Progress in lattice QCD at T>0

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- Fluctuations of conserved charges and equation of state at non-zero baryon density
  HotQCD Collaboration, arXiv:2212.09043, PRD 105(2022) 074511,
  PRD 105 (2021 074512; D. Biswas, PP, S. Sharma, work in progress

- Microscopic origin of universal scaling near the chiral transition
  H.T. Ding, W.-P. Huang, S. Mukherjee, PP, arXiv:2305.10916

- Heavy quark diffusion coefficient from lattice QCD
  HotQCD Collaboration, arXiv:2302.08501, to be published in PRL

QCD thermodynamics at non-zero chemical potential

Taylor expansion :

$$\frac{p(T,\mu_B,\mu_Q,\mu_S,\mu_C)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{BQSC} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k \left(\frac{\mu_C}{T}\right)^l$$ hadronic

$$\frac{p(T,\mu_u,\mu_d,\mu_s,\mu_c)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{udsc} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \left(\frac{\mu_c}{T}\right)^l$$ quark

$$\chi_{ijkl}^{abcd} = T^{i+j+k+l} \frac{\partial^i}{\partial \mu_b^i} \frac{\partial^j}{\partial \mu_b^j} \frac{\partial^k}{\partial \mu_c^i} \frac{\partial^l}{\partial \mu_d^l} \ln Z(T,V,\mu_a,\mu_b,\mu_c,\mu_d) \,|_{\mu_a=\mu_b=\mu_c=\mu_d=0}$$

Taylor expansion coefficients give the fluctuations and correlations of conserved charges, e.g.

$$\chi_2^X = \chi_X = \frac{1}{VT^3}(\langle X^2\rangle - \langle X\rangle^2) \qquad \chi_{11}^{XY} = \frac{1}{VT^3}(\langle XY\rangle - \langle X\rangle\langle Y\rangle)$$

↓

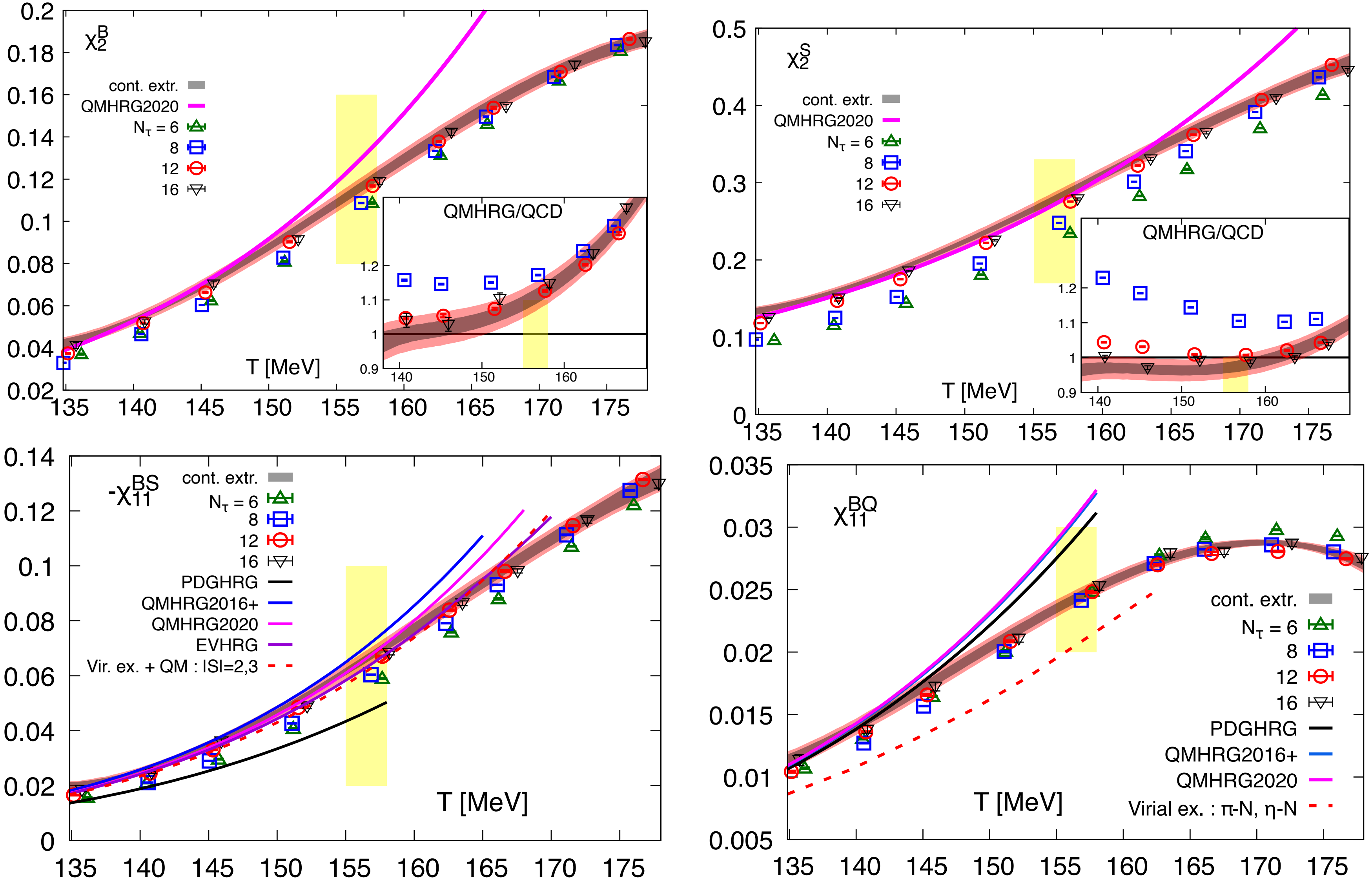
information about carriers of the conserved charges (hadrons or quarks)

↓

probes of deconfinement

Second order Taylor expansion coefficients and HRG

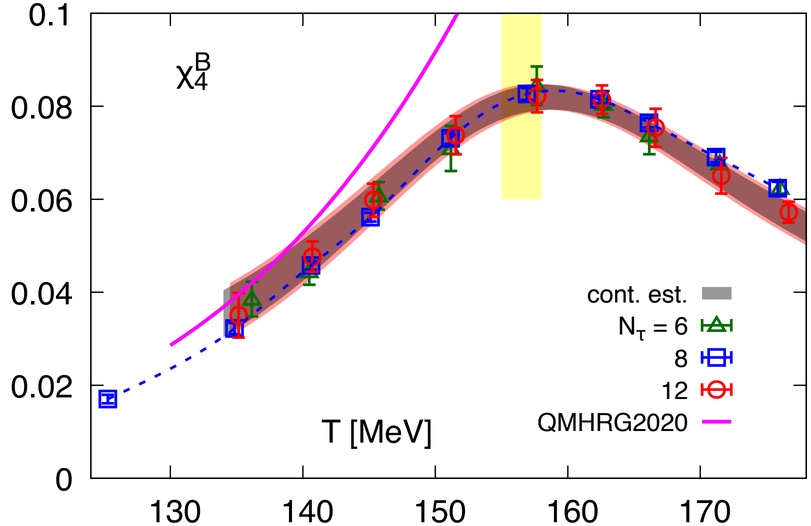
HISQ, m_π^{phys}, $a = 1/(TN_\tau)$

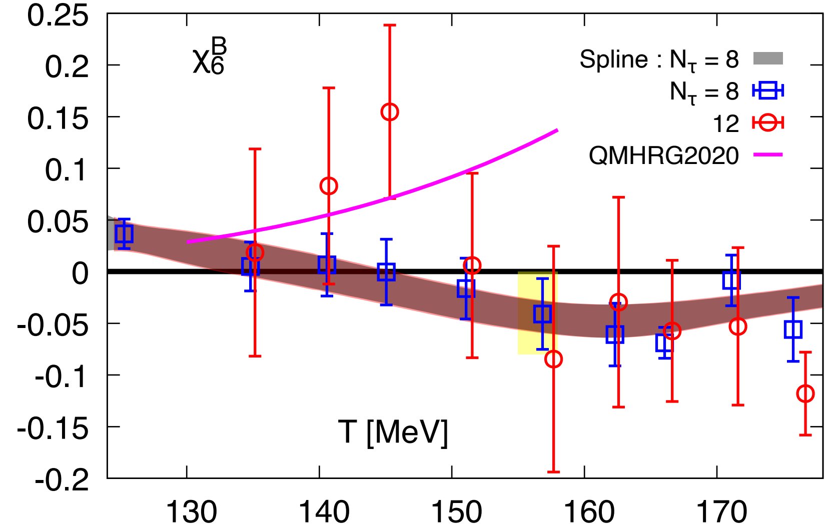


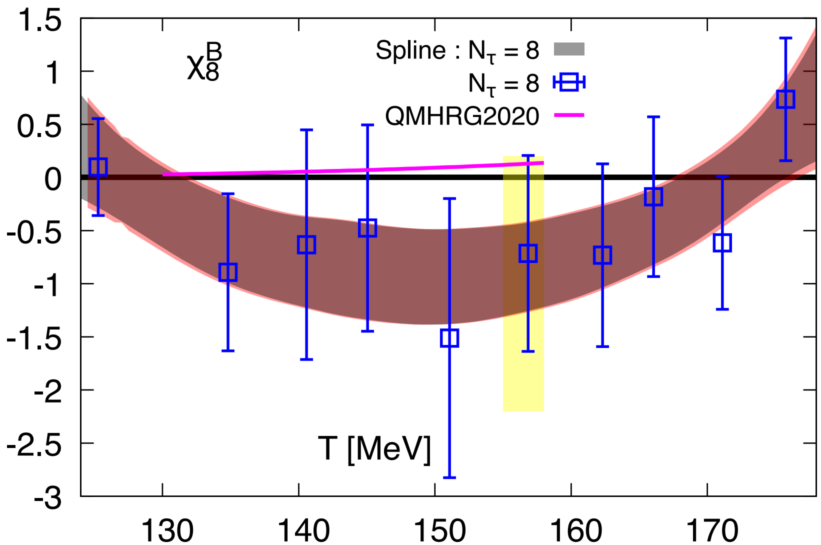
HRG works up to temperatures ≃145-150 MeV

Higher order Taylor expansion coefficients and HRG

HISQ, m_π^{phys}, $a = 1/(TN_\tau)$





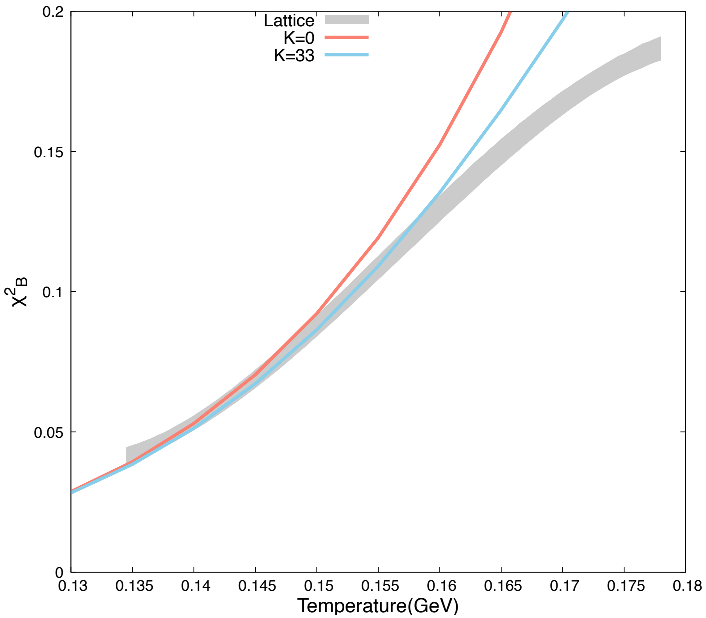


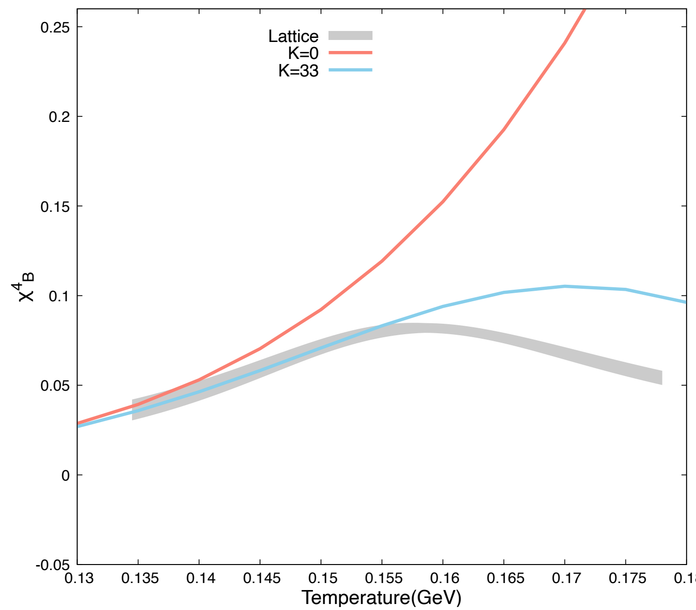
For 4[th] order expansion coefficient HRG may work only for $T<140$ MeV

For 6[th] and 8[th] order expansion coefficients turn negative around T_c HRG, only works for $T<135$ MeV

Possibly no singularity for real values of baryon chemical potential.

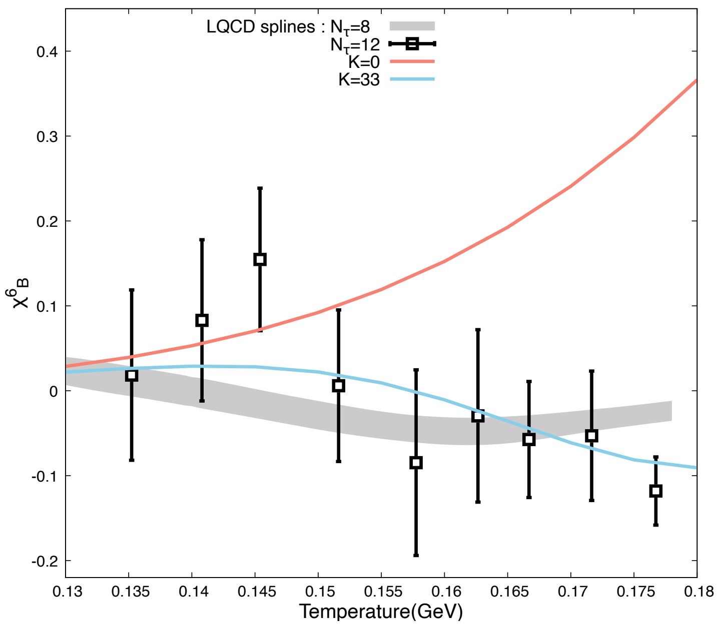
HRG with repulsive mean field

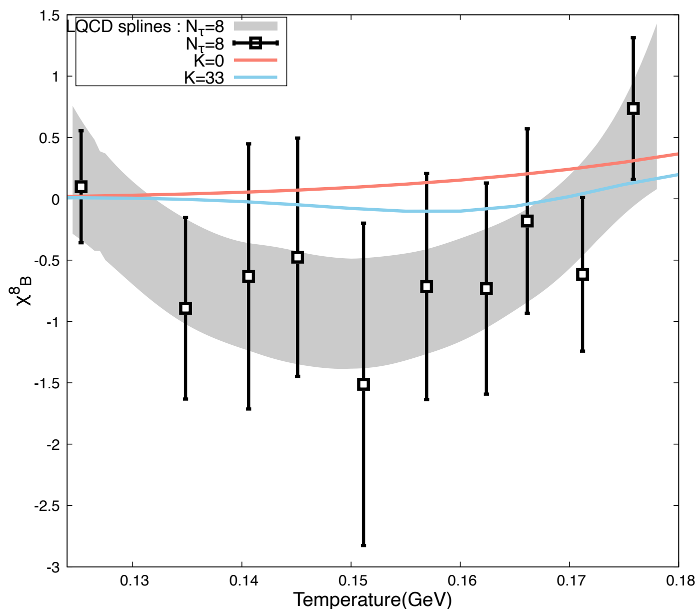




D. Biswas, PP,
S. Sharma,
work in progress

Improved agreement between lattice and HRG.





Pade approximation and radius of convergence

$$\Delta P(T,\mu_B) = P(T,\mu_B) - P(T,0) = \sum_{k=1}^{\infty} P_{2k}\mu_B^{2k} \qquad \bar{x} = (\mu_B/T)\cdot\sqrt{P_4/P_2}$$

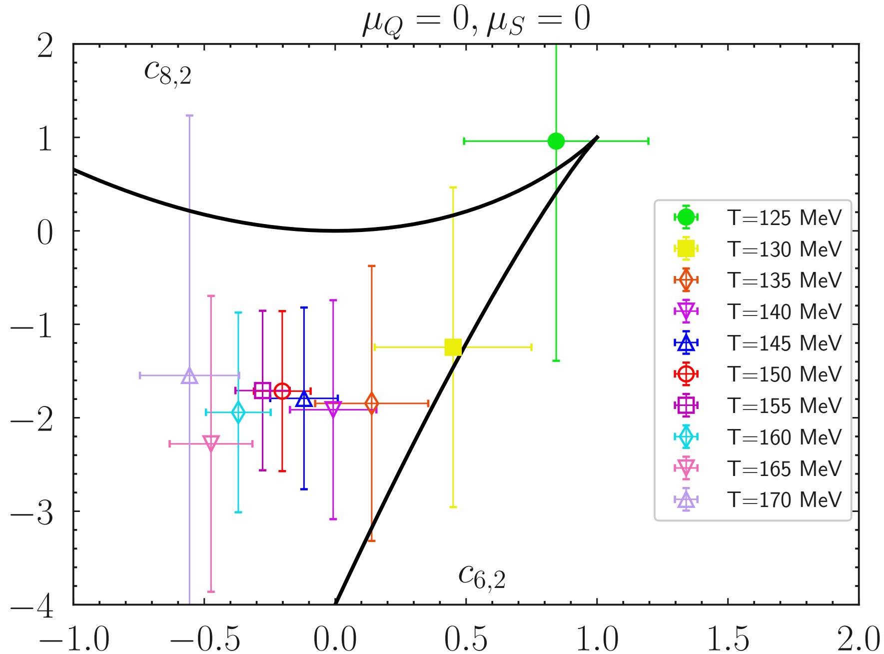
$$\frac{\Delta P(T,\mu_B)}{T^4} = \frac{P_2^2}{P_4}\sum_{k=1}^{\infty} c_{2k,2}\bar{x}^{2k} = \frac{P_2^2}{P_4}(\bar{x}^2 + \bar{x}^4 + c_{6,2}\bar{x}^6 + c_{8,2}\bar{x}^8 + \ldots) \rightarrow \frac{P_2^2}{P_4}P_{[4,4]}$$

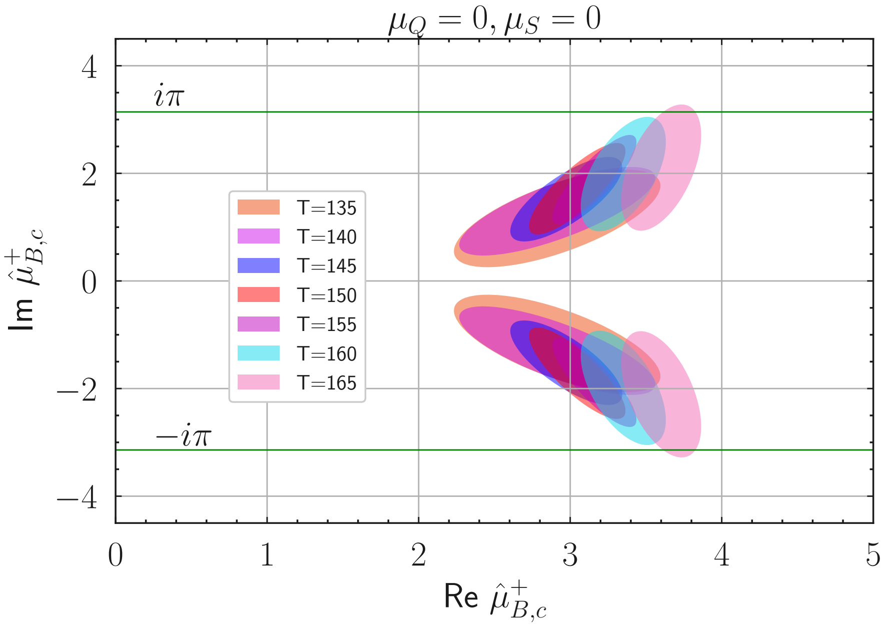
$$c_{6,2} = \frac{P_6 P_2}{P_4^2} = \frac{2}{5}\frac{\chi_6^B\chi_2^B}{(\chi_4^B)^2}\ ,$$

$$c_{8,2} = \frac{P_8 P_2^2}{P_4^3} = \frac{3}{35}\frac{\chi_8^B(\chi_2^B)^2}{(\chi_4^B)^3}\ .$$

$$P_{[4,4]} = \frac{(1-c_{6,2})\bar{x}^2 + (1-2c_{6,2}+c_{8,2})\,\bar{x}^4}{(1-c_{6,2}) + (c_{8,2}-c_{6,2})\bar{x}^2 + (c_{6,2}^2 - c_{8,2})\bar{x}^4}.$$

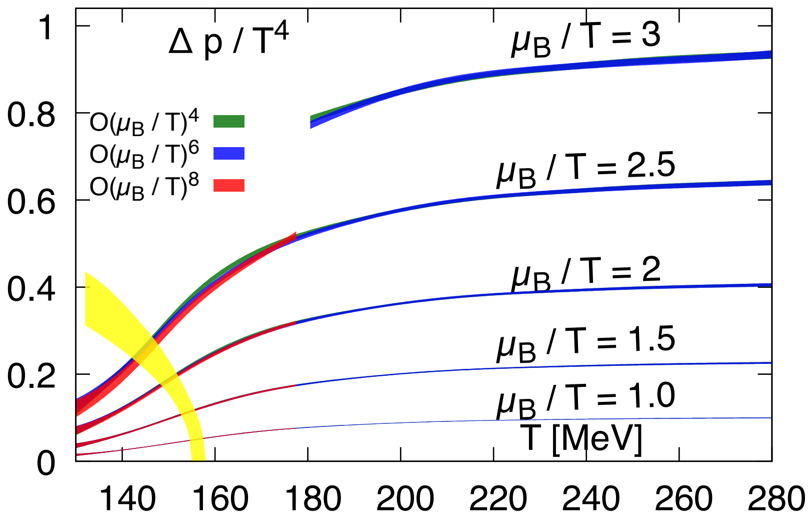
Padé poles: Mercer-Roberts estimators of radius of convergence

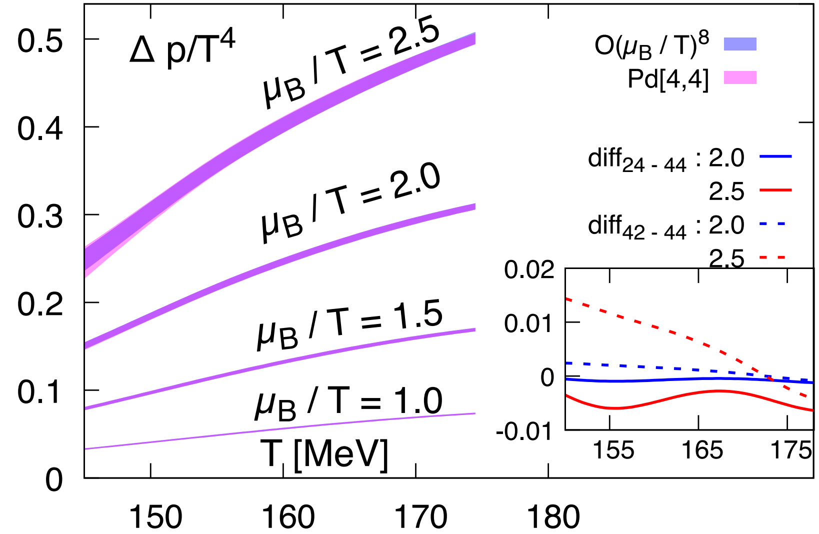


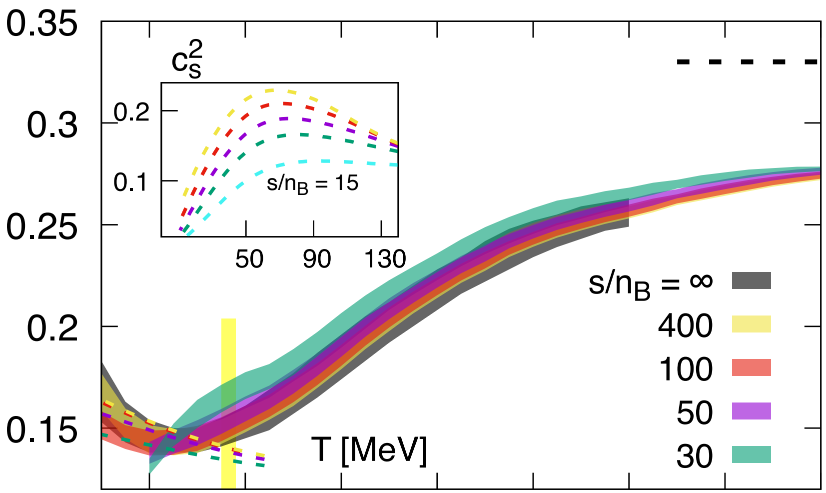


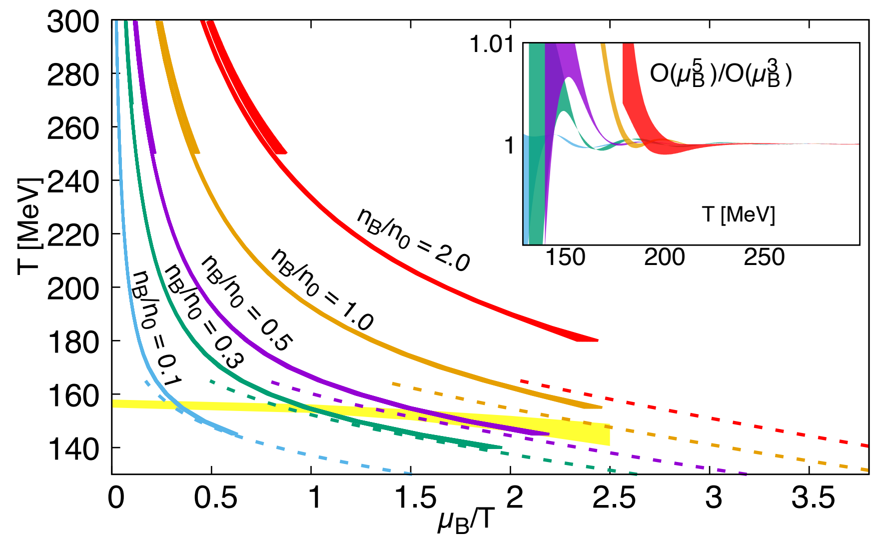
For 135 MeV $< T <$ 165 MeV only complex poles for $|\mu_B/T| > 2.5$

Equation of State at non-zero baryon density









$$\epsilon(T_{\rm pc}(\mu_B)) = \begin{cases} 370(40)(30)\ \mathrm{MeV/fm^3}\ ,\ \mu_B/T = 0 \\ 330(28)(53)\ \mathrm{MeV/fm^3}\ ,\ \mu_B/T = 2.5 \end{cases}$$

Chiral transition and spectrum of Dirac eigenvalues

Macroscopic

Microscopic

$$\bar{\psi}\psi(m) \equiv 2\mathrm{Tr}(\not{D}[\mathcal{U}] + m)^{-1} = 2\sum_j (i\lambda_j + m)^{-1}$$

$$P_{\mathcal{U}}(\lambda; m) = \frac{4m\rho_{\mathcal{U}}(\lambda)}{\lambda^2 + m^2}, \text{ and } \rho_{\mathcal{U}}(\lambda) = \sum_j \delta(\lambda - \lambda_j)$$

$$\mathbb{K}_1(\bar{\psi}\psi) = \frac{T}{V}\langle(\bar{\psi}\psi)\rangle = \int_0^\infty K_1[P_{\mathcal{U}}(\lambda; m_l)]\ \mathrm{d}\lambda = \frac{T}{V}\int_0^\infty \mathrm{d}\lambda \frac{4m_l \langle\rho_{\mathcal{U}}(\lambda)\rangle}{\lambda^2 + m_l^2} = \int_0^\infty P_1(\lambda) d\lambda$$

$$\mathbb{K}_2(\bar{\psi}\psi) = \frac{T}{V}\langle[(\bar{\psi}\psi) - \langle\bar{\psi}\psi\rangle]^2\rangle = \int_0^\infty K_1[P_{\mathcal{U}}(\lambda_1; m_l), P_{\mathcal{U}}(\lambda_2; m_l)]\ \mathrm{d}\lambda_1 \mathrm{d}\lambda_2$$

Chiral susceptibility

$$= \frac{T}{V}\int_0^\infty \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \frac{(4m_l)^2}{(\lambda_1^2 + m_l^2)(\lambda_2^2 + m_l^2)} \times$$

$$[\langle\rho_{\mathcal{U}}(\lambda_1)\rho_{\mathcal{U}}(\lambda_2)\rangle - \langle\rho_{\mathcal{U}}(\lambda_1)\rangle\langle\rho_{\mathcal{U}}(\lambda_2)\rangle] = \int_0^\infty P_2(\lambda) d\lambda$$

$$\mathbb{K}_3(\bar{\psi}\psi) = \frac{T}{V}\langle[(\bar{\psi}\psi) - \langle\bar{\psi}\psi\rangle]^3\rangle = \int_0^\infty P_3(\lambda) d\lambda$$

Binder cumulant

Approaching the chiral limit

$$m_l \to 0: \ \frac{m}{\lambda^2+m^2} \to \pi\delta(\lambda)$$

$$\lim_{m_l \to 0} \mathbb{K}_1(\bar{\psi}\psi) = \lim_{m_l \to 0} \frac{T}{V}\langle(\bar{\psi}\psi)\rangle = 2\pi\mathbb{K}_1[\rho_{\mathcal{U}}(0)] = 2\pi\langle\rho_{\mathcal{U}}(0)\rangle$$

Banks-Casher relation

$$\lim_{m_l \to 0} \mathbb{K}_n(\bar{\psi}\psi) = \lim_{m_l \to 0} = (2\pi)^n\mathbb{K}_n[\rho_{\mathcal{U}}(0)]$$

Universal *O(N)* scaling of the cumulants of the chiral condensate:

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)d\lambda \sim m_l^{1/\delta-n+1} f_n(z)$$

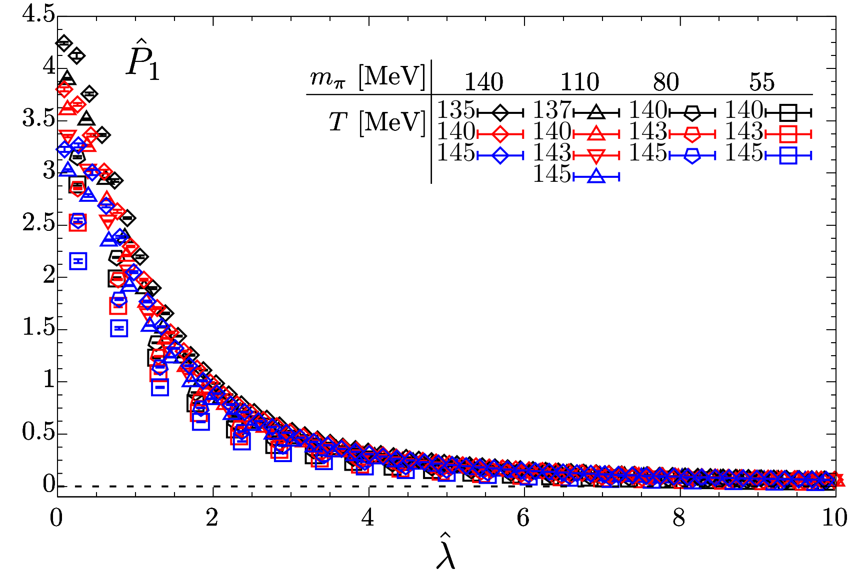
$$z \propto z_0 m_l^{-1/\beta\delta}(T-T_c)/T_c$$

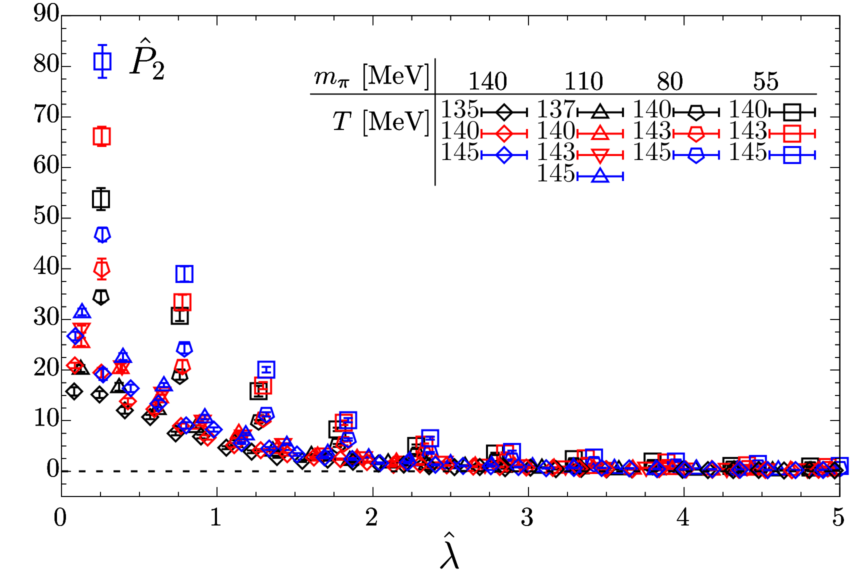
O(N) scaling function

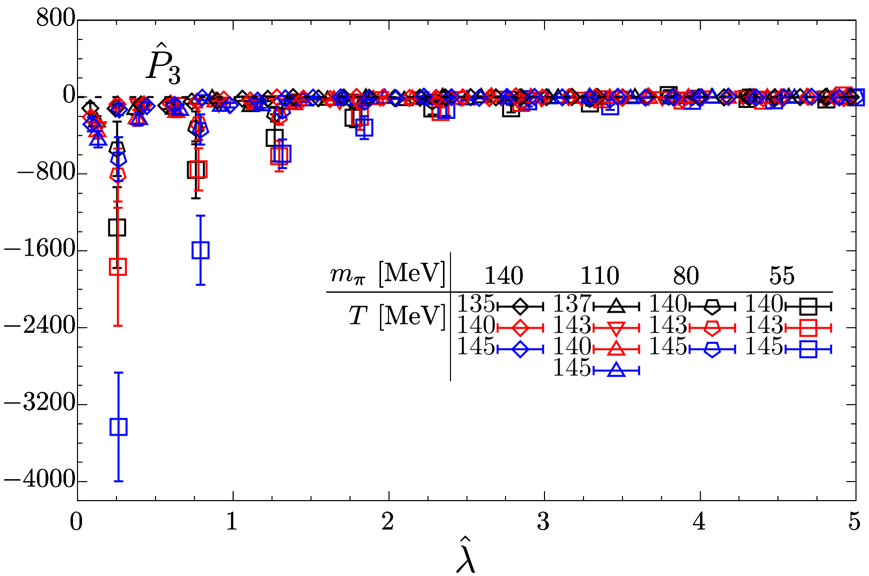
Conjecture: $P_n(\lambda) = m_l^{1/\delta-n+1} f_n(z) g_n(\lambda)$

non-universal scaling function

Chiral observables and spectrum of Dirac eigenvalues







HISQ, m_π^{phys}, $N_\tau = 8$, $N_\sigma = 32 - 56$

$m_l/m_s = 1/27,\ 1/40,\ 1/80,\ 1/160$

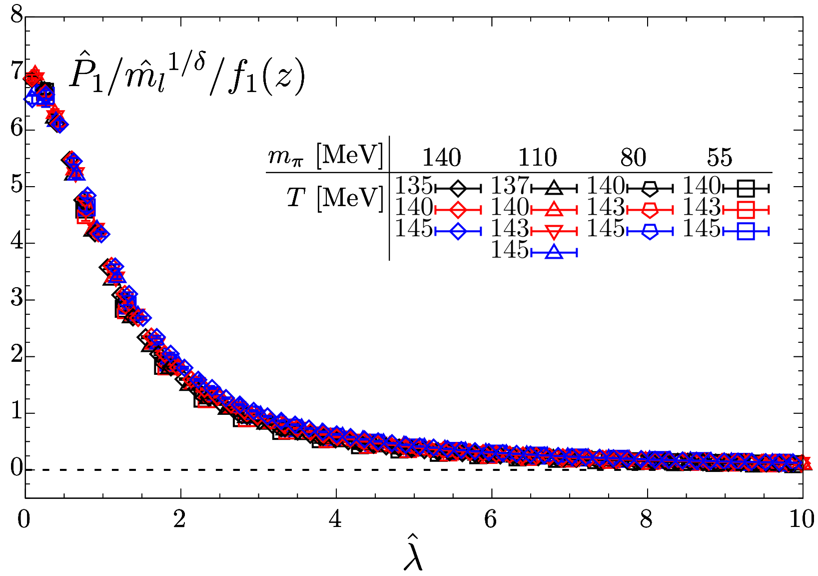
$m_\pi = 140,\ 110,\ 80,\ 55$ MeV

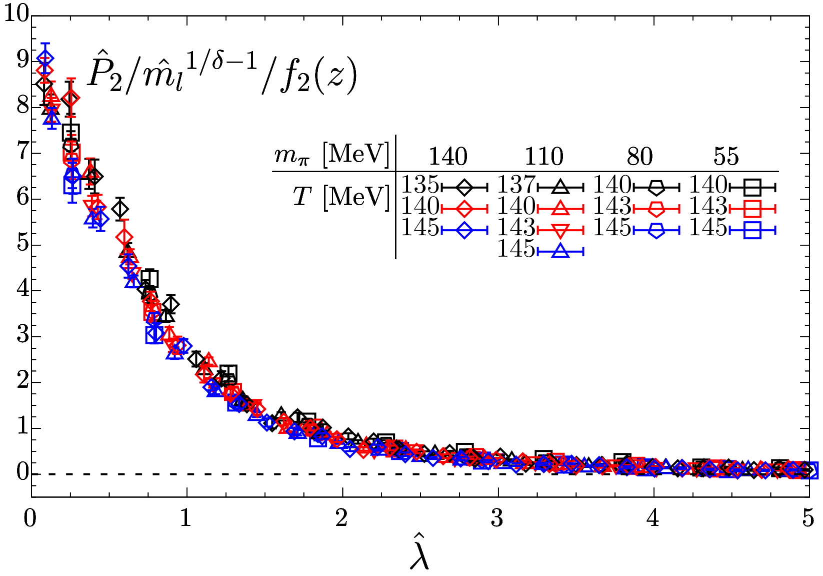
$T = 135 - 176$ MeV

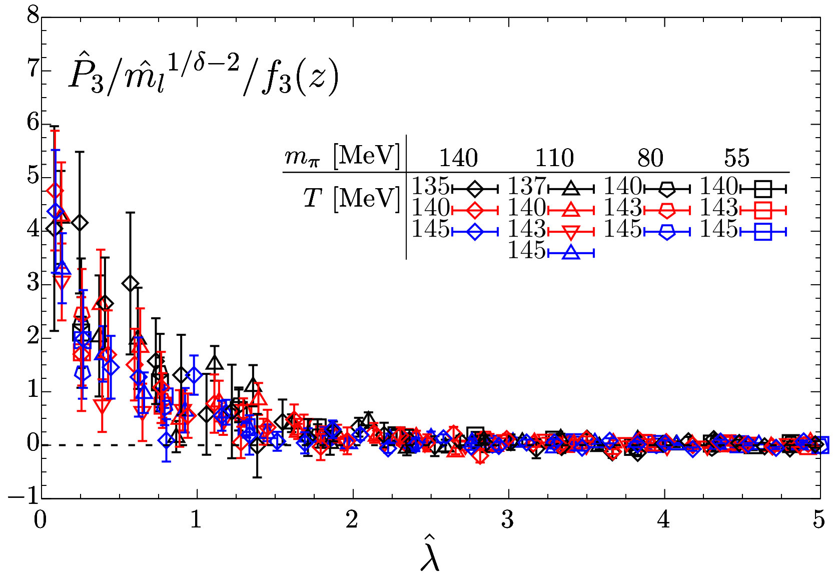
$$\hat{\lambda} = \lambda/m_l$$

Strong quark mass and temperature dependence

Chiral observables and spectrum of Dirac eigenvalues







Scaling works for $T<T_c \simeq 144$ MeV !

Dirac eigenvalues (energy levels of quarks) know about universality class of the QCD chiral transition

Heavy quark diffusion and lattice QCD

Obtain the momentum heavy quark transport coefficient through the force correlator

$$\langle f_i(t) f_j(t)\rangle = \langle E_i(t) E_j(t')\rangle + \frac{1}{3}\langle \mathbf{v}^2\rangle \langle \delta_{ij} B_k(t) B_k(t') - B_i(t') B_j(t)\rangle \qquad \langle \mathbf{v}^2\rangle = \frac{3T}{M}$$

$$t \to i\tau$$

Can be rigorously derived in Heavy Quark Effective Theory

Casalderrey-Solana, Teaney, PRD 74 (2006) 085012; Caron-Huot, Laine, Moore, JHEP 0904 ('09) 053

Bouttefeux, Laine, JHEP 12 (2020) 150

$$G_E(\tau) = -\frac{1}{3}\sum_{i=1}^{3} \frac{\left\langle \mathrm{ReTr}\left[U(\beta,\tau)\, gE_i(\tau,\vec{0})\, U(\tau,0)\, gE_i(0,\vec{0})\right]\right\rangle}{\left\langle \mathrm{ReTr}[U(\beta,0)]\right\rangle} \qquad \kappa_E = \lim_{\omega\to 0}\frac{2T}{\omega}\rho_E(\omega)$$

$$G_B(\tau) = -\frac{1}{3}\sum_{i=1}^{3} \frac{\left\langle \mathrm{ReTr}\left[U(\beta,\tau)\, gB_i(\tau,\vec{0})\, U(\tau,0)\, gB_i(0,\vec{0})\right]\right\rangle}{\left\langle \mathrm{ReTr}[U(\beta,0)]\right\rangle} \qquad \kappa_B = \lim_{\omega\to 0}\frac{2T}{\omega}\rho_B(\omega)$$

$$G_{E,B}(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\rho_{E,B}(\omega)\frac{\cosh\left(\tau - \frac{1}{2T}\right)\omega}{\sinh\frac{\omega}{2T}}$$

$$\kappa = \kappa_E + \frac{2}{3}\langle \mathbf{v}^2\rangle \kappa_B$$

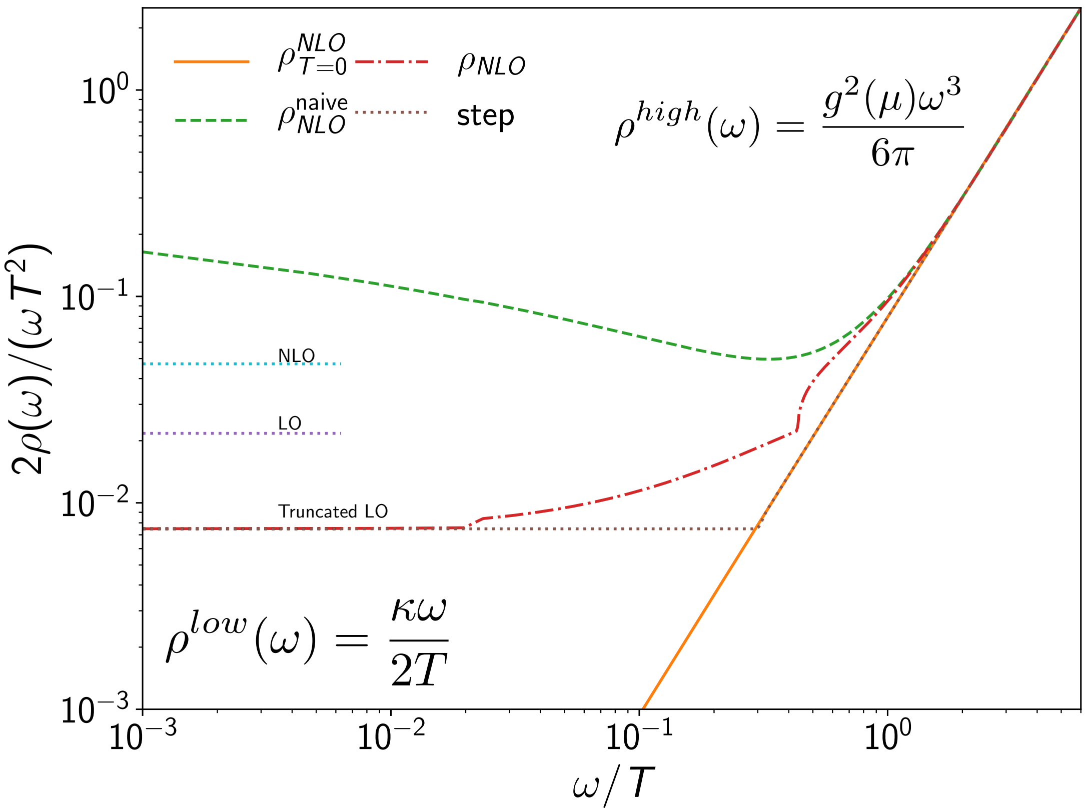
Extracting momentum diffusion coefficient from the lattice

Challenge 1: obtain precise results for chromo-electric and chromo-magnetic (very noisy)

⇒ Noise reduction via multi-level algorithm, applicable to quenched QCD (pure glue plasma)

⇒ Noise reduction by gradient flow method (new development !) , also applicable in full QCD

Challenge 2: reconstruct the spectral function from the Euclidean time lattice correlator



⇒ use known large and small energy behavior of the spectral

Parameterize $\rho(\omega, T)$ as smooth interpolation between $\rho^{low}(\omega, T)$ and $\rho^{high}(\omega)$, and treat κ as well as the additional nuisance parameters of interpolation as fit parameters

Extracting momentum diffusion coefficient from the lattice

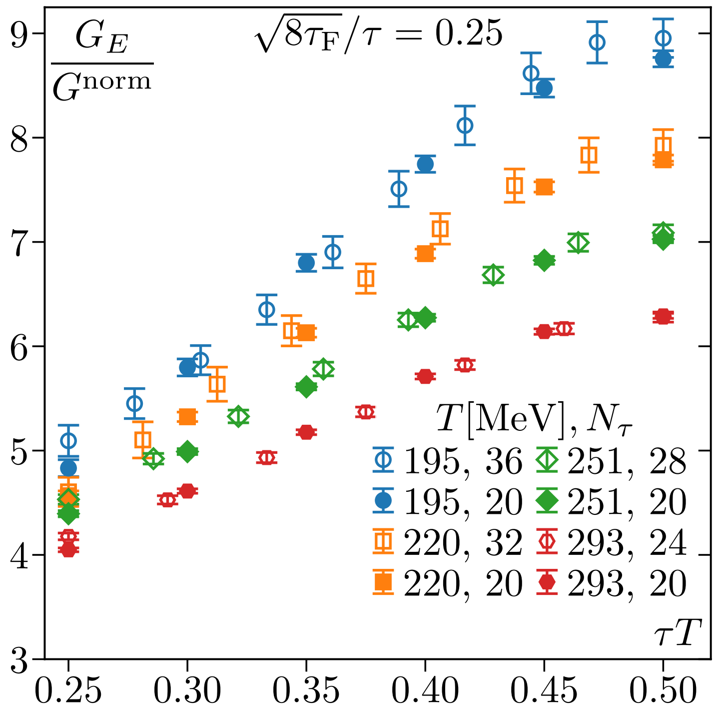
2+1 flavor QCD with $m_l = m_s/5$ ($m_\pi = 320$ MeV), $T = 195 - 354$ MeV, $96^3 \times N_\tau$ lattices with $N_\tau = 36,\ 32,\ 28,\ 24,\ 20$; additional $64^3 \times N_\tau$ lattices with $N_\tau = 20,\ 22,\ 24 \Rightarrow 3$ lattice spacings at each T; Gradient flow for noise reduction

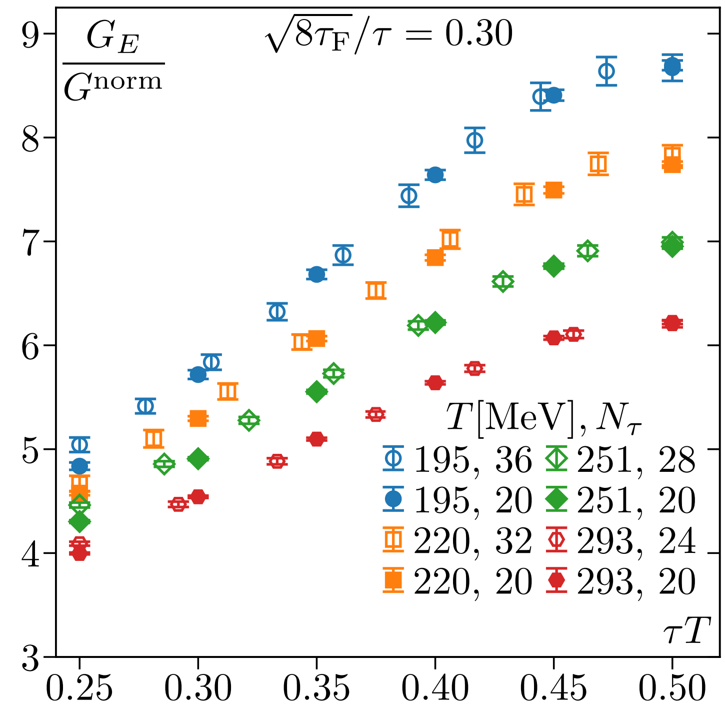
$$A_\mu(x) \to B_\mu(\tau_F, x) \qquad \partial_{\tau_F} B_\mu(\tau_F, x) = -g_0^2 \frac{\delta S_{\rm YM}[B]}{\delta B_\mu(\tau_F, x)}$$
$$B_\mu(0, x) = A_\mu(x)$$

Symanzik gauge action and Zeuthen flow

Gauge fields are smeared in the radius $\sqrt{8\tau_F}$

$$a < \sqrt{8\tau_F} < \tau/3$$





We see small cutoff effects thanks improved actions

Analysis and modeling the chromo-electric correlator

Analysis of the chromo-electric correlator:

- Extrapolate the lattice results on the chromo-electric correlator to the continuum limit
- Perform the zero flow time extrapolation

Fits to model spectral function:

$$\rho^{low}(\omega,T)=\frac{\kappa\omega}{2T} \qquad \rho^{high}(\omega)=\rho^{LO,NLO}(\omega)$$

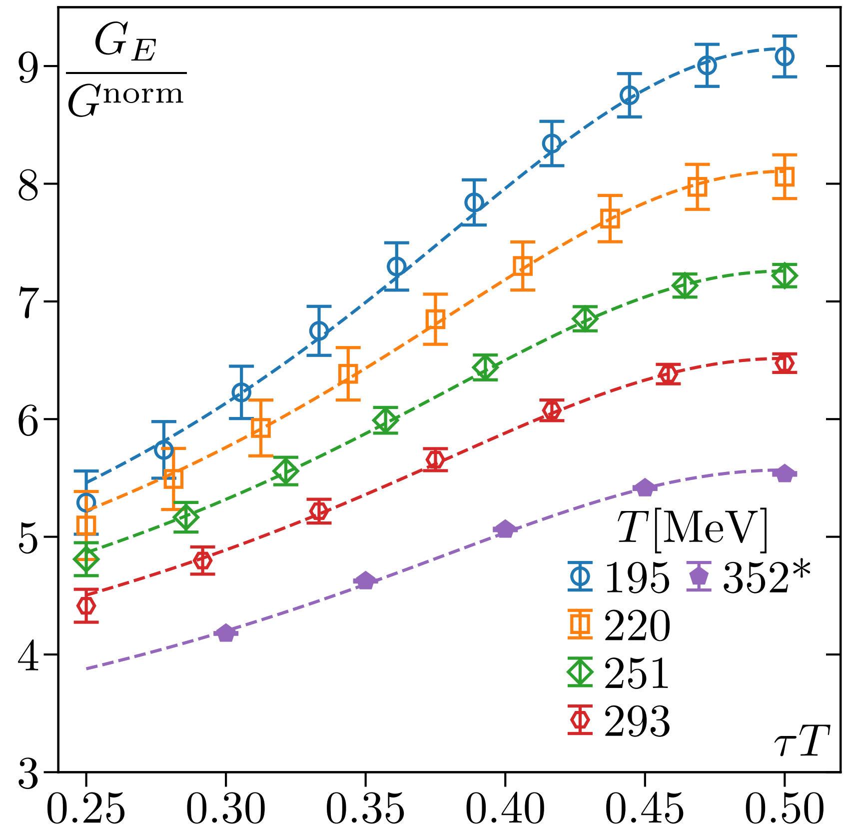
$$\rho^{max}(\omega,T)=max(\rho^{low}(\omega,T),\rho^{high}(\omega))$$

$$\rho^{smax}(\omega,T)=\sqrt{(\rho^{low})^2+(\rho^{high})^2}$$

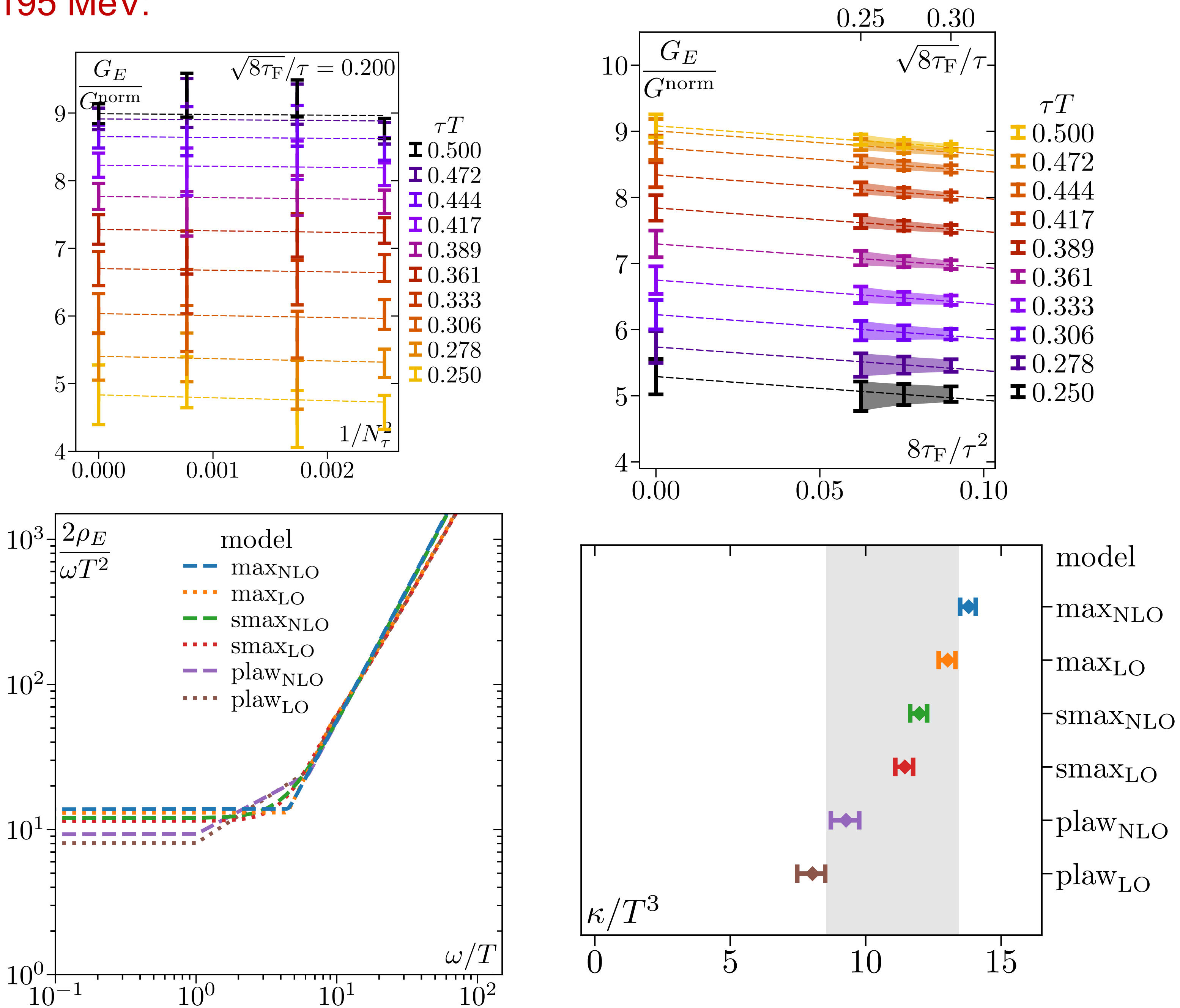
$$\rho^{pow}(\omega,T)=\rho^{low}(\omega,T),\ \omega\leq\omega_{IR}$$

$$\rho^{pow}(\omega,T)=A\omega^{\alpha},\ \omega_{IR}<\omega<\omega_{UV}$$

$$\rho^{pow}(\omega)=\rho^{high}(\omega),\ \omega\geq\omega_{UV}$$

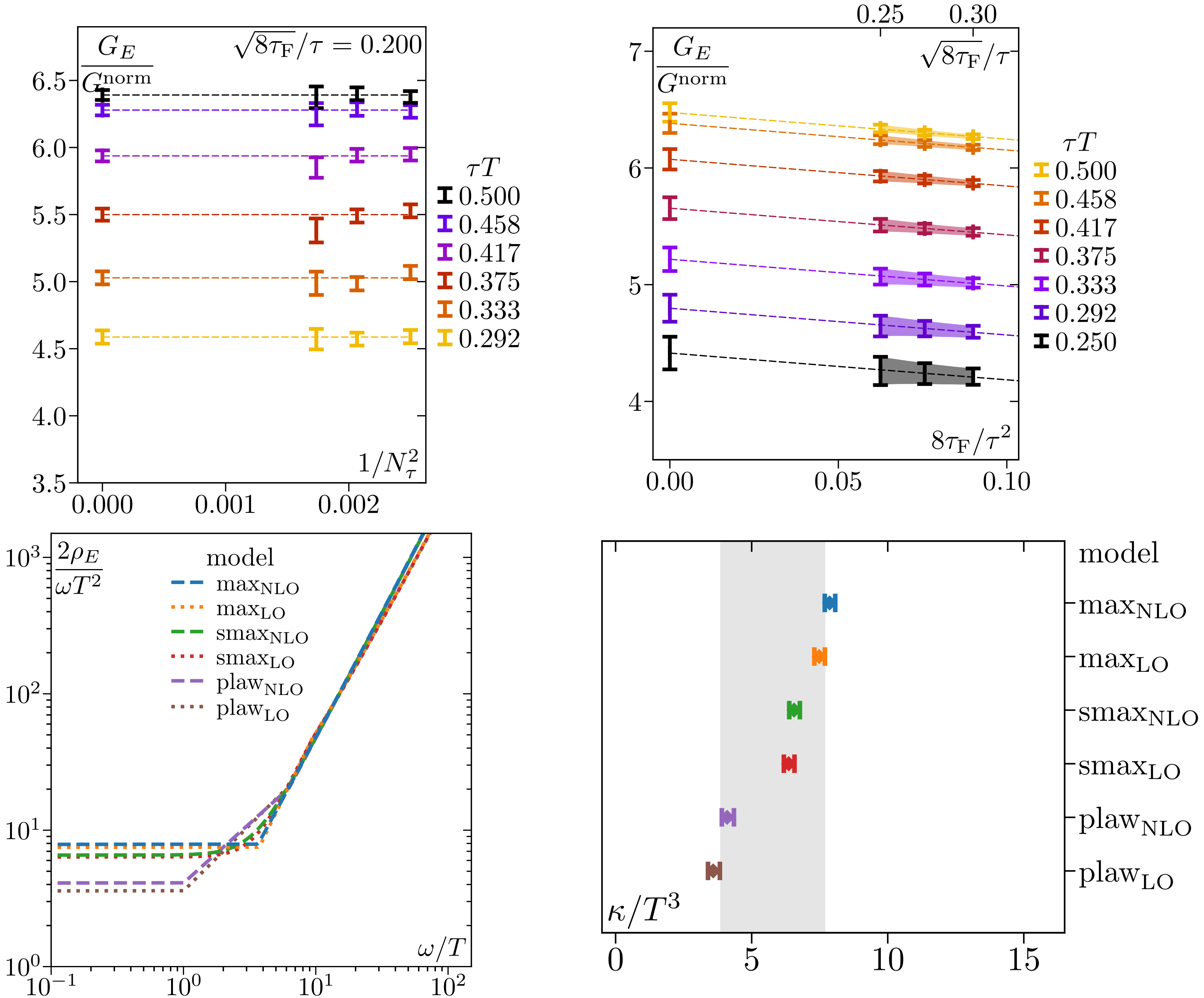


T=195 MeV:



$\frac{G_E}{G^{\mathrm{norm}}}$
$\sqrt{8\tau_F}/\tau = 0.200$
$1/N_\tau^2$
τT
0.500
0.472
0.444
0.417
0.389
0.361
0.333
0.306
0.278
0.250
$\sqrt{8\tau_F}/\tau$
0.25
0.30
$8\tau_F/\tau^2$
$\frac{2\rho_E}{\omega T^2}$
model
$\mathrm{max_{NLO}}$
$\mathrm{max_{LO}}$
$\mathrm{smax_{NLO}}$
$\mathrm{smax_{LO}}$
$\mathrm{plaw_{NLO}}$
$\mathrm{plaw_{LO}}$
ω/T
κ/T^3

T=293 MeV:

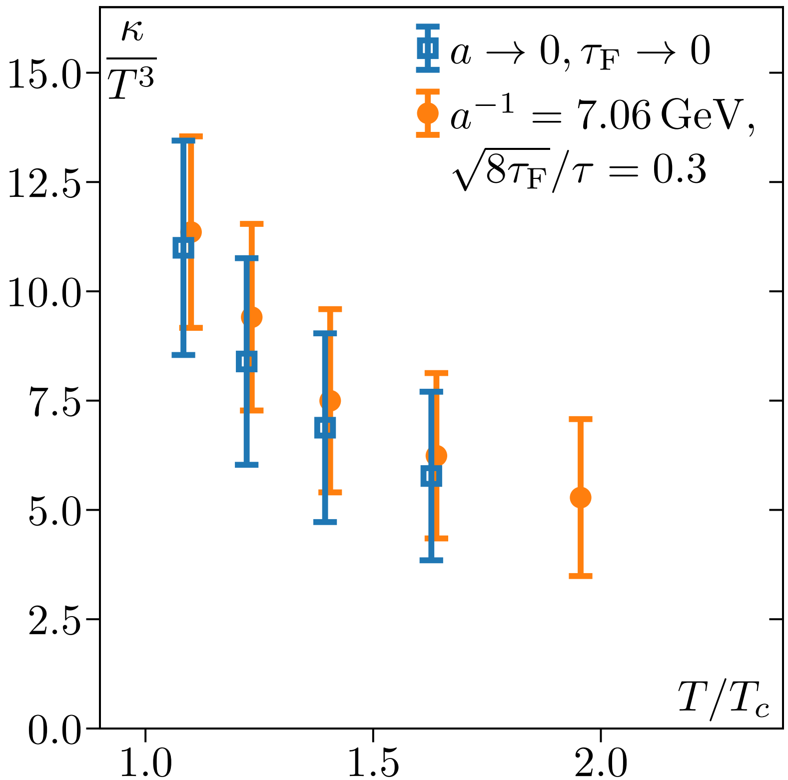


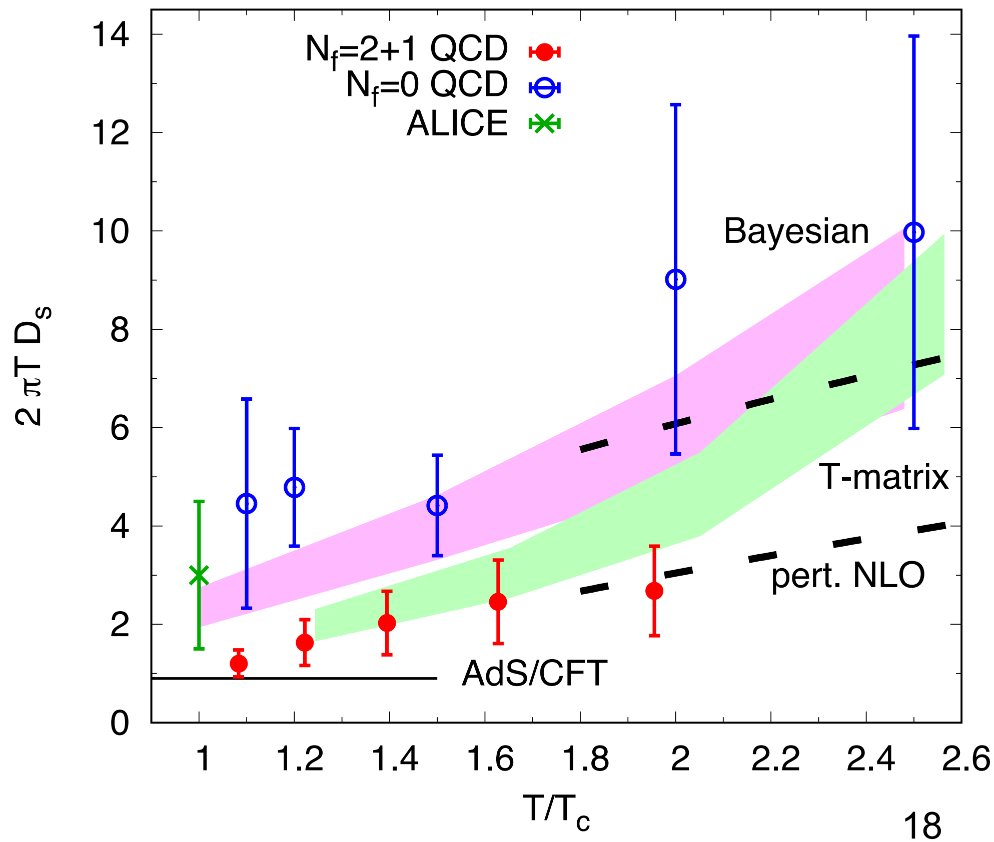
$\frac{G_E}{G^{\mathrm{norm}}}$
$\sqrt{8\tau_F}/\tau = 0.200$
τT
0.500
0.458
0.417
0.375
0.333
0.292
$1/N_\tau^2$
$\frac{G_E}{G^{\mathrm{norm}}}$
$\sqrt{8\tau_F}/\tau$
τT
0.500
0.458
0.417
0.375
0.333
0.292
0.250
$8\tau_F/\tau^2$
$\frac{2\rho_E}{\omega T^2}$
model
$\mathrm{max_{NLO}}$
$\mathrm{max_{LO}}$
$\mathrm{smax_{NLO}}$
$\mathrm{smax_{LO}}$
$\mathrm{plaw_{NLO}}$
$\mathrm{plaw_{LO}}$
ω/T
model
$\mathrm{max_{NLO}}$
$\mathrm{max_{LO}}$
$\mathrm{smax_{NLO}}$
$\mathrm{smax_{LO}}$
$\mathrm{plaw_{NLO}}$
$\mathrm{plaw_{LO}}$
κ/T^3

Heavy quark diffusion coefficient in QCD

- κ/T^3 has significant temperature dependence
- D_s is significantly smaller in 2+1 flavor QCD than in quenched QCD and is close to the AdS/CFT limit

$$D_s = \frac{2T^2}{\kappa}$$



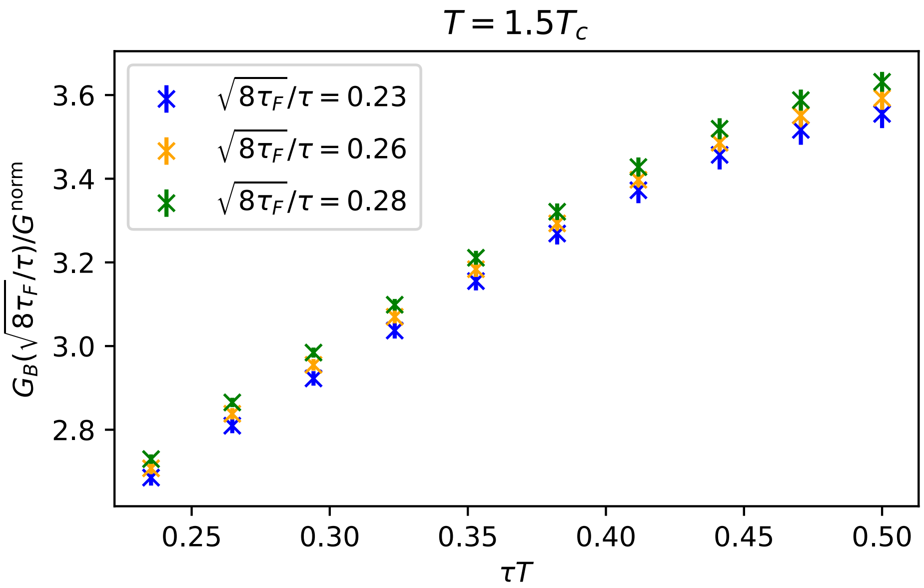


Mass suppressed correction to heavy quark diffusion coefficient

$1/M$ correction to the momentum heavy quark diffusion coefficient $\Rightarrow G_B(\tau, T)$
$G_B(\tau, T)$ has anomalous dimension $\Rightarrow$ additional matching to $\overline{MS}$ is needed

Quenched QCD

Gradient flow + incomplete 1-loop matching



Multi-level algorithm + non-perturbative matching via Schrödinger functional

$1.5T_c : \kappa_B = (1.23 - 2.54)T^3,$

Brambilla, Leino, Mayer-Steudte, PP (TUMQCD), PRD 107 (2023) 054508

$\kappa_B = (1.0 - 2.1)T^3$

Banerjee, Datta, Laine JHEP 08 (2022) 128

$\langle v^2 \rangle$ is taken from PP, EPJC 62 (2009) 85

10-20% correction for bottom quark, ~30% correction for charm quark

Summary

- Hadron resonance gas (HRG) model can describe fluctuations of conserved charges in the low temperature region; The range of validity of HRG can be extended by including repulsive baryon-baryon interactions.
- Thermodynamic quantity can be obtained using Taylor expansion for $\mu_B/T < 2.5$ and hint for critical point can be seen for these values of μ_B.
- Universal aspect of the chiral transition and O(N) scaling can be seen in terms of spectral density of the eigenvalues of the Dirac operator, since cumulants of spectral density are related to the cumulants of the chiral condensate
- First full QCD calculation of the heavy quark diffusion coefficient become available now and indicate that κ/T^3 is larger than un quenched QCD and close to the AdS/CFT bound
- The quark mass suppressed effects in the heavy quark diffusion coefficient have been estimated in quenched QCD to be around $10 - 20\%$ for bottom quarks and around 30% for charm quarks