Revised helicity evolution at small x: exact analytic solution of the large-N_c equations

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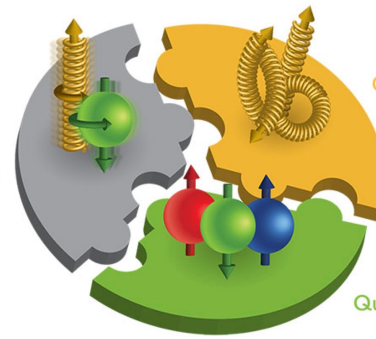
BASED ON WORK DONE WITH DAN PITONYAK AND MATT SIEVERT (2015-2018, 2021), FLORIAN COUGOULIC (2019-2022), JOSH TAWABUTR (2020-2022), ANDREY TARASOV (2021-2022), DANIEL ADAMIAK, NICK BALDONADO, WALLY MELNITCHOUK, NOBUO SATO (2021, 2023), JEREMY BORDEN (2022)

Outline

- Proton spin puzzle
- Quark and gluon helicity evolution at small x:
  - Quark and gluon helicity distributions at small x & polarized dipoles
  - Small-x evolution equations for quark and gluon helicity
- Solving the new evolution equations:
  - Exact solution at large N_C: small-x asymptotics of quark and gluon helicity distributions
  - Phenomenology: first fit of small-x polarized DIS data using evolution in x, predictions for EIC
- Sivers and Boer-Mulders functions at small x.
- Conclusions and Outlook

Proton Spin Puzzle: an Introduction

Proton Spin Puzzle



- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$S_q \approx 0.03$$

- It appeared (constituent) quarks do not carry all of the proton spin (which would have naively corresponded to $S_q = 1/2$).

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

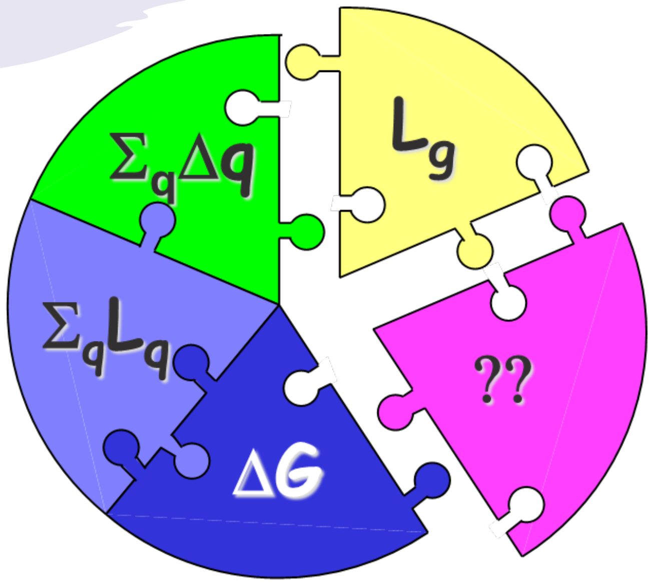
- Missing spin can be
  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x:

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

Can't integrate down to zero, use x_{min} instead!

  - Or all of the above!

Current Knowledge of Proton Spin



- The proton spin carried by the quarks is estimated to be (for $0.001 < x < 1$)

$$S_q(Q^2 = 10 \text{ GeV}^2) \in [0.15, 0.2]$$

- The proton spin carried by the gluons is (for $0.01 < x < 1$, STAR+PHENIX+COMPASS +HERMES+… , analyzed by DSSV, JAM, NNPDF…)

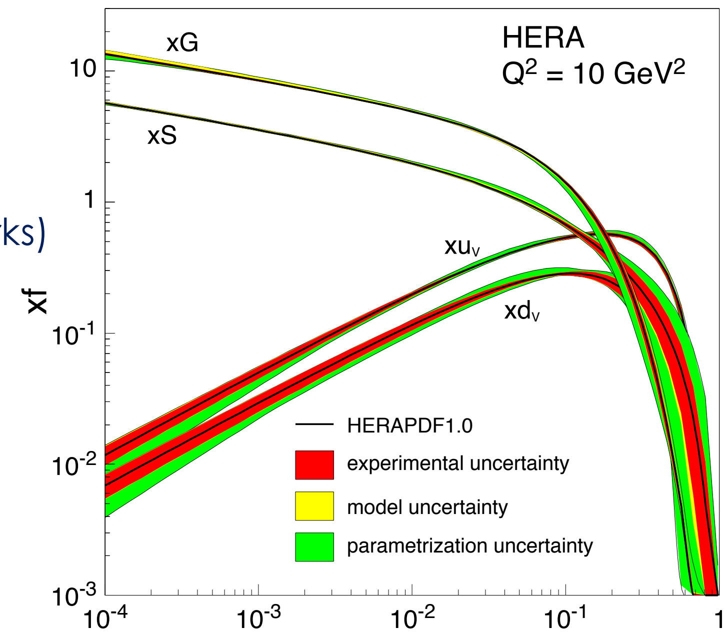
$$S_G(Q^2 = 10 \text{ GeV}^2) \in [0.13, 0.26]$$

- Negative S_G is also possible (JAM '22).
- Unfortunately, the uncertainties are large. Note also that the x-ranges are limited, with more spin (positive or negative) possible at small x.

Gluons and Quarks at Small-x

- There is a large number of small-x gluons (and sea quarks) in a proton:

x (# gluons or quarks)

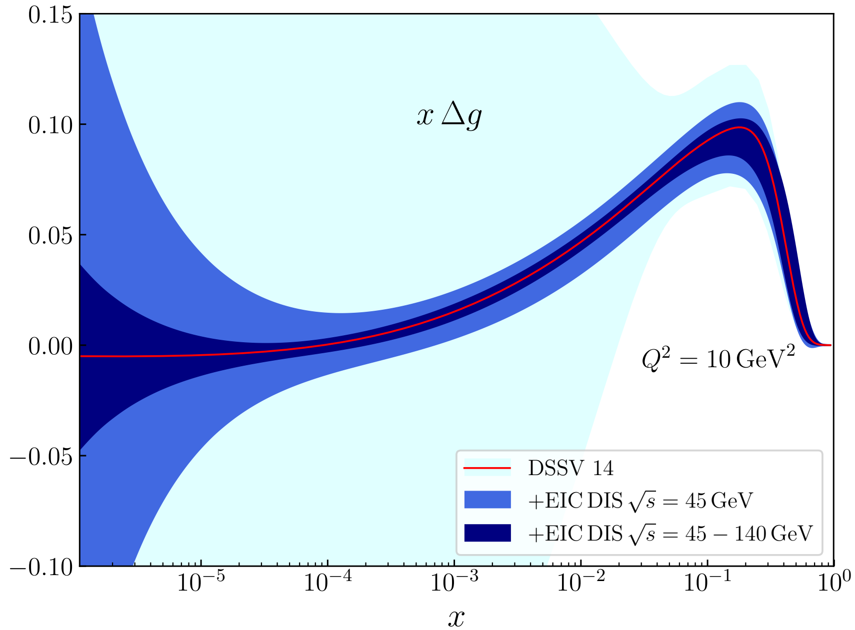


x =fraction of the proton momentum they carry

Can the small-x gluons and quarks carry a lot of the proton spin?

- $G(x, Q^2)$, $q(x, Q^2)$ = gluon and quark number densities / parton distribution functions (q=u,d, or S for sea).

How much spin is there at small x?



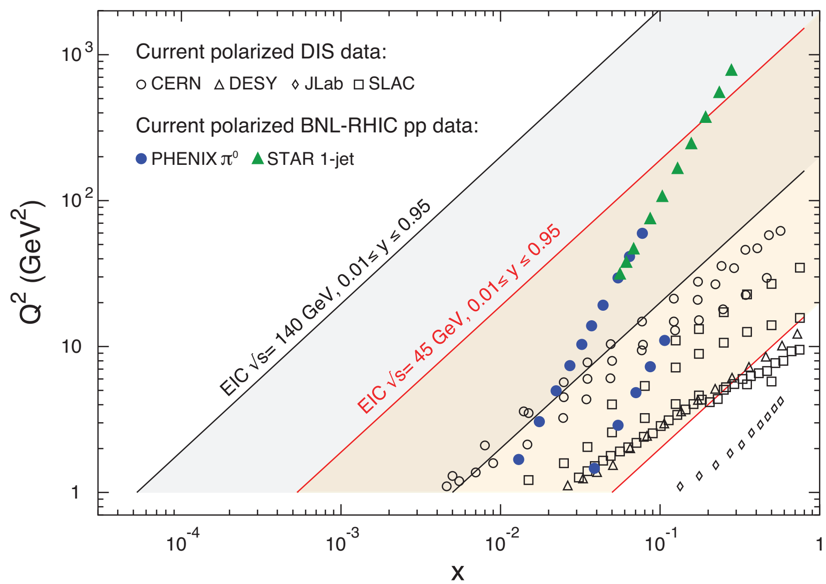
- E. Aschenauer et al, 2020 (DGLAP-based helicity PDF extraction from data)

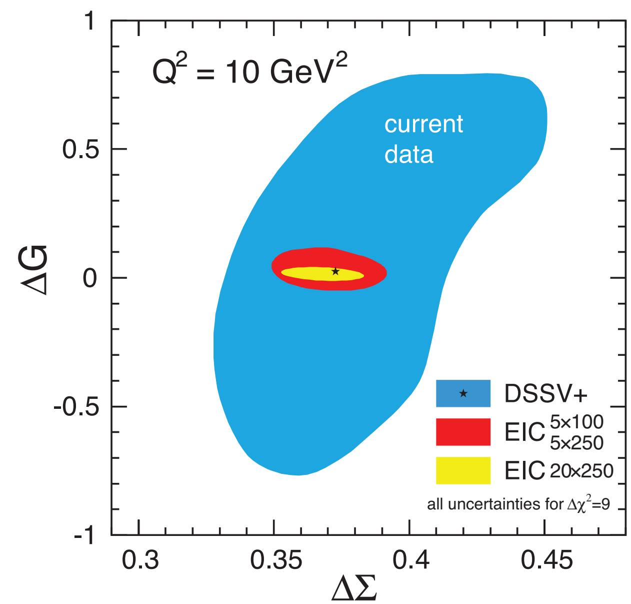
- Uncertainties are very large at small x. Note that this is xΔG, the uncertainties for ΔG are 1/x = 100-10000 times larger! EIC will reduce them, but only where there will be data.

EIC & Spin Puzzle

Still, even with the EIC data we'll need to extrapolate quark and gluon spin down to smaller x.

- Parton helicity distributions are sensitive to low-x physics.
- EIC would have an unprecedented low-x reach for a polarized DIS experiment, allowing to pinpoint the values of quark and gluon contributions to proton's spin (EIC WP '12):





- ΔG and $\Delta\Sigma$ are integrated over x in the $0.001 < x < 1$ interval.

Quark and Gluon Helicity at Small x

YK, D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph], arXiv:1610.06197 [hep-ph] (KPS);
YK, M. Sievert, arXiv:1505.01176 [hep-ph], arXiv:1808.09010 [hep-ph];
F. Cougoulic, YK, A. Tarasov, and Y. Tawabutr, arXiv:2204.11898 [hep-ph] (KPS-CTT).

Sub-Eikonal Operators

Dipole picture of DIS

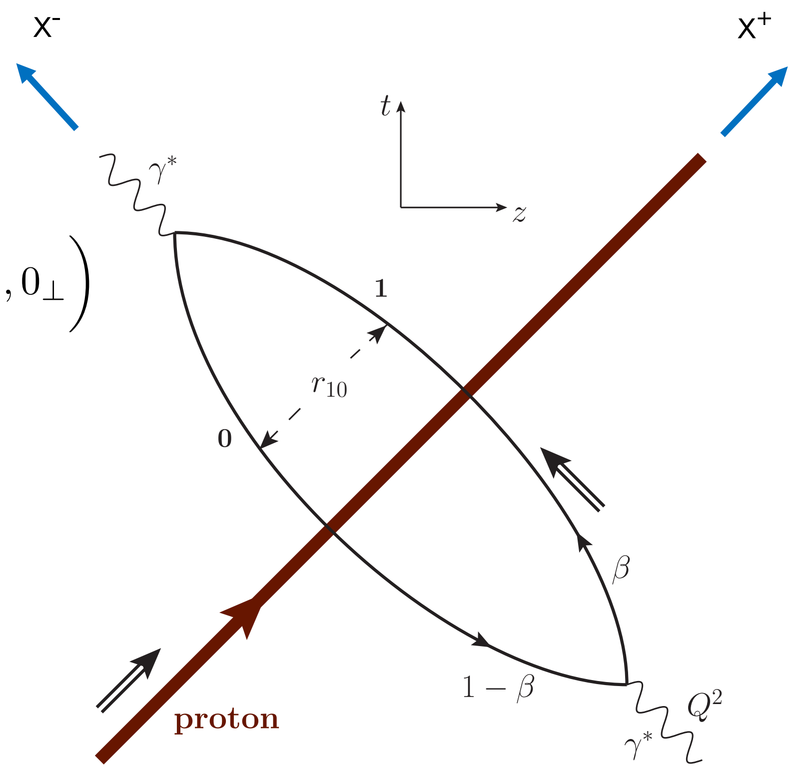
$$W^{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{iq\cdot x} \langle P | j^\mu(x) \, j^\nu(0) | P \rangle$$

$$q^\mu = \left(\frac{Q^2}{2q^-}, q^-, 0_\perp \right)$$

Large q⁻ → large x⁻ separation

$$e^{iq\cdot x} = e^{i\frac{Q^2}{2q^-}x^- + iq^- x^+}$$

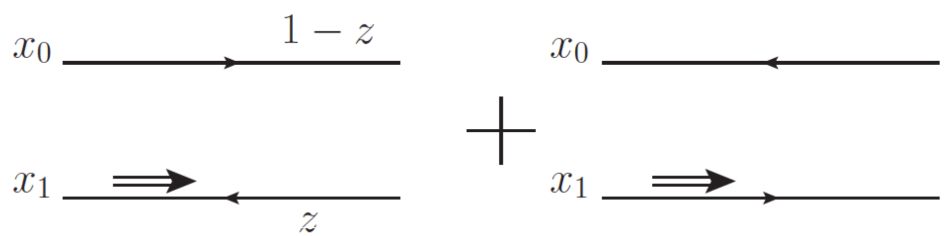
$$x^\pm = \frac{t \pm z}{\sqrt{2}}$$



aka the "shock wave"

Polarized Dipole: non-eikonal small-x physics

- All flavor-singlet small-x helicity observables depend on "polarized dipole amplitudes":



$$G_{10}(z) \equiv \frac{1}{2N_c} \mathrm{Re} \left\langle\!\!\left\langle \mathrm{T\,tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] + \mathrm{T\,tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

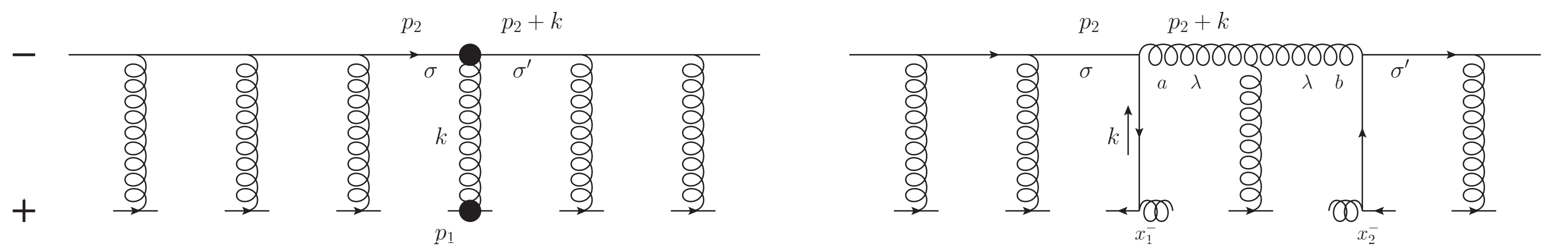
$$V_{\underline{x}} = \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^- A^+(0^+, x^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

Polarized fundamental "Wilson line"

- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized "Wilson line" V^{pol}, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal *t*-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two *t*-channel quarks, as shown above.
- We employ a blend of Brodsky & Lepage's LCPT and background field method-inspired operator treatment. We refer to the latter as the **light-cone operator treatment (LCOT)**.

Notation

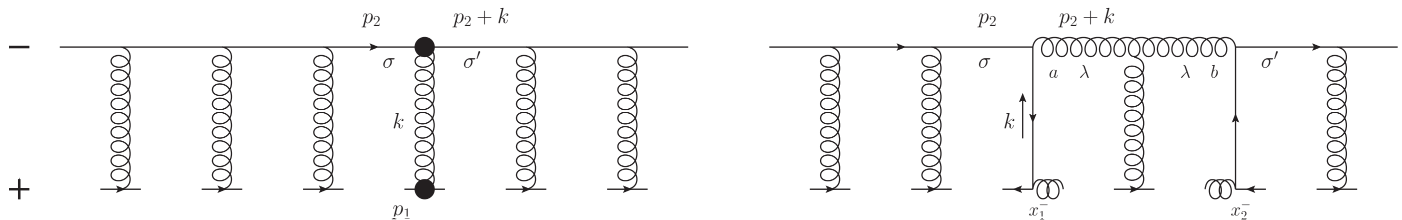
- Fundamental light-cone Wilson line:

$$V_{\underline{x}}[b^-, a^-] = \mathrm{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$

- Adjoint light-cone Wilson line:

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

Sub-eikonal quark S-matrix in background gluon and quark fields



- The full sub-eikonal S-matrix for massless quarks is (Balitsky&Tarasov '15; KPS '17; YK, Sievert, '18; Chirilli '18; Altinoluk et al, '20; YK, Santiago '21)

$$V_{\underline{x},\underline{y};\sigma',\sigma} = V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\sigma,\sigma'}$$

"helicity independent" "helicity dependent" $-\vec{\mu}\cdot\vec{B} = -\mu_z\,B_z = \mu_z\,F^{12}$

$$+\frac{i\,P^+}{s}\int\limits_{-\infty}^{\infty} dz^- d^2z\; V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\left[-\delta_{\sigma,\sigma'}\,\overleftarrow{D}^i\;D^i + g\,\sigma\,\delta_{\sigma,\sigma'}\,F^{12}\right](z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z})$$

$$-\frac{g^2P^+}{2s}\delta^2(\underline{x}-\underline{y})\int\limits_{-\infty}^{\infty} dz_1^-\int\limits_{z_1^-}^{\infty} dz_2^-\,V_{\underline{x}}[\infty,z_2^-]t^b\psi_\beta(z_2^-,\underline{x})U_{\underline{x}}^{ba}[z_2^-,z_1^-]\left[\delta_{\sigma,\sigma'}\gamma^+ - \sigma\delta_{\sigma,\sigma'}\gamma^+\gamma^5\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-,\underline{x})t^aV_{\underline{x}}[z_1^-,-\infty]$$

"helicity independent" "helicity dependent"

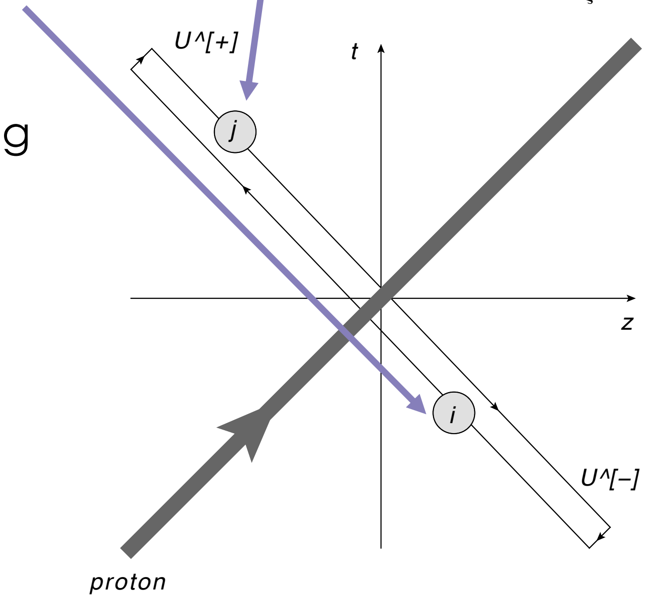
Helicity Observables at Small x

Dipole Gluon Helicity TMD

- We start with the definition of the gluon dipole helicity TMD, corresponding to the Jaffe-Manohar PDF ΔG,

$$g_1^G(x, k_T^2) = \frac{-2i\, S_L}{x\, P^+} \int \frac{d\xi^- \, d^2\xi}{(2\pi)^3}\, e^{ixP^+\, \xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P, S_L | \epsilon_T^{ij}\, \mathrm{tr}\left[F^{+i}(0)\, \mathcal{U}^{[+]\dagger}[0,\xi]\, F^{+j}(\xi)\, \mathcal{U}^{[-]}[\xi, 0]\right] |P, S_L \right\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past-pointing Wilson line staples (hence the name 'dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Gluon Helicity

- A calculation gives

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right) G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right) \right]_{x_{10}^2 = \frac{1}{Q^2}}$$

$$g_{1L}^{G\,dip}(x, k_T^2) = \frac{N_c}{\alpha_s 2\pi^4} \int d^2 x_{10} \, e^{-i\underline{k}\cdot\underline{x}_{10}} \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right)$$

- Here we defined a new dipole amplitude G_2 (cf. Hatta et al, 2016; KPS 2017)

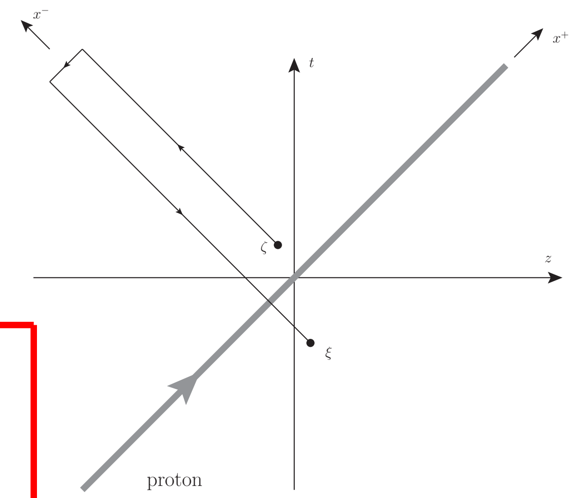
$$\int d^2 \left(\frac{x_1 + x_0}{2} \right) G_{10}^i(zs) = (x_{10})_\perp^i \, G_1(x_{10}^2, zs) + \epsilon^{ij} \, (x_{10})_\perp^j \, G_2(x_{10}^2, zs)$$

$$G_{10}^j(zs) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^\dagger \, V_{\underline{1}}^{j\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{j\,\mathrm{G}[2]} \right)^\dagger V_{\underline{0}} \right] \right\rangle\!\!\right\rangle$$

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} \equiv \frac{P^+}{2s} \int\limits_{-\infty}^{\infty} dz^- \, V_{\underline{z}}[\infty, z^-] \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i (z^-, \underline{z}) \right] V_{\underline{z}}[z^-, -\infty]$$

What is this D-D operator? Turns out it is related to the DD operator from before.

Quark Helicity PDF and TMD



- The flavor-singlet quark helicity PDF and SIDIS TMD are

$$\Delta\Sigma(x,Q^2) = -\frac{N_c\,N_f}{2\pi^3}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\right\}}\frac{dx_{10}^2}{x_{10}^2}\left[Q(x_{10}^2,zs)+2\,G_2(x_{10}^2,zs)\right]$$

$$g_{1L}^S(x,k_T^2) = \frac{8\,i\,N_c\,N_f}{(2\pi)^5}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int d^2x_{10}\,e^{i\underline{k}\cdot\underline{x}_{10}}\,\frac{\underline{x}_{10}}{x_{10}^2}\cdot\frac{\underline{k}}{\underline{k}^2}\left[Q(x_{10}^2,zs)+2\,G_2(x_{10}^2,zs)\right]$$

- G_2 was defined before. This is the gluon admixture to quark helicity distributions.
- The dipole amplitude Q is due to F^{12} & axial current.
- The contribution of G_2 comes from the DD operator in the quark S-matrix.
- Hence, the DD operator is related to the Jaffe-Manohar distribution.

Amplitude Q

$$Q(x_{10}^2, zs) \equiv \int d^2 \left(\frac{x_0 + x_1}{2} \right) Q_{10}(zs)$$

- The amplitude Q is defined by

$$Q_{10}(zs) \equiv \frac{1}{2N_c} \text{Re} \left\langle\!\!\left\langle \text{T}\,\text{tr} \left[V_{\underline{0}}\, V_{\underline{1}}^{\text{pol}[1]\,\dagger} \right] + \text{T}\,\text{tr} \left[V_{\underline{1}}^{\text{pol}[1]}\, V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle$$

with $V_{\underline{x}}^{\text{pol}[1]} = V_{\underline{x}}^{\text{G}[1]} + V_{\underline{x}}^{\text{q}[1]}$, where

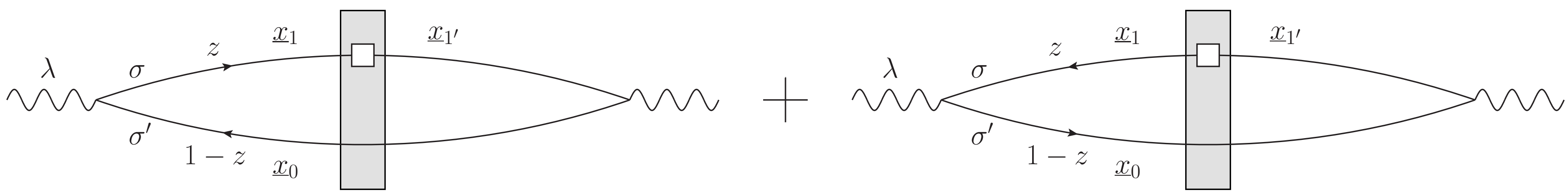
$$V_{\underline{x}}^{\text{G}[1]} = \frac{i\, g\, P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\ V_{\underline{x}}[x^-, -\infty]$$

$$V_{\underline{x}}^{\text{q}[1]} = \frac{g^2 P^+}{2\, s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U_{\underline{x}}^{ba}[x_2^-, x_1^-]\ \left[\gamma^+ \gamma^5\right]_{\alpha\beta}\ \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty]$$

- This is the 'old' KPS polarized dipole amplitude. U = adjoint light-cone Wilson line.

g_1 structure function

- g_1 structure function is obtained similarly, using DIS in the dipole picture:



- One gets

$$g_1(x, Q^2) = -\sum_f \frac{N_c\, Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

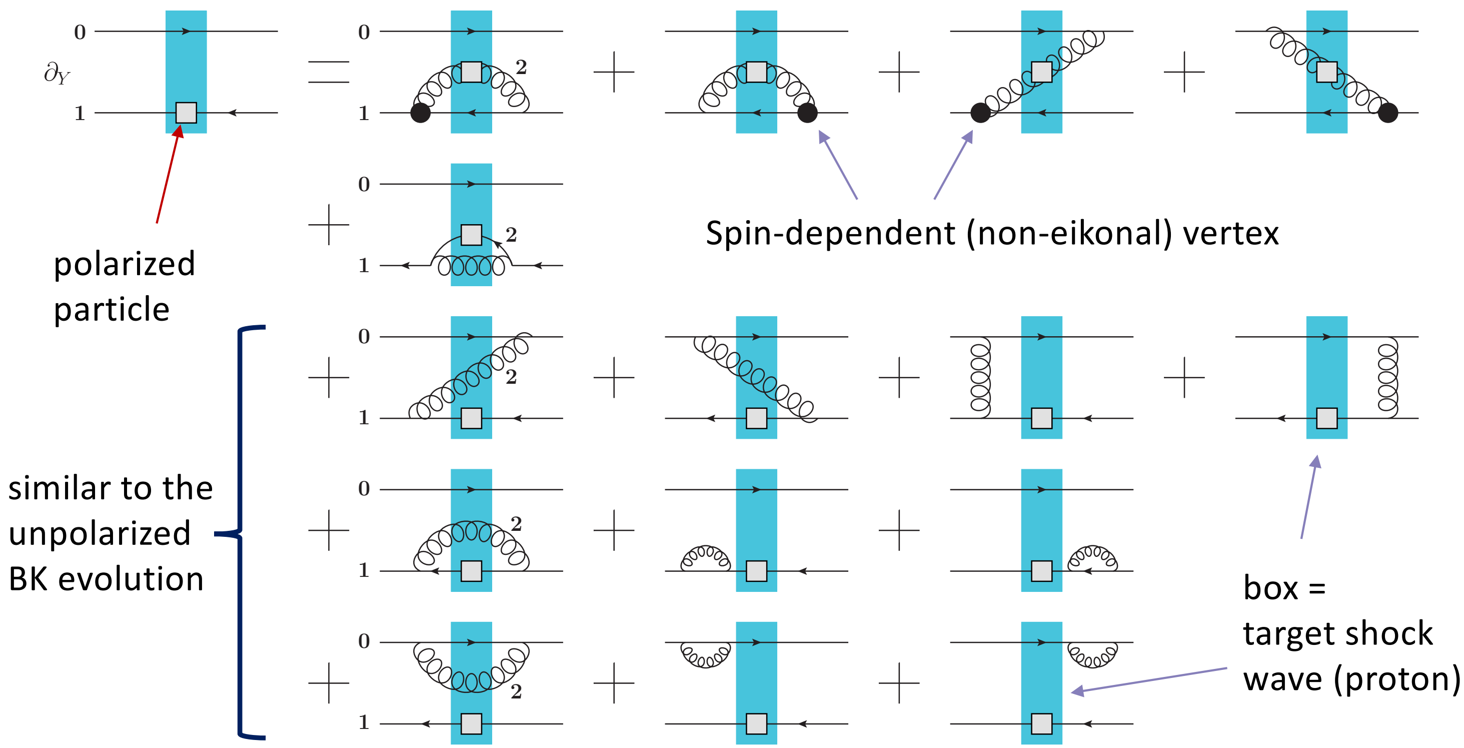
such that one reproduces the standard result $g_1(x, Q^2) = \frac{1}{2} \sum_f Z_f^2\, \Delta q_f^+(x, Q^2)$

$$\Delta q_f^+ = \Delta q_f + \Delta \bar{q}_f$$

Helicity Evolution

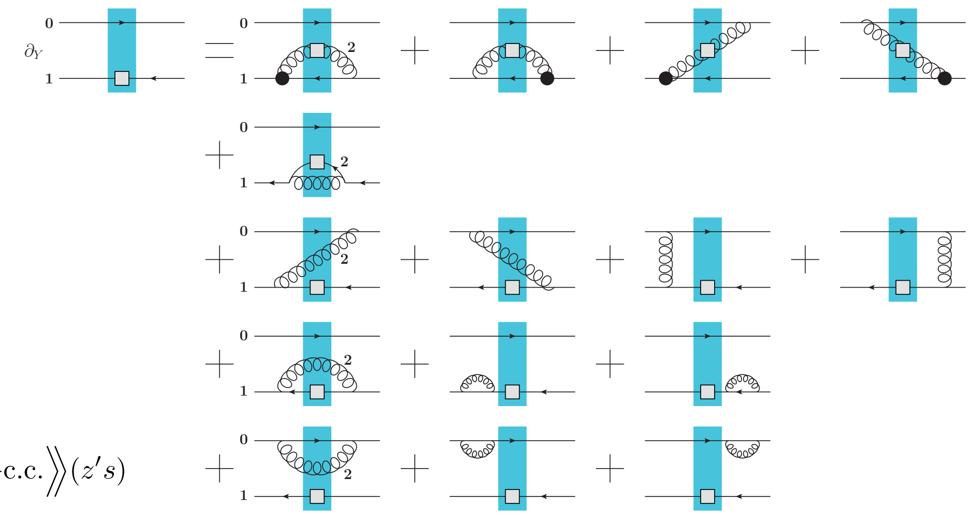
Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole

$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$



$$\frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.} \right\rangle\!\!\right\rangle(zs) = \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.} \right\rangle\!\!\right\rangle_0(zs)$$

$$+\frac{\alpha_s\, N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \Bigg\{ \left[\frac{1}{x_{21}^2} - \frac{\underline{x}_{21}}{x_{21}^2} \cdot \frac{\underline{x}_{20}}{x_{20}^2}\right] \frac{1}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right] \left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.} \right\rangle\!\!\right\rangle(z's)$$

$$+\left[2\frac{\epsilon^{ij}\, x_{21}^j}{x_{21}^4} - \frac{\epsilon^{ij}\,(x_{20}^j + x_{21}^j)}{x_{20}^2\, x_{21}^2} - \frac{2\,\underline{x}_{20} \times \underline{x}_{21}}{x_{20}^2\, x_{21}^2}\left(\frac{x_{21}^i}{x_{21}^2} - \frac{x_{20}^i}{x_{20}^2}\right)\right] \frac{1}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right] \left(U_{\underline{2}}^{i\,\mathrm{G}[2]}\right)^{ba} \right\rangle\!\!\right\rangle(z's) \Bigg\}$$

$$+\frac{\alpha_s N_c}{4\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int \frac{d^2 x_2}{x_{21}^2} \Bigg\{ \frac{1}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{1}}^{ba} \right\rangle\!\!\right\rangle(z's) + 2\, \frac{\epsilon^{ij}\, \underline{x}_{21}^j}{x_{21}^2} \frac{1}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{i\,\mathrm{G}[2]\,\dagger}\right] U_{\underline{1}}^{ba} \right\rangle\!\!\right\rangle(z's) \Bigg\}$$

$$+\frac{\alpha_s\, N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2\, \frac{x_{10}^2}{x_{21}^2\, x_{20}^2} \Bigg\{ \frac{1}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{2}}^{ba} \right\rangle\!\!\right\rangle(z's) - \frac{C_F}{N_c^2} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] \right\rangle\!\!\right\rangle(z's) \Bigg\} +$$

c.c.

Equation does not close!

Large N_c

- At large-N_c the equations close (Q → G).
- Everything with 2 in the subscript (e.g., G_2 and Γ_2) is new compared to the KPS ('15-'18) papers.

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big],$$

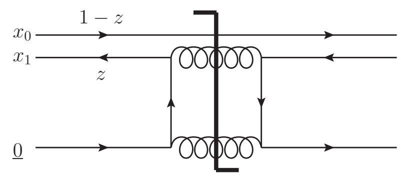
$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big],$$

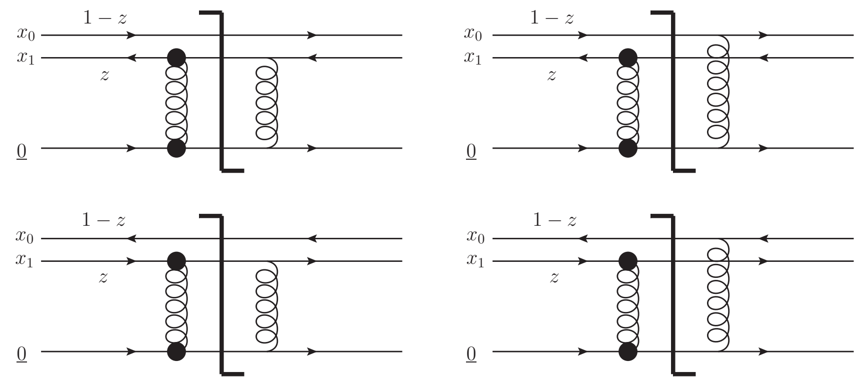
$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]$$

Initial Conditions

- The initial conditions are given by the Born-level graphs





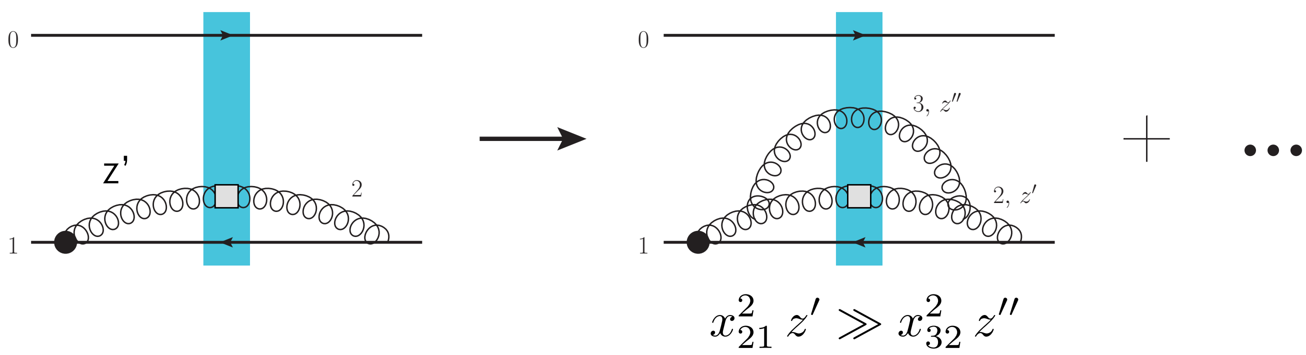
$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2 \ln(zs\, x_{10}^2) \right]$$

$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

- Similar Born-level calculation is done for G_2 and Γ_2.

"Neighbor" dipole

- There is a new object in the evolution equation – **the neighbor dipole amplitude**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to lifetime ordering, may 'know' about another dipole:



- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

Resummation Parameter

- For helicity evolution the leading resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.

- This was known before: Kirschner and Lipatov '83; Kirschner '84; Bartels, Ermolaev, Ryskin '95, '96; Griffiths and Ross '99; Itakura et al '03; Bartels and Lublinsky '03.

Large-N_c&N_f

- Equations also close in the large-N_c&N_f (Veneziano) limit.
- This is more realistic as it includes quarks.
- Everything with 2 in the subscript (e.g., G_2 and Γ_2) is new compared to the KPS ('15-'18) papers.

$$
\begin{aligned}
Q(x_{10}^2, zs) = {} & Q^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[2\,\widetilde{G}(x_{21}^2, z's) + 2\,\widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) \\
& + Q(x_{21}^2, z's) - \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) + 2\,\Gamma_2(x_{10}^2, x_{21}^2, z's) + 2\,G_2(x_{21}^2, z's) \Big] \\
& + \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2\,G_2(x_{21}^2, z's) \right],
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Gamma}(x_{10}^2, x_{21}^2, z's) = {} & Q^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \Big[2\,\widetilde{G}(x_{32}^2, z''s) \\
& + 2\,\widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + Q(x_{32}^2, z''s) - \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2\,\Gamma_2(x_{10}^2, x_{32}^2, z''s) + 2\,G_2(x_{32}^2, z''s) \Big] \\
& + \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2\,G_2(x_{32}^2, z''s) \right],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}(x_{10}^2, zs) = {} & \widetilde{G}^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[3\,\widetilde{G}(x_{21}^2, z's) + \widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) \\
& + 2\,G_2(x_{21}^2, z's) + \left(2 - \frac{N_f}{2N_c}\right) \Gamma_2(x_{10}^2, x_{21}^2, z's) - \frac{N_f}{4N_c}\, \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) \Big] \\
& - \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{\max\{x_{10}^2,\, 1/z's\}}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2\,G_2(x_{21}^2, z's) \right],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) = {} & \widetilde{G}^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \Big[3\,\widetilde{G}(x_{32}^2, z''s) \\
& + \widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2\,G_2(x_{32}^2, z''s) + \left(2 - \frac{N_f}{2N_c}\right) \Gamma_2(x_{10}^2, x_{32}^2, z''s) - \frac{N_f}{4N_c}\, \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) \Big] \\
& - \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z' x_{21}^2/x_{10}^2} \frac{dz''}{z''} \int\limits_{\max\{x_{10}^2,\, 1/z''s\}}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2\,G_2(x_{32}^2, z''s) \right],
\end{aligned}
$$

$$
G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\frac{z}{z'} x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\widetilde{G}(x_{21}^2, z's) + 2\,G_2(x_{21}^2, z's) \right],
$$

$$
\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\frac{z'}{z''} x_{21}^2} \frac{dx_{32}^2}{x_{32}^2} \left[\widetilde{G}(x_{32}^2, z''s) + 2\,G_2(x_{32}^2, z''s) \right]
$$

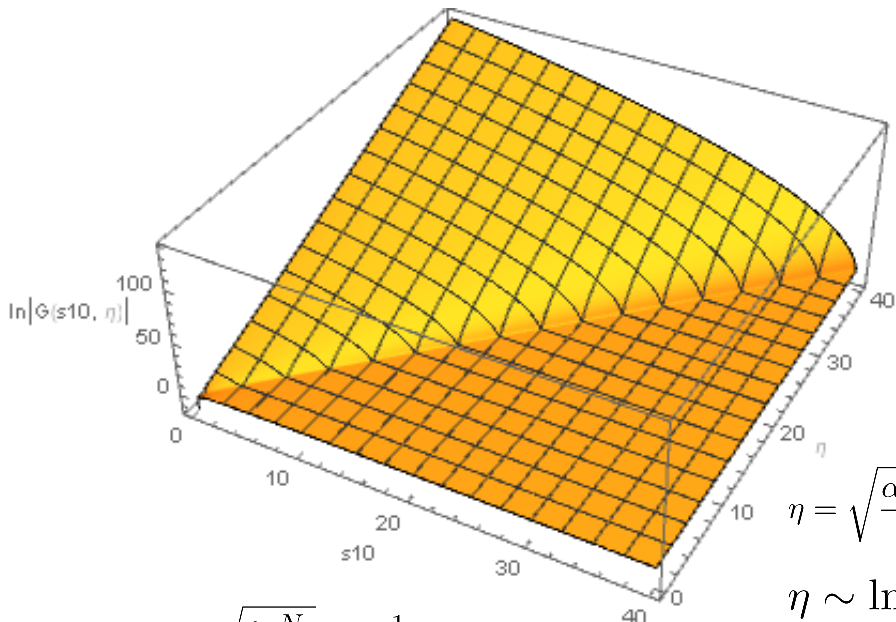
Quark Helicity at Small x: Asymptotics and Phenomenology

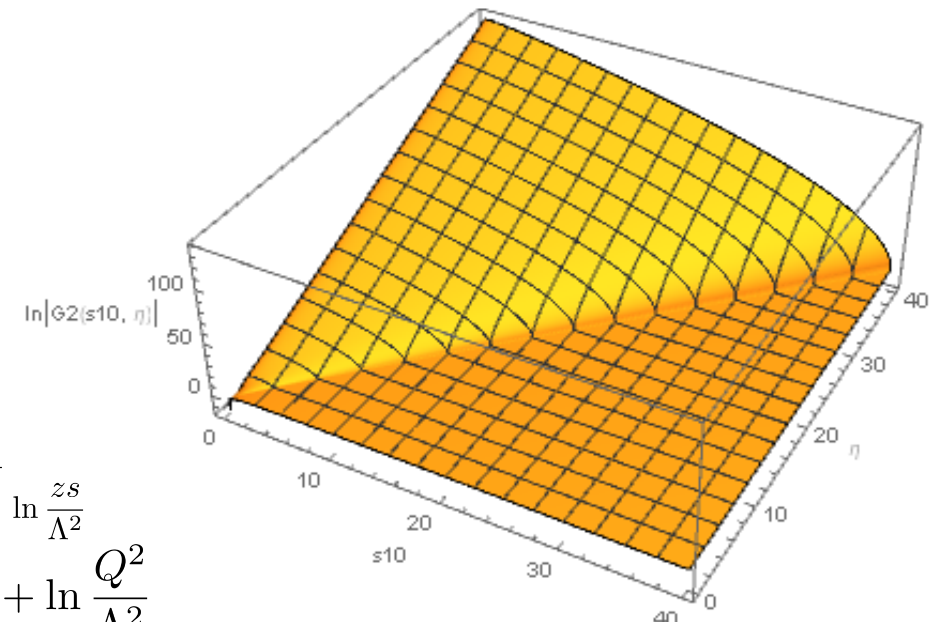
YK, D. Pitonyak, M. Sievert, arXiv:1610.06188 [hep-ph], arXiv:1703.05809 [hep-ph] (KPS);
F. Cougoulic, YK, A. Tarasov, and Y. Tawabutr, arXiv:2204.11898 [hep-ph] (KPS-CTT);
D. Adamiak, W. Melnitchouk, D. Pitonyak, N. Sato, M. Sievert, YK, 2102.06159 [hep-ph];
J. Borden and YK, 2304.06161 [hep-ph].

Small-x Asymptotics

Solution of the Large-N_c Equations

F. Cougoulic, YK, A. Tarasov, Y. Tawabutr, 2022





$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

$$\eta \sim \ln \frac{1}{x} + \ln \frac{Q^2}{\Lambda^2}$$

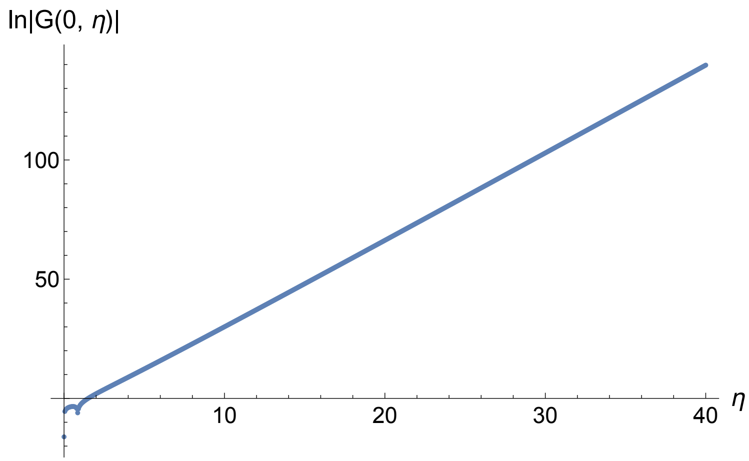
$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

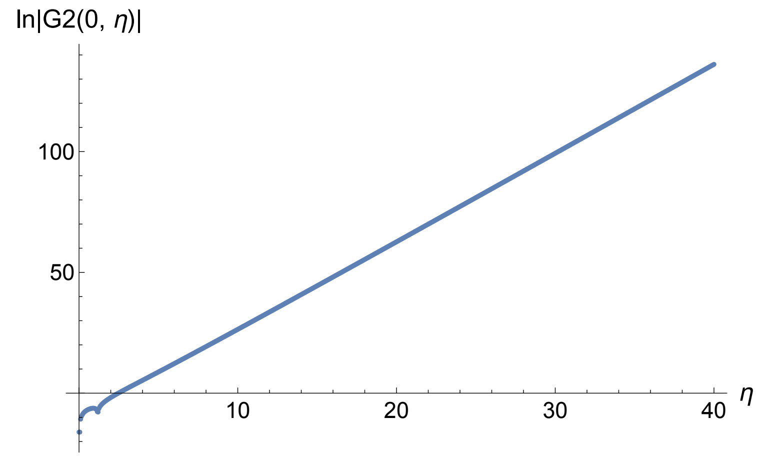
$$s_{10} \sim \ln \frac{Q^2}{\Lambda^2}$$

The large-N_c equations for G and G_2 can be solved numerically (and, possibly, analytically).

Small-x Asymptotics

- Fitting the slope of the log plots of G and G_2 vs e we can read off the small-x intercept (the power of x):





F. Cougoulic, YK, A. Tarasov, Y. Tawabutr, 2022

Small-x Asymptotics for Helicity Distributions

- The resulting small-x asymptotics for helicity PDFs and the g_1 structure function at large N_c is

$$\Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim g_1(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

- This power (aka the intercept) is in complete agreement with the work by J. Bartels, B. Ermolaev, and M. Ryskin (BER, 1996) using infrared evolution equations (with the analytic intercept constructed by KPS in 2016):

$$\alpha_h = \sqrt{\frac{17+\sqrt{97}}{2}}\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}} \approx 3.664\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}$$

- "Peace in the valley."
- **Right?**

Analytic Solution of the Large-N_c Equations

- We want to solve these equations:

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big],$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big],$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]$$

Analytic Solution of the Large-N_c Equations

- The strategy is to use the double Laplace transform,

$$\bar{\alpha}_s \equiv \frac{\alpha_s \, N_c}{2\pi}$$

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \, e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \, G_{2\omega\gamma}$$

- One gets the expressions for all the other dipole amplitudes this way, for instance

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \, e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{\omega\gamma}{2\, \bar{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) - 2\, G_{2\omega\gamma} \right]$$

- Neighbor amplitudes involve several different double Laplace transforms:

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \left[e^{\omega \ln(z'sx_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) + e^{\omega \ln(z'sx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \, G_{2\omega\gamma}^{(0)} \right]$$

Analytic Solution of the Large-N_C Equations

- In the end, all the amplitudes in the double-Laplace space can be expressed in terms of the initial conditions/inhomogeneous terms, e.g.,

$$G_{2\omega\gamma} = G^{(0)}_{2\omega\gamma} + \frac{\bar{\alpha}_s}{\omega\left(\gamma - \gamma^-_\omega\right)\left(\gamma - \gamma^+_\omega\right)} \Bigg[2\left(\gamma - \delta^+_\omega\right)\left(G^{(0)}_{\delta^+_\omega \gamma} + 2\, G^{(0)}_{2\,\delta^+_\omega \gamma}\right) - 2\left(\gamma^+_\omega - \delta^+_\omega\right)\left(G^{(0)}_{\delta^+_\omega \gamma^+_\omega} + 2\, G^{(0)}_{2\,\delta^+_\omega \gamma^+_\omega}\right) + 8\, \delta^-_\omega \left(G^{(0)}_{2\omega\gamma} - G^{(0)}_{2\omega\gamma^+_\omega}\right)\Bigg]$$

with

$$\delta^\pm_\omega = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}\right] \qquad \gamma^\pm_\omega = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{16\,\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- More details in J. Borden, YK, 2304.06161 [hep-ph].

Small-x Asymptotics for Helicity Distributions

- Let's take a closer look at the anomalous dimension:

$$\Delta G(x, Q^2) = \int \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} \left(\frac{Q^2}{\Lambda^2}\right)^{\Delta\gamma_{GG}(\omega)} \Delta G_\omega(\Lambda^2)$$

- In the pure-glue case, Bartels, Ermolaev and Ryskin's (BER) anomalous dimension can be found analytically. It reads (KPS '16)

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1 - \frac{3\,\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \bar{\alpha}_s = \frac{\alpha_s\, N_c}{2\pi}$$

- Our evolution's anomalous dimension can be found analytically at large-N_c (J. Borden, YK, 2304.06161 [hep-ph]):

$$\Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

A Tale of Two Anomalous Dimensions

- The two anomalous dimensions look similar enough but are not the same function.

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1-\frac{3\,\bar{\alpha}_s}{\omega^2}}{1-\frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s\sqrt{1-\frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- Their expansions in α_S start out the same, then differ at four (!) loops (the first 3 terms agree with the existing finite-order calculations, the four-loop result is unknown):

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5} + \frac{504\,\bar{\alpha}_s^4}{\omega^7} + \dots$$

$$\Delta\gamma_{GG}^{us}(\omega) = \frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5} + \frac{496\,\bar{\alpha}_s^4}{\omega^7} + \dots$$

A Tale of Two Intercepts

$$\Delta G(x, Q^2) = \int \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} \left(\frac{Q^2}{\Lambda^2}\right)^{\Delta\gamma_{GG}(\omega)} \Delta G_\omega(\Lambda^2)$$

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1 - \frac{3\,\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- The intercept (largest power Re[ω]) is given by the right-most singularity (branch point) of the anomalous dimension.
- For BER this gives $\alpha_h = \sqrt{\frac{17 + \sqrt{97}}{2}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.664 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$
- For us $\alpha_h = \frac{4}{3^{1/3}} \sqrt{\mathrm{Re}\left[(-9 + i\,\sqrt{111})^{1/3}\right]} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.661 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$

A Tale of Two Intercepts

$$\Delta\Sigma(x,Q^2)\Big|_{x\ll 1} \sim \Delta G(x,Q^2)\Big|_{x\ll 1} \sim \left(\frac{1}{x}\right)^{\alpha_h}$$

- BER: $$\alpha_h = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.664\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$
- Us: $$\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[(-9+i\,\sqrt{111})^{1/3}\right]}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.661\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$
- Our numerical solution also gave the intercept of 3.660 or 3.661, but we believed we had larger error bars.
- We (still) disagree with BER. Albeit in the 3rd decimal point…

Phenomenology

D. Adamiak, N. Baldonado, YK, W. Melnitchouk, D. Pitonyak,
N. Sato, M. Sievert, A. Tarasov, Y. Tawabutr, = JAMsmallx in preparation

Polarized DIS and SIDIS data

- We can use the large-N_c&N_f version of the evolution to fit all the existing world polarized DIS and SIDIS data.
- Why not large-N_c? Have to distinguish a true quark dipole from the subset of the gluon one. Then can fit all helicity PDFs for light quark flavors, in addition to the gluon helicity PDF.
- Hence, the quark amplitudes Q_f come with the flavor index f.
- Drawback: many dipole amplitudes, hard to constrain all.
- LO intercept is large: had to include running coupling (not shown) into the evolution.

$$
\begin{aligned}
Q(x_{10}^2, zs) = Q^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[2\, \widetilde{G}(x_{21}^2, z's) + 2\, \widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) \\
+ Q(x_{21}^2, z's) - \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \Big] \\
+ \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Gamma}(x_{10}^2, x_{21}^2, z's) = Q^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \Big[2\, \widetilde{G}(x_{32}^2, z''s) \\
+ 2\, \widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + Q(x_{32}^2, z''s) - \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \Big] \\
+ \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}(x_{10}^2, zs) = \widetilde{G}^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[3\, \widetilde{G}(x_{21}^2, z's) + \widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) \\
+ 2\, G_2(x_{21}^2, z's) + \left(2 - \frac{N_f}{2N_c} \right) \Gamma_2(x_{10}^2, x_{21}^2, z's) - \frac{N_f}{4N_c}\, \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) \Big] \\
- \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{\max\{x_{10}^2,\, 1/z's\}}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],
\end{aligned}
$$

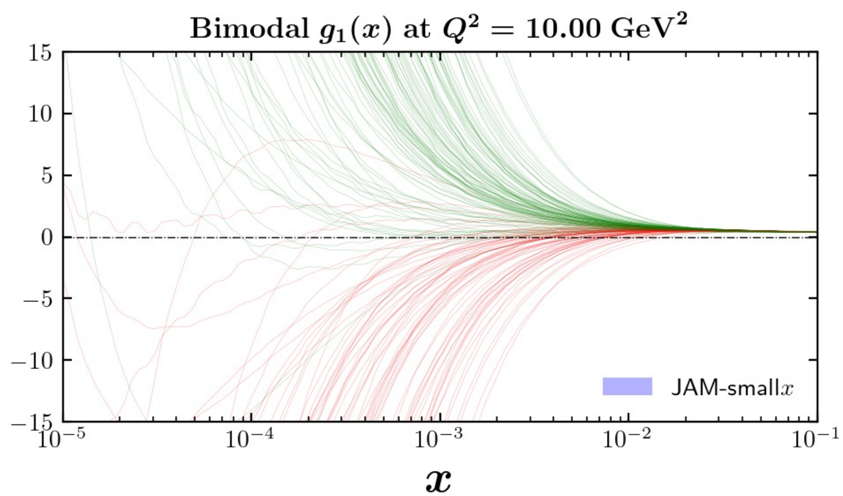
$$
\begin{aligned}
\widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) = \widetilde{G}^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \Big[3\, \widetilde{G}(x_{32}^2, z''s) \\
+ \widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + \left(2 - \frac{N_f}{2N_c} \right) \Gamma_2(x_{10}^2, x_{32}^2, z''s) - \frac{N_f}{4N_c}\, \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) \Big] \\
- \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z' x_{21}^2/x_{10}^2} \frac{dz''}{z''} \int\limits_{\max\{x_{10}^2,\, 1/z''s\}}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right],
\end{aligned}
$$

$$
G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\frac{z}{z'} x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\widetilde{G}(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],
$$

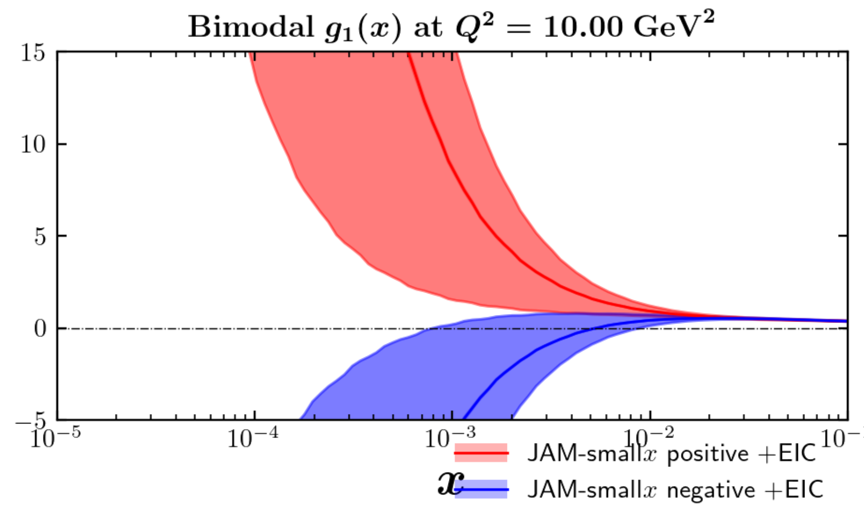
$$
\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\frac{z'}{z''} x_{21}^2} \frac{dx_{32}^2}{x_{32}^2} \left[\widetilde{G}(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]
$$

Proton g_1 structure function

JAM-smallx preliminary



Replicas



Error bands

- JAM is based on a Bayesian Monte-Carlo: it uses replicas.
- Due to the lack of constraints, everything is bi-modal!
- This lines in the middle – PDF extractions using EIC pseudo-data, it would resolve the bi-modality.

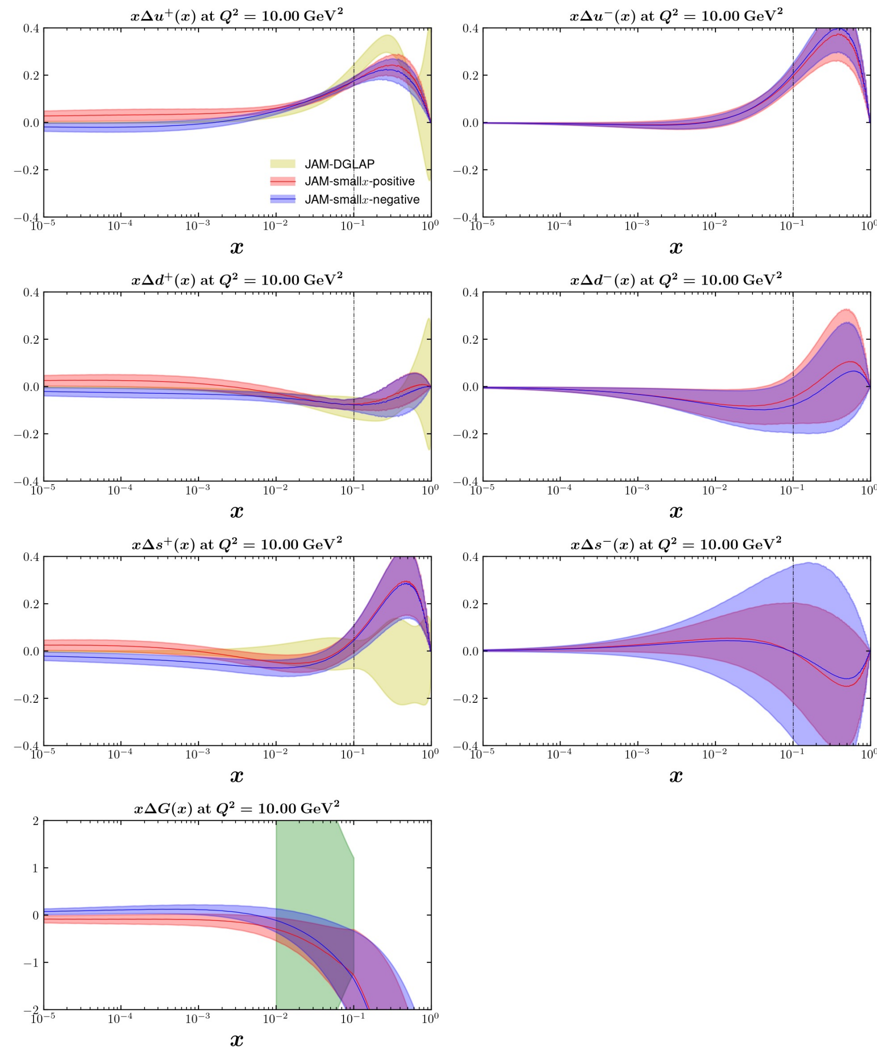
Helicity PDFs:

JAM-smallx preliminary

- Here too, everything is bi-modal.
- Again, lines in the middle are the EIC pseudo-data.

$$\Delta q^{+} = \Delta q + \Delta \bar{q}$$

$$\Delta q^{-} = \Delta q - \Delta \bar{q}$$



Sivers and Boer-Mulders Functions at Small x

YK, M. G. Santiago, 2209.03538 [hep-ph]

Sivers Function

- Similar technique can be applied to the Sivers function: first, simplify it at small x. This gives (for the flavor non-singlet Sivers function, at the sub-eikonal order)

$$
\begin{aligned}
& -\frac{\underline{k}\times \underline{S}_P}{M_P}\, f_{1\,T}^{\perp\,NS}(x,k_T^2)\Big|_{\text{sub-eikonal}} = \frac{16N_c}{(2\pi)^3}\int d^2x_{10}\,\frac{d^2k_{1\perp}}{(2\pi)^3}\,\frac{e^{i(\underline{k}+\underline{k}_1)\cdot\underline{x}_{10}}}{\underline{k}_1^2\,\underline{k}^2}\int\limits_{\frac{\Lambda^2}{s}}^{1}\frac{dz}{z} \\
& \times\Big\{\underline{k}_1\cdot\underline{k}\,(k-k_1)^i\,\Big[\epsilon^{ij}\,S_P^j\,x_{10}^2\,F_A^{NS}(x_{10}^2,z) + x_{10}^i\,\underline{x}_{10}\times\underline{S}_P\,F_B^{NS}(x_{10}^2,z) + \epsilon^{ij}\,x_{10}^j\,\underline{x}_{10}\cdot\underline{S}_P\,F_C^{NS}(x_{10}^2,z)\Big] \\
& + i\,\underline{k}_1\cdot\underline{k}\;\underline{x}_{10}\times\underline{S}_P\,F^{NS\,[2]}(x_{10}^2,z) - i\,\underline{k}\times\underline{k}_1\,\underline{x}_{10}\cdot\underline{S}_P\,F_{\text{mag}}^{NS}(x_{10}^2,z)\Big\}\,.
\end{aligned}
$$

- The dipole amplitudes F_A, F_B, F_C and F_{mag} are defined in terms of the same operators as the helicity dipole amplitudes. Except now the proton is transversely polarized.

Sivers Function

YK, M. G. Santiago, 2209.03538 [hep-ph]

- These amplitudes obey the following evolution equations at large N_c (plus 4 similar equations for the neighbor dipole amplitudes Γ):

$$F_A^{NS}(x_{10}^2, z) = F_A^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\frac{z}{z'}x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[6\, F_A^{NS}(x_{21}^2, z') - F_B^{NS}(x_{21}^2, z') + F_C^{NS}(x_{21}^2, z')\right], \tag{57a}$$

$$F_B^{NS}(x_{10}^2, z) = F_B^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\frac{z}{z'}x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[-2\, F_A^{NS}(x_{21}^2, z') + 5\, F_B^{NS}(x_{21}^2, z') - F_C^{NS}(x_{21}^2, z')\right], \tag{57b}$$

$$F_C^{NS}(x_{10}^2, z) = F_C^{NS\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\frac{z}{z'}x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\, F^{NS\,\mathrm{mag}}(x_{21}^2, z') + 6\, F_C^{NS}(x_{21}^2, z')\right], \tag{57c}$$

$$F^{NS\,\mathrm{mag}}(x_{10}^2, z) = F^{NS\,\mathrm{mag}\,(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\max\left\{\frac{\Lambda^2}{s}, \frac{1}{s x_{10}^2}\right\}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \tag{57d}$$

$$\times \left[\Gamma^{NS\,\mathrm{mag}}(x_{10}^2, x_{21}^2, z') + 2\,\Gamma_A^{NS}(x_{10}^2, x_{21}^2, z') - \Gamma_B^{NS}(x_{10}^2, x_{21}^2, z') + 3\,\Gamma_C^{NS}(x_{10}^2, x_{21}^2, z')\right].$$

Sivers Function: Small-x Asymptotics

- Solving those equations numerically, and adding the eikonal spin-dependent odderon contribution, we arrive at the following small-x asymptotics:

$$f_{1T}^{\perp NS}(x, k_T^2) = C_O(x, k_T^2)\,\frac{1}{x} + C_1(x, k_T^2)\left(\frac{1}{x}\right)^{3.4\sqrt{\frac{\alpha_s N_c}{4\pi}}} + \ldots$$

Spin-dependent Odderon (Boer, Echevarria, Mulders, Zhou, 2015)

Our sub-eikonal DLA contribution

- While the sub-eikonal contribution is nominally suppressed by a power of x (it is suppressed for zero α_s), the actual power in it for realistic α_s is close to 1.
- Thus, the DLA correction is almost as large as the leading eikonal spin-dependent odderon contribution.

YK, M. G. Santiago, 2209.03538 [hep-ph]

Boer-Mulders Function

- Similar procedure yields, for the Boer-Mulders function,

$$h_1^{\perp NS}(x \ll 1, k_T^2) \sim \left(\frac{1}{x}\right)^{-1}$$

- Boer-Mulders function is sub-eikonal at small x. Moreover, the DLA correction to the sub-eikonal power of x is zero.

- We observe that the intercepts of the leading small-x asymptotics terms for the T-odd leading-twist quark TMDs, the Sivers function and Boer-Mulders function, appear to receive no perturbative corrections (no α_s corrections). Could this be an exact statement in QCD?

YK, M. G. Santiago,
2209.03538 [hep-ph]

Conclusions

We have constructed new evolution equations in x for the quark and gluon helicity distributions. We can now predict the x-dependence of helicity PDFs at small x.

These equations have now been solved exactly analytically at large-N_c yielding the small-x asymptotics of $\Delta\Sigma(x, Q^2)$ and $\Delta G(x, Q^2)$. The study of the large-N_c&N_f equations and OAM distributions are under way.

First successful fit (JAMsmallx) of polarized world DIS data for x<0.1 done using solely the small-x helicity evolution. There is a clear possibility of a significant amount of proton spin to be found at small x, about 20% in the quark sector alone. This should help us solve the spin puzzle.

The same technique can be applied to other distribution functions. We have obtained the small-x asymptotics for the Sivers and Boer-Mulders functions, beyond the already known spin-dependent odderon contribution.

Backup Slides

Eikonality

- One can classify various TMDs by their small-x asymptotics.
- Eikonal behavior corresponds to (up to ~α_S corrections in the power)

$$f(x, k_T^2) \sim \frac{1}{x}$$

  Examples: unpolarized TMDs, Sivers function for gluons and quarks.
- Sub-eikonal behavior corresponds to

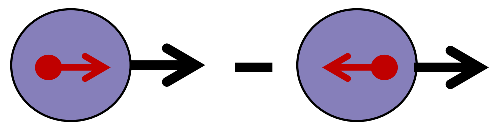
$$g(x, k_T^2) \sim \left(\frac{1}{x}\right)^0 = \text{const}$$

  Example: helicity TMDs.
- Sub-sub-eikonal behavior is $h(x, k_T^2) \sim x$
  Example: transversity.

- We've been calling the leading power of x "eikonality".

Helicity Distributions

- To quantify the contributions of quarks and gluons to the proton spin on defines helicity distribution functions: number of quarks/gluons with spin parallel to the proton momentum minus the number of quarks/gluons with the spin opposite to the proton momentum:



- The helicity parton distributions are

$$\Delta f(x, Q^2) \equiv f^+(x, Q^2) - f^-(x, Q^2)$$

with the flavor-singlet quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

and $\Delta G(x, Q^2)$ the gluon helicity distribution.

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

Small-x Spin Challenge

- Can we constrain theoretically the amount of proton spin and OAM coming from small x?

- Any existing and future experiment probes the helicity distributions and OAM down to some x_{min} .

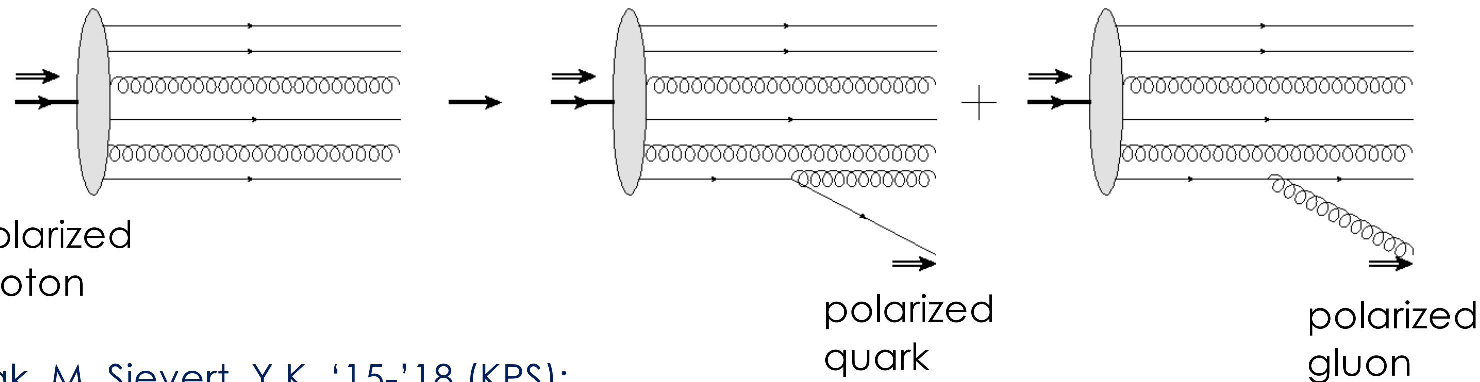
$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx \, \Delta G(x, Q^2)$$

$$L_{q+\bar{q}}(Q^2) = \int_0^1 dx \, L_{q+\bar{q}}(x, Q^2) \qquad L_G(Q^2) = \int_0^1 dx \, L_G(x, Q^2)$$

- At very small x (for the proton), saturation sets in: that region likely carries a negligible amount of proton spin. But what happens at larger (but still small) x?

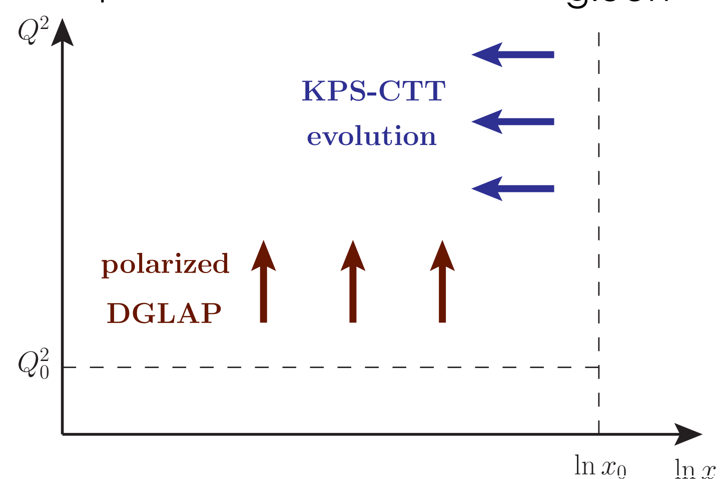
Helicity Evolution at Small x

- To understand how much of the proton's spin is at small x one can construct a helicity analogue of the BFKL equation:



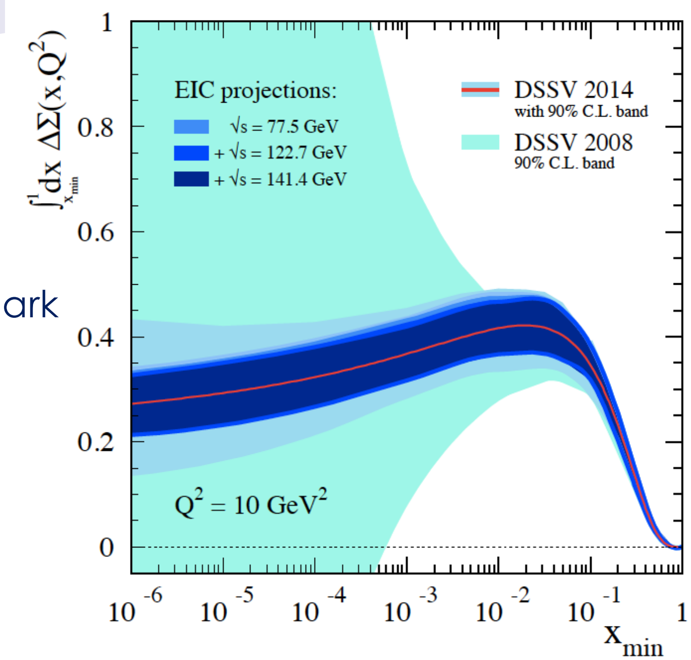
D. Pitonyak, M. Sievert, Y.K. '15-'18 (KPS);
F. Cougoulic, YK, A. Tarasov, Y. Tawabutr '22

- These new helicity evolution equations are subtle, since they must keep track of both quark and gluon helicities. (Other small-x evolution equations, BFKL/BK/JIMWLK, only have gluons at leading order.)

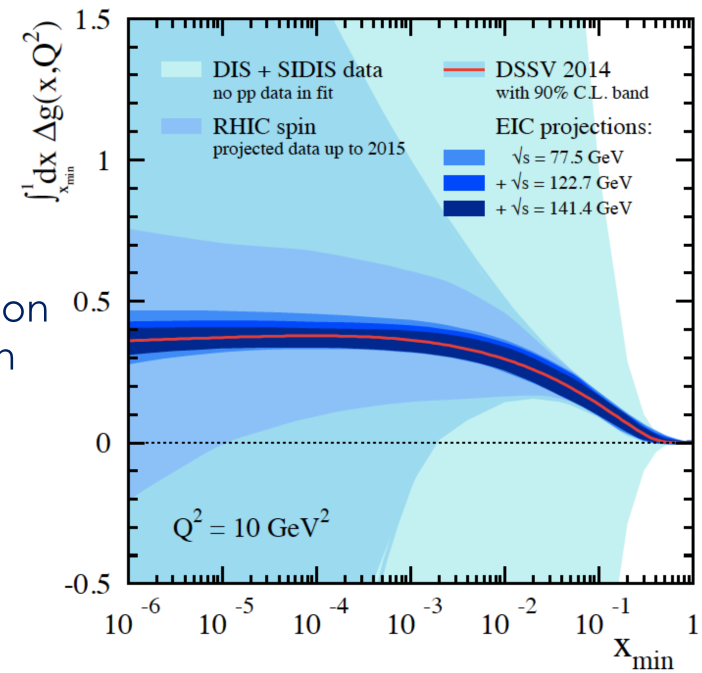


How much spin is there at small x?

2 x (quark spin)



gluon spin



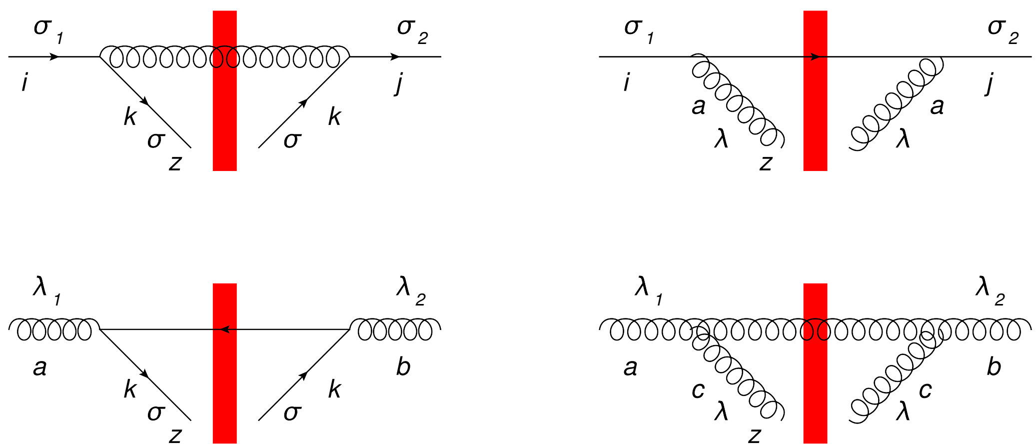
- E. Aschenauer et al, arXiv:1509.06489 **[hep-ph]**, (DSSV = de Florian, Sassot, Stratmann, Vogelsang, DGLAP-based helicity PDF extraction from data)
- Uncertainties are very large at small x! (EIC may reduce them.)

Philosophy of small-x approach to spin

- DGLAP equation evolves in Q^2, it does not evolve in x.
- Hence, DGLAP-based analyses (DSSV, NNPDF, standard JAM) cannot predict the x-dependence of PDFs.
- If we want to predict helicity PDFs at small x, we need a different evolution equation that evolves in x.
- Such equations were constructed by D. Pitonyak, M. Sievert, and YK, (KPS, 2015-2018) along with a recent important correction by F. Cougoulic, YK, A. Tarasov, Y. Tawabutr (KPS-CTT) in 2022; both works use an approach similar to the BK/JIMWLK evolution.
- Other important work on spin at small x: J. Bartels, B. Ermolaev, M. Ryskin '95-'96; Y. Hatta et al '16; R. Boussarie, Y. Hatta, F. Yuan '19; G. Chirilli '21. Significant related works by R. Kirschner and L. Lipatov '83; T. Altinoluk, G. Beuf et al '20, '21.

Helicity Evolution Ingredients

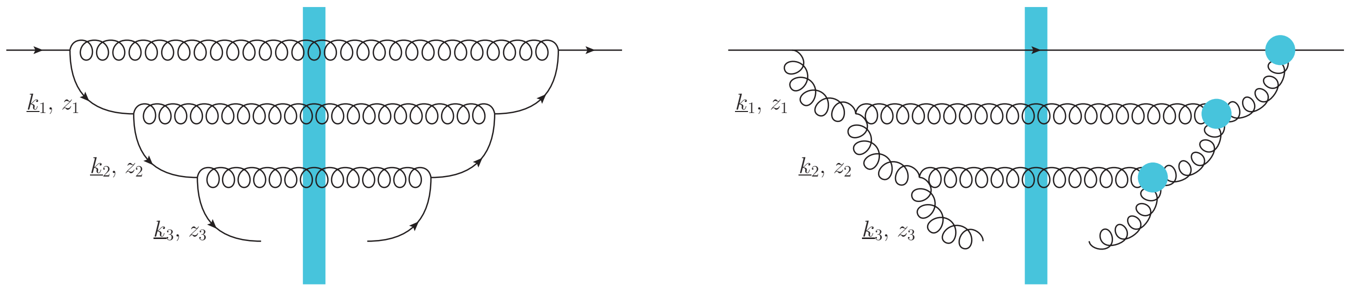
- Unlike the unpolarized evolution, in one step of helicity evolution we may emit a soft gluon or a soft quark (all in $A^{-}=0$ LC gauge of the projectile):



- When emitting gluons, one emission is eikonal, while another one is soft, but non-eikonal, as is needed to transfer polarization down the cascade/ladder.

Helicity Evolution: Ladders

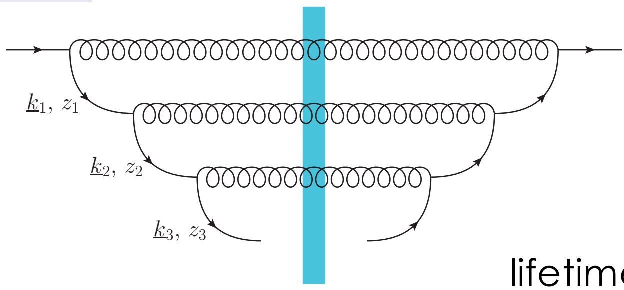
- To get an idea of how the helicity evolution works let us try iterating the splitting kernels by considering ladder diagrams (circles denote non-eikonal gluon vertices):

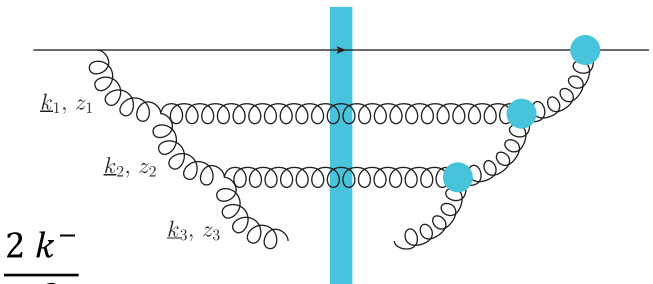


- To get the leading-energy asymptotics we need to order the longitudinal momentum fractions of the quarks and gluons (just like in the unpolarized evolution case) $1 \gg z_1 \gg z_2 \gg z_3 \gg \ldots$

  obtaining a nested integral $$\alpha_s^3 \int\limits_{z_i}^{1} \frac{dz_1}{z_1} \int\limits_{z_i}^{z_1} \frac{dz_2}{z_2} \int\limits_{z_i}^{z_2} \frac{dz_3}{z_3}\, z_3 \otimes \frac{1}{z_3\, s} \sim \frac{1}{s}\, \alpha_s^3\, \ln^3 s$$

Helicity Evolution: Ladders



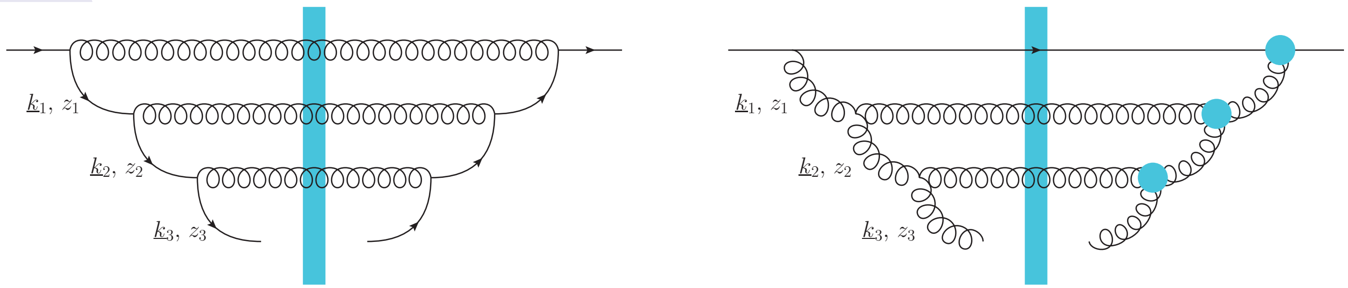


lifetime = $\frac{2\,k^-}{k_\perp^2}$

- However, these are not all the logs of energy one can get here. Transverse momentum (or distance) integrals have UV and IR divergences, which lead to logs of energy as well.
- If we order gluon/quark lifetimes as (Sudakov-β ordering) $\frac{2k_1^-}{\underline{k}_1^2} \gg \frac{2k_2^-}{\underline{k}_2^2} \gg \frac{2k_3^-}{\underline{k}_3^2} \gg \ldots$ then ($z_i = k_i^-/p^-$). $\frac{\underline{k}_1^2}{z_1} \ll \frac{\underline{k}_2^2}{z_2} \ll \frac{\underline{k}_3^2}{z_3} \ll \ldots$ and $z_1\,\underline{x}_1^2 \gg z_2\,\underline{x}_2^2 \gg z_3\,\underline{x}_3^2 \gg \ldots$

  we would get integrals like $\int\limits_{1/(z_n\,s)}^{x_{n-1,\perp}^2\,z_{n-1}/z_n} \frac{dx_{n,\perp}^2}{x_{n,\perp}^2}$ also generating logs of energy.

Helicity Evolution: Ladders



- To summarize, the above ladder diagrams are parametrically of the order

$$\frac{1}{s}\,\alpha_s^3\,\ln^6 s$$

- Note two features:
  - 1/s suppression due to non-eikonal exchange
  - two logs of energy per each power of the coupling!

Cross Checks

Cross Checks: spin-dependent DGLAP

- Our evolution at large Q^2 can be cross-checked against the small-x limit of the spin-dependent DGLAP equation.
- At DLA, ΔG is simply proportional to G_2:

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right) G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right) \right]_{x_{10}^2 = \frac{1}{Q^2}} \approx \frac{2N_c}{\alpha_s \pi^2} G_2 \left(\frac{1}{Q^2}, zs = \frac{Q^2}{x} \right)$$

- We solved our large-N_c equations iteratively, comparing $\ln(Q^2)$ terms with small-x spin-dependent DGLAP evolution.

$$\frac{\partial \Delta G(x, Q^2)}{\partial \ln Q^2} = \int\limits_x^1 \frac{dz}{z} \, \Delta P_{GG}(z) \, \Delta G \left(\frac{x}{z}, Q^2 \right)$$

- We obtained a complete agreement with LO+NLO+NNLO DGLAP at small x (R. Mertig, W.L. van Neerven '95; S. Moch, J.A.M. Vermaseren, A. Vogt, '14).

$$\Delta P_{GG}(z) = \frac{\alpha_s}{2\pi} 4N_c + \left(\frac{\alpha_s}{2\pi} \right)^2 4N_c^2 \ln^2 z + \left(\frac{\alpha_s}{2\pi} \right)^3 \frac{7}{3} N_c^3 \ln^4 z + \ldots .$$

We can predict higher orders.

Cross Checks: two methods, ΔG in g_1

- We rederived our evolution equations using the background field method.
- This is in addition to a blend of Brodsky & Lepage's LCPT and background field method-inspired operator treatment which I presented above (LCOT).
- Finally, one can even verify that ΔG enters the expression for g_1 with the right coefficient using $\Delta G \sim G_2$ and

$$g_1(x,Q^2) = -\sum_f \frac{N_c\, Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

(Adamiak, Melnitchouk, Pitonyak, Sato, Sievert, YK, 2102.06159 [hep-ph] = JAMsmallx)

Small-x Polarized DIS Data

$$g_1(x,Q^2) = \frac{1}{2}\sum_f e_f^2 \left[\Delta q_f + \Delta q_{\bar f}\right]$$

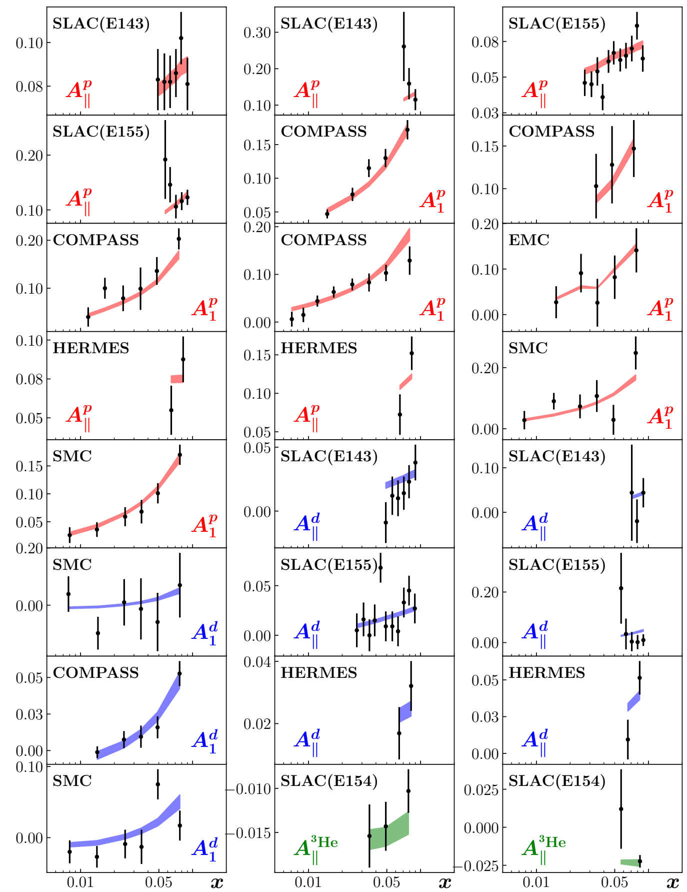
$$A_1 \sim A_{\parallel} = \frac{\sigma_{+-} - \sigma_{++}}{\sigma_{+-} + \sigma_{++}} \sim \frac{g_1}{F_1}$$

- We have analyzed all existing world polarized DIS data with x<0.1=x_0 , $Q^2 > m_c^2$ (122 data points) using the large-N_C KPS evolution with the Born-inspired initial conditions (8 parameters for 2 flavors, 11 parameters for 3 flavors).

$$G^{(0)}(x_{10}^2, z) \propto a_q \ln \frac{zs}{\Lambda^2} + b_q \ln \frac{1}{x_{10}^2 \Lambda^2} + c_q$$

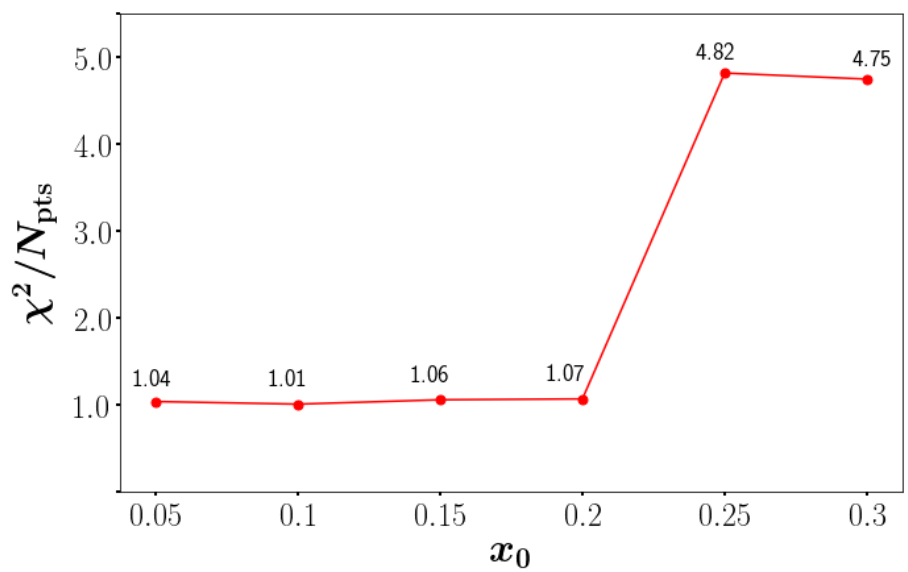
- It worked well, with χ^2/N_{pts} =1.01 (cf. JAM16: χ^2/N_{pts} =1.07)

- Small-x evolution starts at x_0=0.1 ! (cf. x_0=0.01 for unpolarized BK/JIMWLK evolution) Our approach fails at larger x as expected (x_0=0.3 gives χ^2/N_{pts} =4.75).

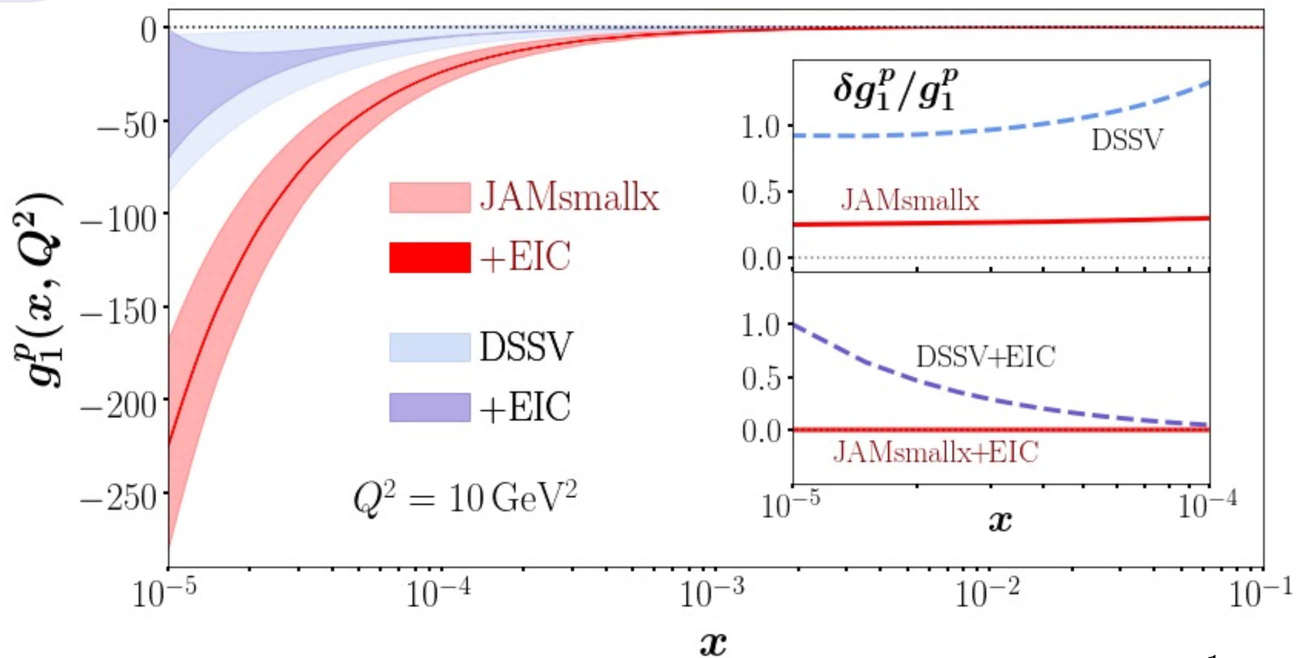


Where to start small-x evolution

- The evolution starts at $x=x_0$, and continues toward smaller x.
- The quality of our fit rapidly deteriorates for $x_0>0.2$, as expected from a small-x approach.
- In unpolarized BK/JIMWLK evolution, typically $x_0=0.01$, so the fact that our fit works up to such a high x_0 is quite remarkable.



Prediction for g_1 structure function

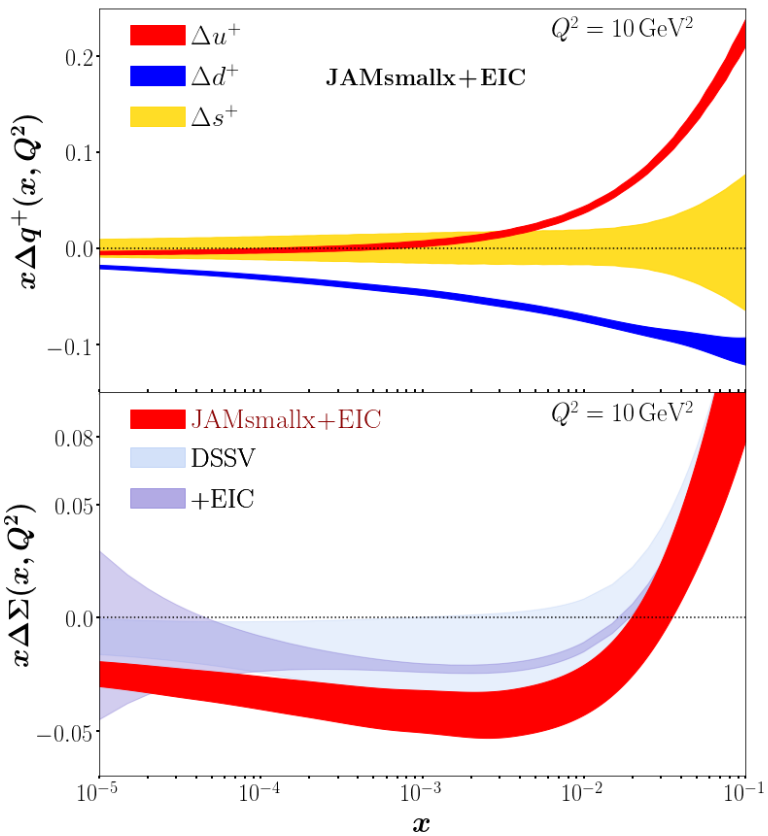


g_1 structure function ≈ spin-dependent part of σ^{e+p}

$$g_1(x, Q^2) = \frac{1}{2} \sum_f e_f^2 \left[\Delta q_f + \Delta q_{\bar{f}}\right]$$

Thick band: 1σ CL; thin band: impact of EIC data. With the EIC pseudo-data we have 1096 data points.

Predictions for helicity PDFs



D. Adamiak, W. Melnitchouk, D. Pitonyak, N. Sato, M. Sievert & YK, 2102.06159 [hep-ph], in the JAM Collaboration framework.

$$\Delta q^+ = \Delta q + \Delta \bar{q}$$

$$\Delta\Sigma(x, Q^2) = \sum_f \left[\Delta q_f + \Delta q_{\bar{f}}\right]$$

Our (red) error band does not explode in the unmeasured region. We can **predict** spin at small x.

If we plug in $\alpha_s = 0.25$ we get $\alpha_h^q = 0.80$

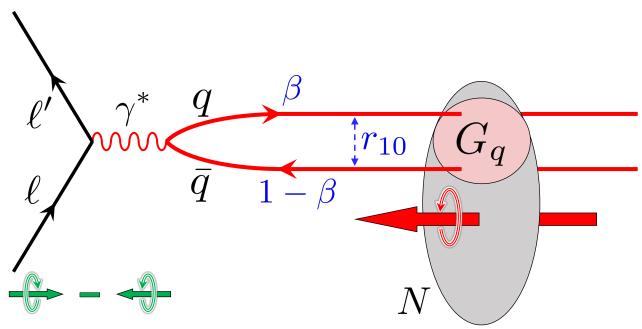
old intercept!

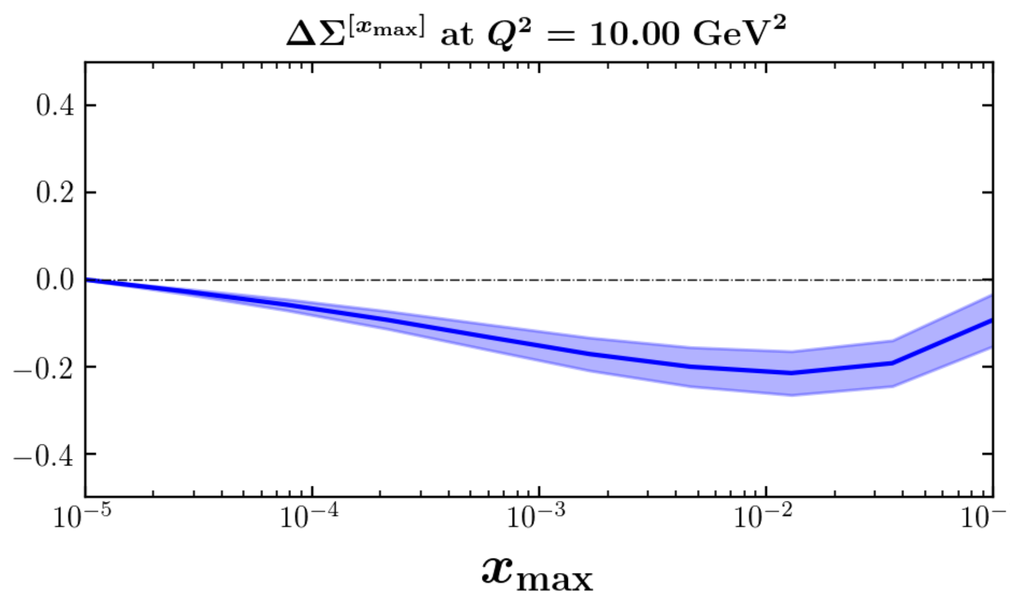
$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Small-x quarks impact on the proton spin

- Potentially negative 10-20% of the proton spin may be carried by small-x quarks (JAMsmallx, preliminary):

$$\Delta\Sigma^{[x_{max}]}(Q^2) = \int_{10^{-5}}^{x_{max}} dx\, \Delta\Sigma(x, Q^2)$$





Speculation on a path to resolving the spin puzzle

- Above we discussed quark helicity at small x. Let's add the orbital angular momentum (OAM) (Hatta & Yang, '18; YK '19):

$$\frac{1}{2}\,\Delta\Sigma(x,Q^2) + L_{q+\bar{q}}(x,Q^2) = -\frac{1}{2}\,\Delta\Sigma(x,Q^2)$$

- So, the net quark (1/2) helicity+OAM = (-1/2) helicity.
- For x<0.001 we thus expect (preliminary!)

$$\left[\frac{1}{2}\Delta\Sigma + L_{q+\bar{q}}\right]_{Q^2=10\,\mathrm{GeV}^2,\ x<0.001} \approx -\frac{1}{2}\,(-0.2) = 0.1$$

- Add to this the larger-x numbers

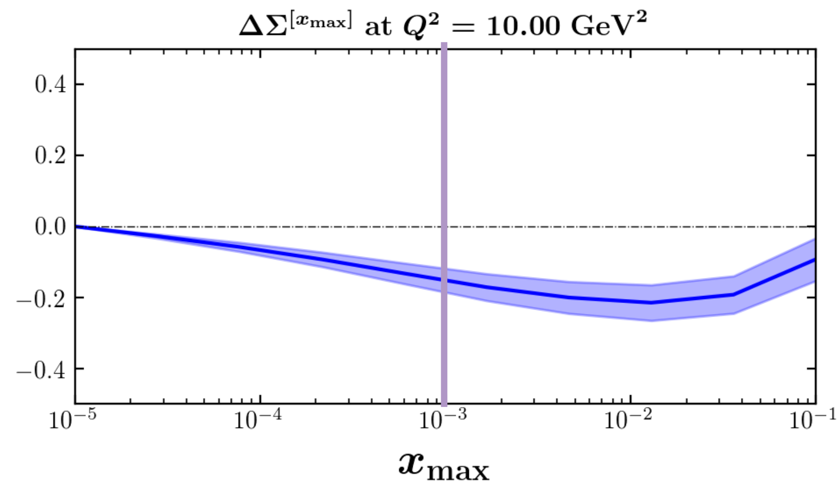
$$S_q(Q^2 = 10\,\mathrm{GeV}^2, x > 0.001) \approx 0.18$$

$$S_G(Q^2 = 10\,\mathrm{GeV}^2, x > 0.05) \approx 0.2$$

- We get

$$0.18 + 0.2 + 0.1 = 0.48$$

JAMsmallx, preliminary, Adamiak, Melnitchouk, Pitonyak, Sato, Sievert, YK



A warning

- Note that the above phenomenology was done using the old version of the evolution equations. Consider it a proof of principle.

- Phenomenology with the new corrected evolution equations is being developed as we speak.