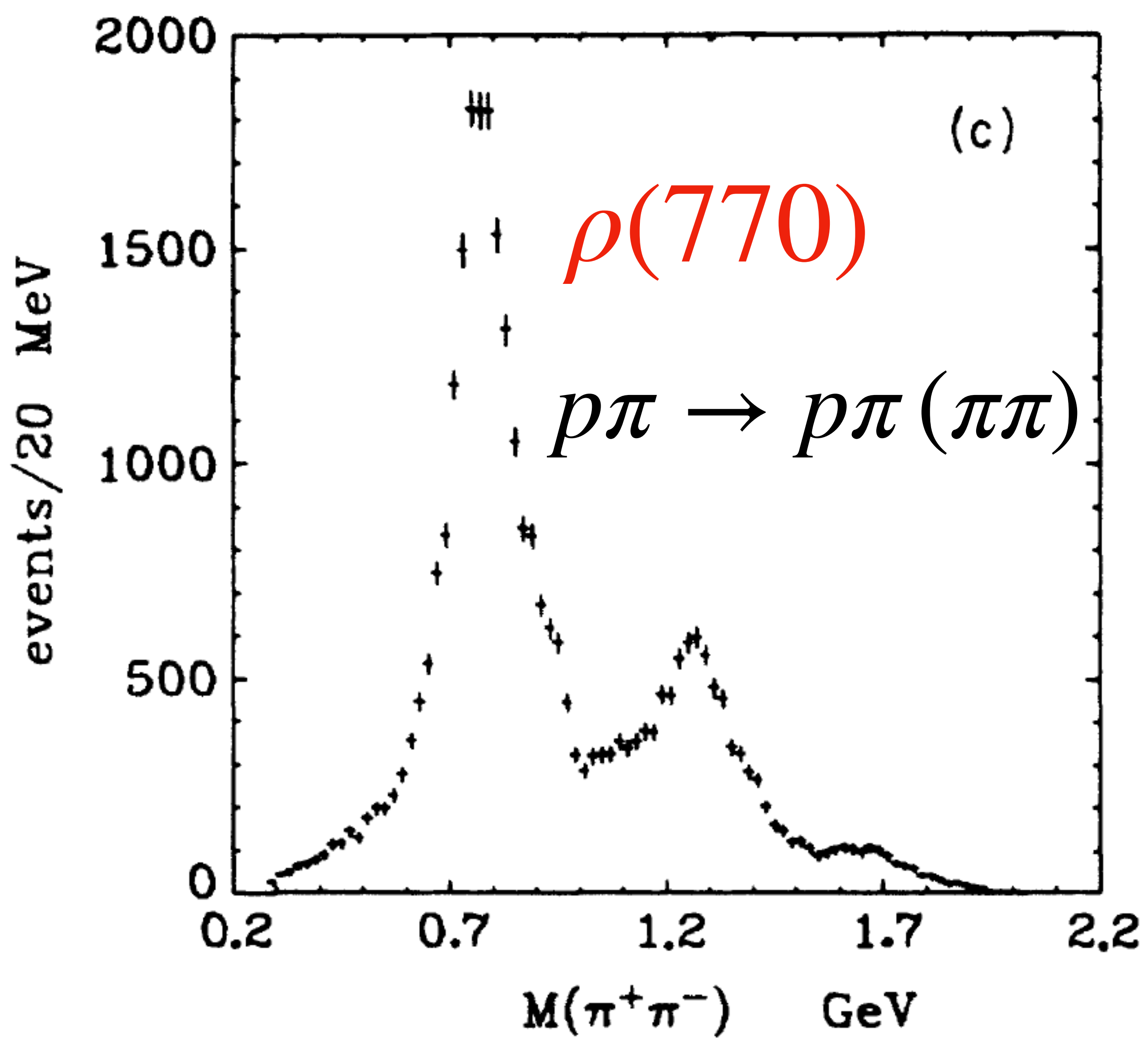


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Finite-volume formalism

[Lüscher, 89']

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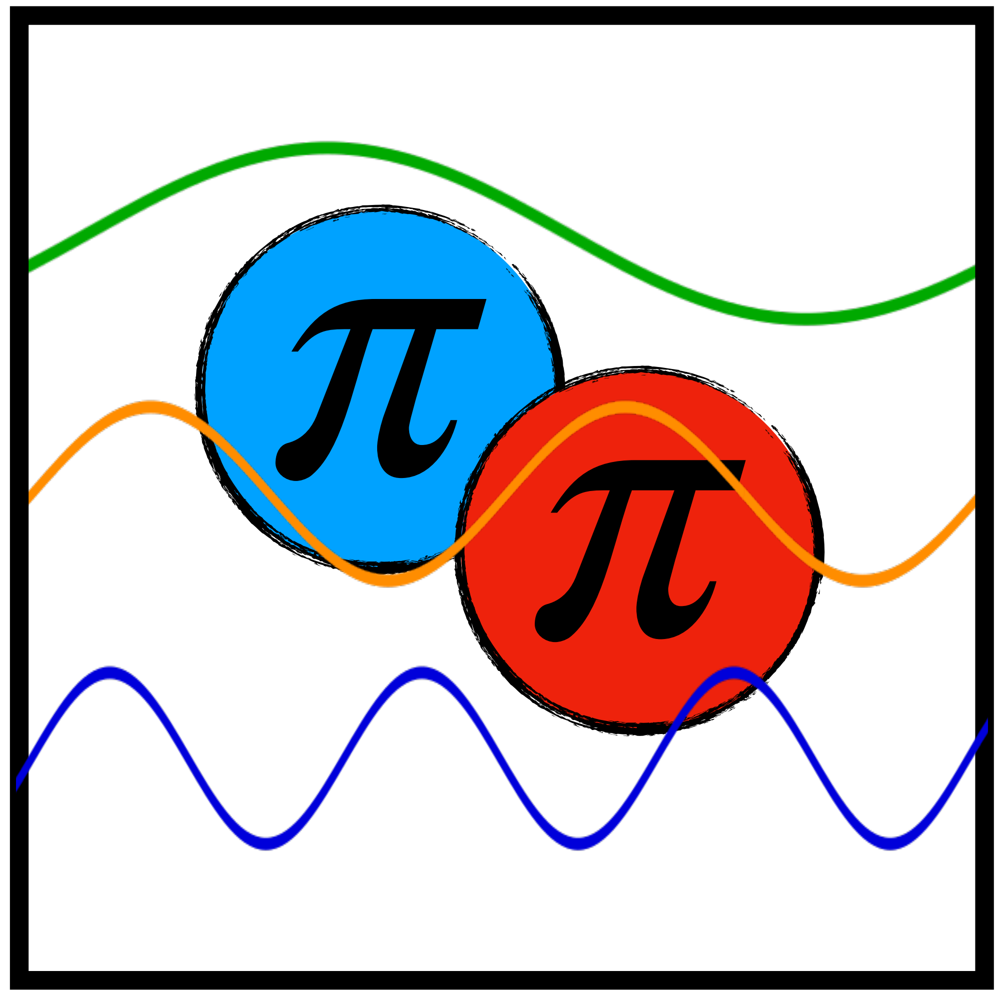

# Towards the QCD S-Matrix

**The S-Matrix contains the physical information of the theory:**

$$S_{ab}(E) \equiv \langle \mathrm{out} | \mathcal{S} | \mathrm{in} \rangle$$

Lattice QCD → QCD S-matrix

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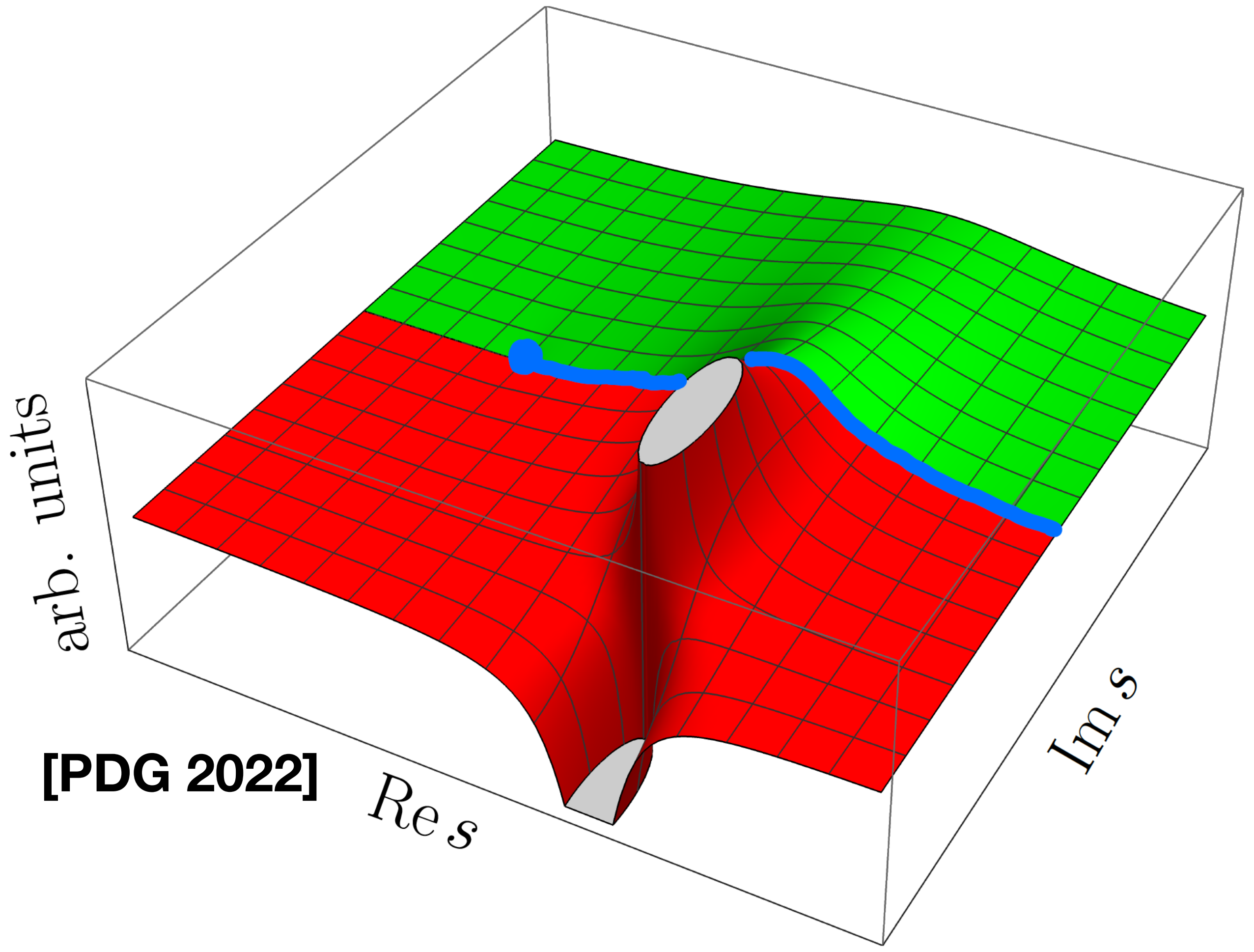
- Resonances as poles in the S-matrix (or scattering amplitude)




[PDG 2022]

$$\sim \frac{g}{E^2 - E_R^2}$$

$E_R = M_R - i\,\Gamma/2$

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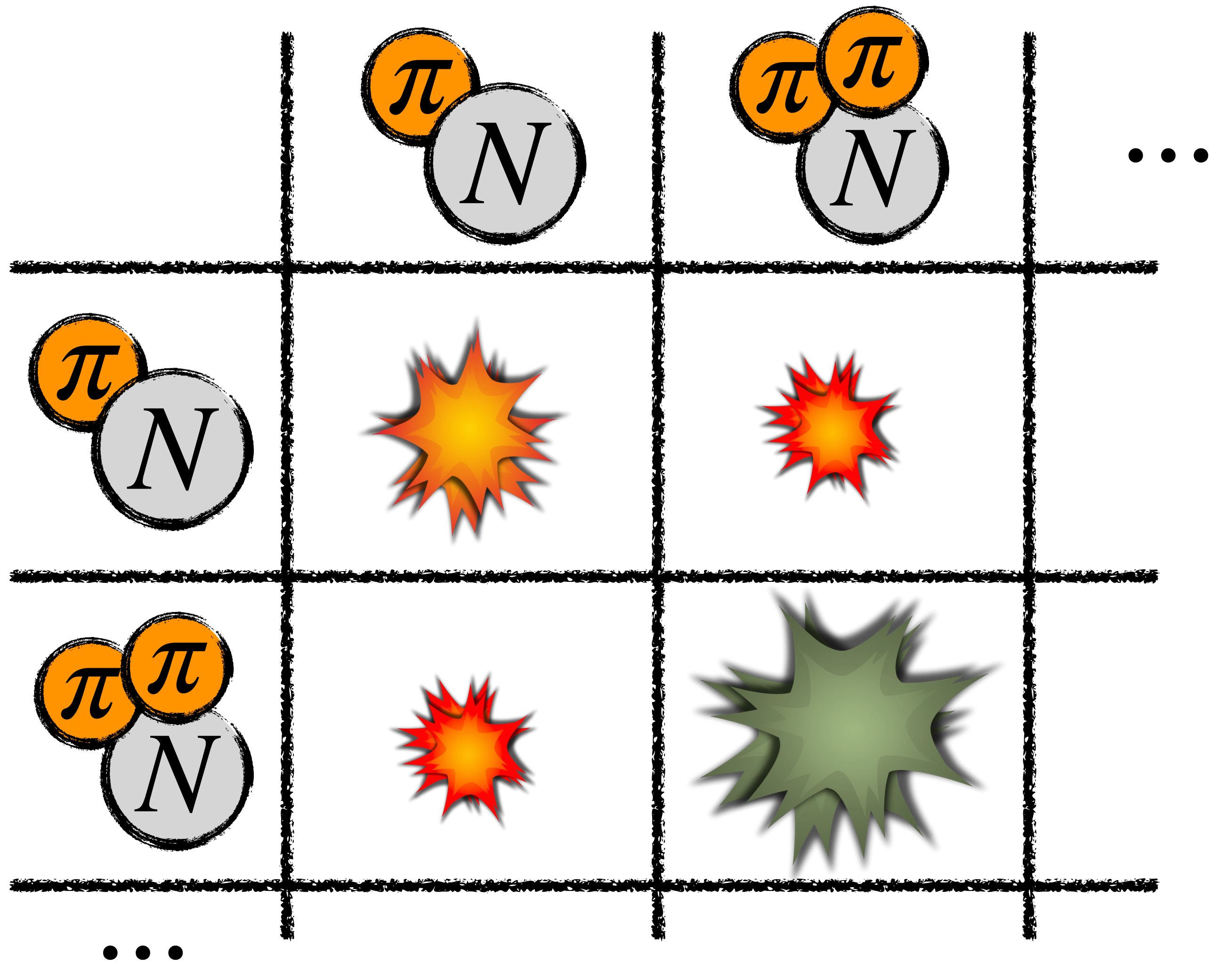


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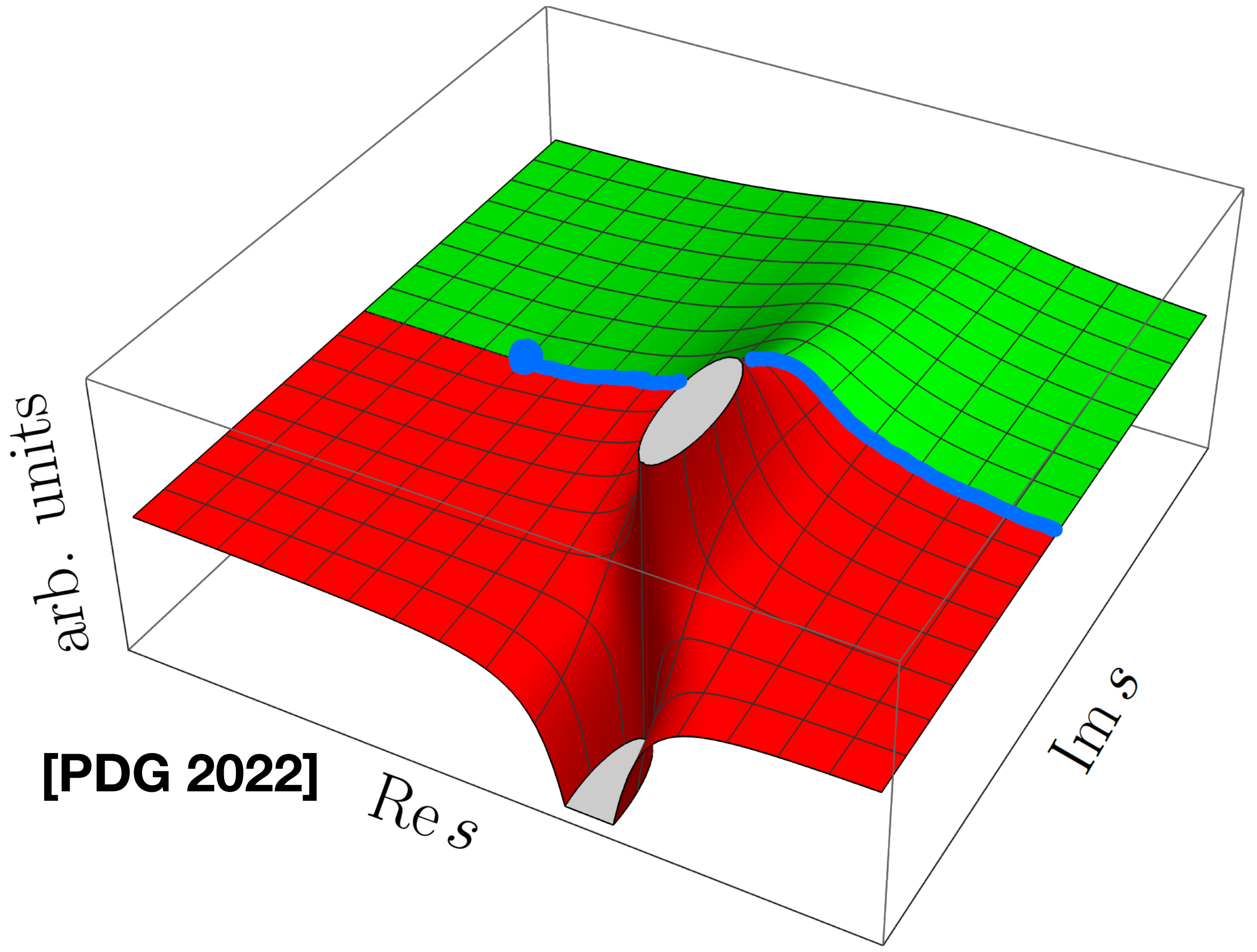


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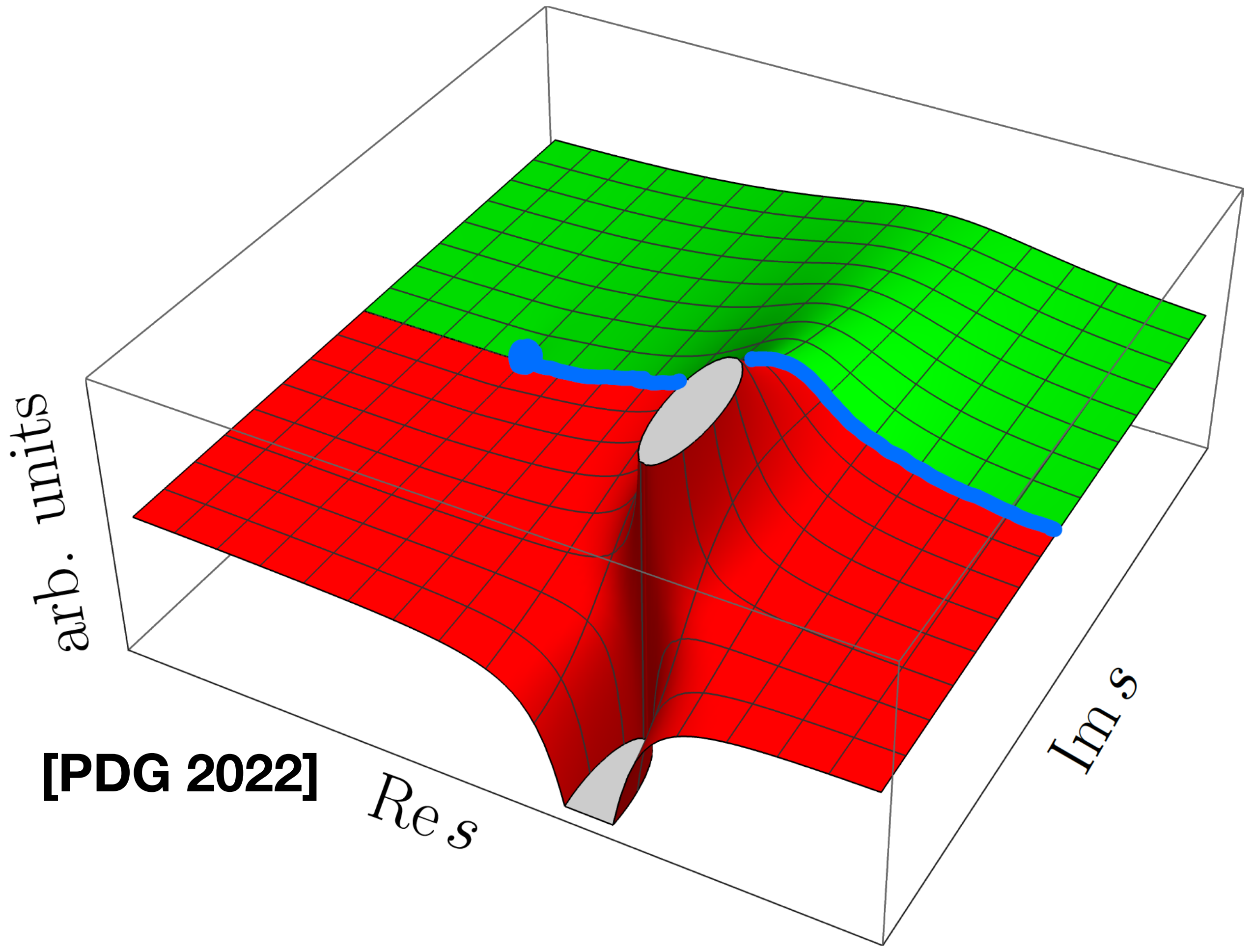


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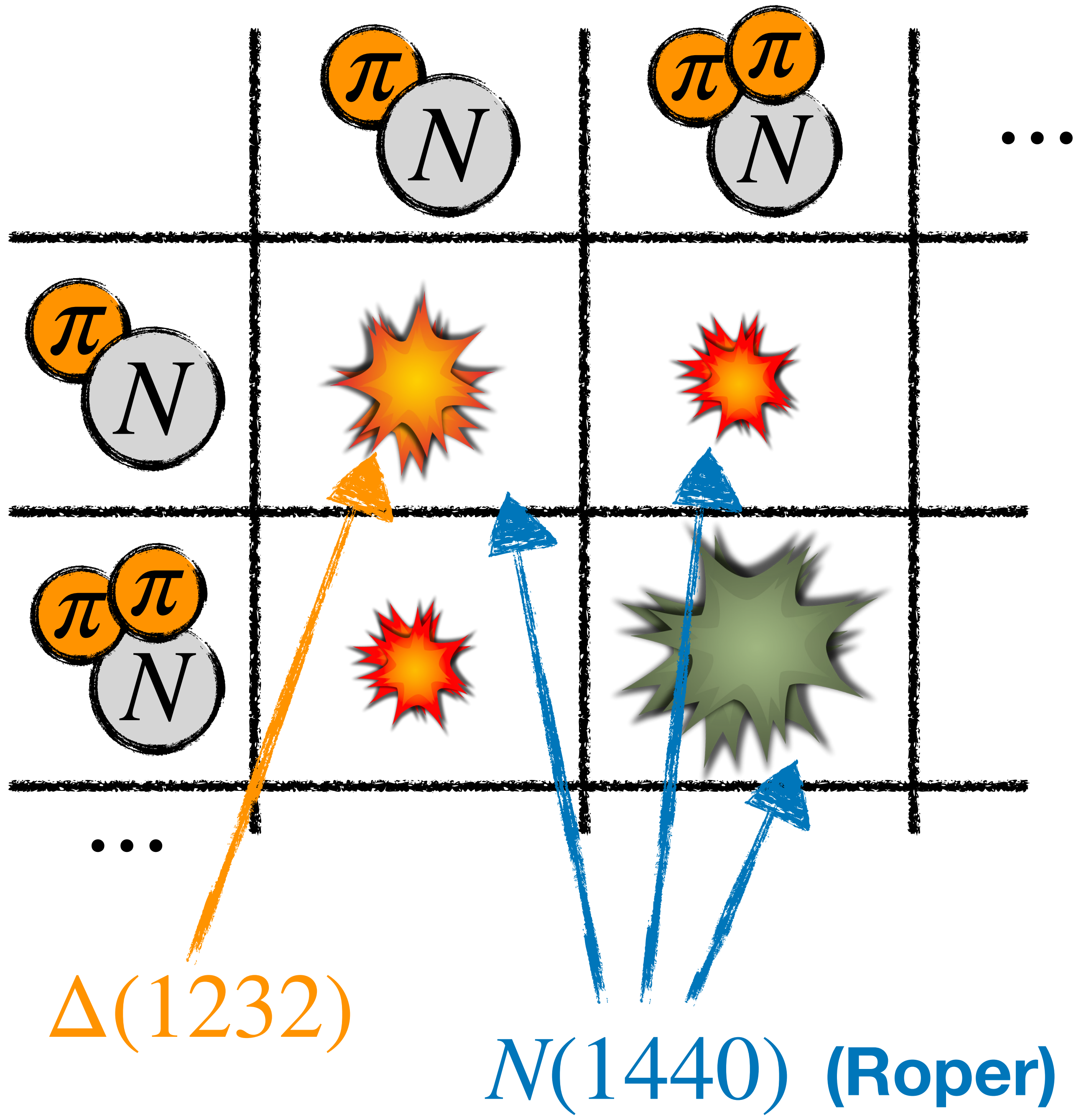

# Outline

1. Scattering amplitudes from lattice QCD
2. Meson-Baryon scattering: $\Delta(1232)$ and $\Lambda(1405)$
3. Three-particle systems

# Scattering amplitudes from lattice QCD

# Energy levels from lattice QCD

- The energy levels of the theory are measured from Euclidean correlation functions

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \sum_n \left|\langle 0|\mathcal{O}(0)|n\rangle\right|^2 e^{-E_n t}$$

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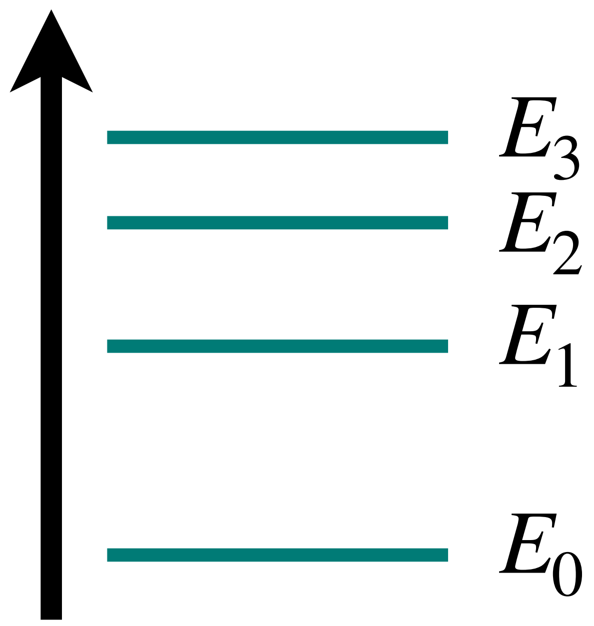
- Multiple operators with the same quantum names to obtain several energy levels

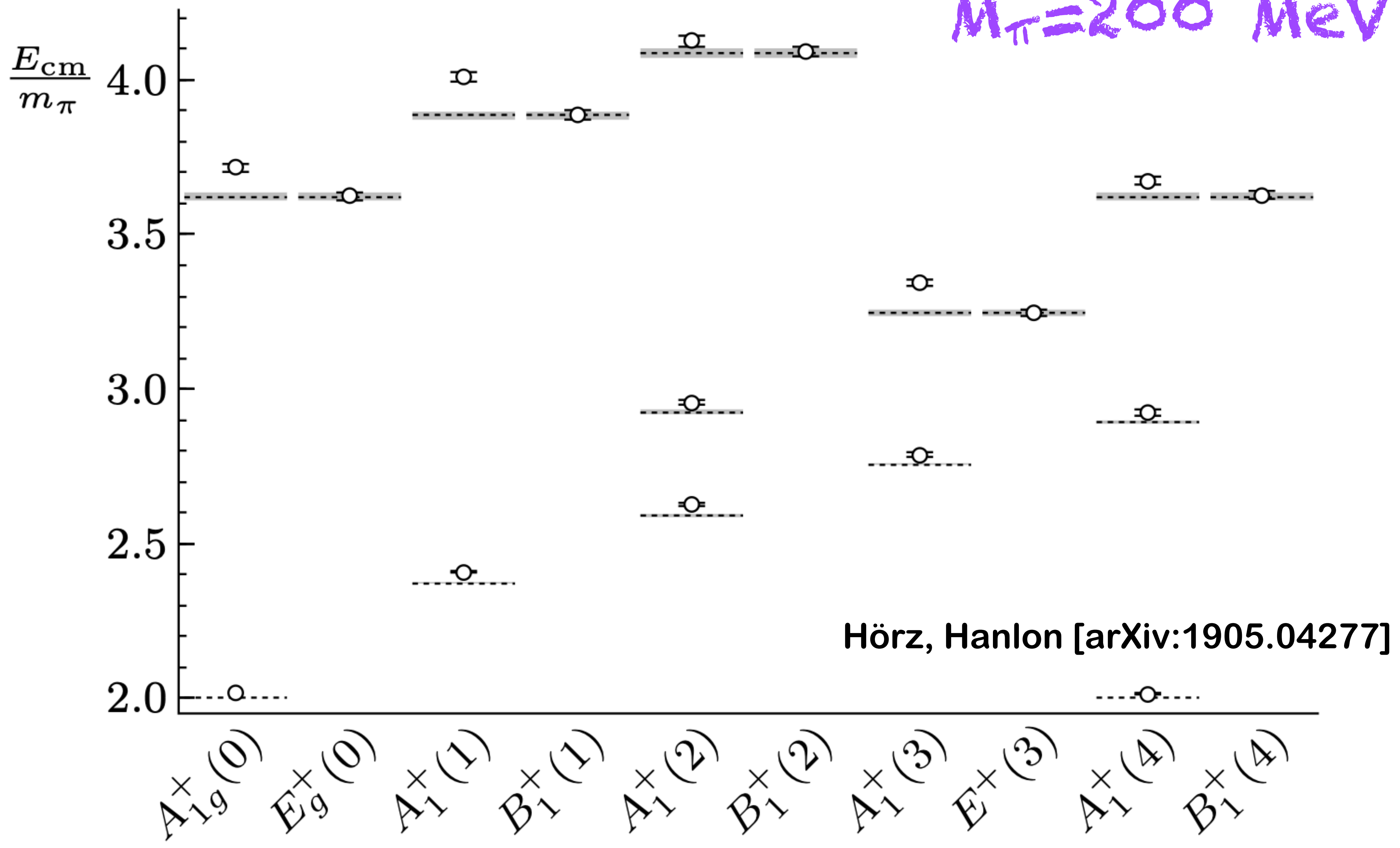
▸ Variational techniques
(Generalized EigenValue Problem, GEVP)

The Spectrum



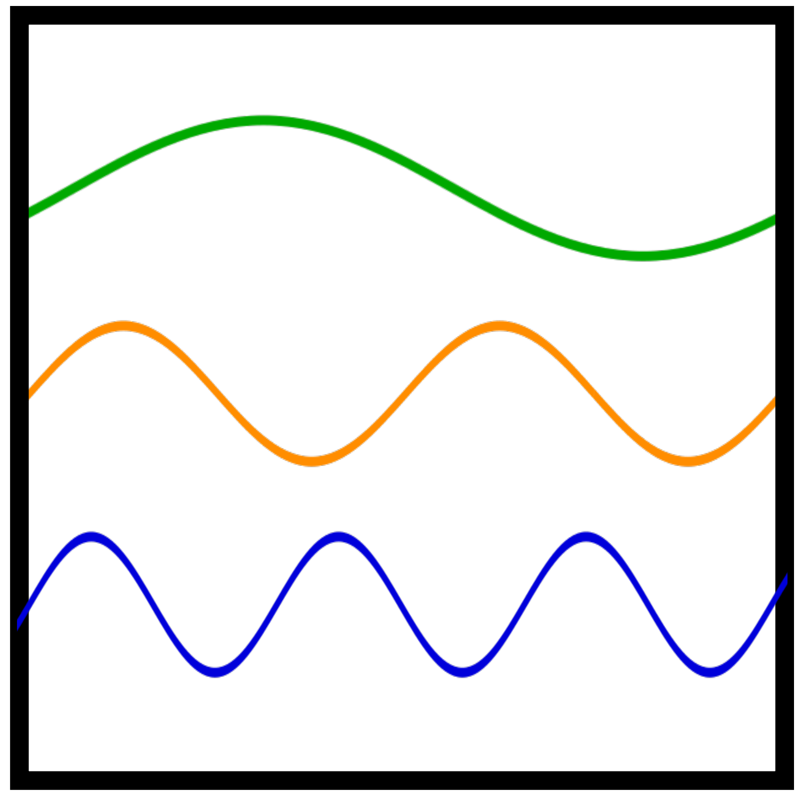

$\mathcal{O} \sim \pi^+\pi^+$

$M_\pi$=200 MeV



# Finite-Volume Spectrum

**Free scalar particles in finite volume with periodic boundaries**

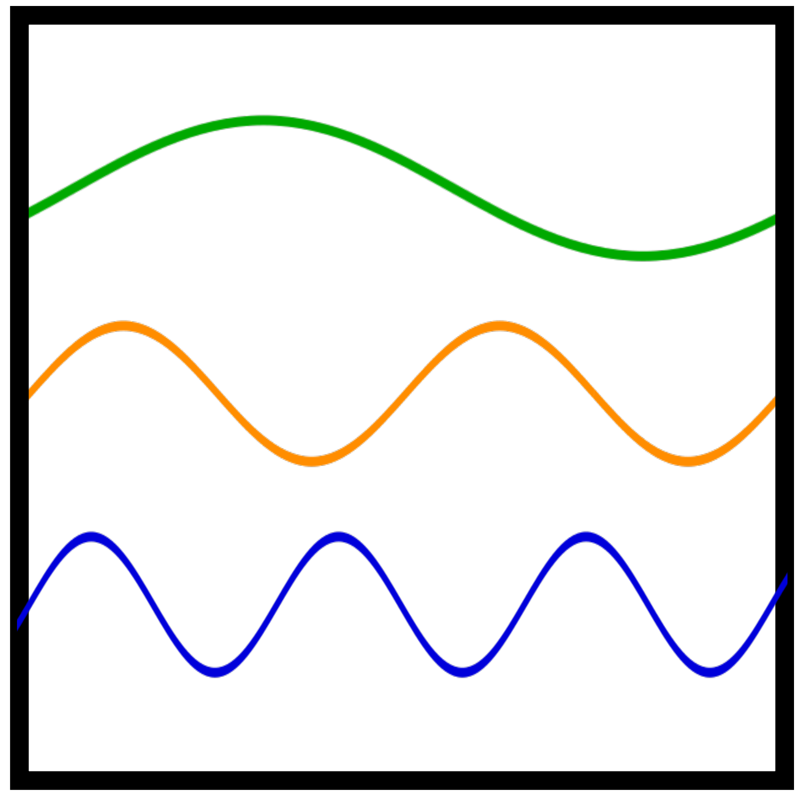


$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

**Two particles:** $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$

# Finite-Volume Spectrum

Free scalar particles in finite volume with periodic boundaries → Interactions change the spectrum: it can be treated as a perturbation



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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

$$\Delta E_2 = \frac{\mathcal{M}_2(E=2m)}{8m^2L^3} + O(L^{-4})$$

[Huang, Yang, 1958]

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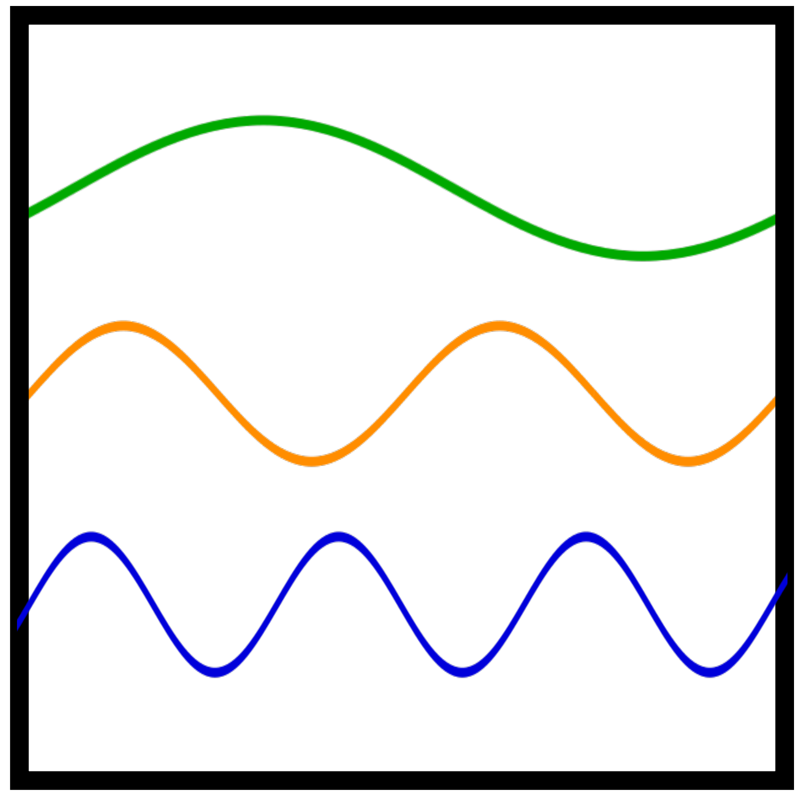

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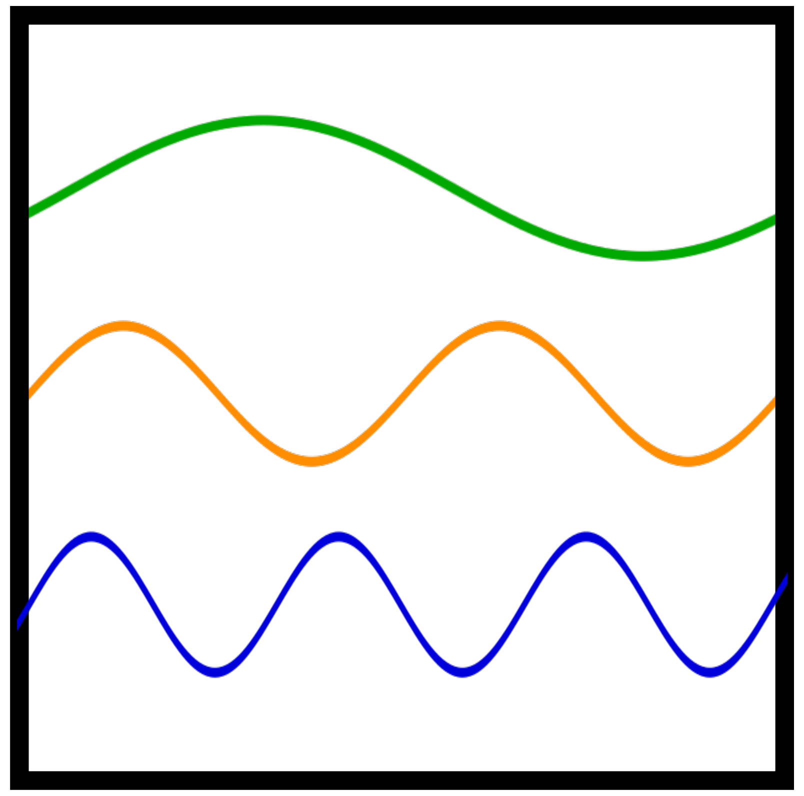

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In general a problem of
Quantum Field Theory
in finite volume

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# Finite-Volume formalism

- Indirect connection between the spectrum and the two-particle scattering amplitude [Lüscher 89']

**Two-particle Quantization Condition**

$$\det_{\ell m}\left[\mathcal{K}_2(E) + F^{-1}(E,\vec{P},L)\right]\Bigg|_{E=E_n} = 0$$

Scattering K-Matrix — Known kinematic function

"QC2"

! Note: only valid for two particles below inelastic thresholds.

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K-matrix parametrized in terms of phase shift

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**Finite-volume information**

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int\frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2-q^2}$$

! Note: only valid for two particles below inelastic thresholds.

# Example I=2 ππ scattering

Two pions in s-wave

$$\mathcal{K}_2^{s-wave}(E_n) = \frac{-1}{F_{00}(E_n, \overrightarrow{P}, L)}$$

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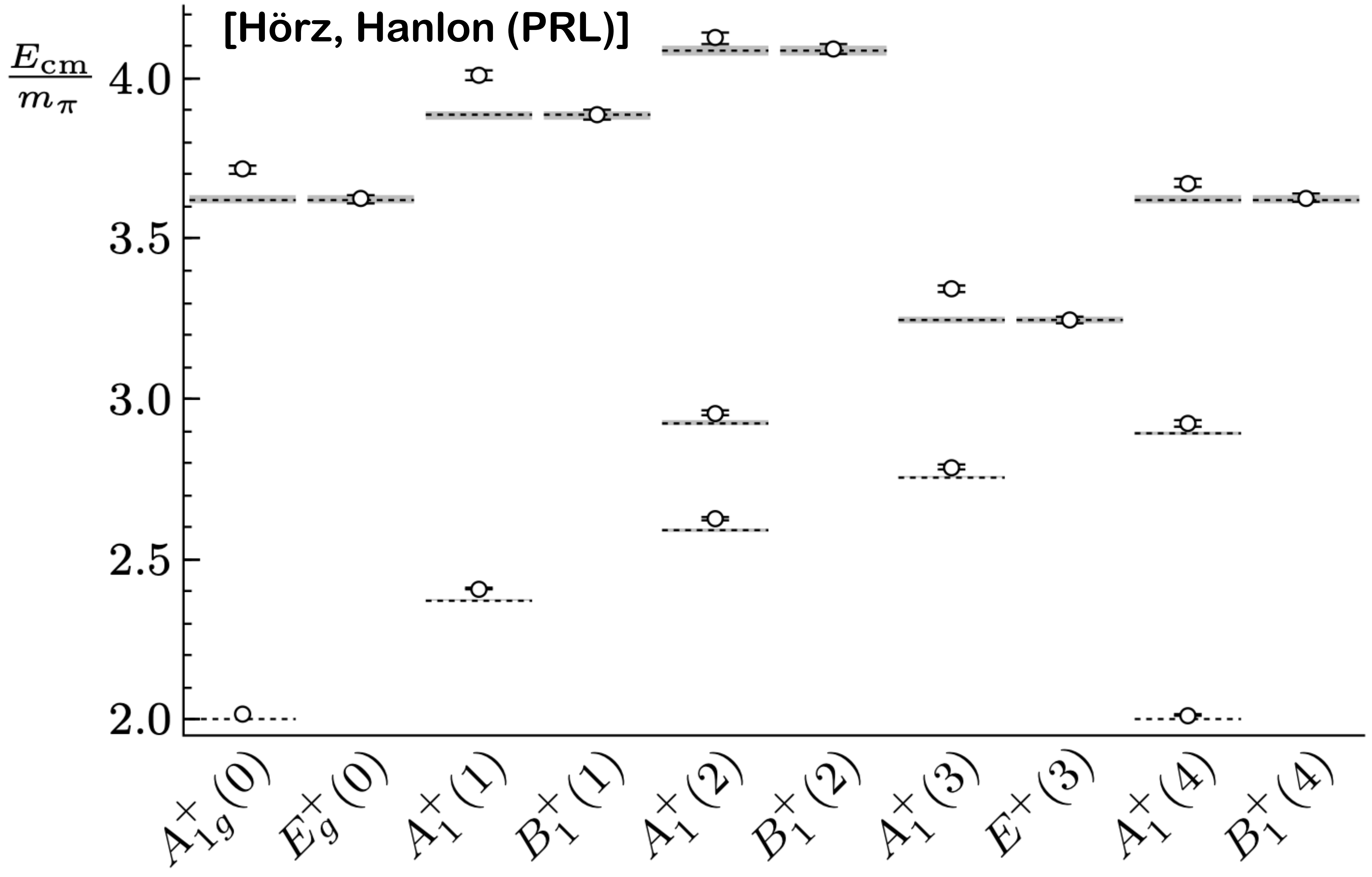


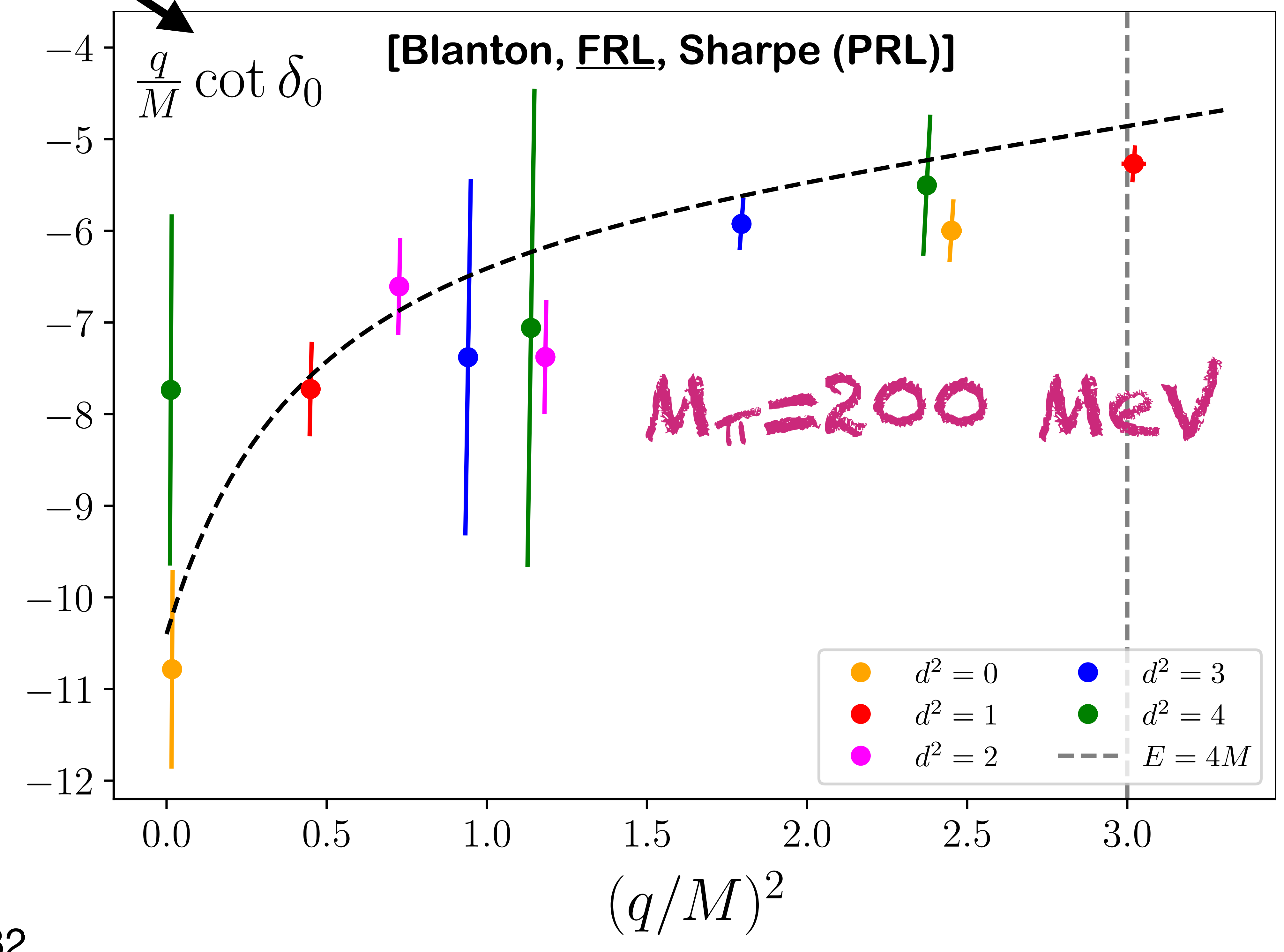




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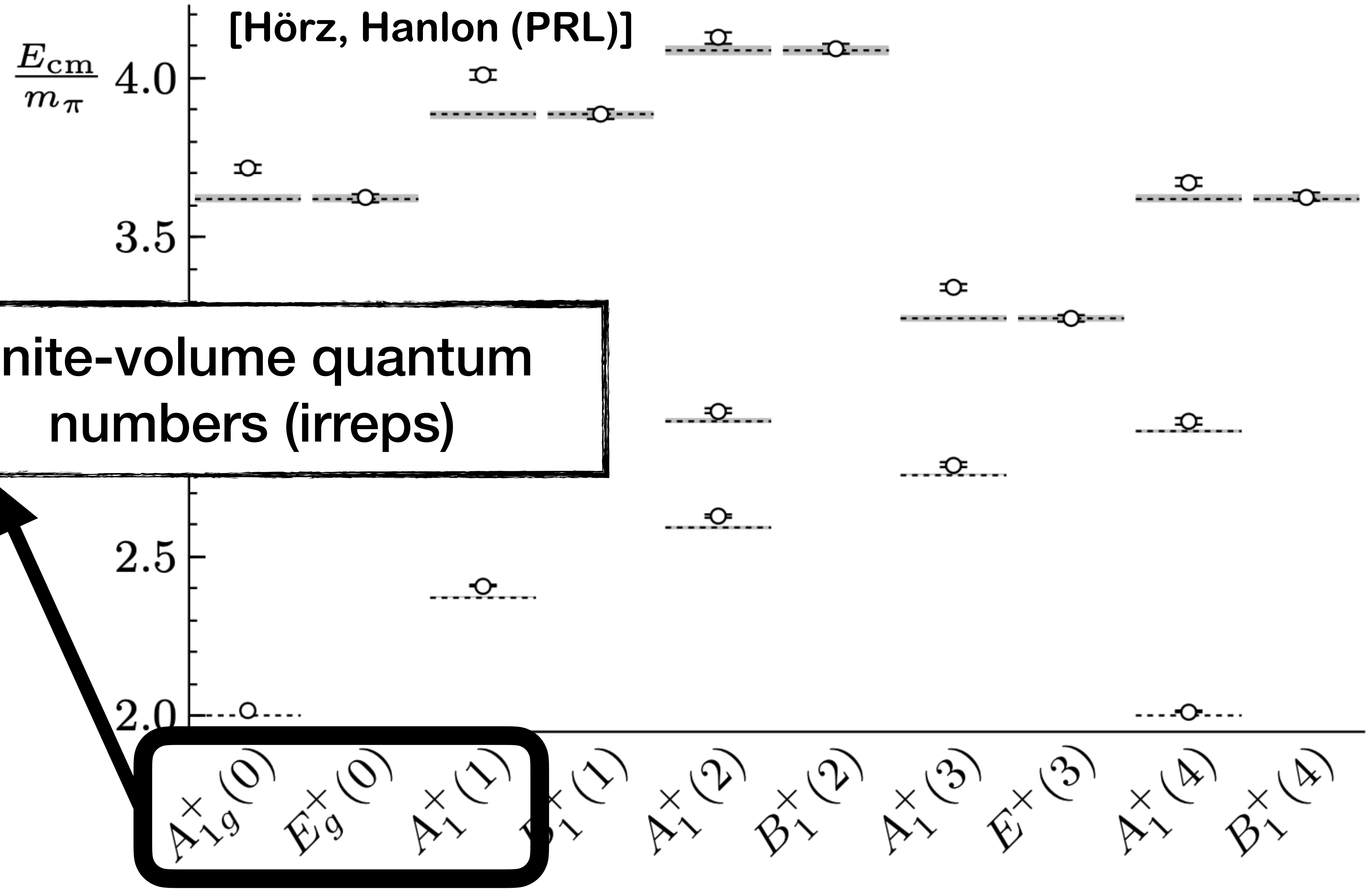
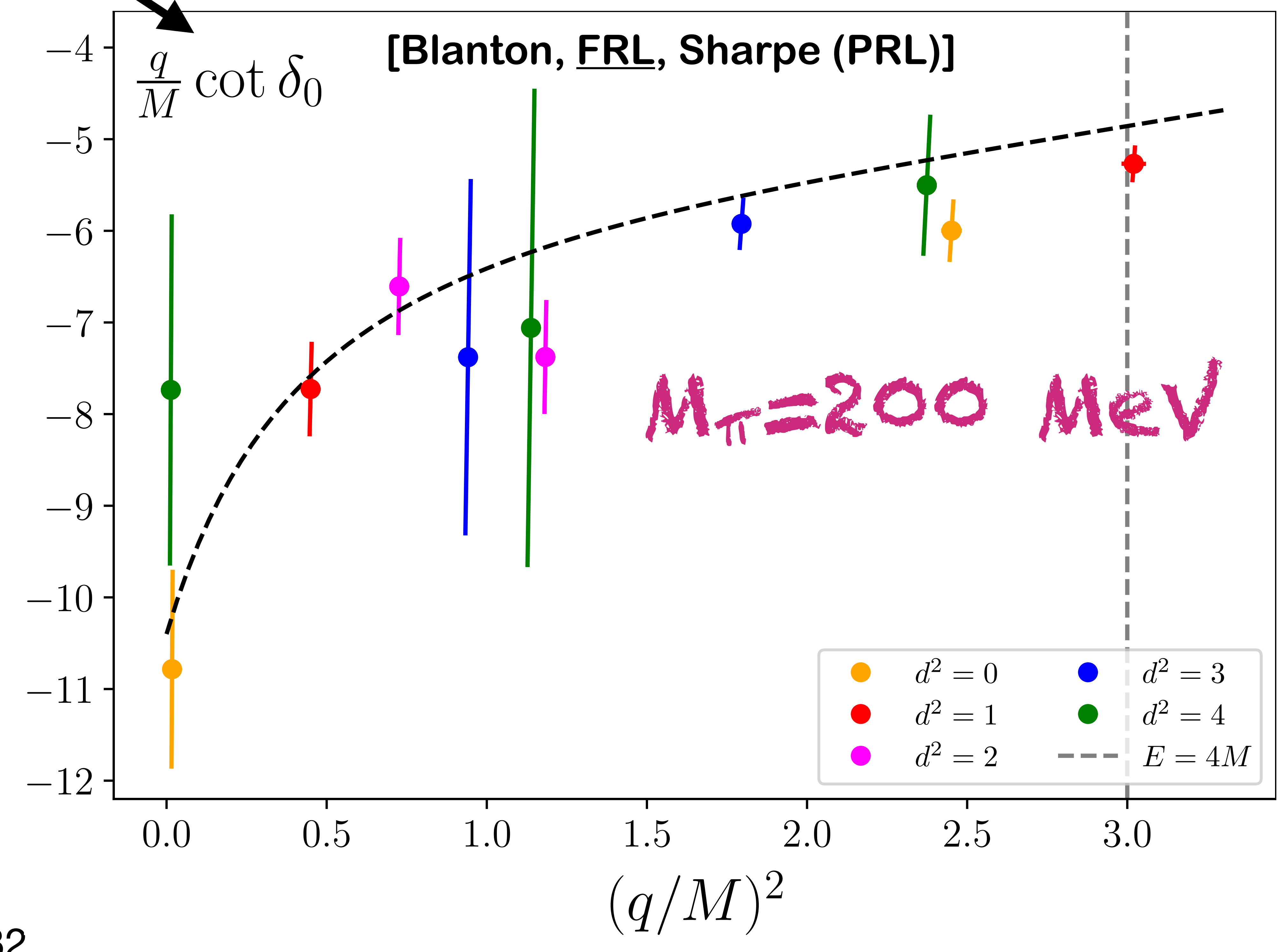



finite-volume quantum numbers (irreps)



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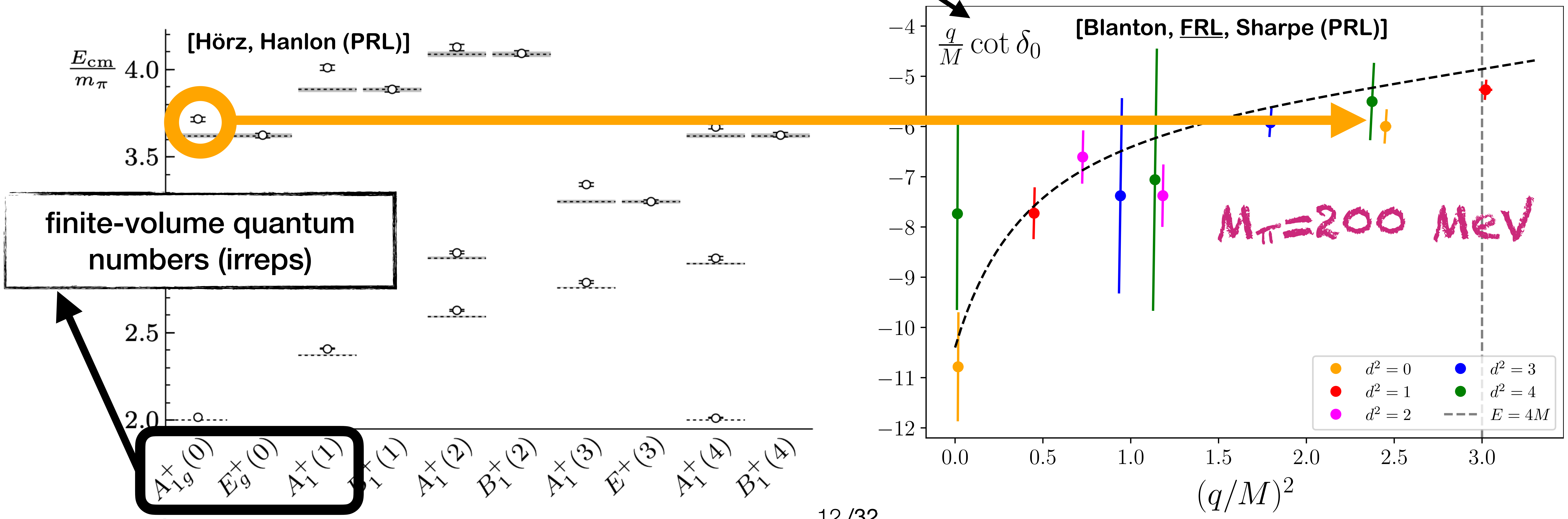

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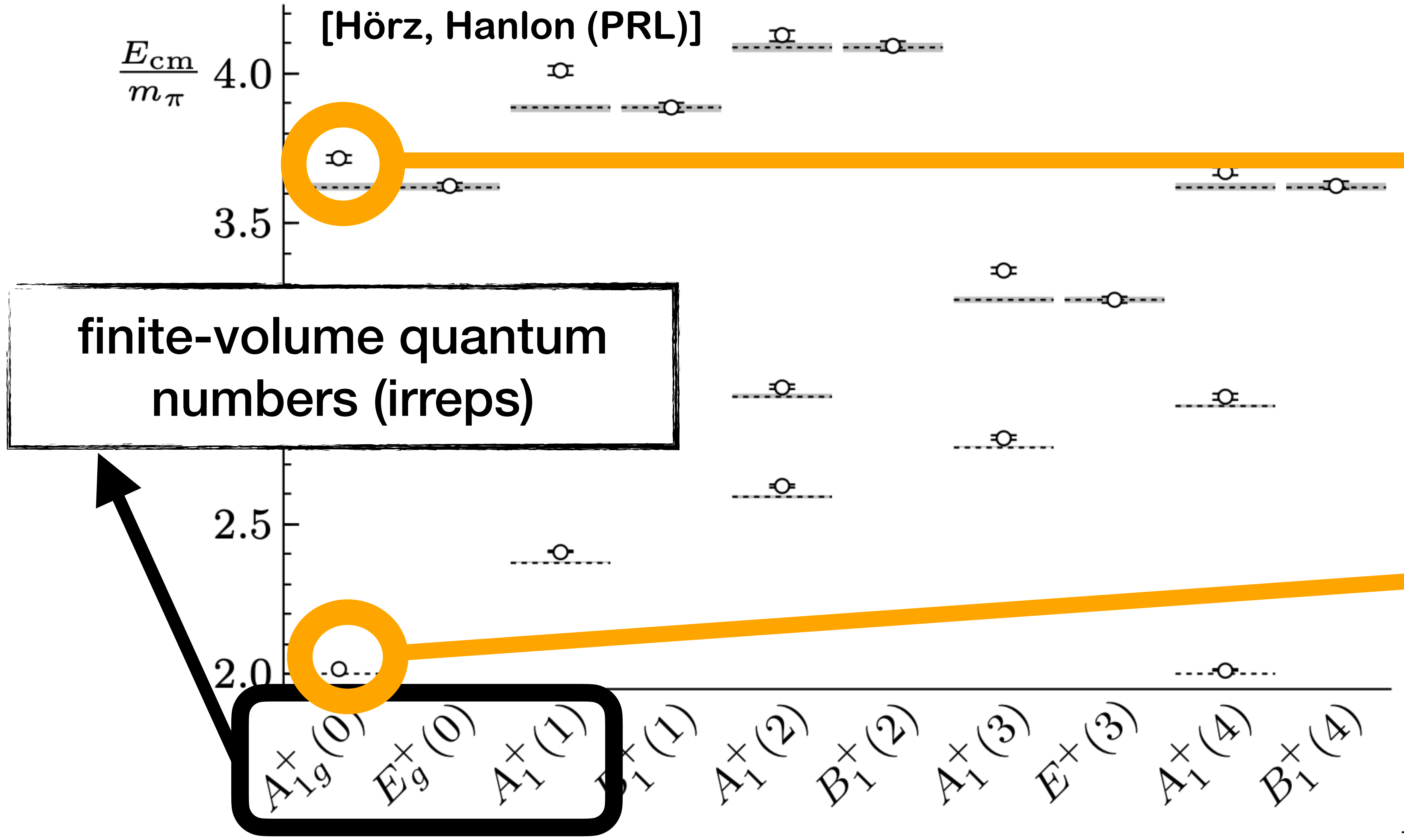


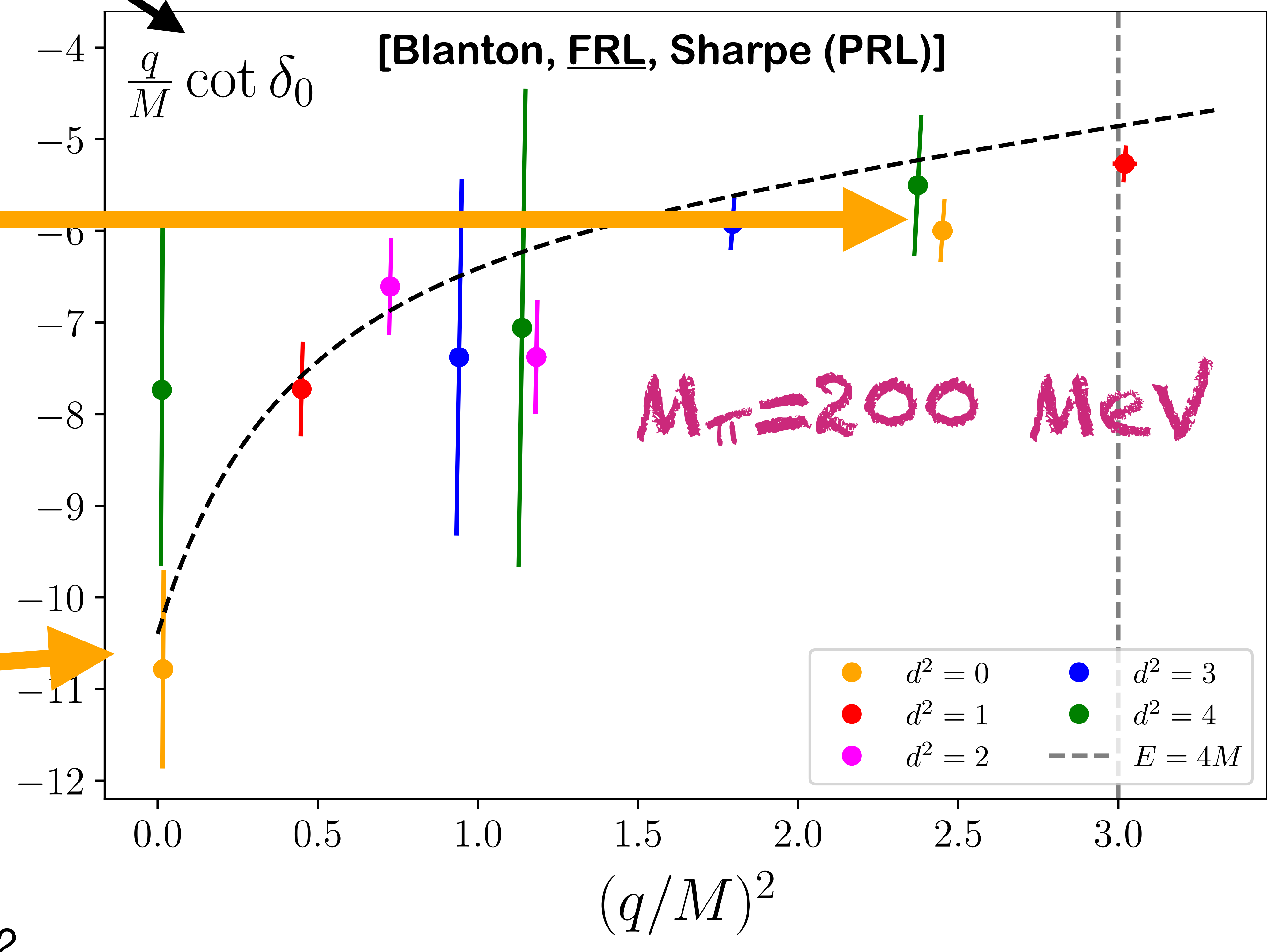

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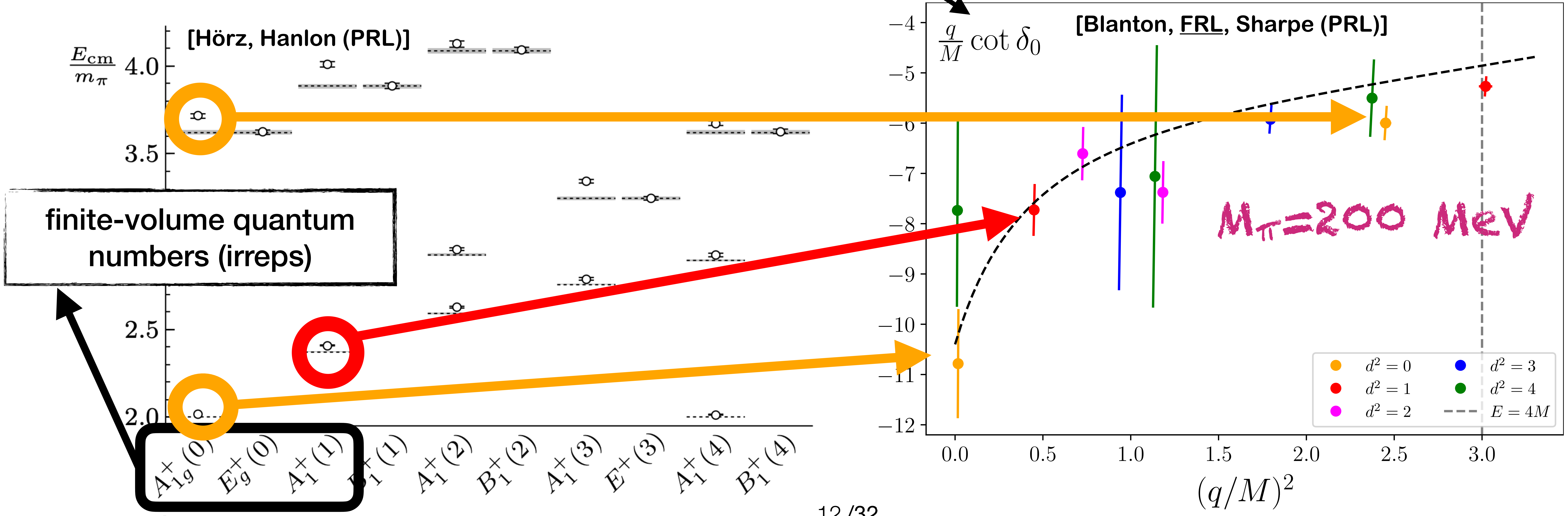
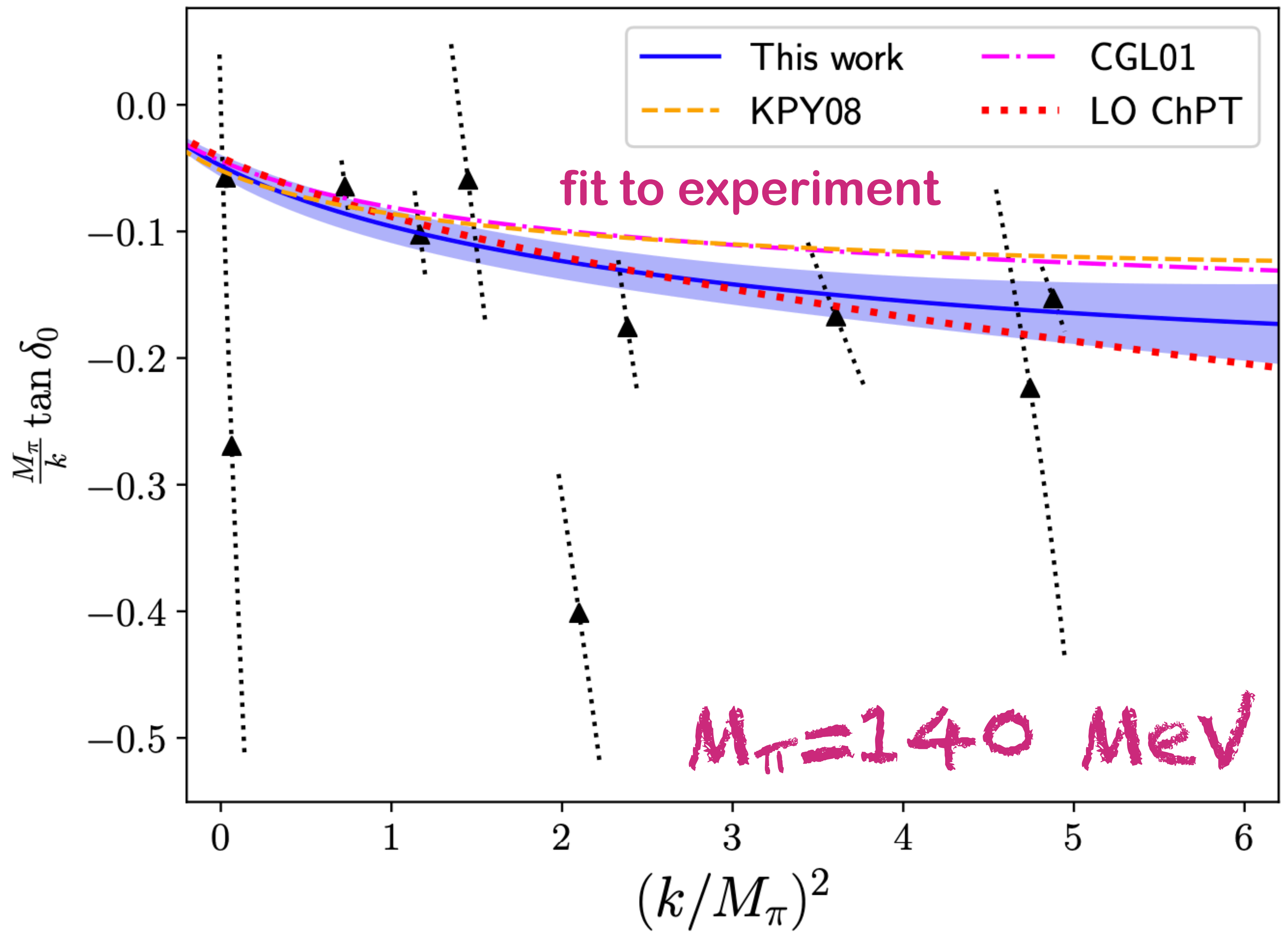


# Towards the physical point

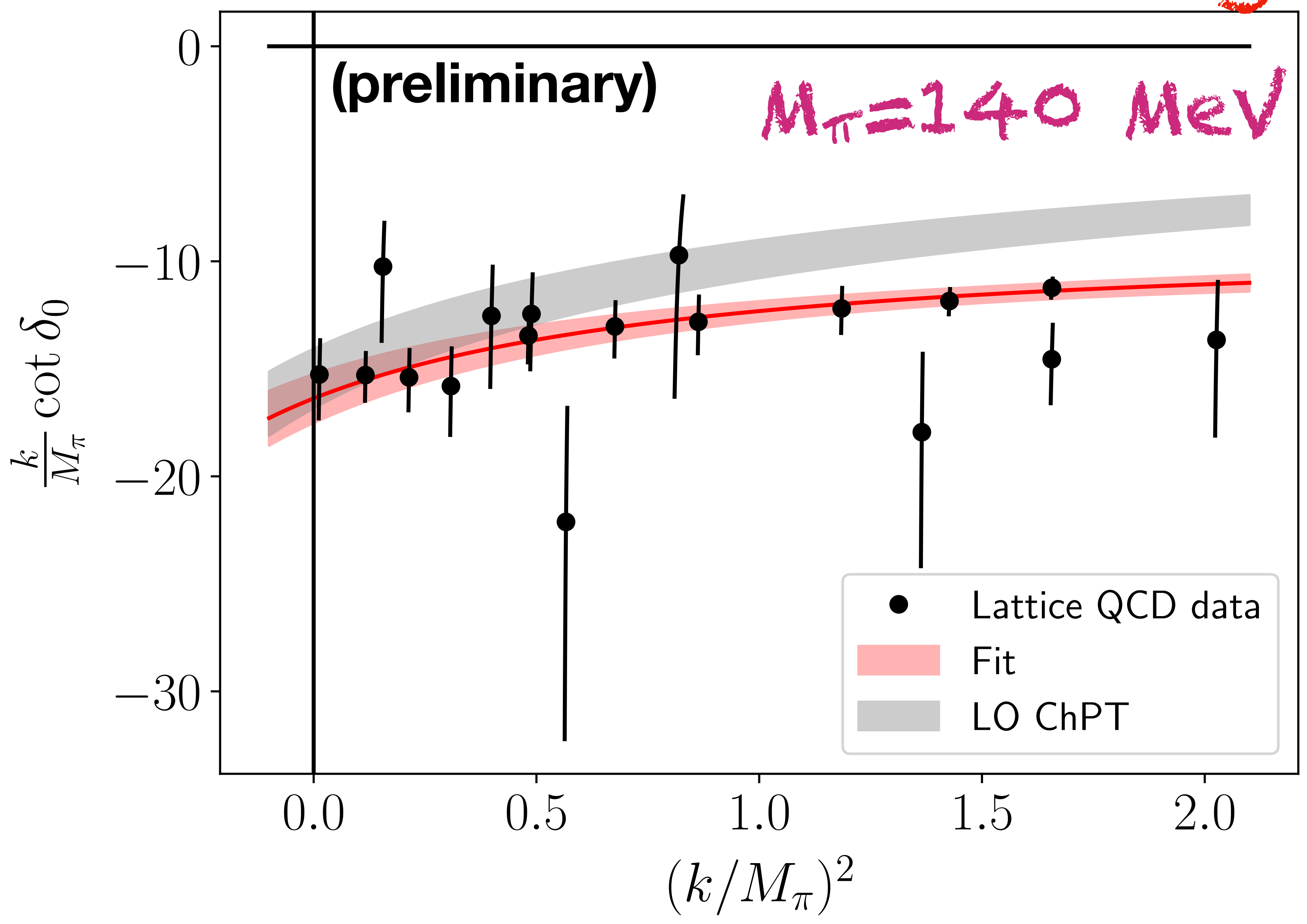
- Some systems already being studied at the physical point!

## I=2 ππ scattering




[Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC), EPJC 2021]

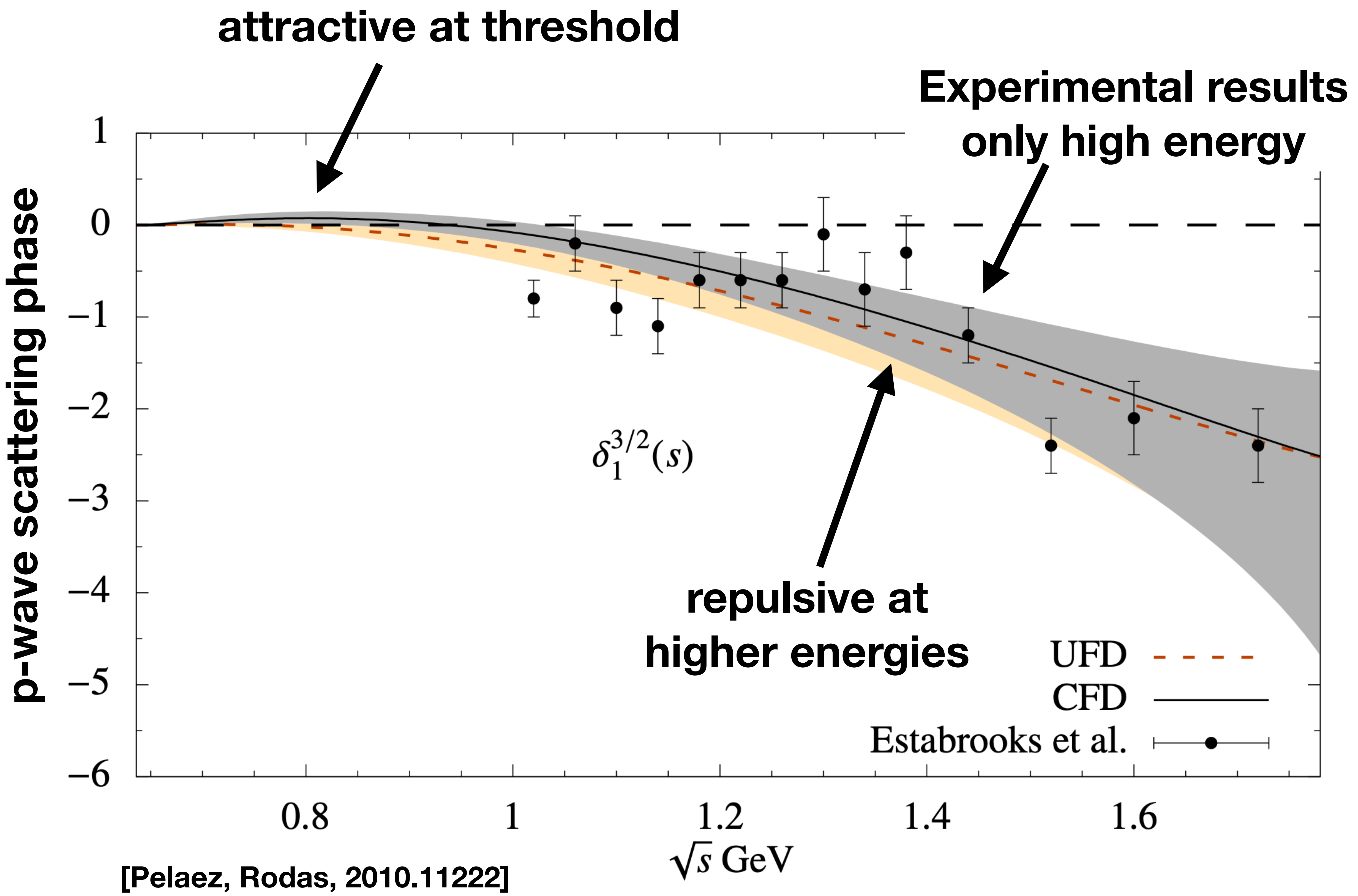
## I=3/2 πK scattering




[Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe (in preparation) ]

# p-wave $\pi^+K^+$ scattering

- Lattice QCD results are complementary to experiment!




[Pelaez, Rodas, 2010.11222]

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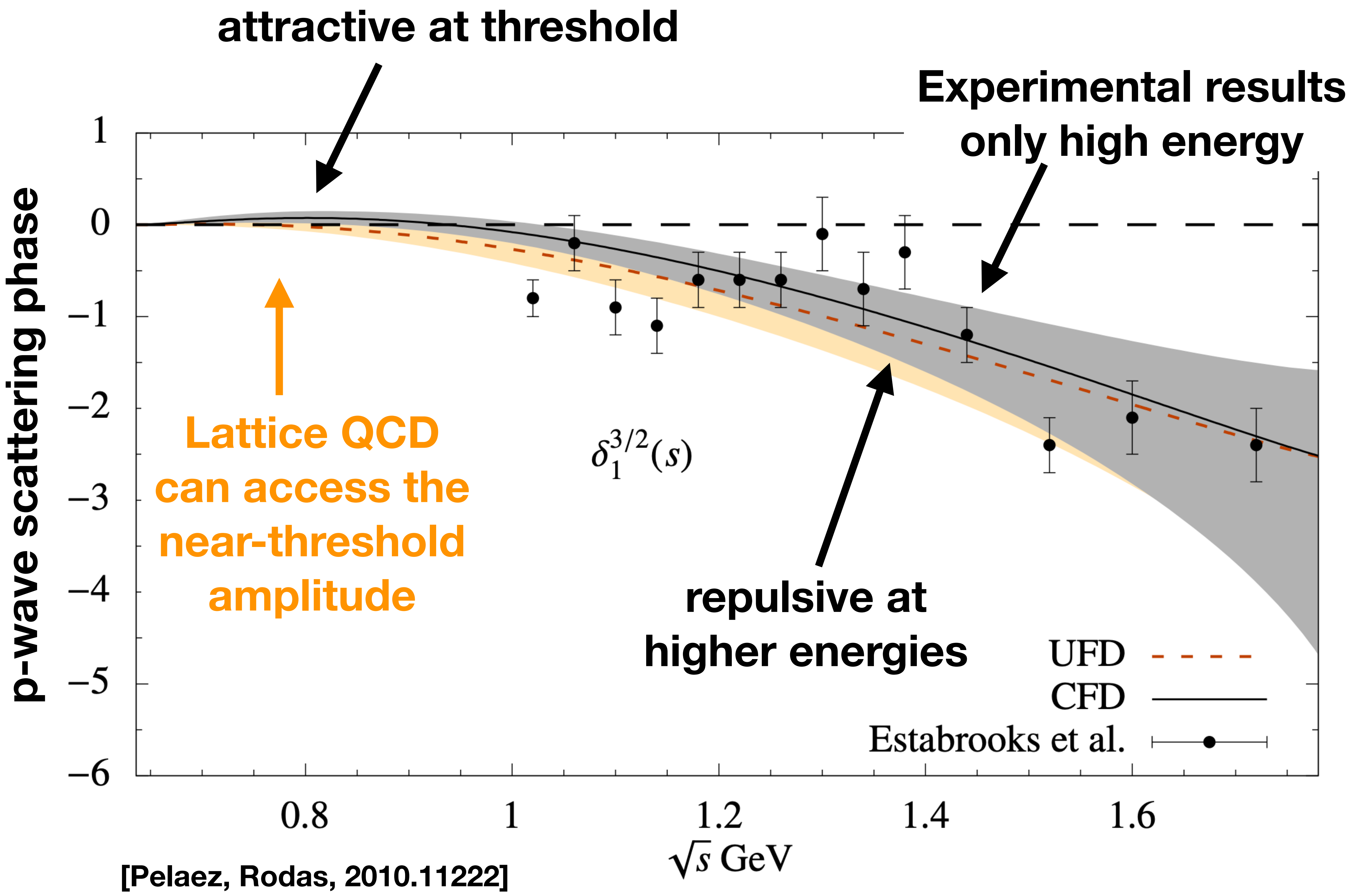
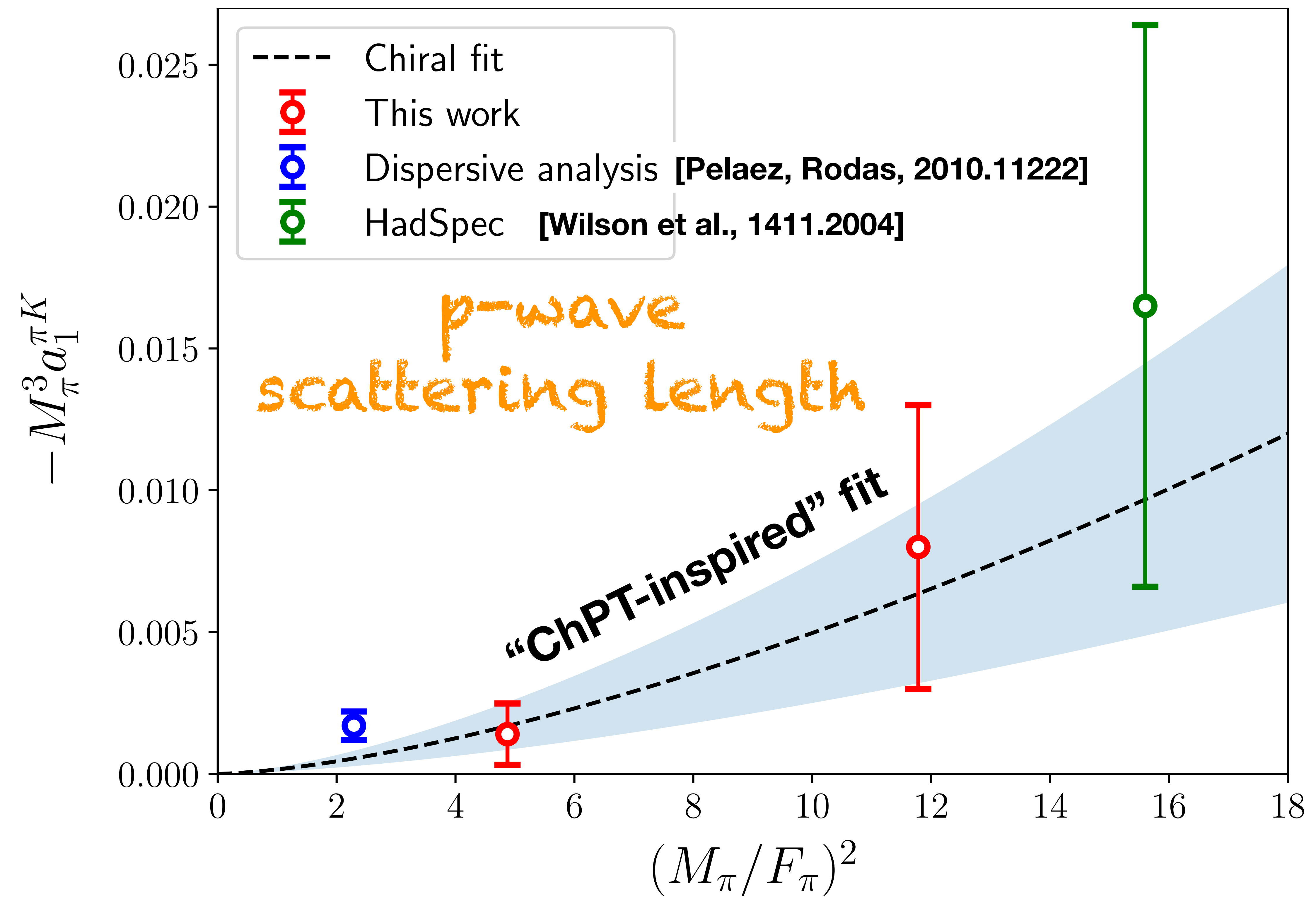



[Pelaez, Rodas, 2010.11222]

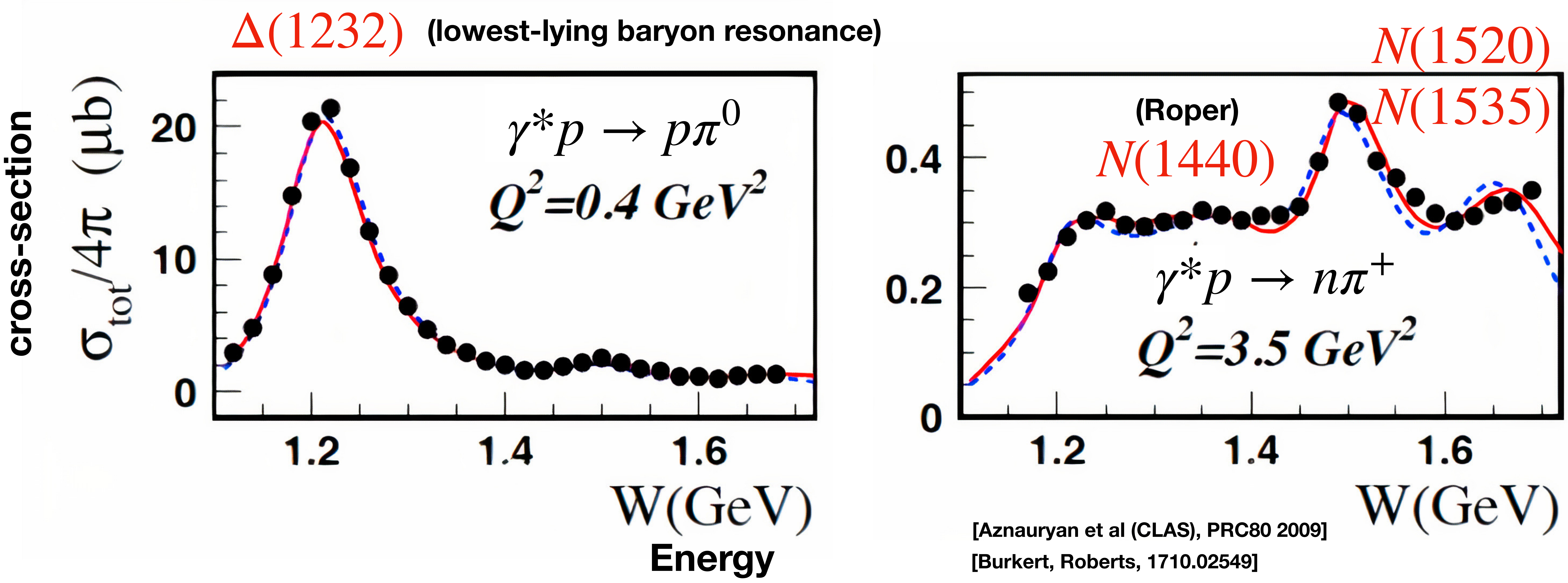
[Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2023]



# The $\pi N$ scattering amplitude

# πN scattering

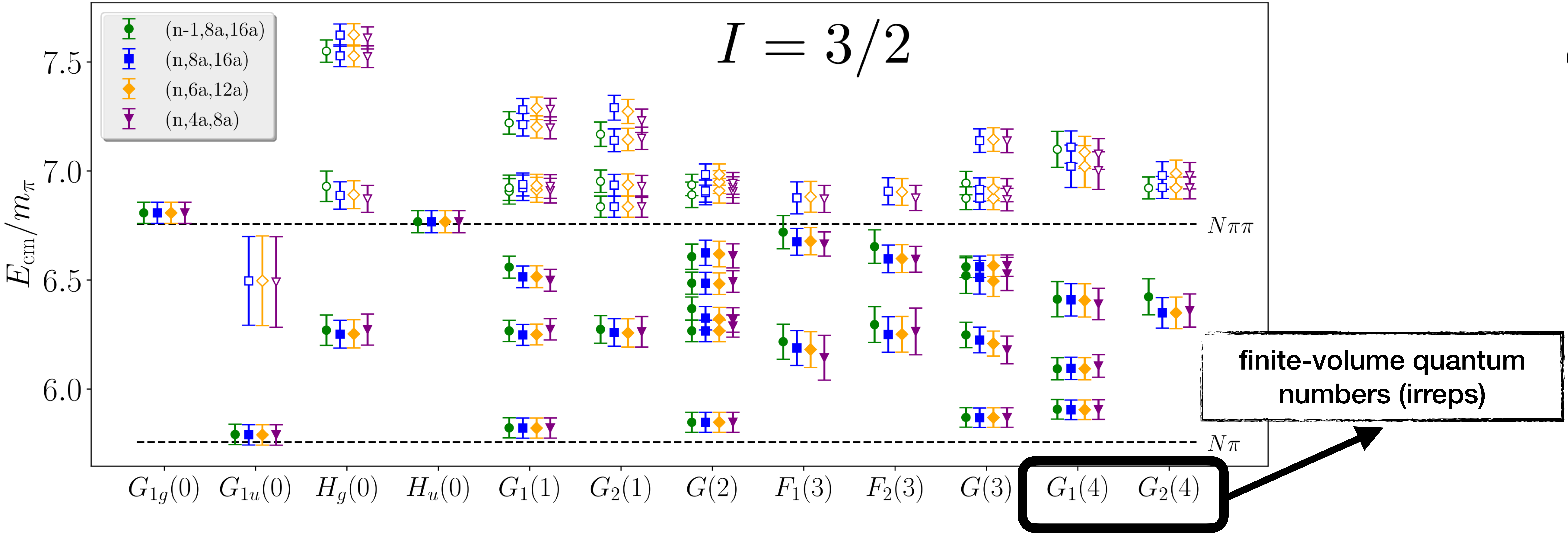
- Pion-nucleon scattering is an important process in QCD




[Aznauryan et al (CLAS), PRC80 2009]
[Burkert, Roberts, 1710.02549]

# Lattice QCD πN spectrum

- Key ingredient: reliable variational extractions of the lattice QCD energy levels: GEVP + stability






Fernando Romero-López, MIT

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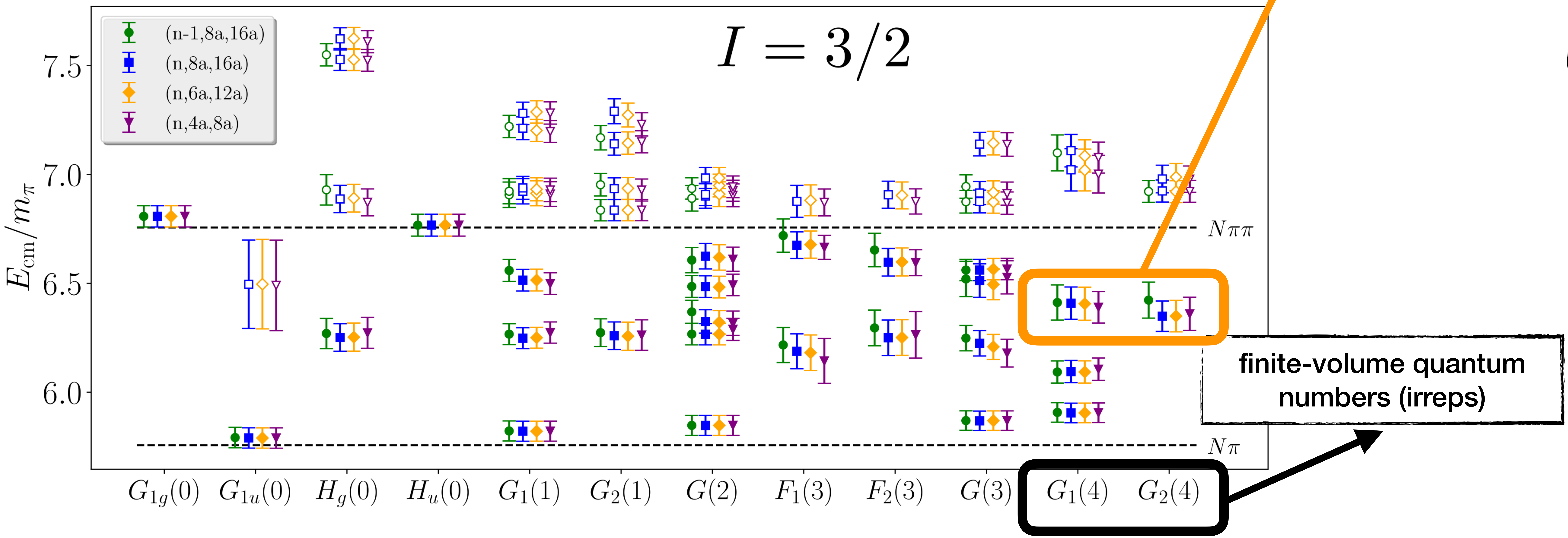




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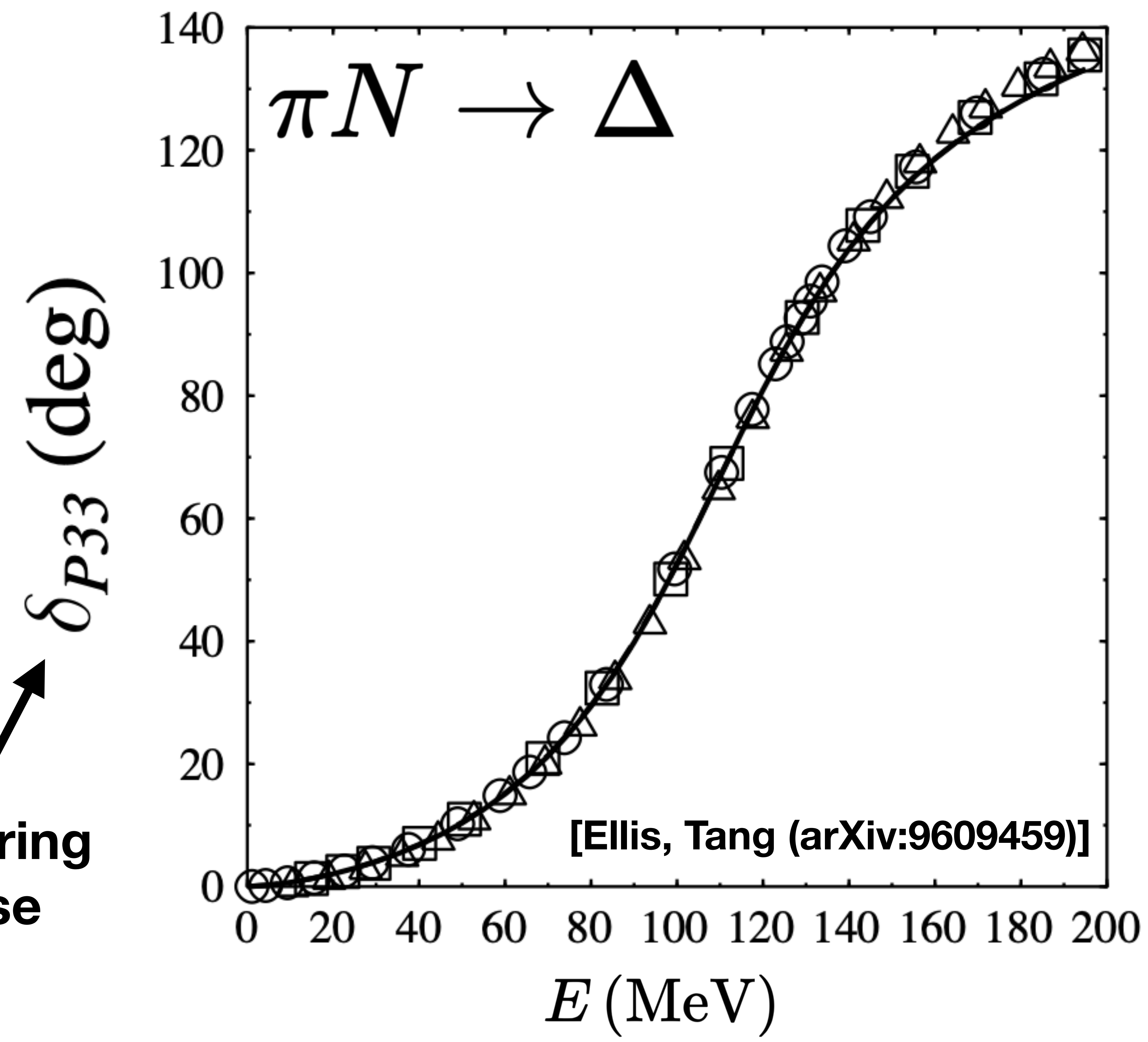
# The Δ(1232) resonance

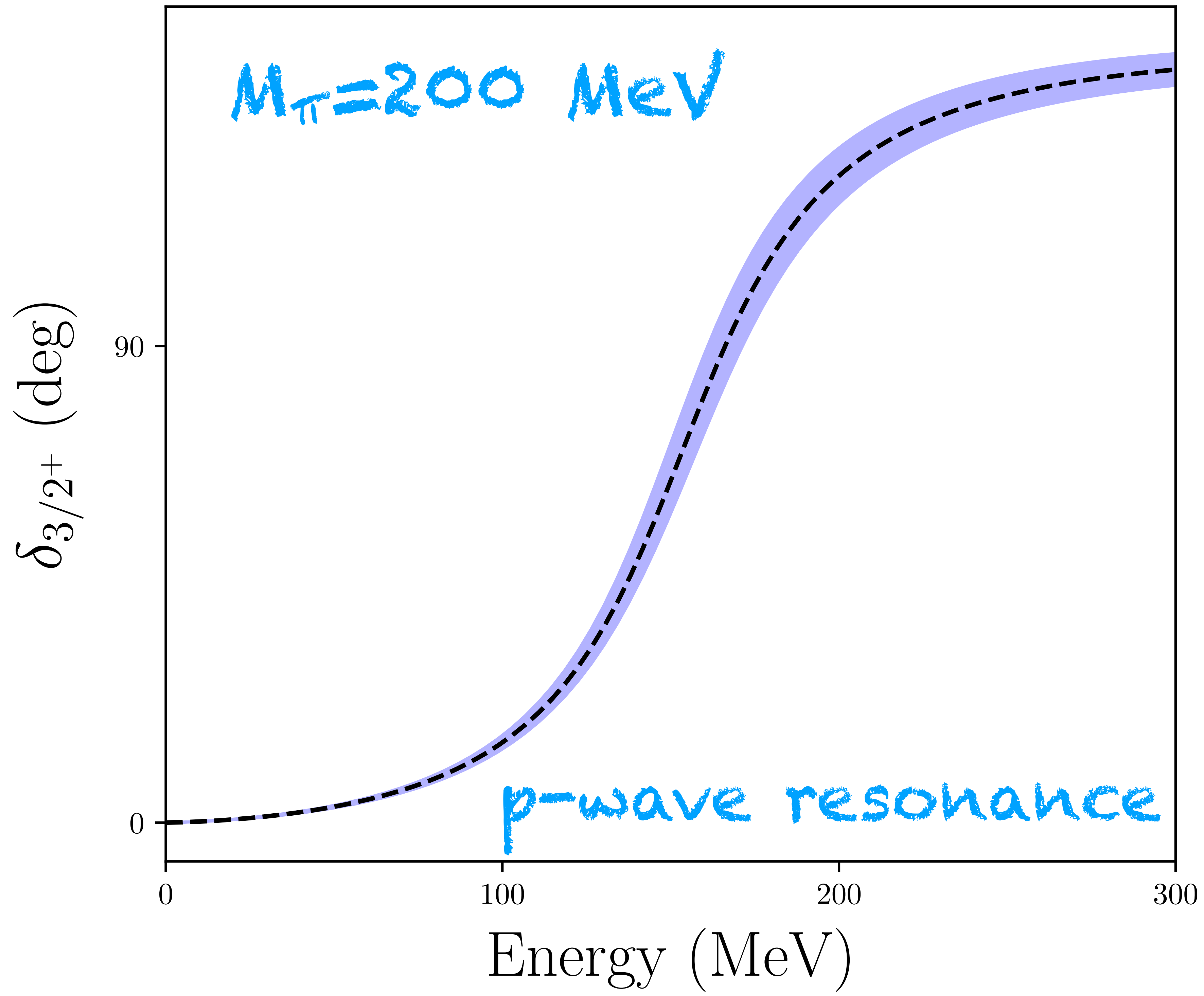
## Experiment




scattering phase

## Lattice QCD

[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867]

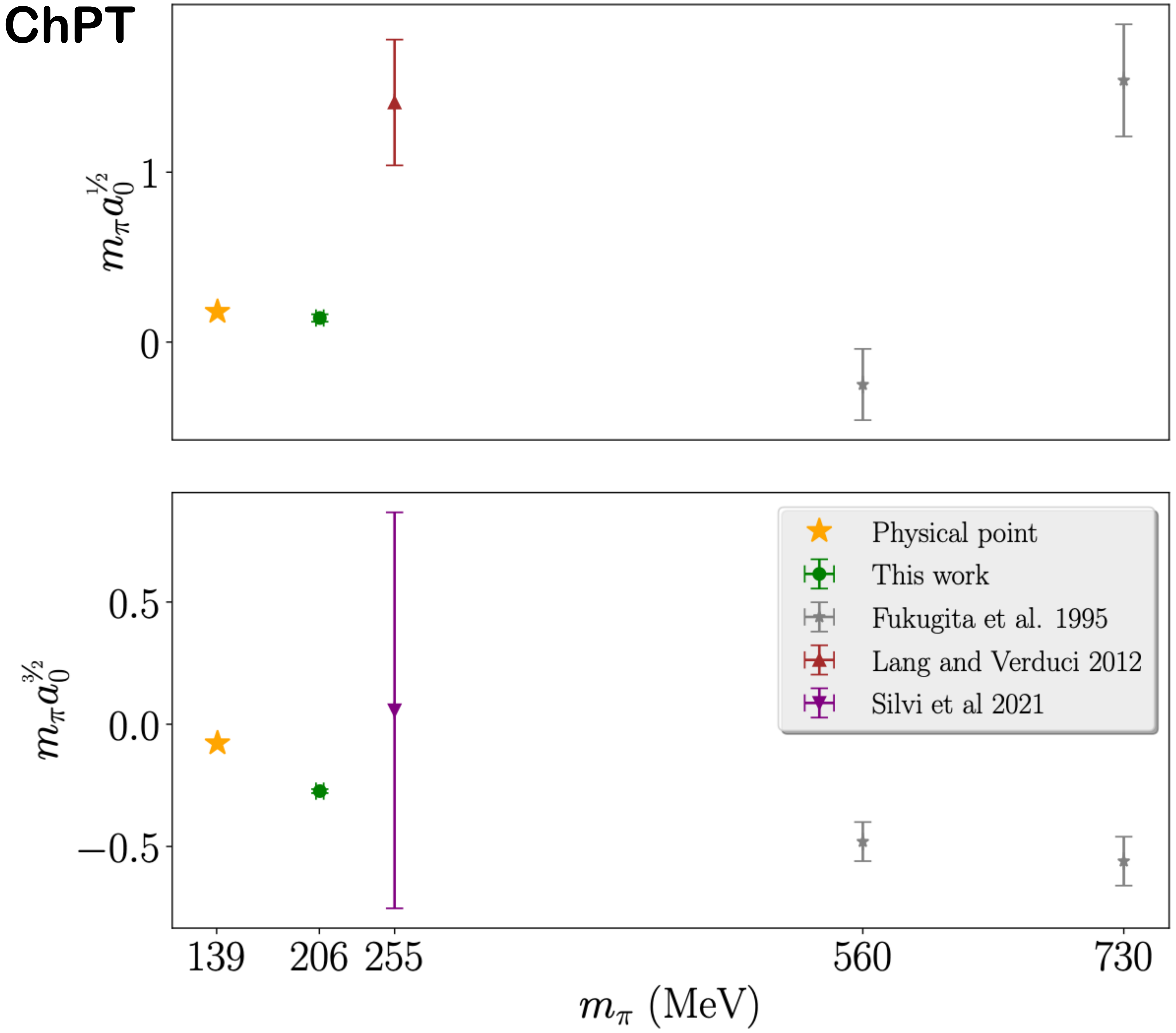


# The πN scattering lengths

[Bulava, Hanlon, Hörz, Morningstar, Nicholson, FRL, Skinner, Vranas, Walker-Loud, 2208.03867]

- Our results can be used to test the convergence of baryon ChPT

| | $m_\pi$ (MeV) | $m_\pi a_0^{1/2}$ | $m_\pi a_0^{3/2}$ |
|---|---|---|---|
| This work | 200 | 0.142(22) | −0.2735(81) |
| LO $\chi$PT | 200 | 0.321(04)(57) | −0.161(02)(28) |



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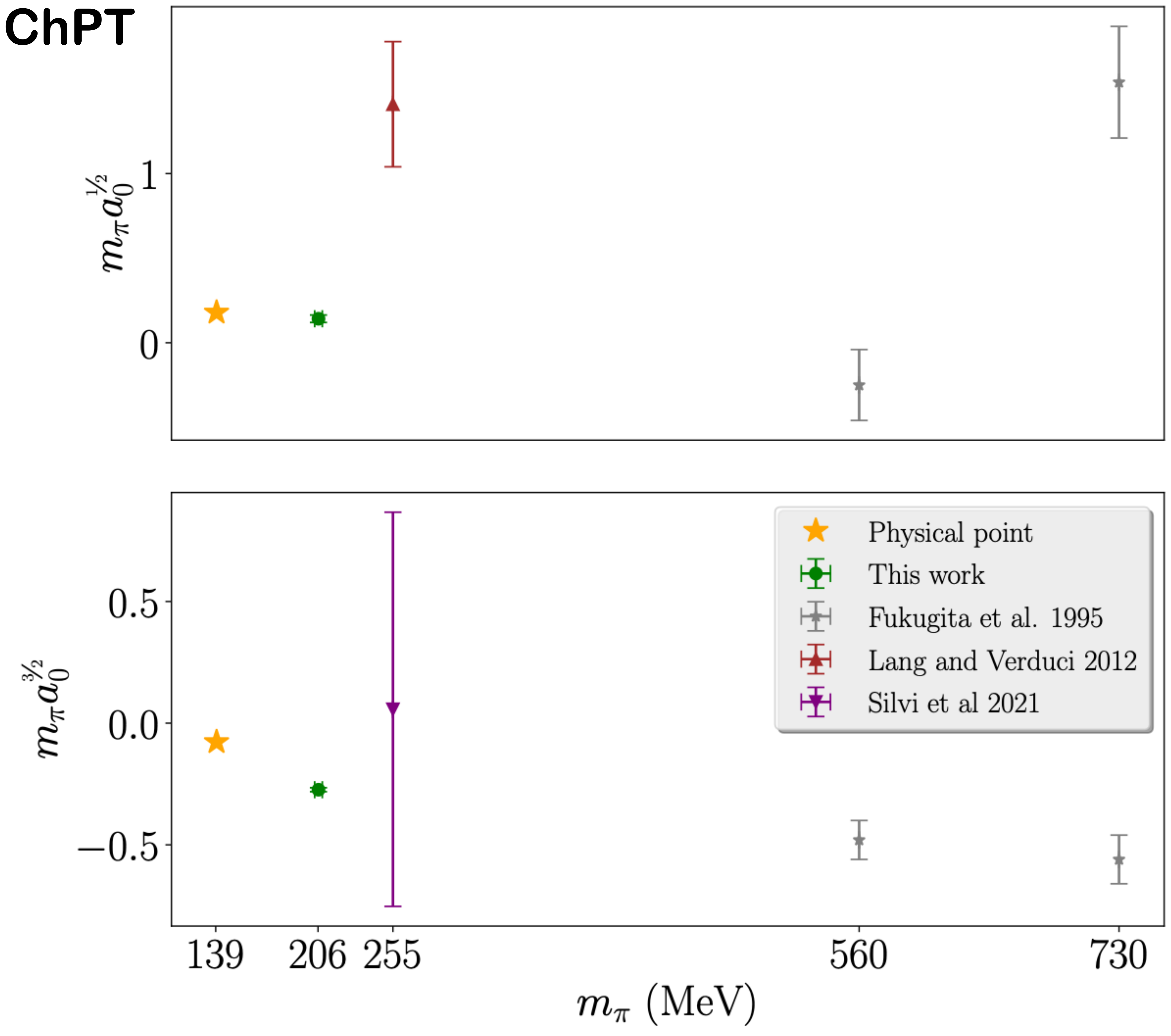
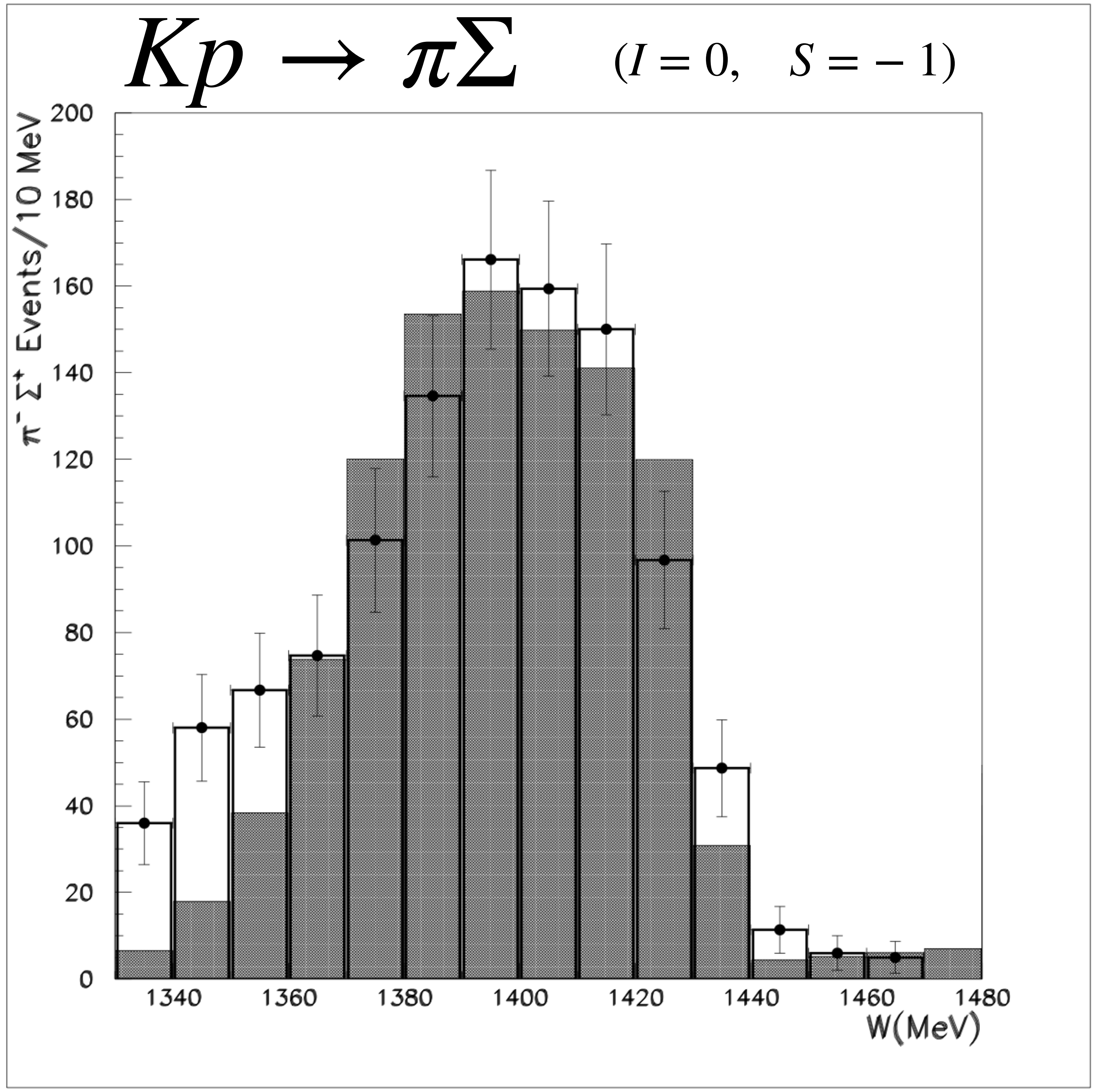
We find poor convergence at $M_\pi$ = 200 MeV




- Additional values of the pion mass are needed!

$\Lambda(1405)$

# The Λ(1405) resonance




[Oller, Meißner, 0011146], [Hemingway, NPB 1985]

- **PDG (4 star resonance):**

**Λ(1405) 1/2⁻** $I(J^P) = 0(\frac{1}{2}^-)$

Mass $m = 1405.1^{+1.3}_{-1.0}$ MeV
Full width $\Gamma = 50.5 \pm 2.0$ MeV
Below $\overline{K}N$ threshold

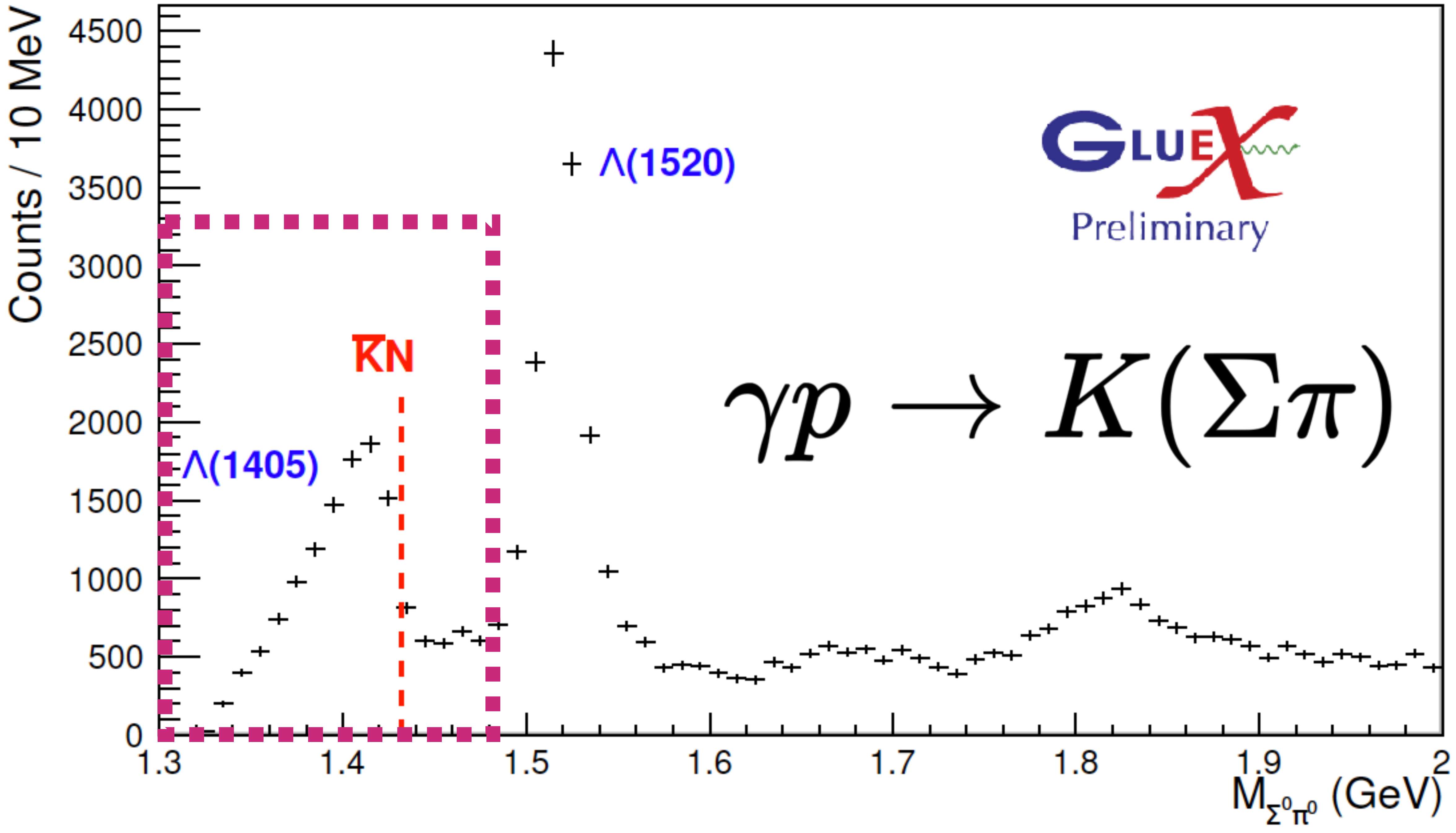
| Λ(1405) DECAY MODES | Fraction $(\Gamma_i/\Gamma)$ | $p$ (MeV/c) |
|---|---|---|
| $\Sigma\pi$ | 100 % | 155 |

- **Resonance appearing in multi-channel scattering:**

$$\begin{pmatrix} \pi\Sigma \to \pi\Sigma & \pi\Sigma \to Kp \\ Kp \to \pi\Sigma & Kp \to Kp \end{pmatrix}$$

# The Λ(1405) resonance

[Wickramaarachchi et al, 2209.06230]



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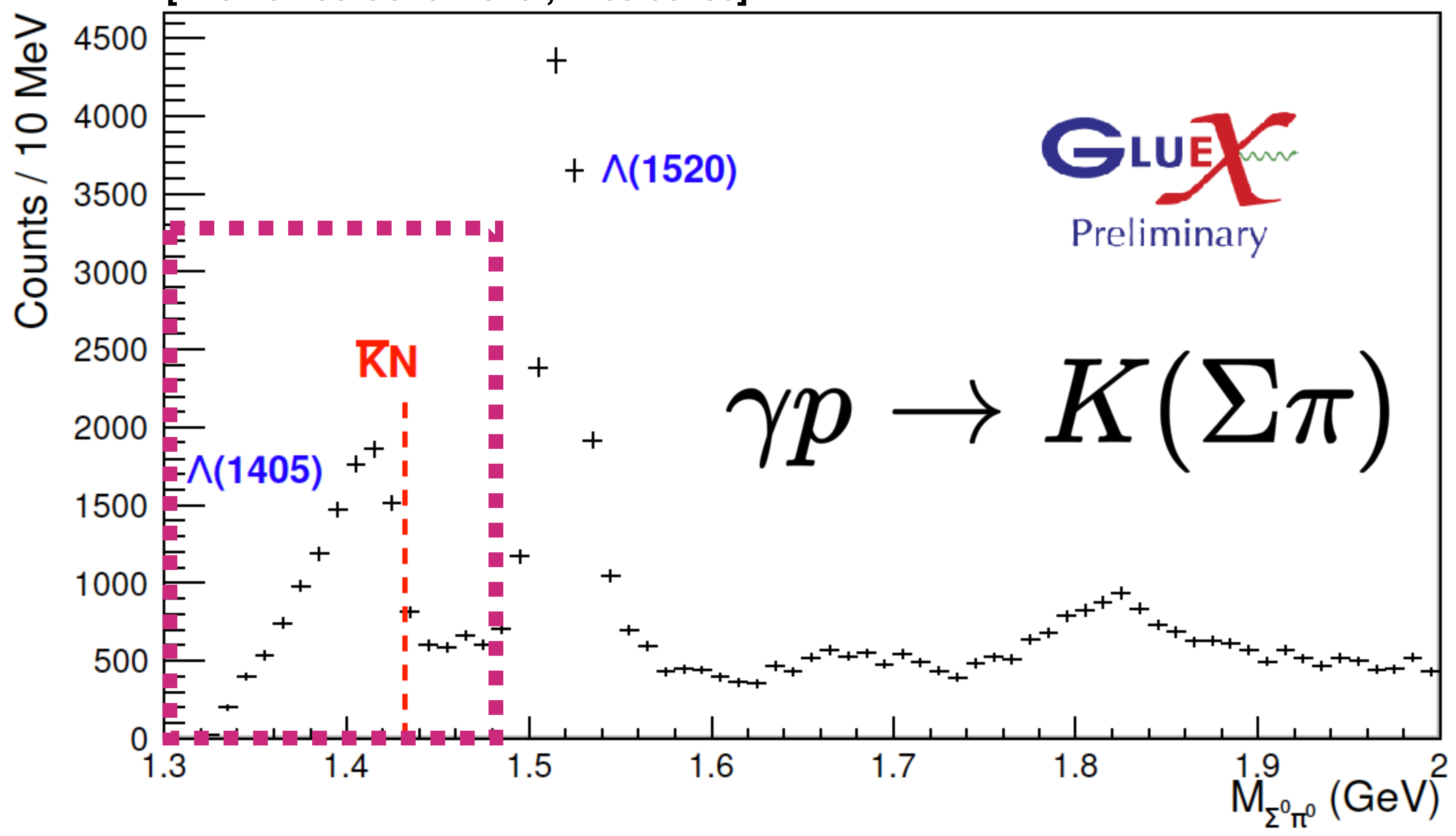


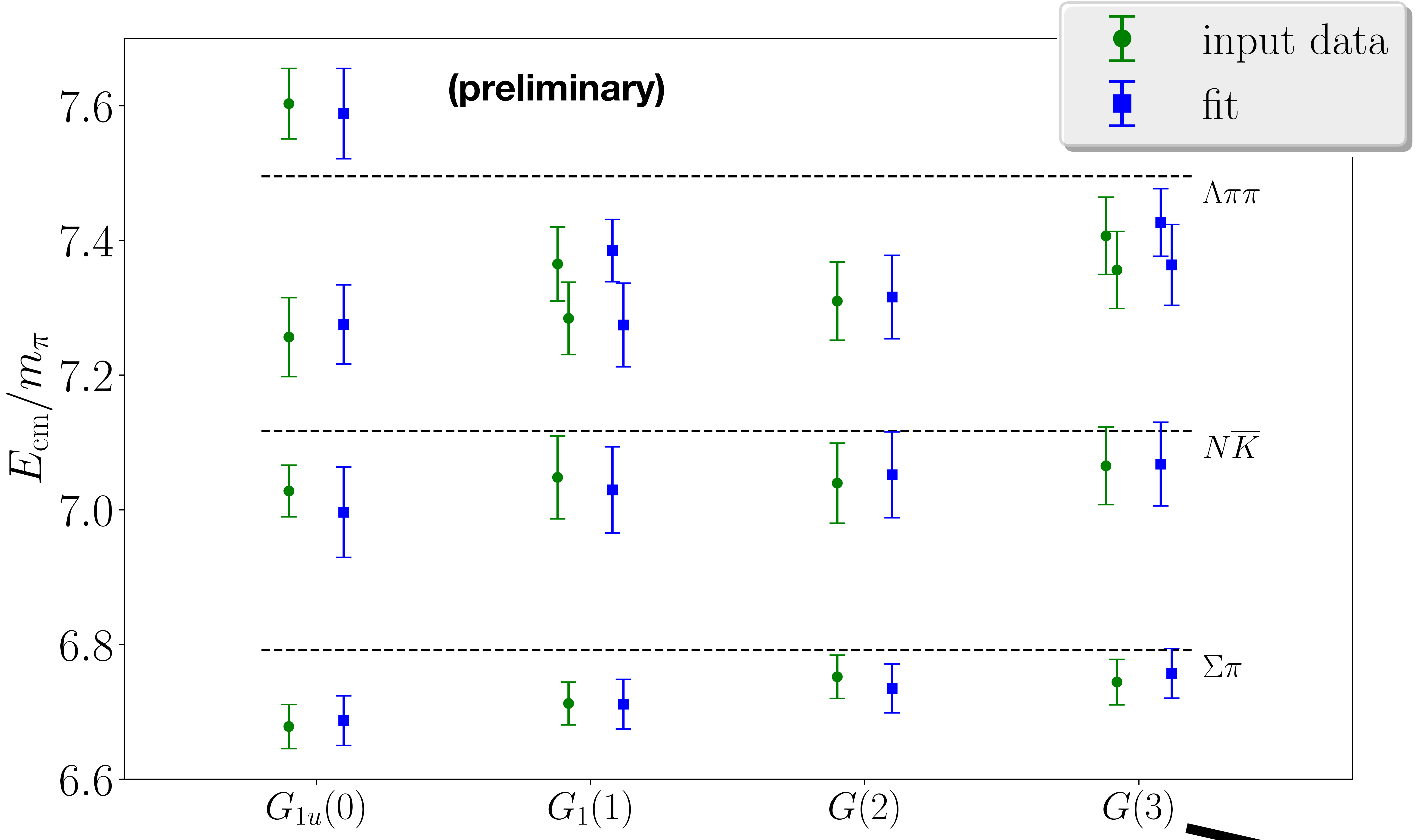

- **Long-standing puzzle about its fundamental nature:**
  - ➡ **One or two resonance poles?**
  - ➡ **Chiral EFT analysis indicate two poles**

| approach **[PDG 2022]** | pole 1 [MeV] | pole 2 [MeV] | |
|---|---|---|---|
| Refs. [14, 15], NLO | $1424^{+7}_{-23} - i\ 26^{+3}_{-14}$ | $1381^{+18}_{-6} - i\ 81^{+19}_{-8}$ | [1109.3005, 1201.6549] |
| Ref. [17], Fit II | $1421^{+3}_{-2} - i\ 19^{+8}_{-5}$ | $1388^{+9}_{-9} - i\ 114^{+24}_{-25}$ | [1210.3485] |
| Ref. [18], solution #2 | $1434^{+2}_{-2} - i\ 10^{+2}_{-1}$ | $1330^{+4}_{-5} - i\ 56^{+17}_{-11}$ | [1411.7884] |
| Ref. [18], solution #4 | $1429^{+8}_{-7} - i\ 12^{+2}_{-3}$ | $1325^{+15}_{-15} - i\ 90^{+12}_{-18}$ | |

well resolved (pole 1) — more uncertain (pole 2)

- **Lattice QCD: essential non-perturbative verification**

# The Λ(1405) in Lattice QCD




- Fit with ERE in 2x2 K matrix

$$(K_2)_{ij} = A_{ij} + B_{ij}E_{\rm cm}^2 + \dots$$

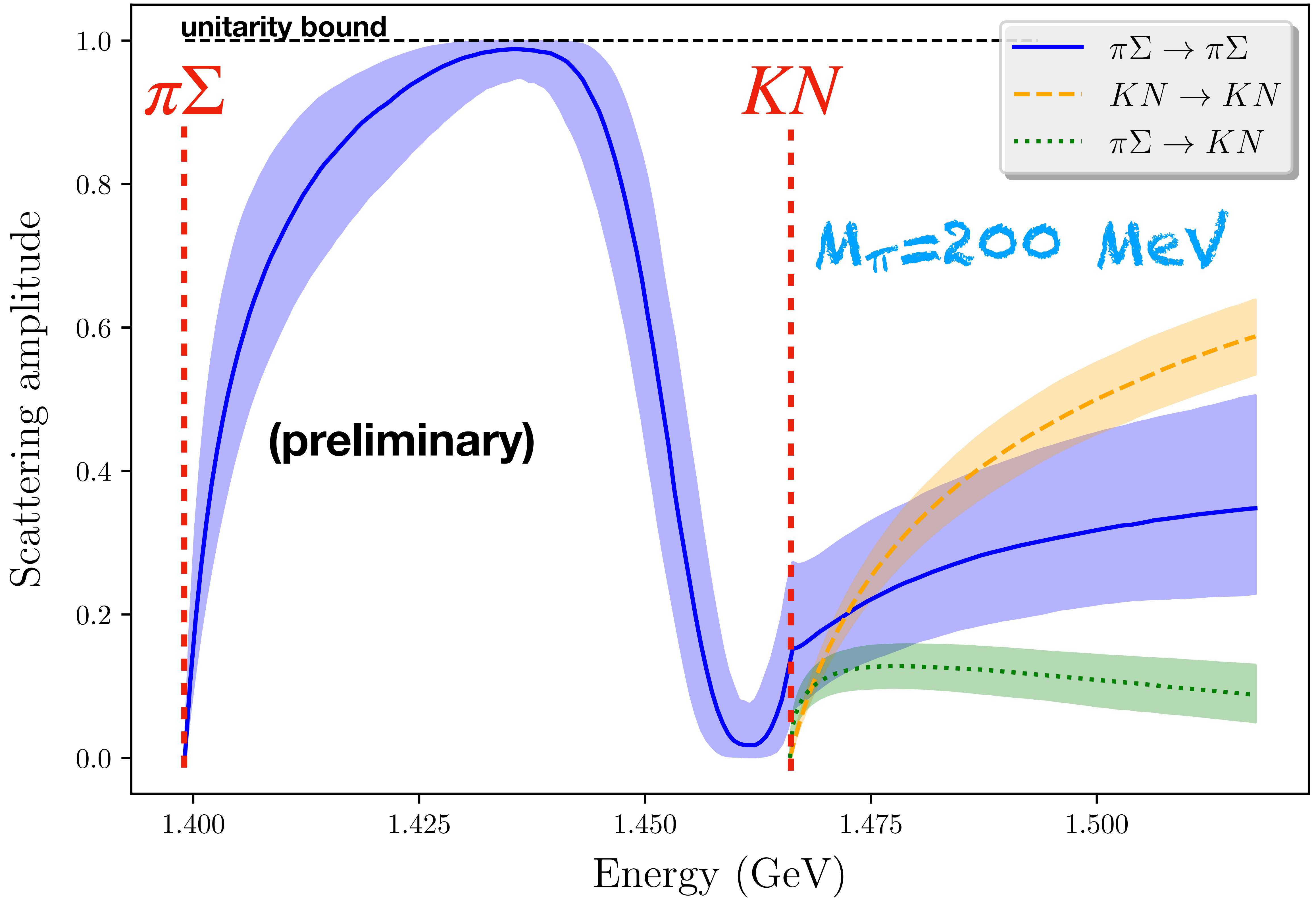
**4 parameters**
**15 energies**

$$\chi^2/\text{dof} = 1.02$$

finite-volume quantum numbers (irreps)

# The Λ(1405) in lattice QCD

☑ **First lattice QCD study of its analytic structure using multi-channel scattering**




**[Bulava, Cid Mora, Hanlon, Hörz, Mohler, Morningstar, Moscoso, Nicholson, FRL, Skinner, Walker-Loud, (in prep)]**

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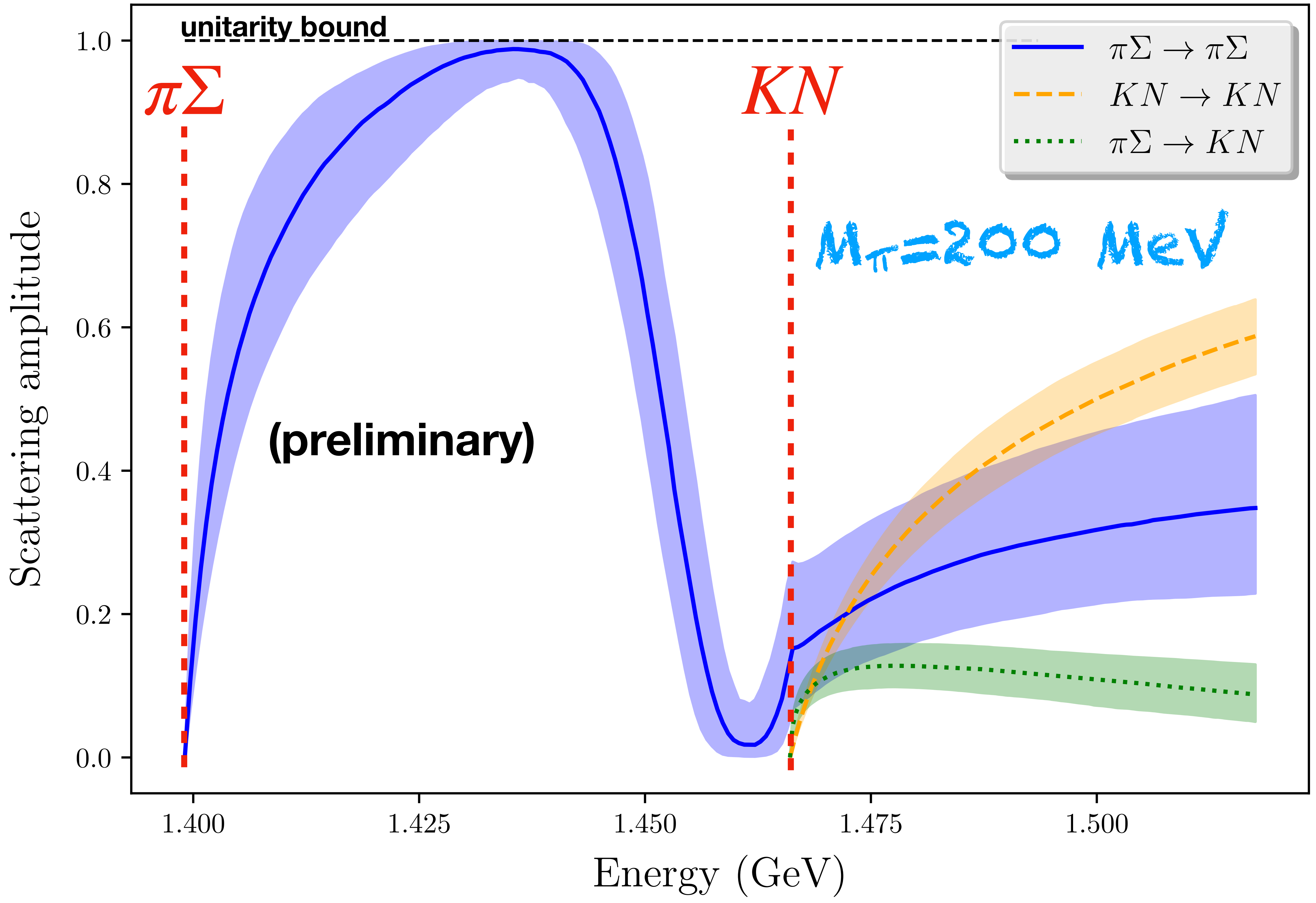


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☑ **We find two resonance poles (preliminary)**

$$\sqrt{s}_{\text{pole 1}} = 1456(21) - i\,12(6)\ \text{MeV} \qquad \sqrt{s}_{\text{pole 2}} = 1395(18)\ \text{MeV}$$

**(qualitative agreement with chiral EFTs)**

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☑ **Qualitative picture stable under model variations**

☑ **Results close to the physical point**

➡ **Work in progress for a physical point calculation**



Fernando Romero-López, MIT

# Three-particle processes

# Why three particles?

- The two-body formalism is restricted to few interesting resonances
  - **Exotics:** $T_{cc} \to DD^*, DD\pi$
  - **Roper:** $N(1440) \to \Delta\pi \to N\pi\pi$

| Resonance | $I_{\pi\pi\pi}$ | $J^P$ |
|---|---|---|
| $\omega(782)$ | 0 | $1^-$ |
| $h_1(1170)$ | 0 | $1^+$ |
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(with $\geq 3\pi$ decay modes)

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- ☑ Major developments in the three-particle finite-volume formalism

[Hansen, Sharpe, PRD 2014 & 2015], [Hammer, Pang, Rusetsky, JHEP 2017] x 2
[Mai, Döring, EPJA 2017]
[...]
[Blanton, FRL, Sharpe, JHEP 2019], [Hansen, FRL, Sharpe, JHEP 2020]
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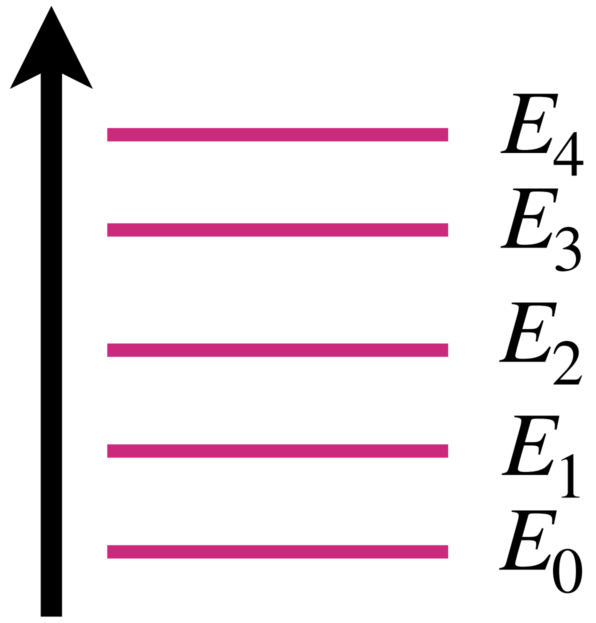
# Relativistic three-particle formalism for identical particles

[Hansen, Sharpe, PRD 2014 & 2015]

2π Spectrum

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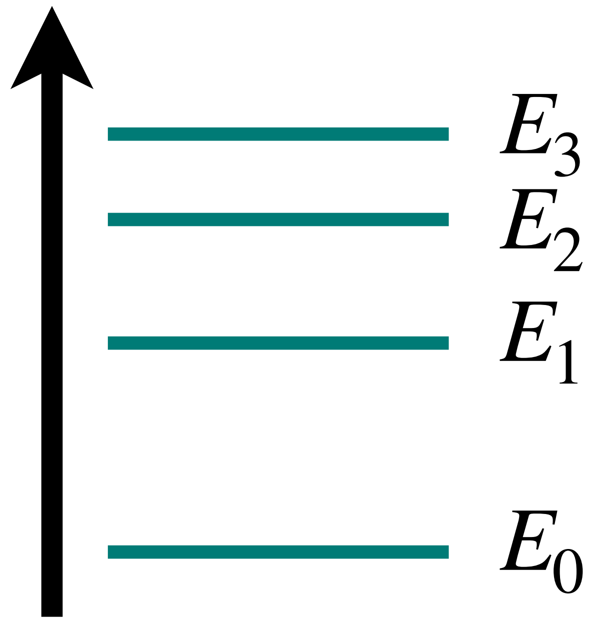

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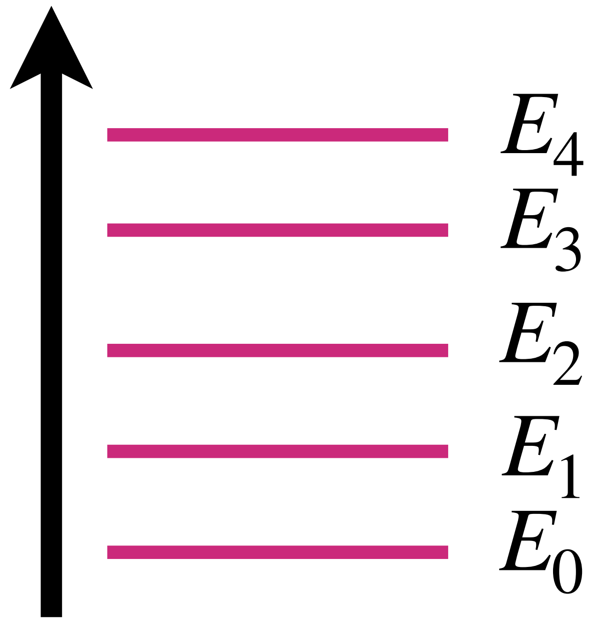


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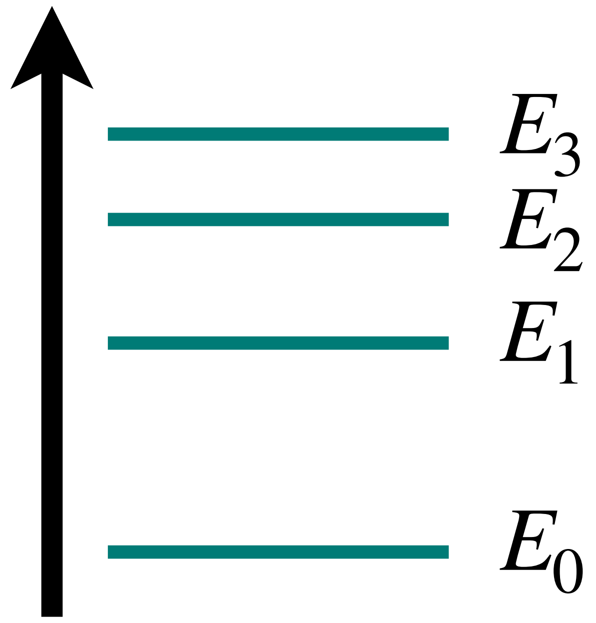


Quantization conditions

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

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**Matrix indices describe three on-shell particles**

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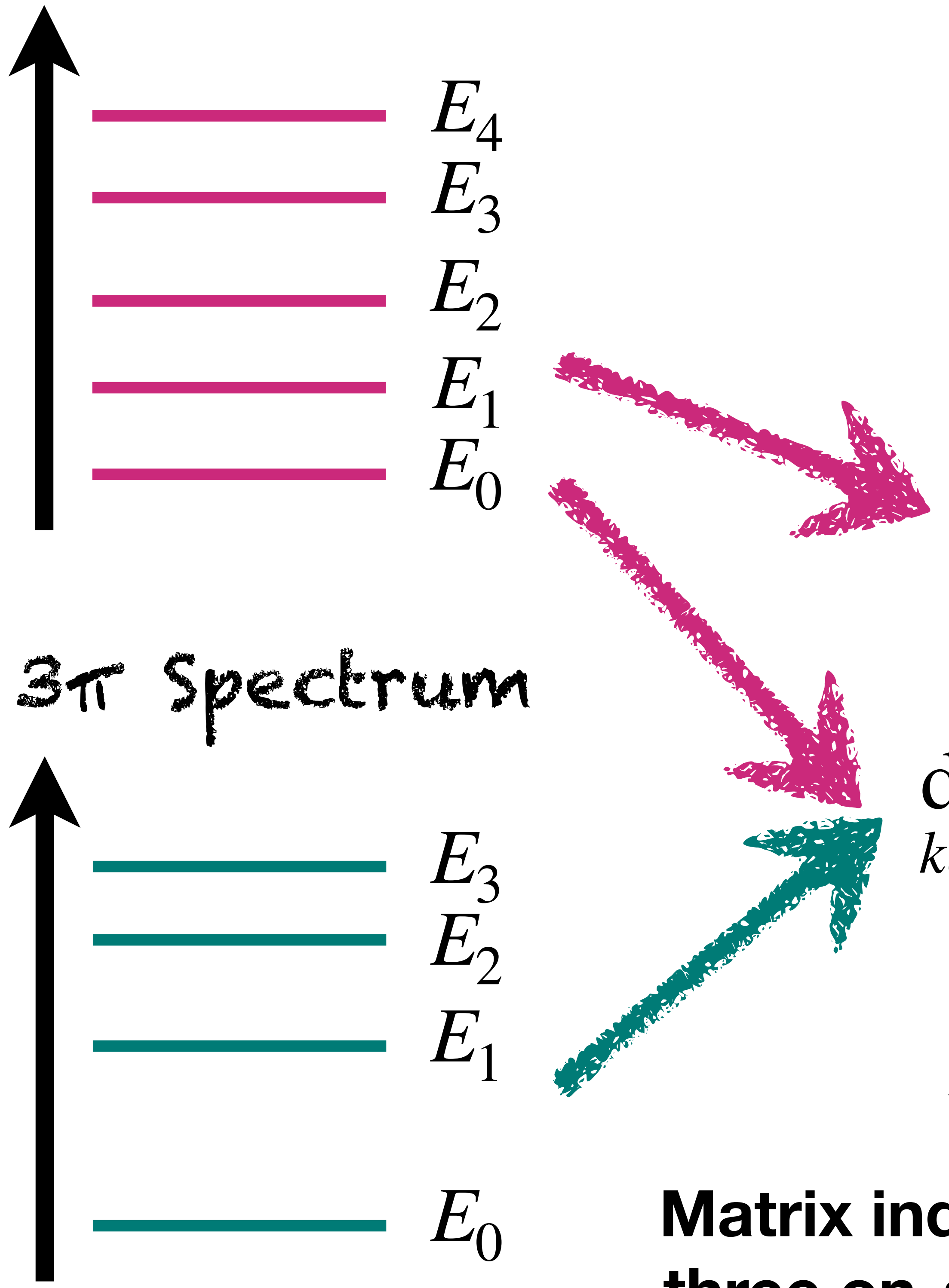


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Fit

K-matrices

$$\mathcal{K}_2$$

$$\mathcal{K}_{df,3}$$

Parametrize:

$$\mathcal{K}_2 = c_0 + c_1 k^2 + \dots$$

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**[Blanton, FRL, Sharpe, JHEP 2019]**




**Fernando Romero-López, MIT**

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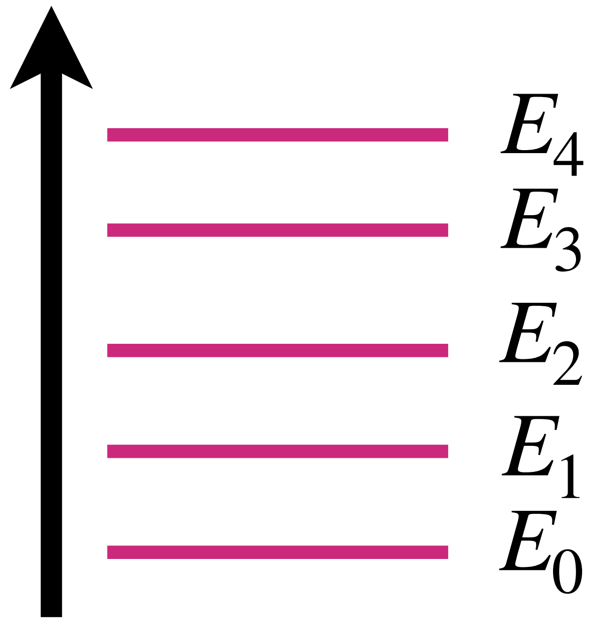


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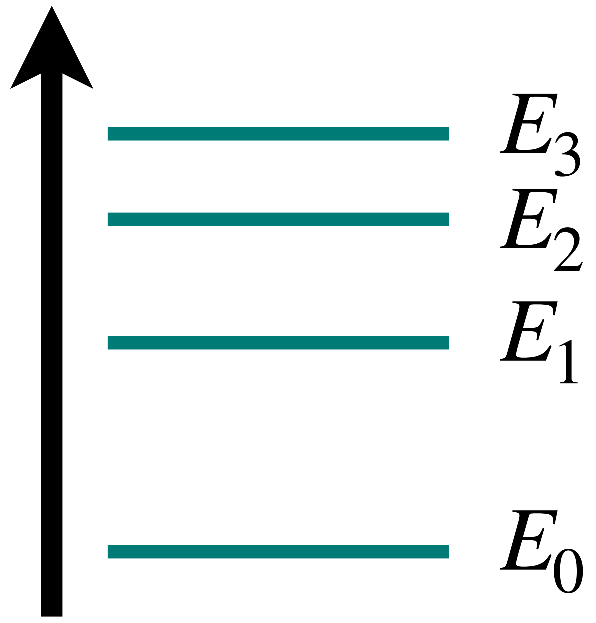


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Integral equations

[Briceño et al., PRD 2018]
[Hansen et al., PRL 2021]
[Jackura et al., PRD 2021]
[Dawid et al., 2303.04394]

Scattering amplitudes

$\mathcal{M}_2$

$\mathcal{M}_3$

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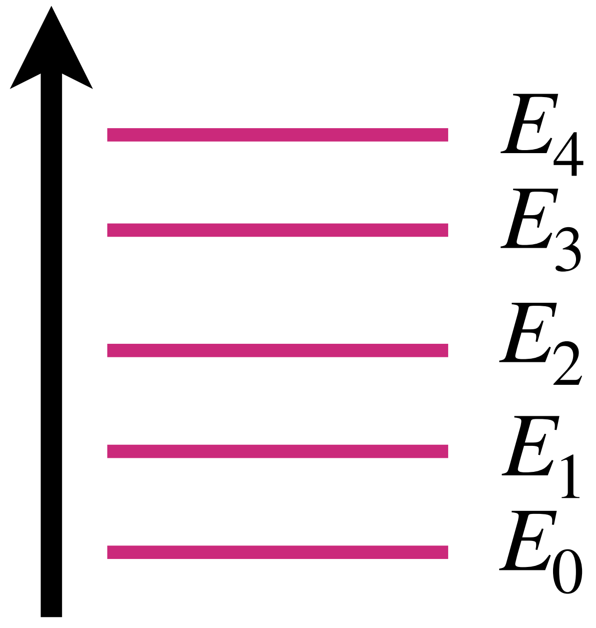


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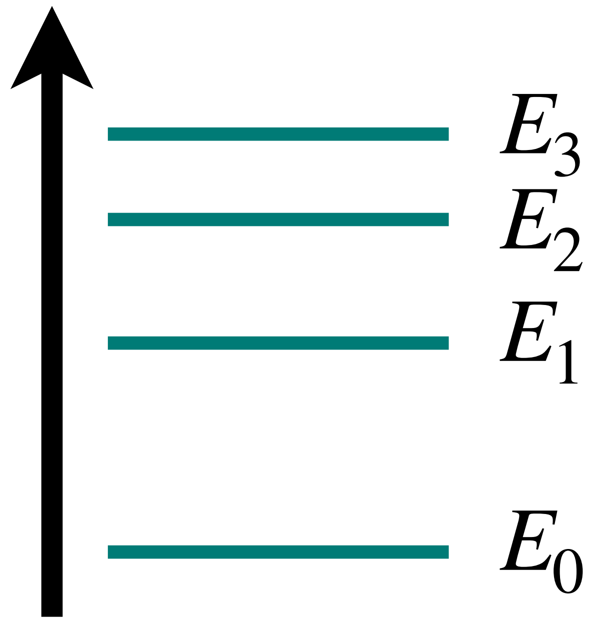


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↕

Experiments & resonance properties

# Three-meson systems

- **Three-particle formalism applied to weakly-interacting (non-resonant) systems:** $\pi^+\pi^+\pi^+, \ \pi^+\pi^+K^+$

  **[Blanton ... FRL ... et al., PRL 2020 & JHEP 2021], [Draper ... FRL ... et al., JHEP 2023], [Fischer ... FRL ... et al, EPJC 2021]**

  **[Alexandrou et al, Brett et al, Culver et al, Hansen et al, Mai et al]**

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[Alexandrou et al, Brett et al, Culver et al, Hansen et al, Mai et al]

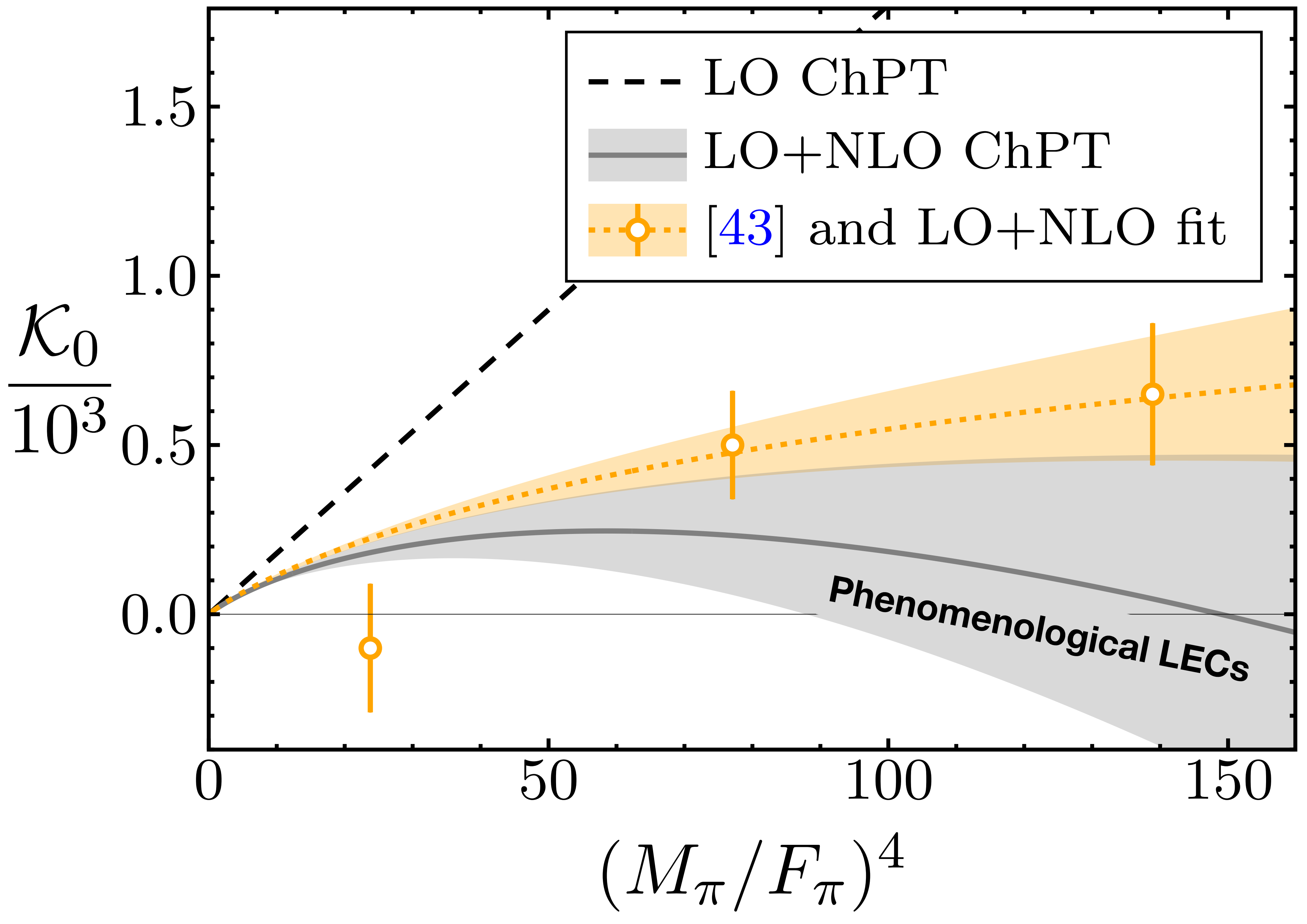
## Example: $\pi^+\pi^+\pi^+$ scattering

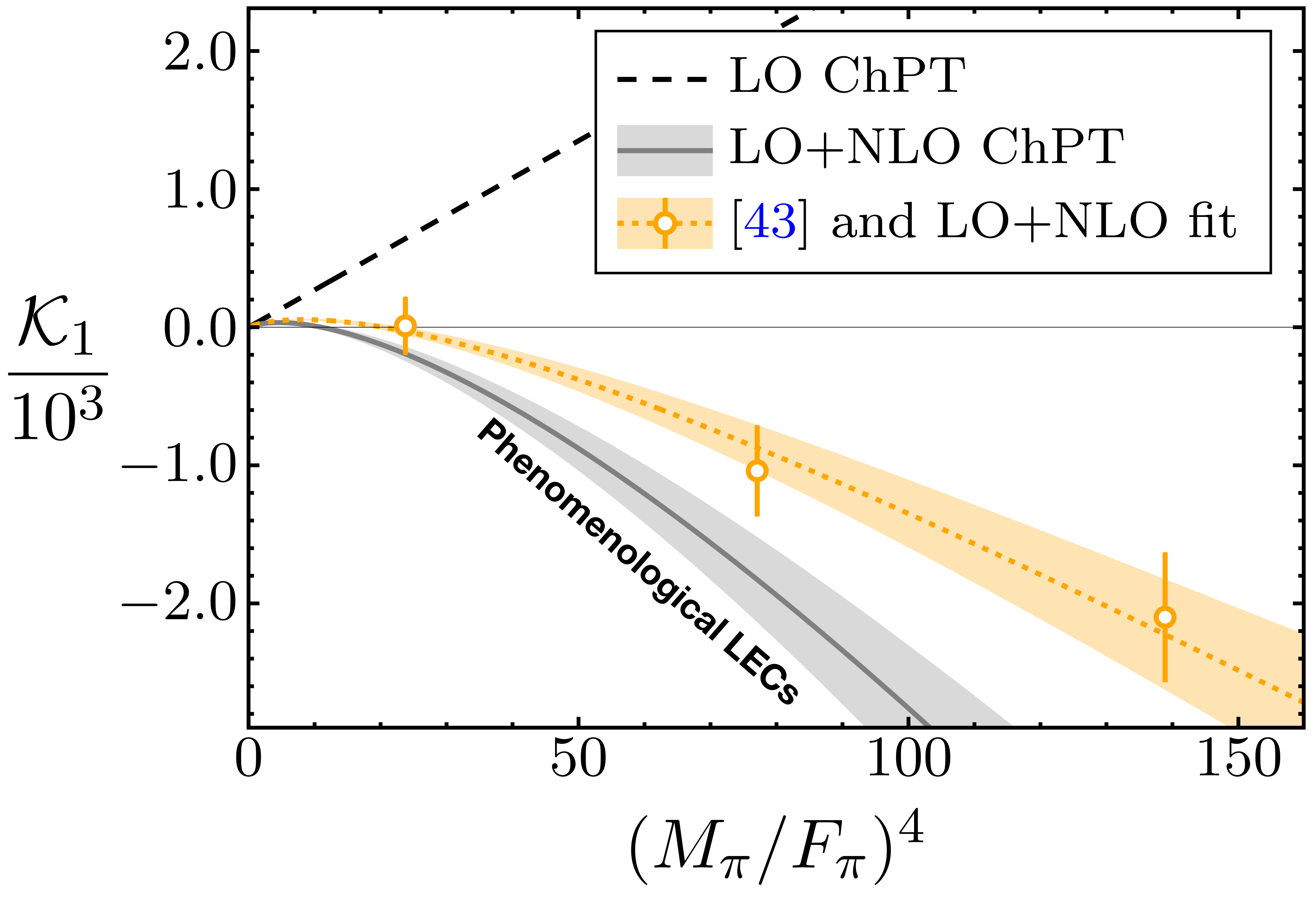
Lattice data: [Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

NLO ChPT: [Baeza-Ballesteros, Bijnens, Husek ,FRL, Sharpe, Sjö, JHEP 2023]

parametrized by the three-particle K-matrix

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\left(\frac{s-9M_\pi^2}{9M_\pi^2}\right) + \cdots$$






Fernando Romero-López, MIT

# Three-body resonances

- Formalism to study relevant three-pion resonances is available
  [Blanton, Sharpe, PRD 2021] x 2, [Hansen, FRL, Sharpe, JHEP 2021], [Mai et al (GWQCD), PRL 2021]
  - Preparing formalism for $T_{cc}$ [Draper, Hansen, FRL, Sharpe, (in prep)]
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- Extensive lattice QCD data is still not available
- Extract resonance properties on a toy model [Garofalo, Mai, FRL, Rusetsky, Urbach (2211.05605)]

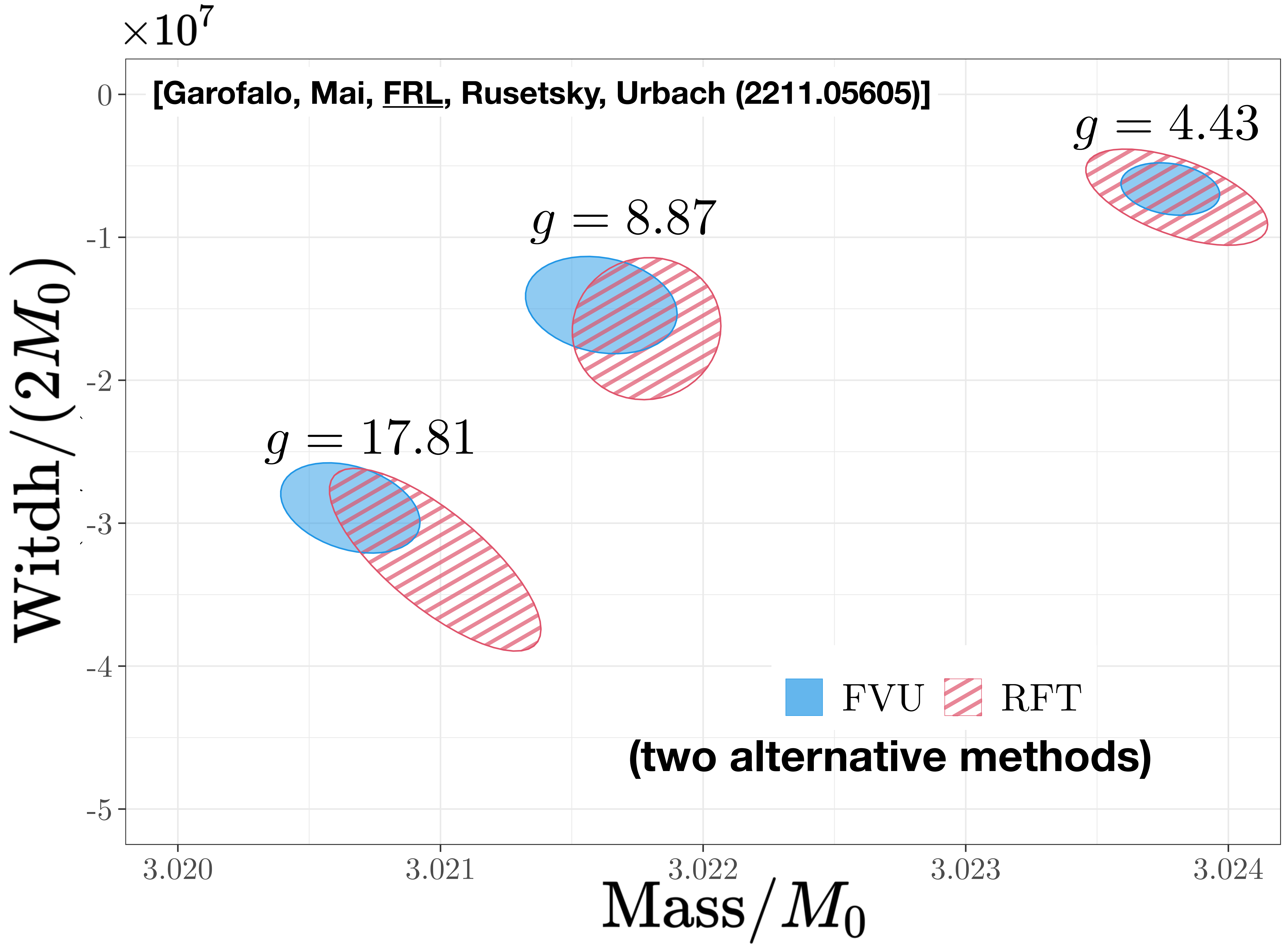
$$\mathcal{L} = \sum_{i=0,1} \left( \partial_\mu \phi_i^\dagger \partial_\mu \phi_i + m_i \phi_i^\dagger \phi_i + \lambda_i \phi_i^4 \right) + \frac{g}{2} \phi_1^\dagger \phi_0^3 + h.c.$$

Induces transitions:

$$\phi_1 \to 3\phi_0$$
$$(M_1 > 3M_0)$$

  - Test formalism in a controlled setup
  - Computationally cheaper
  - Allows access to several volumes & energy levels

# A toy three-body resonance



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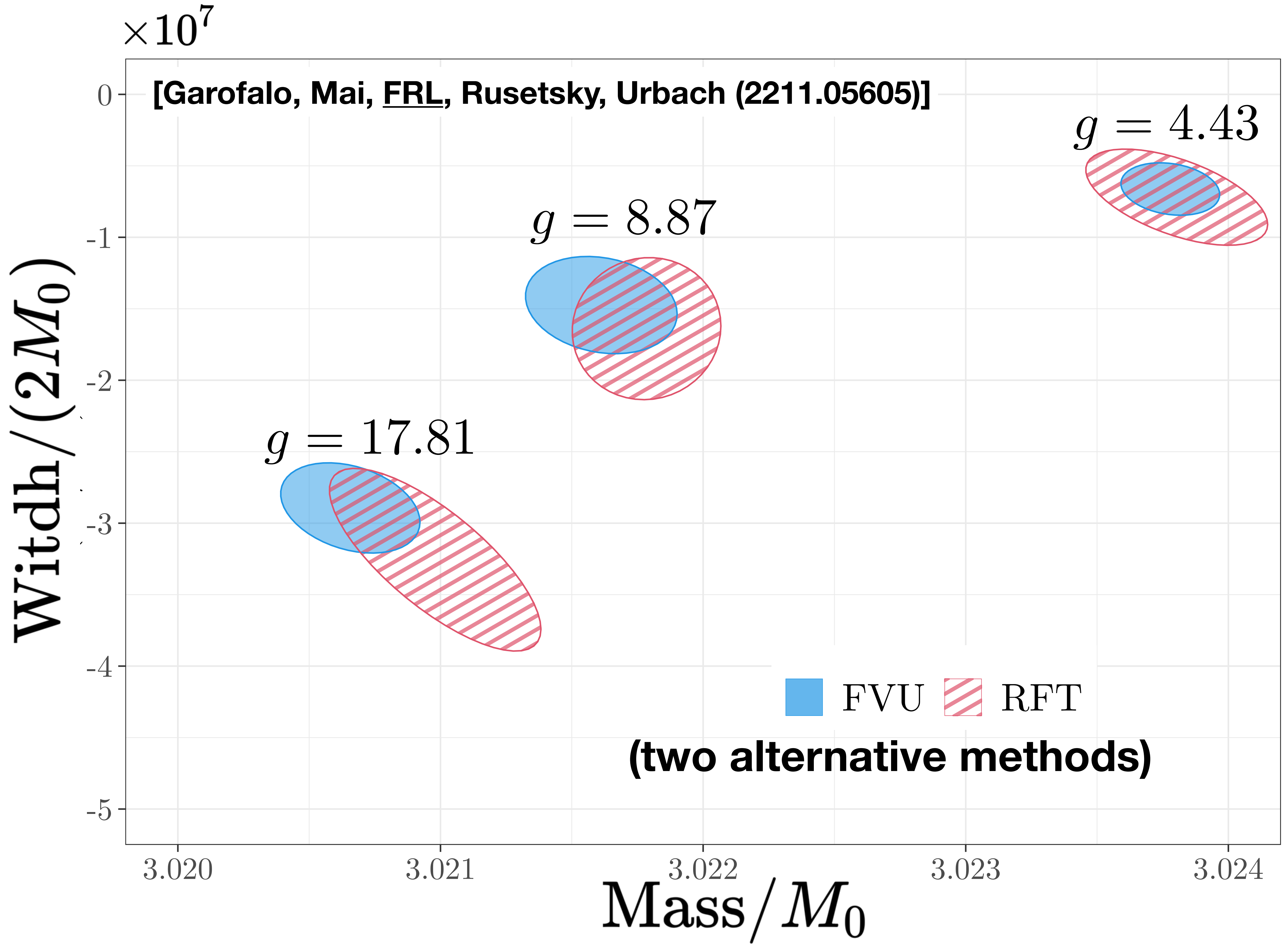


(two alternative methods)

- ☑ Successfully determined properties of three-particle resonance for the first time!
- ☑ Good agreement between methods
- ☐ Next: QCD resonances.
  (e.g. $h_1(1170)$, $T_{cc}^+$ and Roper )

# Summary & Outlook

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- ☑ Lattice QCD provides a first-principle tool to investigate the hadron spectrum
- ☑ Access to quantities that are hard experimentally, e.g. hyperons
- ☑ Several studies of scattering lattice QCD: $\Delta(1232), \Lambda(1405)$
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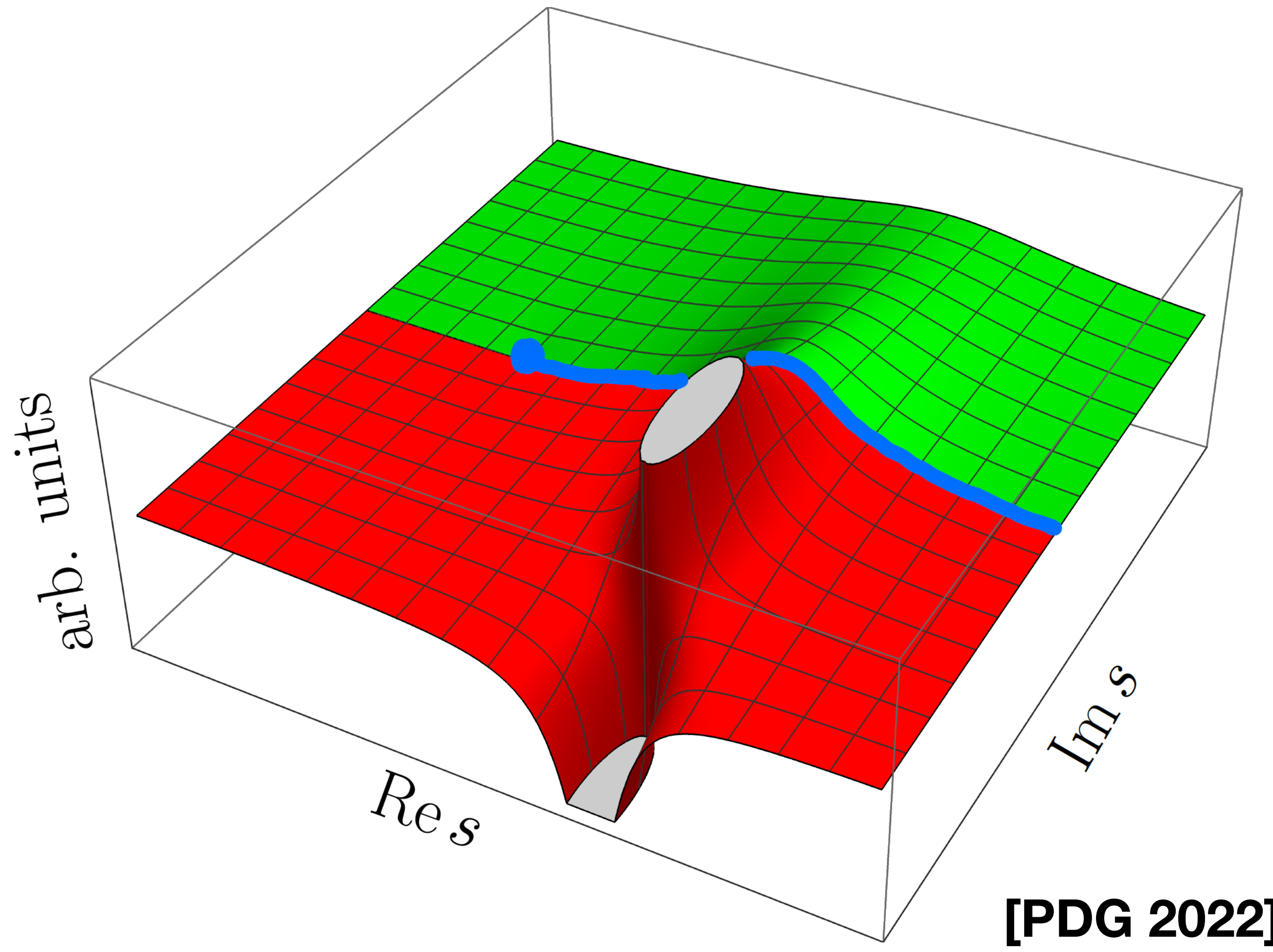


[PDG 2022]

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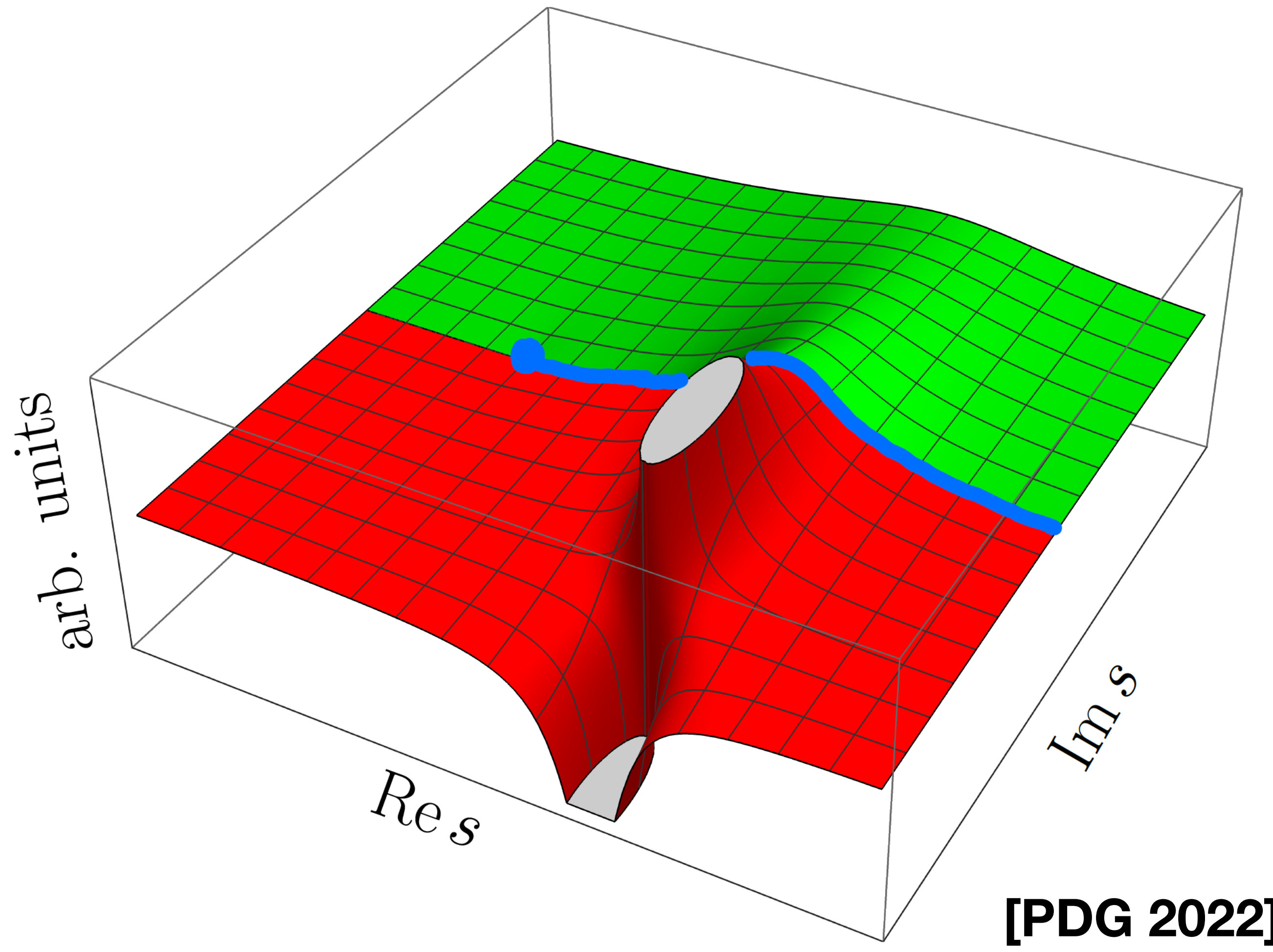
- ☐ Present frontier: lattice data for three-particle meson resonances
- ☐ Further theoretical developments necessary
  - ➡ Roper resonance $N(1440) \to N\pi, N\pi\pi$
  - ➡ Four or more particle resonances




[PDG 2022]

# Summary and outlook

- ☑ Lattice QCD provides a first-principle tool to investigate the hadron spectrum
- ☑ Access to quantities that are hard experimentally, e.g. hyperons
- ☑ Several studies of scattering lattice QCD: $\Delta(1232), \Lambda(1405)$
- ☑ First results on three-particle resonances on a toy model
- ☐ Present frontier: lattice data for three-particle meson resonances
- ☐ Further theoretical developments necessary
  - ➡ Roper resonance $N(1440) \to N\pi, N\pi\pi$
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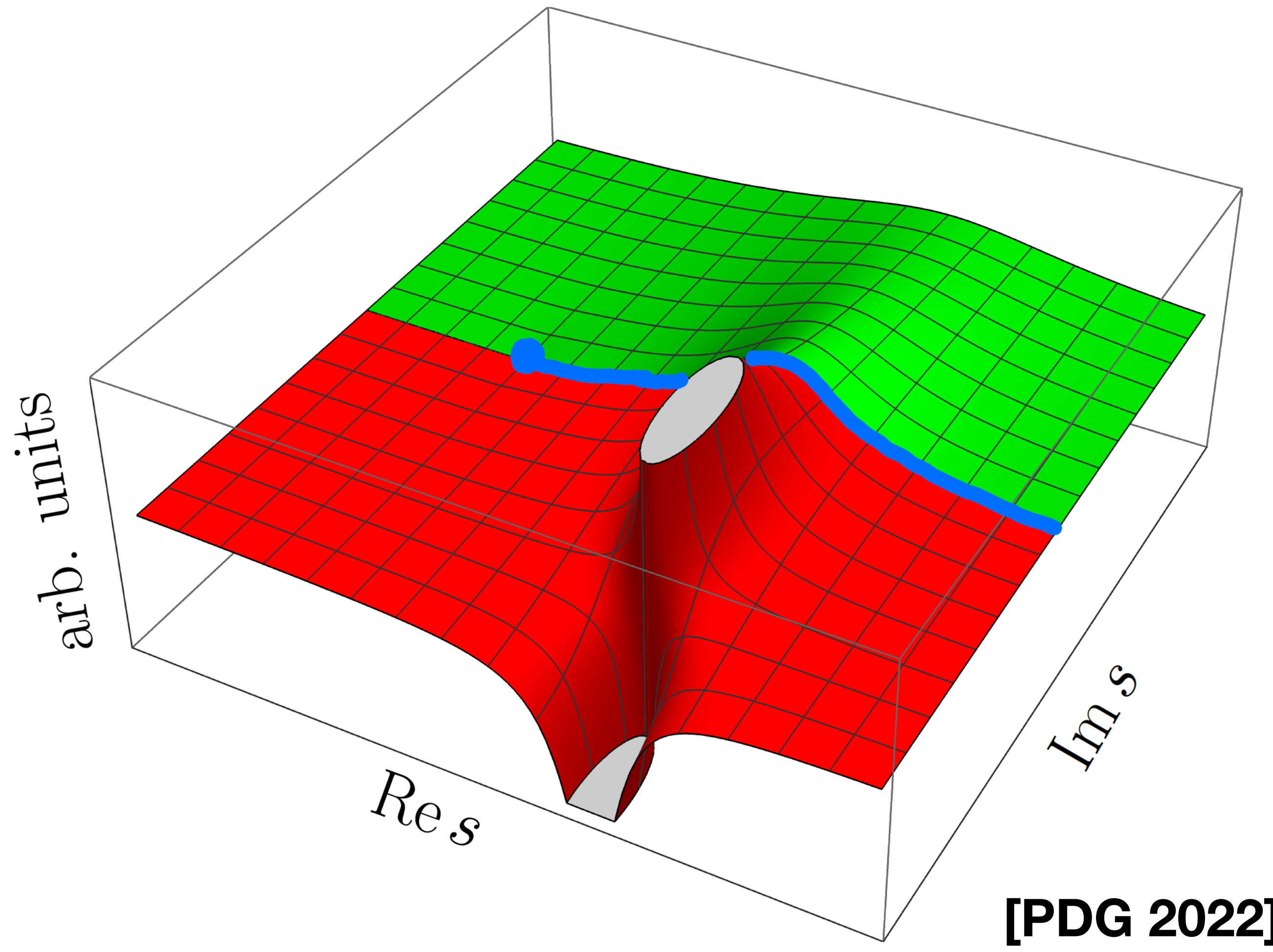


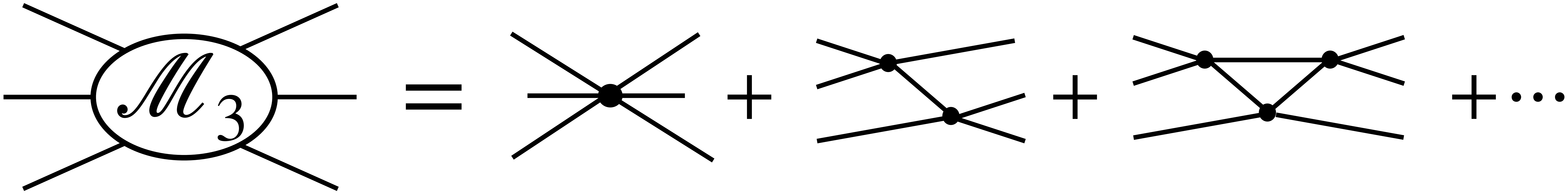

[PDG 2022]

Thanks!

# Back-up

# Three-particle scattering

Qualitatively more complicated than the two-particle case!



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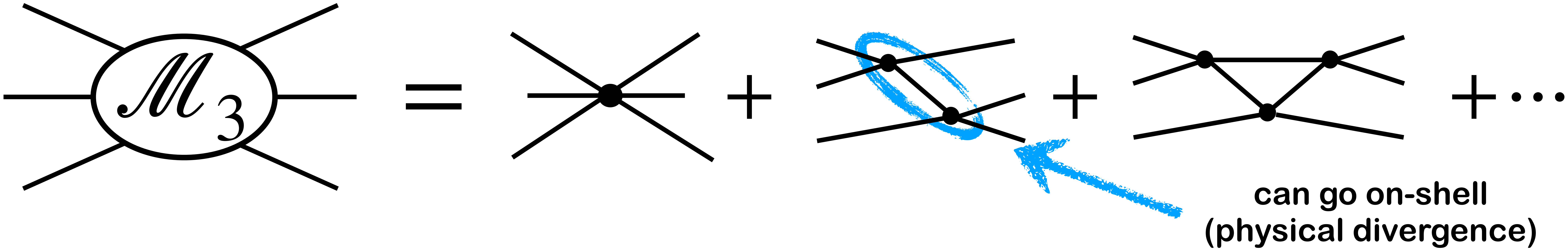

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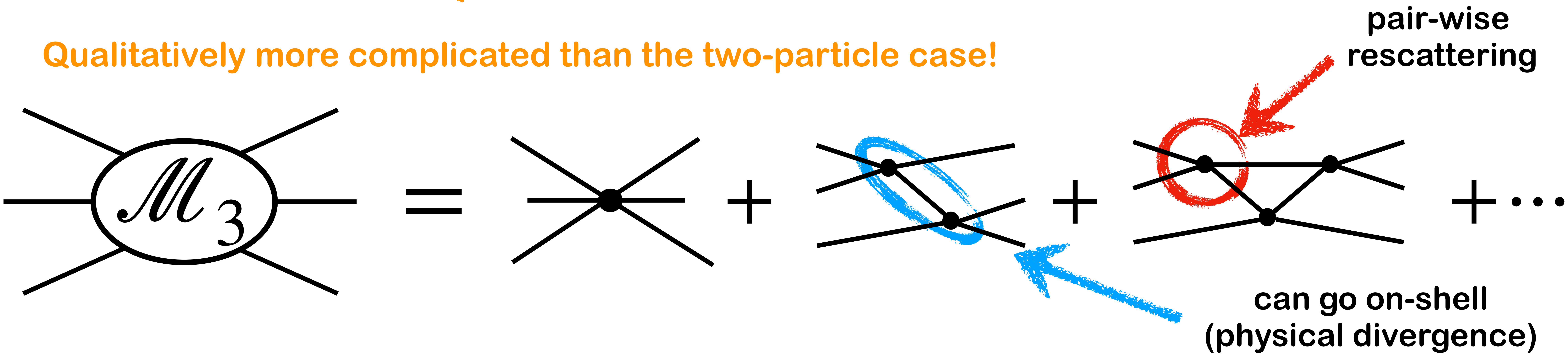

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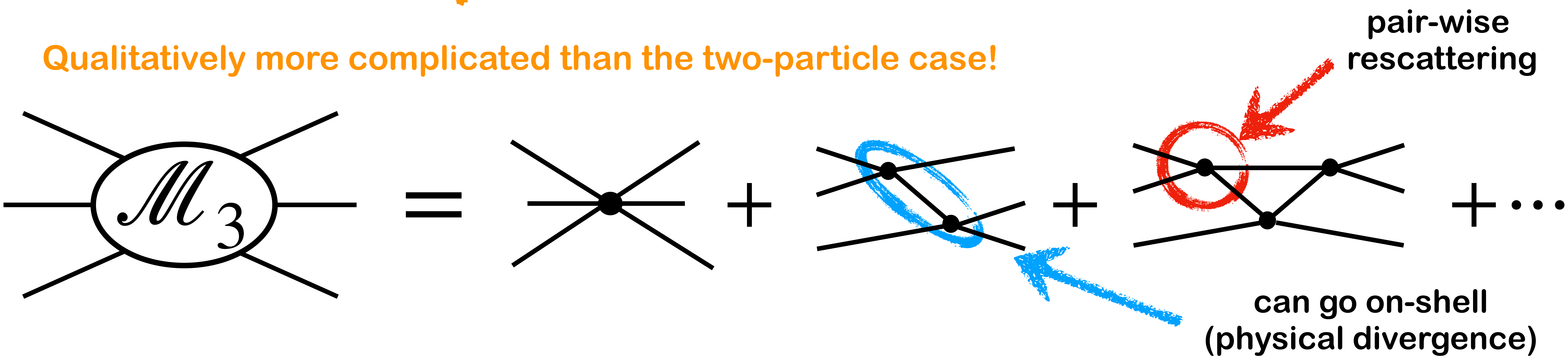
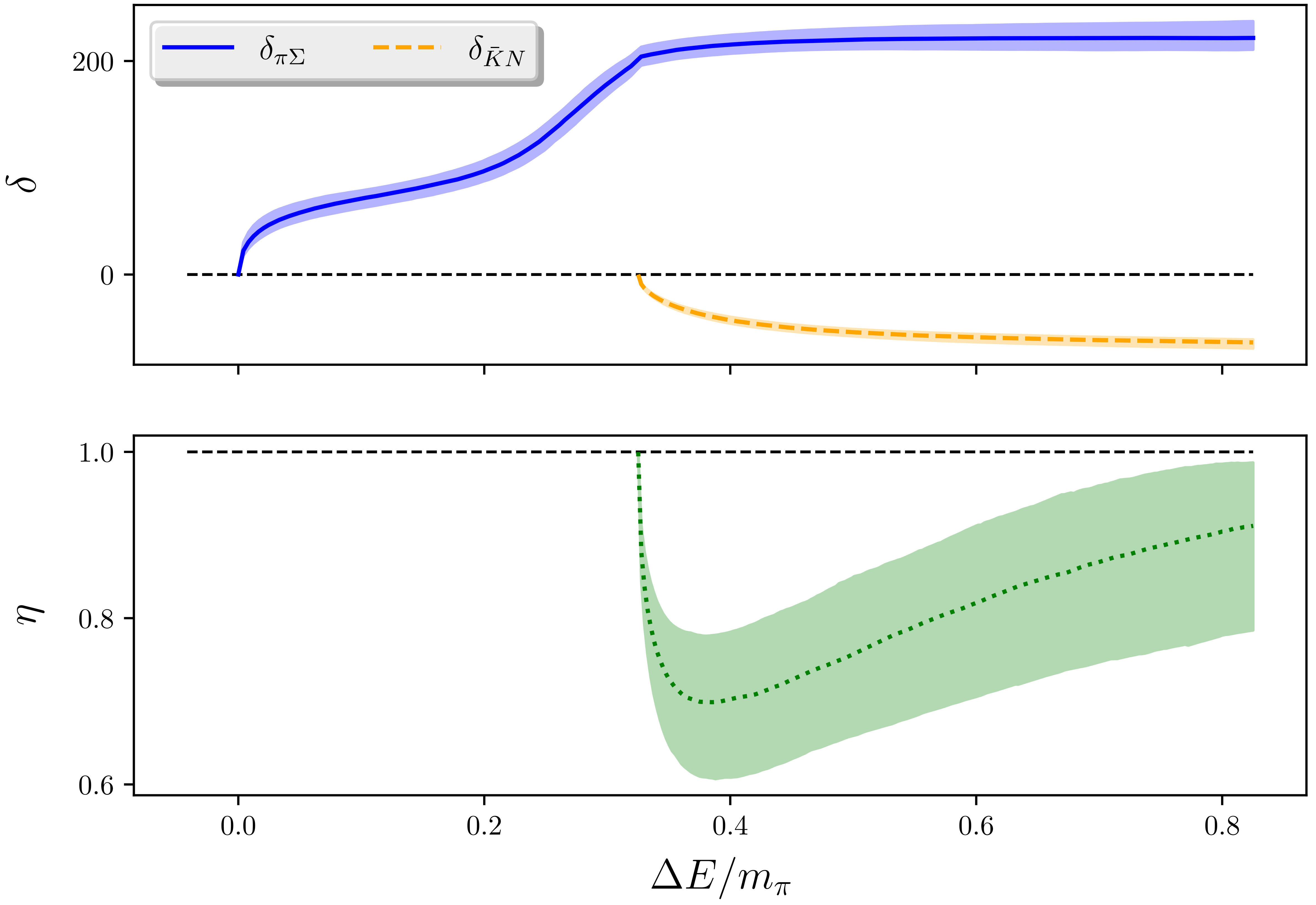
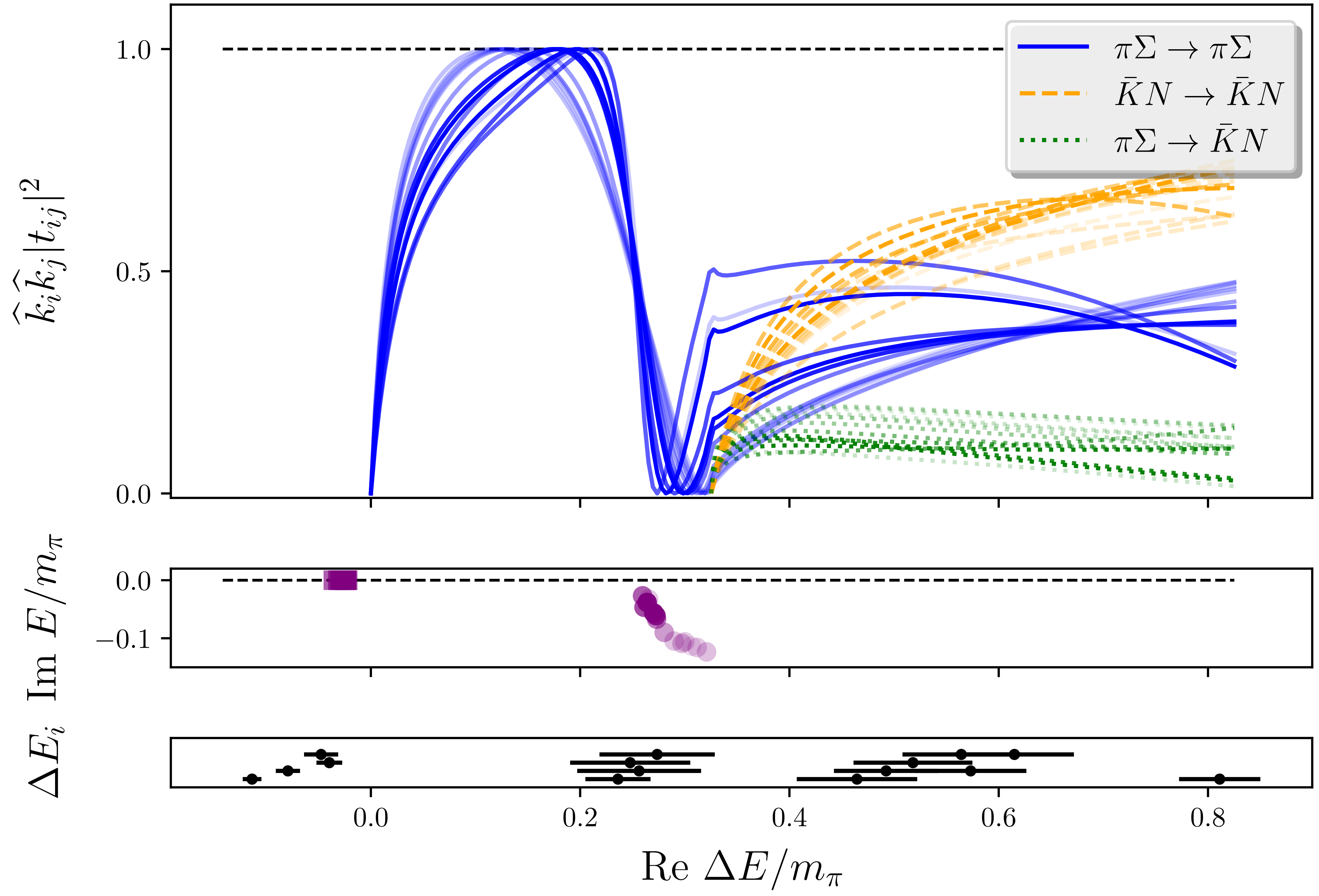



- Finite-volume formalism has been developed independently by three groups
  - Generic Relativistic Field Theory (RFT) [Hansen, Sharpe, PRD 2014 & 2015]
  - Non-Relativistic EFT (NREFT) [Hammer, Pang, Rusetsky, JHEP 2017] x 2
  - Finite-Volume Unitarity (FVU) [Mai, Döring, EPJA 2017]

Equivalence has been established
[Jackura et al. PRD 2019], [Blanton, Sharpe, PRD 2020], [Jackura, 2208.10587]

# The Λ(1405) resonance



# Poles

$$\mathcal{M}_2(s) = \frac{16\pi\sqrt{s}}{k\cot\delta(k) - ik},$$

$$k = \pm\sqrt{k^2}$$

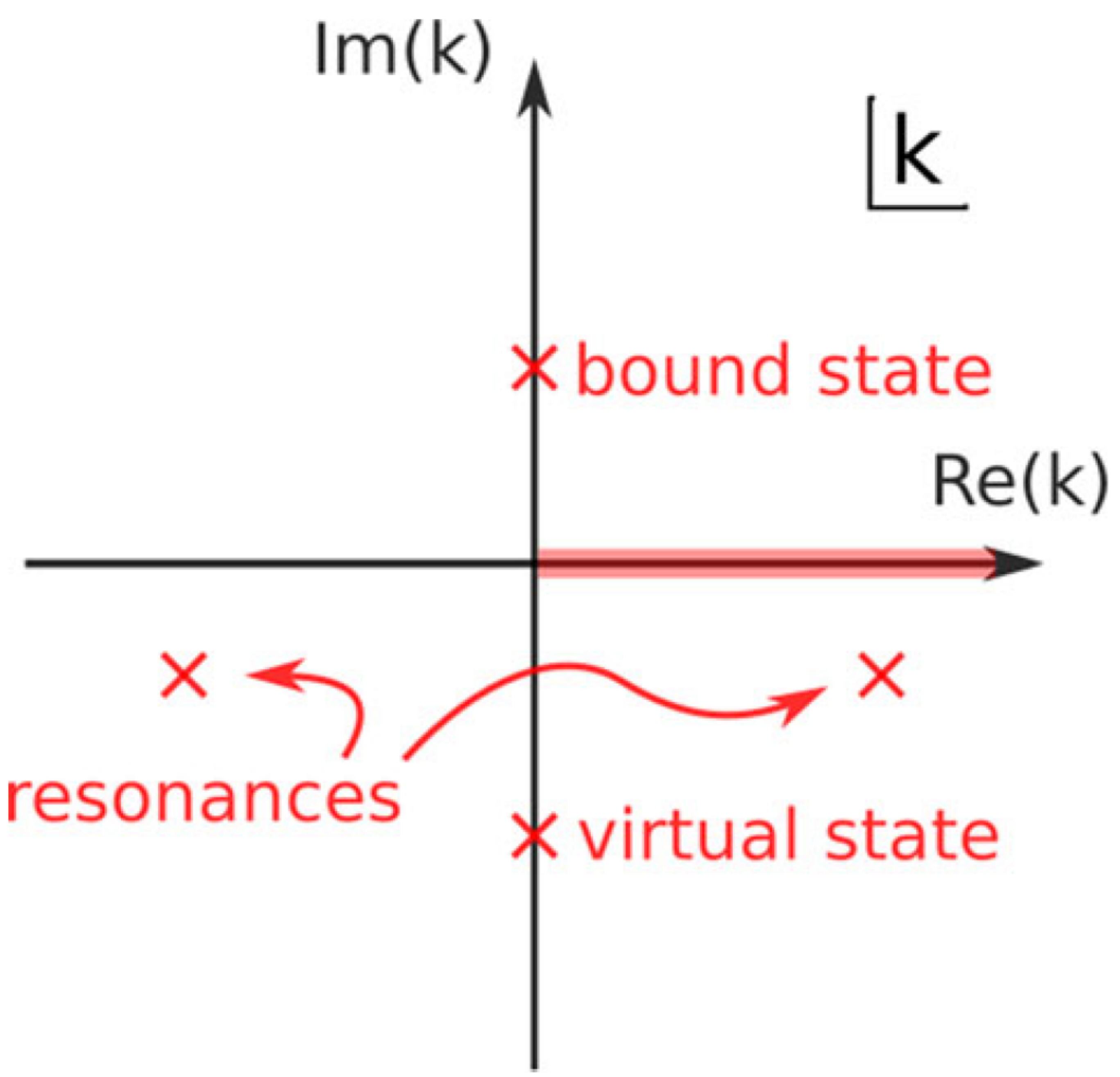


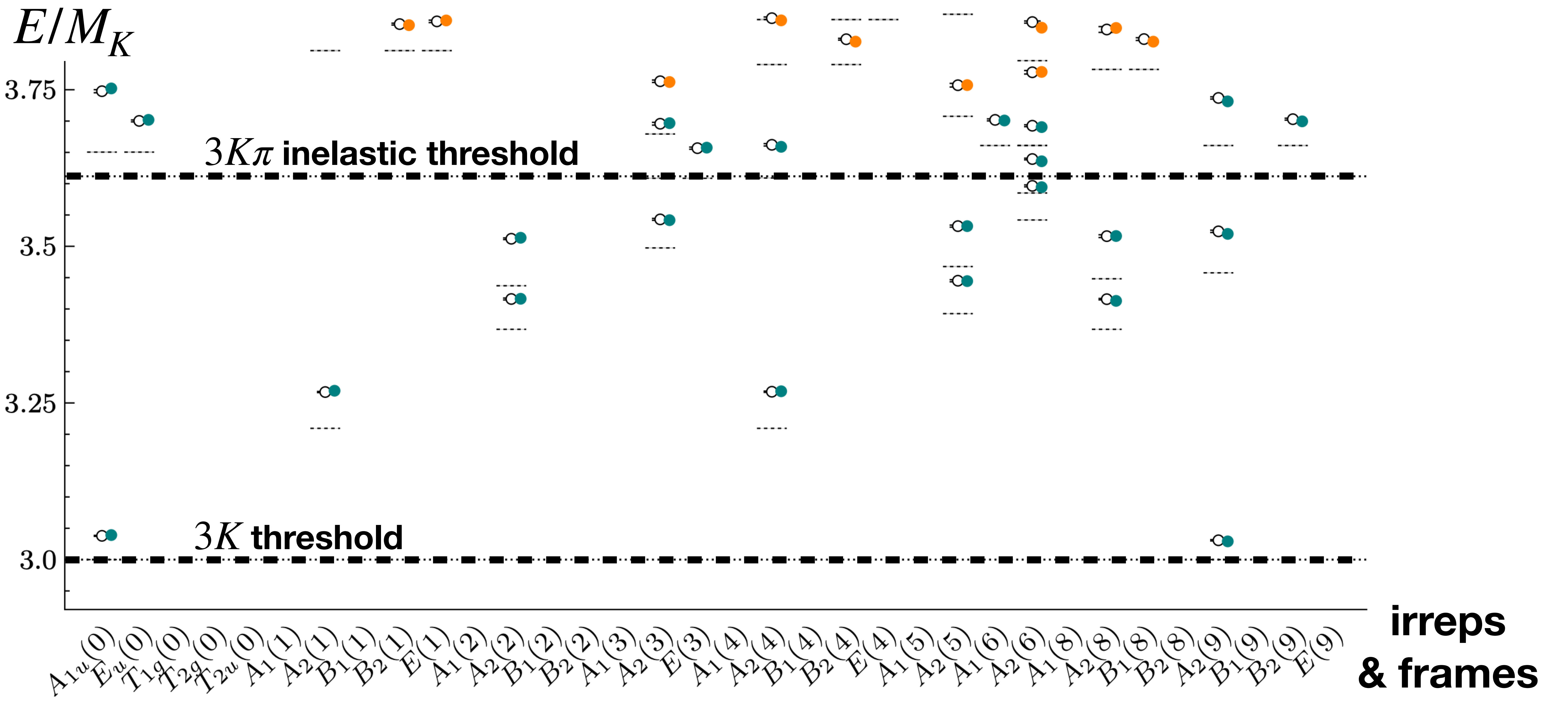

**Fig. 1** Naming convention for the poles in the $k$-plane. The thick red line for positive real valued $k$ marks the physical momenta in the scattering regime

**[Matuschek et al, EPJA 2021]**

# Applying the three-body formalism

- **Three pions and three kaons at maximal isospin have been explored by different groups**
  [Alexandrou et al, PRD 2020], [Blanton et al., PRL 2020 & JHEP 2021], [Brett et al, PRD 2021], [Culver et al, PRD 2021], [Fischer et al, EPJC 2021], [Hansen et al, PRL 2021], [Mai et a PRL 2019 & 2021]
- **Requires large sets of energy levels obtained using variational techniques**
  Using stochastic LapH method [Morningstar et al, PRD 2011]

**$3K^+$ energy levels**




[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

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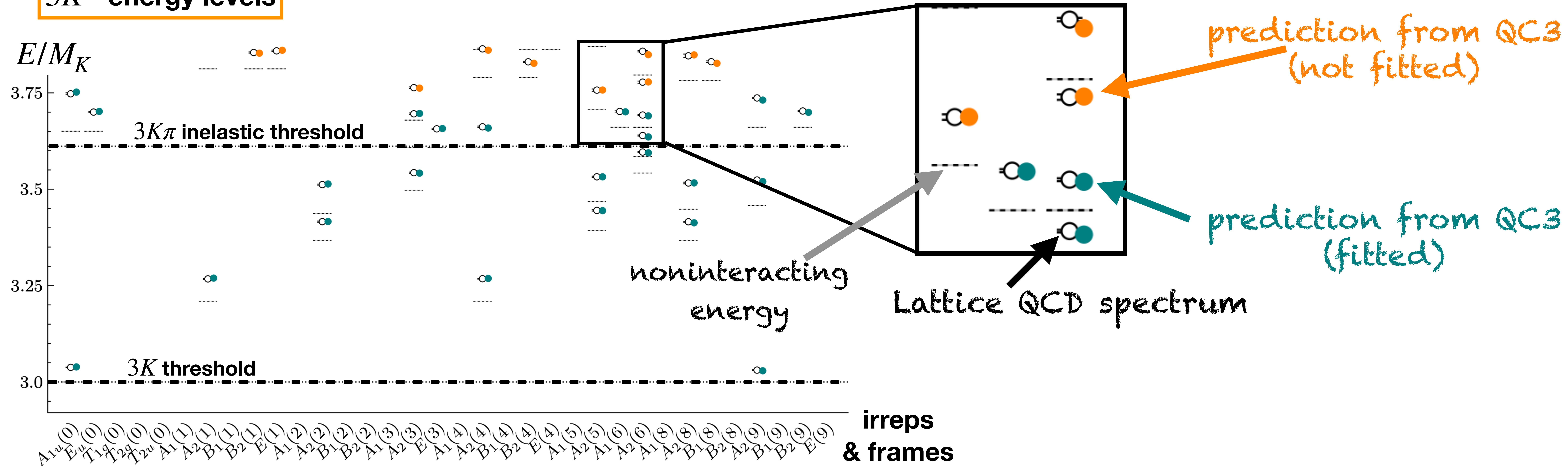


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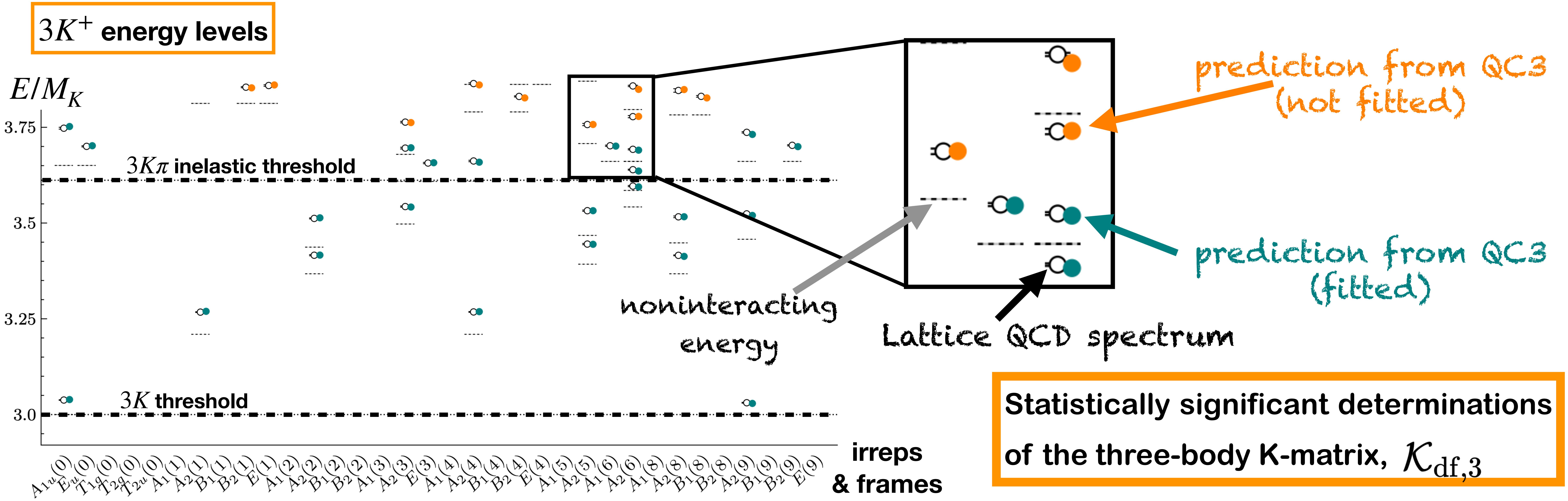


**Statistically significant determinations of the three-body K-matrix, $\mathcal{K}_{\mathrm{df},3}$**

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

# $\mathcal{K}_{df,3}$ from lattice QCD

**Depend of CM energy** | **Angular dependence**

$$\mathcal{K}_{df,3} = \mathcal{K}_{df,3}^{iso,0} + \mathcal{K}_{df,3}^{iso,1}\Delta + \mathcal{K}_{df,3}^{iso,2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B,$$

$$\Delta \equiv \frac{s-9m^2}{9m^2}$$

$$\Delta_B = \sum_{i,j=1}^{3} \widetilde{t}_{ij}^{\,2} - \Delta^2,$$

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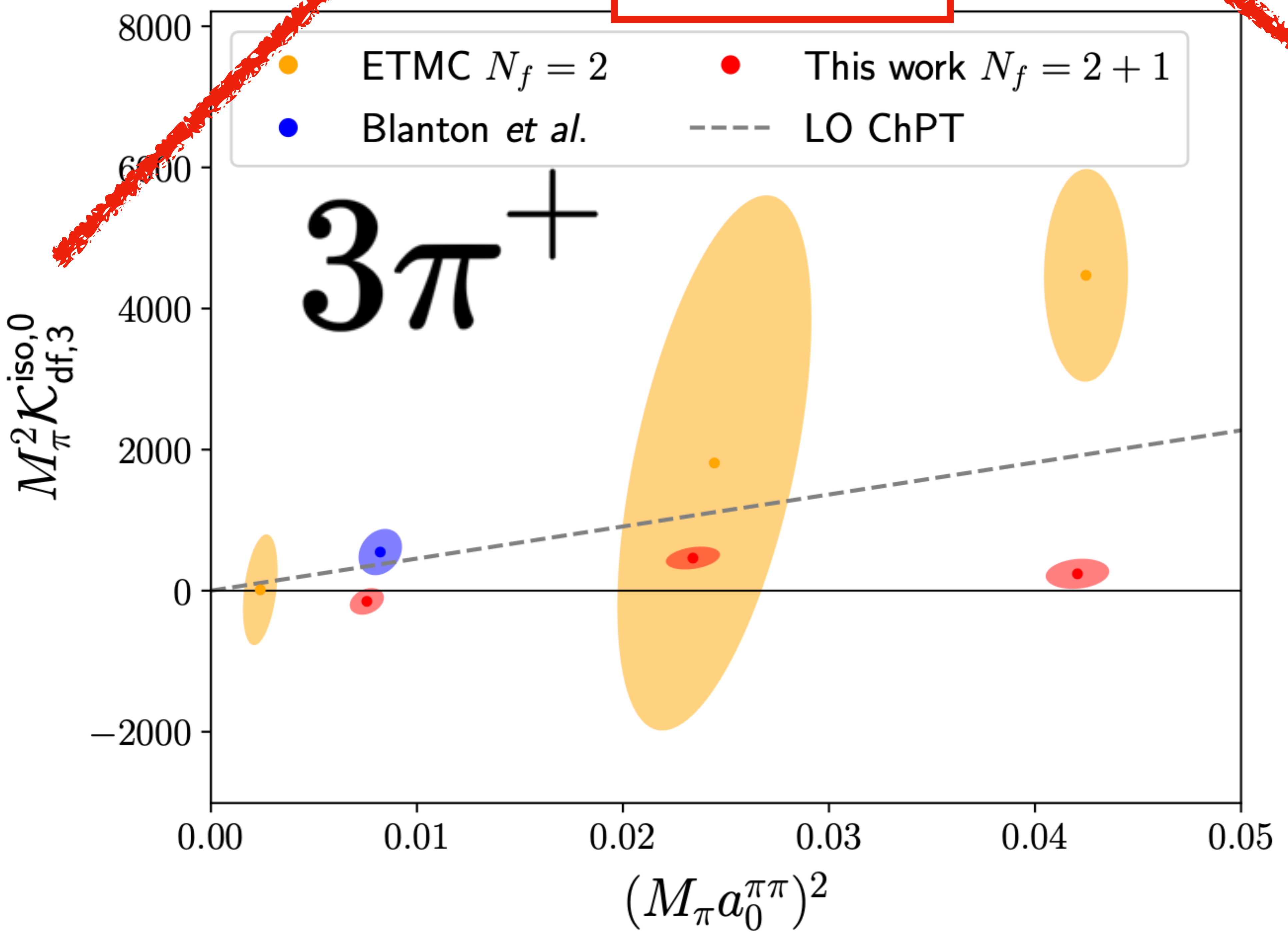


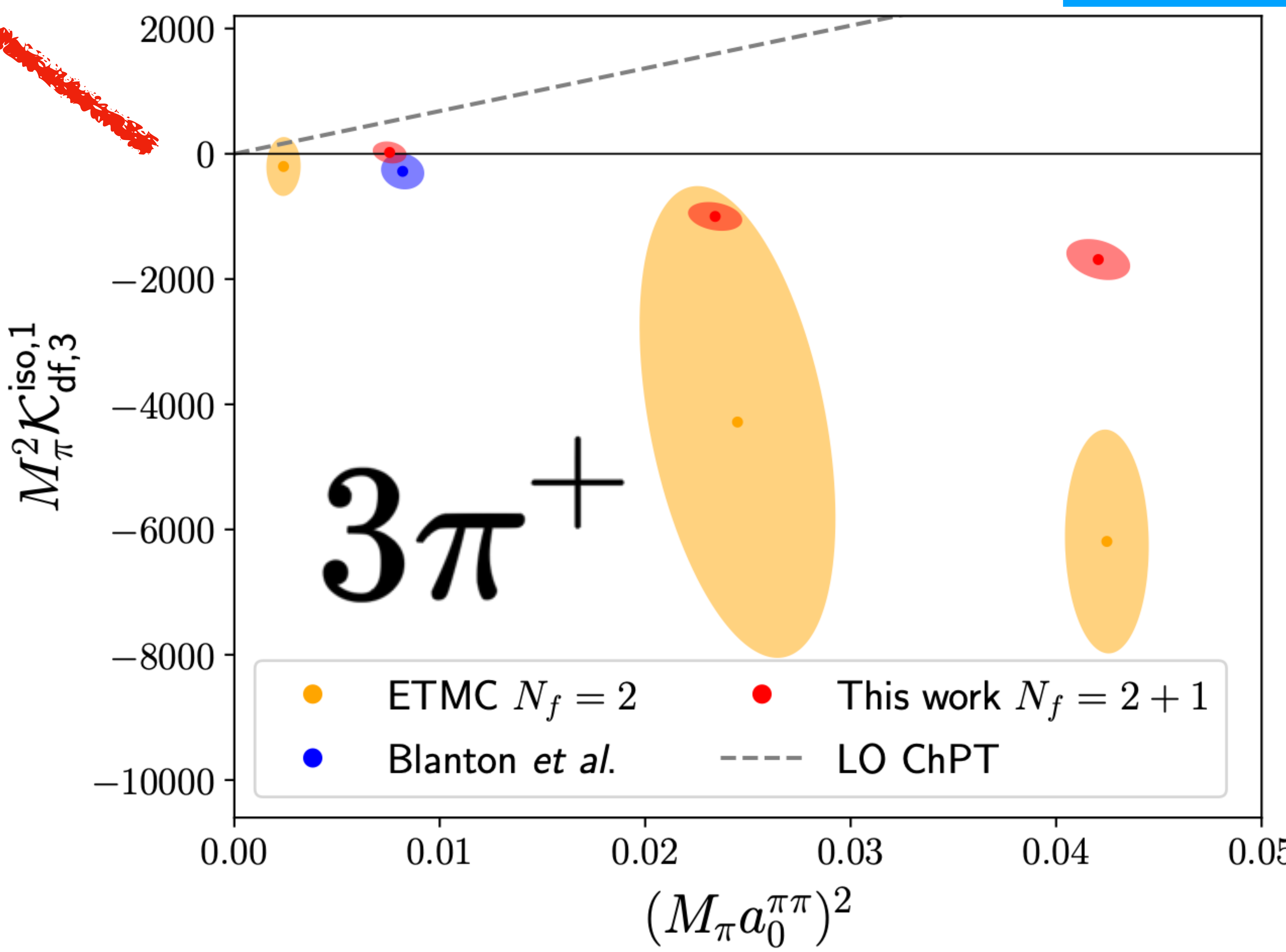




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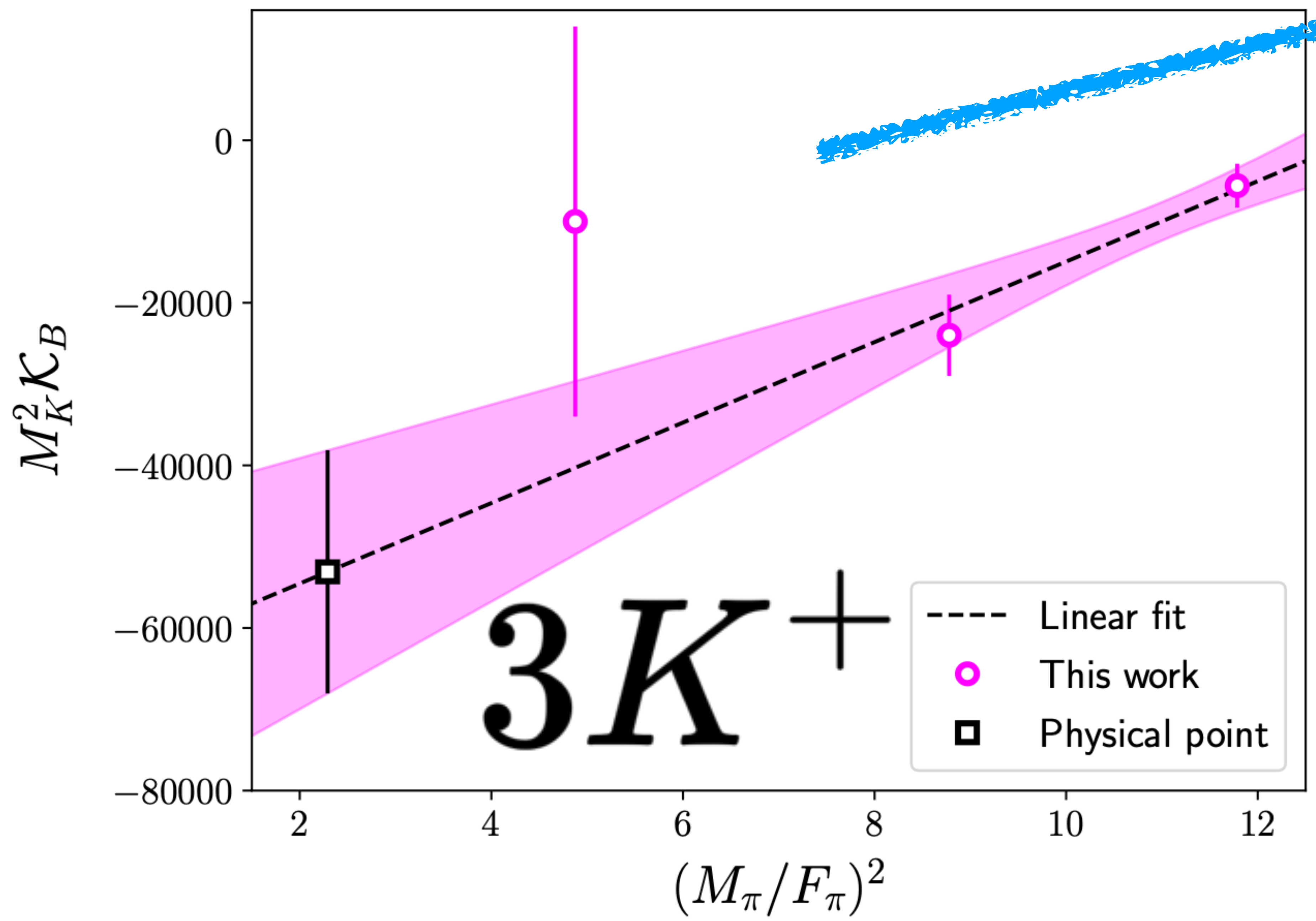


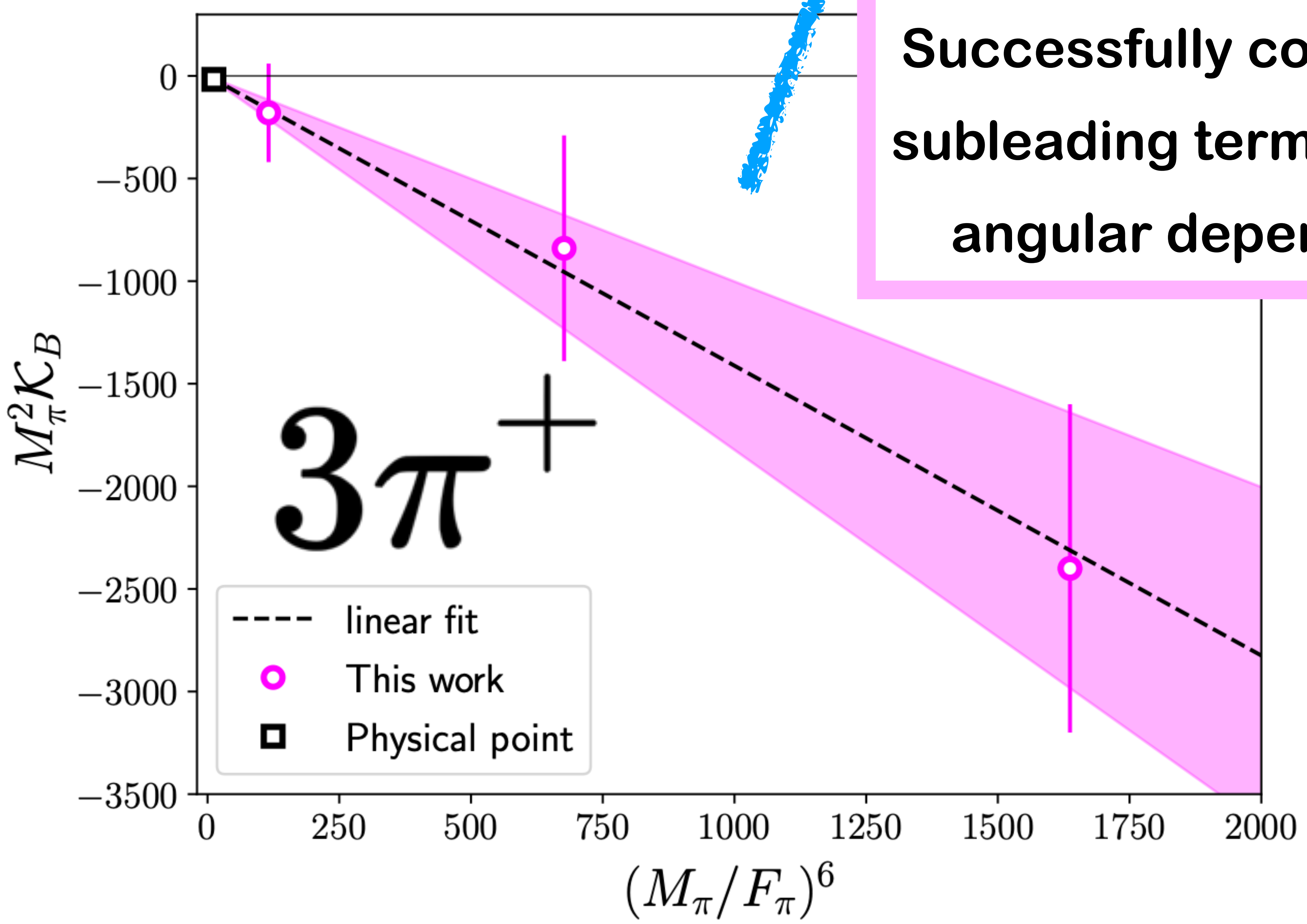


**Successfully constrained subleading term including angular dependence!**



[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

# Nondegenerate systems

- Relevant three-body systems involve nonidentical particles ($\pi\pi N$)
- First step: formalism for three nonidentical scalars

e.g. $\pi^+\pi^0\pi^-$, $K^+K^+\pi^+$, $D_s^+D^0\pi^-$

[Blanton, Sharpe, PRD 2021] x 2, [Hansen, FRL, Sharpe, JHEP 2021], [Mai et al (GWQCD), PRL 2021]

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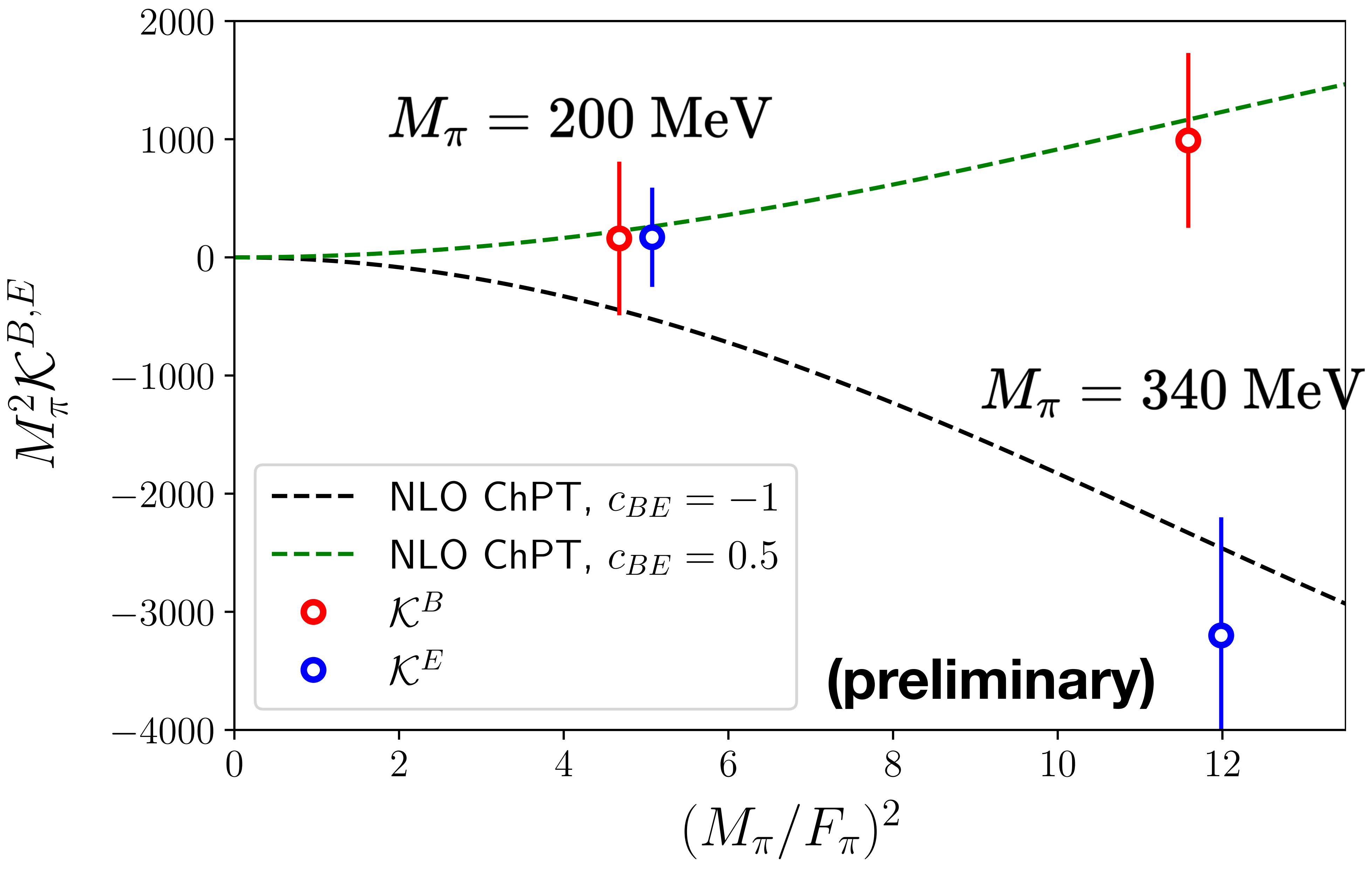
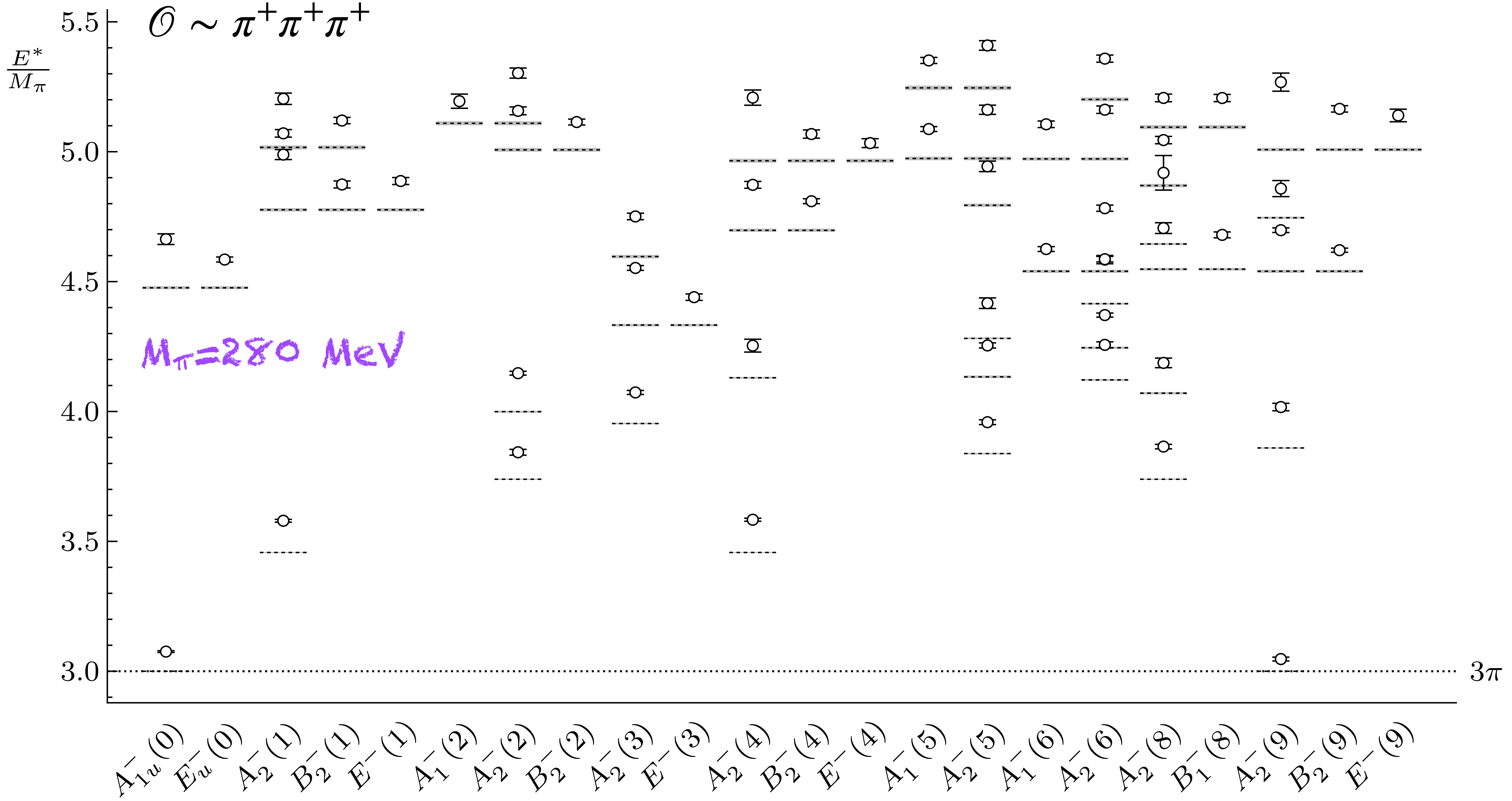
[Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe (in preparation) ]

[Talk by S. Sharpe]



# 3π+ energy levels




[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

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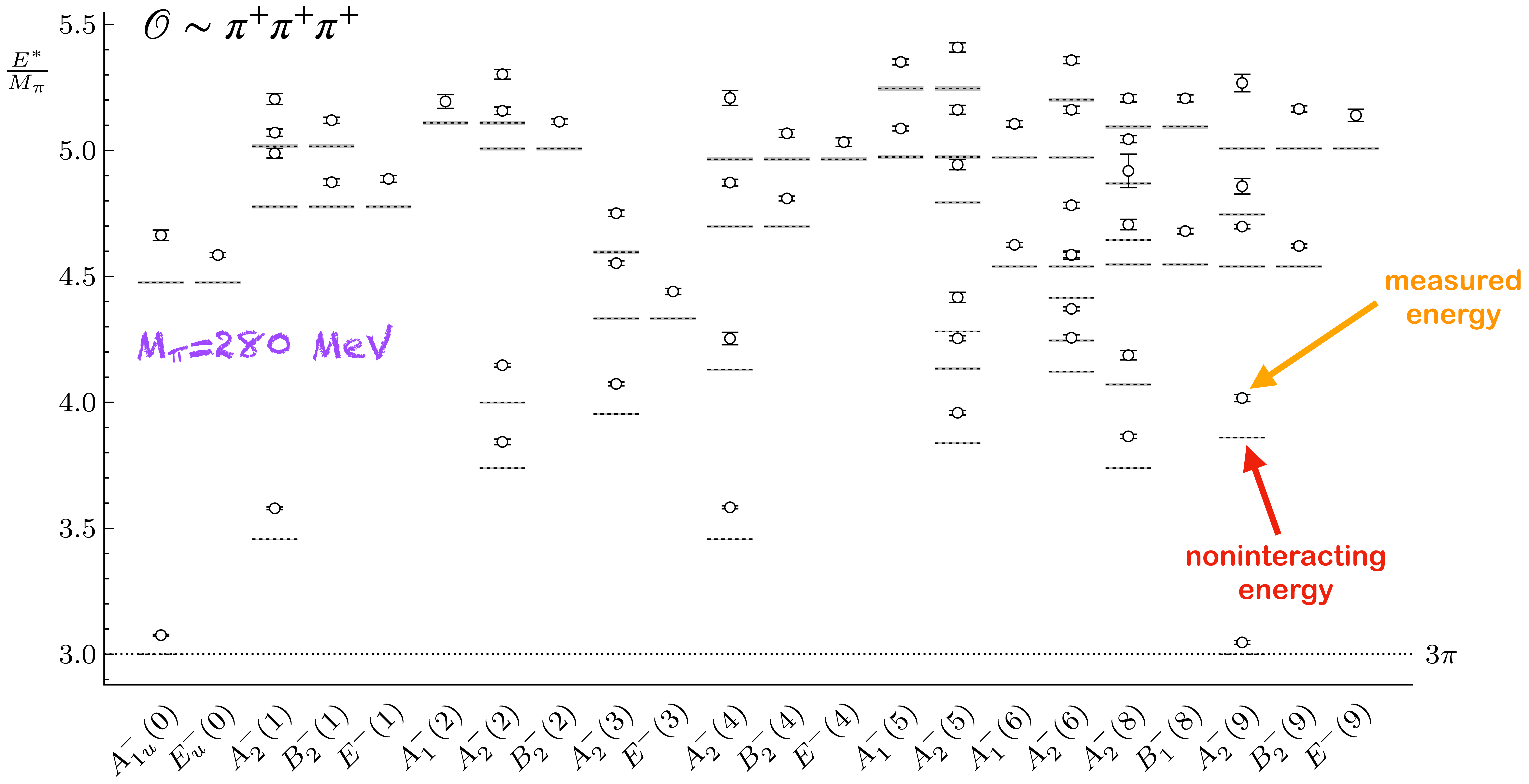


[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

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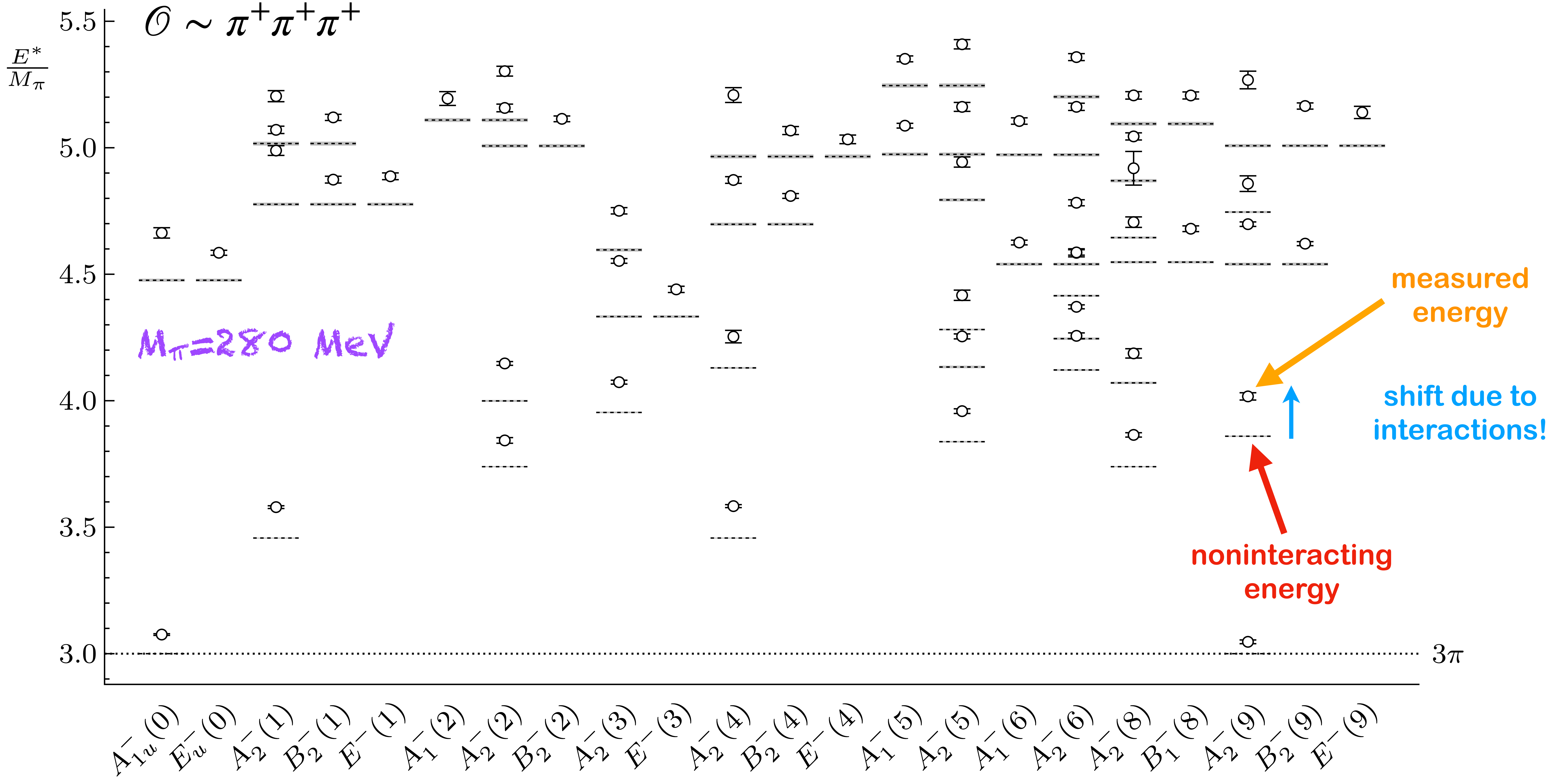


[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

# Integral equations (RFT)

## Final step

Physical 3->3 amplitude

$\mathcal{K}_2, \mathcal{K}_{df,3} \longrightarrow \mathcal{M}_3$

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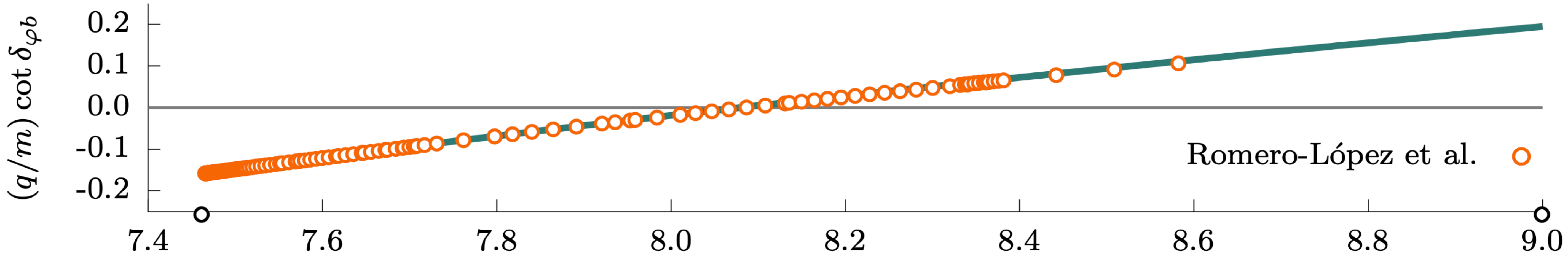
Integral equations

Particle-Dimer phase shift [Jackura et al.]



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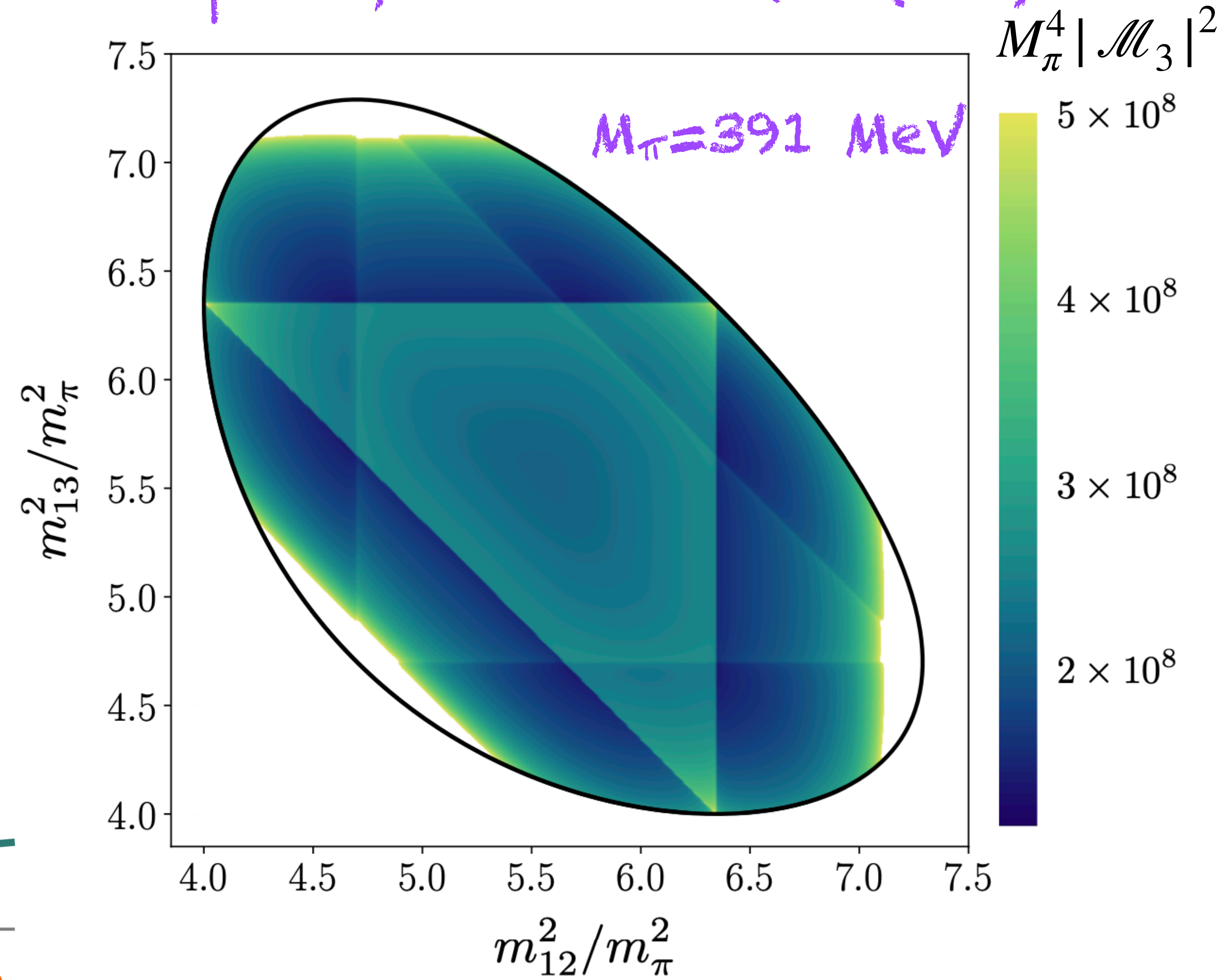
**Final step**

$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow{\text{Integral equations}} \mathcal{M}_3$

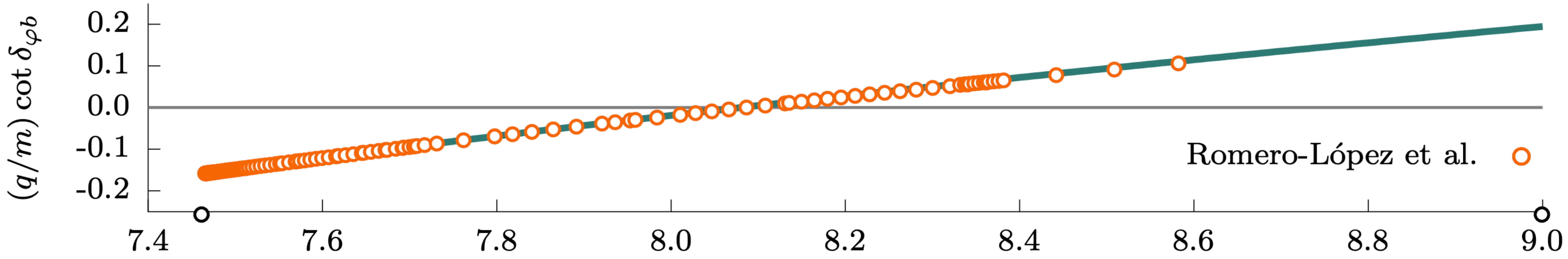
Physical 3->3 amplitude

Dalitz plots from lattice QCD ($3\pi^+$)




[Hansen et al. (HadSpec) ]

Particle-Dimer phase shift [Jackura et al.]



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$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

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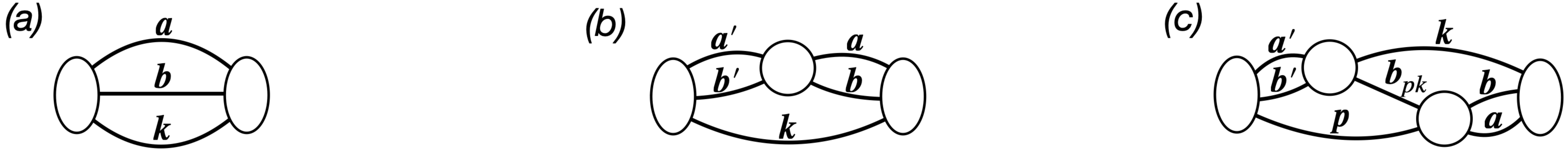


$F$ $\quad$ $\mathcal{K}_2$ $\quad$ $G$

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2-q^2}$$

$$\mathcal{K}_2^\ell = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

$$G_{p00;k00} \equiv \frac{1}{L^3}\frac{1}{2\omega_p}\frac{H(\vec{p})H(\vec{k})}{b_{pk}^2 - m^2}\frac{1}{2\omega_k}$$

# Including isospin

- Relevant three-body systems involve nonidentical particles ($\pi\pi N$)
- Let us consider mass-degenerate pions with different flavor [Hansen, FRL, Sharpe, JHEP 2020]

e.g. $\pi^+\pi^0\pi^-$

- All pions have the same mass
- Overall isospin is conserved
- Presence of resonances
- Example of multi-channel scattering