Gradient flow exact renormalization group

Hiroshi Suzuki (Kyushu University)

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The gradient flow in QCD and other strongly coupled field theories
@ ECT*

- H. Sonoda (Kobe Univ.), H.S.,
  PTEP **2019**, no.3, 033B05 (2019) [arXiv:1901.05169 [hep-th]]
  PTEP **2021**, no.2, 023B05 (2021) [arXiv:2012.03568 [hep-th]]
  PTEP **2022**, no.5, 053B01 (2022) [arXiv:2201.04448 [hep-th]]
- Y. Miyakawa, H.S.,
  PTEP **2021**, no.8, 083B04 (2021) [arXiv:2106.11142 [hep-th]]
- Y. Miyakawa, H. Sonoda, H.S.,
  PTEP **2022**, no.2, 023B02 (2022) [arXiv:2111.15529 [hep-th]]
- and works in progress

K. Wilson's Exact Renormalization Group (ERG)

- Effective interactions under the change of the scale:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{S_\tau} \sim e^{n[(D-2)/2](\tau-\tau_0)} Z(\tau,\tau_0)^n \langle \phi(e^{\tau-\tau_0} x_1) \cdots \phi(e^{\tau-\tau_0} x_n) \rangle_{S_{\tau_0}}$$

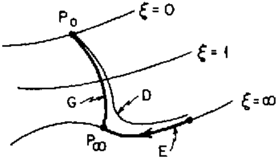

Fig. 12.6. Renormalization group trajector

- Continuum QFT: $\xi = \xi_0 |K - K_c|^{1/y_E}$:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_g$$
$$\equiv \lim_{\tau_0 \to -\infty} e^{n[(D-2)/2](\tau-\tau_0)} Z(\tau,\tau_0)^n \langle \phi(e^{\tau-\tau_0} x_1) \cdots \phi(e^{\tau-\tau_0} x_n) \rangle_{S_{\tau_0}, K = K_c - g e^{-y_E(\tau-\tau_0}}$$

- Non-perturbative fixed point relevant to particle physics?
- Many-flavor gauge theories (Banks-Zaks fixed point)?
- Asymptotically-safe gravity?
- Gauge symmetry is essential. . .

ERG: Polchinski equation in scalar theory

- Smooth momentum cutoff such as

$$K(p/\Lambda) = e^{-p^2/\Lambda^2}$$

- "Integrate out" momentum modes $|p| > \Lambda$ to yield the Wilson action $S_\Lambda[\phi]$
- $S_\Lambda[\phi]$: reaction under the change of the cutoff Λ
- Make everything dimensionless by taking Λ as the unit
- $S_\tau[\phi]$ ($\tau \sim -\ln\Lambda$): reaction under the change of the scale
- Polchinski equation:

$$\frac{\partial}{\partial\tau} e^{S_\tau[\phi]} = \int d^D x \left(-2\partial^2 - \frac{D-2}{2} - \gamma_\tau - x \cdot \frac{\partial}{\partial x} \right) \phi(x) \cdot \frac{\delta}{\delta\phi(x)} e^{S_\tau[\phi]} + \int d^D x \, (-2\partial^2 + 1 - \gamma_\tau) \frac{\delta}{\delta\phi(x)} \cdot \frac{\delta}{\delta\phi(x)} e^{S_\tau[\phi]}$$

(We have generalized as $K(p)[1-K(p)] \to p^2$ and introduced the anomalous dimension by $\gamma_\tau \equiv \partial_\tau \ln Z(\tau,\tau_0)$)

- Huge application in critical phenomena...

ERG in gauge theory

- Local gauge transformation

$$A^a_\mu(k) \to A^a_\mu(k) + ik_\mu \chi^a(k) - g \int_q f^{abc}\chi^b(q) A^c_\mu(k-q)$$

$$\psi(p) \to \psi(p) - g \int_q \chi^a(q) T^a \psi(p-q)$$

  mixes modes with different momenta and the conventional ERG does not keep a manifest gauge symmetry
- ERG keeps a modified gauge symmetry (Becchi, Ellwanger, Bonini-D'Attanasio-Marchesini, Reuter-Wetterich, Higashi-Itou-Kugo, Igarashi-Itoh-Sonoda), but its precise form depends on the Wilson action itself!
- This prevents us to take a gauge-invariant ansatz (truncation) for the Wilson action...
- ... critical exponents can depend on the gauge fixing parameter...
- We want ERG that keeps a manifest gauge symmetry

Representation of the Wilson action by the field diffusion

- "Integral representation" of the Wilson action:

$$e^{S_\tau[\phi]}$$
$$= \hat{s} \int [d\phi'] \prod_x \delta\left(\phi(x) - e^{\int_{\tau_0}^{\tau} d\tau'\,[(D-2)/2+\gamma_{\tau'}]} \phi'(t - t_0, e^{\tau-\tau_0} x)\right) (\hat{s}')^{-1} e^{S_{\tau_0}[\phi']}$$

Here, $\phi'(t, x)$ is the solution to the diffusion equation

$$\partial_t \phi'(t, x) = \partial^2 \phi'(t, x), \qquad \phi'(0, x) = \phi'(x),$$

where the diffusion time is given by

$$t - t_0 = e^{2(\tau-\tau_0)} - 1,$$

and the scrambler

$$\hat{s} \equiv \exp\left[\frac{1}{2} \int d^D x \, \frac{\delta^2}{\delta\phi(x)\delta\phi(x)}\right]$$

- ERG and the field diffusion: Abe-Fukuma, Carosso-Hasenfratz-Neil, Matsumoto-Tanaka-Tsuchiya

What happens with a gauge-covariant diffusion equation?

- Yang-Mills gradient flow (Narayanan-Neuberger, Lüscher):

$$\partial_t A'^a_\mu(t,x) = D'_\nu F'^a_{\nu\mu}(t,x) = \partial^2 A'^a_\mu(t,x) + \cdots, \qquad A'^a_\mu(0,x) = A'^a_\mu(x)$$

- For fermion (Lüscher):

$$\partial_t \psi'(t,x) = D'_\mu D'_\mu \psi'(t,x) \qquad \psi'(0,x) = \psi'(x)$$

$$\partial_t \bar\psi'(t,x) = \bar\psi'(t,x) \overleftarrow{D}'_\mu \overleftarrow{D}'_\mu \qquad \bar\psi'(0,x) = \bar\psi'(x)$$

- Simply imitating the scalar theory,

$$\begin{aligned}
&e^{S_\tau[A,\psi,\bar\psi]} \\
&= \hat{s} \int [dA' d\psi' d\bar\psi'] \\
&\quad \times \prod_{x,\mu,a} \delta\left(A^a_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau'\,[(D-2)/2+\gamma_{\tau'}]} A'^a_\mu(t-t_0, e^{\tau-\tau_0}x) \right) \\
&\quad \times \prod_{x} \delta\left(\psi(x) - e^{\int_{\tau_0}^{\tau} d\tau'\,[(D-1)/2+\gamma_{F\tau'}]} \psi'(t-t_0, e^{\tau-\tau_0}x) \right) \\
&\quad \times \prod_{x} \delta\left(\bar\psi(x) - e^{\int_{\tau_0}^{\tau} d\tau'\,[(D-1)/2+\gamma_{F\tau'}]} \bar\psi'(t-t_0, e^{\tau-\tau_0}x) \right) (\hat{s}')^{-1} e^{S_{\tau_0}[A',\psi',\bar\psi']}
\end{aligned}$$

$$\hat{s} \equiv \exp\left[\frac{1}{2} \int d^D x \frac{\delta^2}{\delta A^a_\mu(x) \delta A^a_\mu(x)} \right] \exp\left[-i \int d^D x \frac{\overrightarrow{\delta}}{\delta\psi(x)} \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} \right]$$

We term this, Gradient Flow Exact Renormalization Group (GFERG)

- Actually, to diffuse the gauge modes (Zwanziger term),

$$\partial_t A'^a_\mu(t,x) = D'_\nu F'^a_{\nu\mu}(t,x) + \alpha_0 D'_\mu \partial_\nu A'^a_\nu(t,x) \qquad \text{etc.}$$

- RG evolution keeps a manifest gauge symmetry: If S_{τ_0} is invariant under ($g_\tau \equiv e^{-\int^\tau d\tau' \, [(D-4)/2+\gamma_{\tau'}]}$)

$$\begin{aligned} A^a_\mu(x) &\to A^a_\mu(x) + \partial_\mu \chi^a(x) + g_\tau f^{abc} A^b_\mu(x)\chi^c(x) \\ \psi(x) &\to \psi(x) - g_\tau \chi^a(x) T^a \psi(x) \\ \bar\psi(x) &\to \bar\psi(x) + g_\tau \chi^a(x) \bar\psi(x) T^a \end{aligned}$$

  then S_τ is invariant too.

- RG evolution keeps a modified chiral symmetry: If S_{τ_0} satisfies

$$\int d^D x \left\{ S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \gamma_5 \psi(x) + \bar\psi(x)\gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau \right. \\ \left. + 2iS_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau - 2i \,\mathrm{tr}\left[\gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \right] \right\} = 0$$

  then S_τ does too. This is a generalization of the Ginsparg-Wilson relation

Polchinski equation in GFERG

- Taking the τ derivative of the integral representation,

$$
\begin{aligned}
&\frac{\partial}{\partial\tau} e^{S_\tau[A,\psi,\bar{\psi}]} \\
&= \int d^D x\, \frac{\delta}{\delta A^a_\mu(x)} \Big[-2 D_\nu F^a_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A^a_\nu(x) \\
&\qquad\qquad - \left(\frac{D-2}{2} + \gamma_\tau + x\cdot\frac{\partial}{\partial x} \right) A^a_\mu(x) \Big] \Big|_{A\to A+\delta/\delta A} e^{S_\tau[A,\psi,\bar{\psi}]} \\
&\qquad + (\text{fermion})
\end{aligned}
$$

- This contains functional derivatives up to 4th order (conventional ERG contains only up to 2nd order)
- The price of the manifest gauge symmetry...

1PI action Γ_τ

- Usually, in non-perturbative studies in ERG, the so-called 1PI action Γ_τ (Nicoll-Chang, Wetterich, Morris, Bonini-D'Attanasio-Marchesini) is employed
- We can define the corresponding Legendre transf. in GFERG:

$$
\begin{aligned}
\mathcal{A}_\mu(x) &\equiv A_\mu(x) + \frac{\delta S_\tau}{\delta A_\mu(x)} = e^{-S_\tau}\hat{s}A_\mu(x)\hat{s}^{-1}e^{S_\tau} \\
\Psi(x) &\equiv \psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}S_\tau = e^{-S_\tau}\hat{s}\psi(x)\hat{s}^{-1}e^{S_\tau} \\
\bar{\Psi}(x) &\equiv \bar{\psi}(x) + iS_\tau\frac{\overleftarrow{\delta}}{\delta\psi(x)} = e^{-S_\tau}\hat{s}\bar{\psi}(x)\hat{s}^{-1}e^{S_\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_\tau[\mathcal{A}_\mu, \Psi, \bar{\Psi}] - \frac{1}{2}\int d^Dx\, \mathcal{A}_\mu(x)\mathcal{A}_\mu(x) + i\int d^Dx\, \bar{\Psi}(x)\Psi(x) \\
&\equiv S_\tau[A_\mu, \psi, \bar{\psi}] + \frac{1}{2}\int d^Dx\, A_\mu(x)A_\mu(x) - i\int d^Dx\, \bar{\psi}(x)\psi(x) \\
&\qquad - \int d^Dx\, \mathcal{A}_\mu(x)A_\mu(x) + i\int d^Dx\, \left[\bar{\Psi}(x)\psi(x) + \bar{\psi}(x)\Psi(x)\right]
\end{aligned}
$$

- Keeps the manifest gauge symmetry and the chiral symmetry
- GFERG equation is however quite involved. . .

(Formal) equivalence to the RG flow defined through the gradient flow

- In the continuum limit, defined by

$$\tau_0 \to -\infty, \qquad g_{\tau_0} \to 0,$$

at least formally,

$$\int [dA]\, e^{S_\tau[A]} \hat{s}^{-1} \left[g_\tau A^{a_1}_{\mu_1}(x_1) \cdots g_\tau A^{a_n}_{\mu_n}(x_n) \right]$$
$$= \Lambda^{-n} \int [d\tilde{A}]\, e^{S_{\Lambda_0}[\tilde{A}]} \tilde{A}^{a_1}_{\mu_1}(1/\Lambda^2, x_1/\Lambda) \cdots \tilde{A}^{a_n}_{\mu_n}(1/\Lambda^2, x_n/\Lambda),$$

where

$$\Lambda \equiv e^{-\tau} e^{\tau_0} \Lambda_0, \qquad \Lambda_0 \to \infty,$$

is kept fixed in the continuum limit, and the dimensionful field

$$\tilde{A}^a_\mu(\tilde{t}, \tilde{x}), \qquad \tilde{A}^a_\mu(\tilde{t} = 0, \tilde{x}) \equiv \tilde{A}^a_\mu(\tilde{x}) = \Lambda_0 g_{\tau_0} A^a_\mu(x),$$

obeys the Yang-Mills gradient flow

- RHS: RG flow defined through the gradient flow around the Gaussian fixed point (Lüscher, Makino-Morikawa-H.S.) and even non-perturbative (Carosso-Hasenfratz-Neil, Carosso)
- This is “formal” because I neglected the issue of gauge fixing. . .

Perturbative analysis

- Perturbative expansion around the Gauss fixed point (w.r.t. g_τ)

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- $D = 4$ Yang-Mills theory to $O(g_\tau^2)$, we had the beta function (Sonoda-H.S., unpublished).

$$\gamma_\tau = -\frac{\beta_\tau}{2g_\tau^2} = -\frac{1}{(4\pi)^2}\frac{7}{2}C_A g_\tau^2, \qquad f^{acd}f^{bcd} = C_A\delta^{ab}$$

This is not the expected one.

$$\gamma_\tau = -\frac{\beta_\tau}{2g_\tau^2} = -\frac{1}{(4\pi)^2}\frac{11}{3}C_A g_\tau^2$$

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- The gauge fixing is necessary in S_{τ_0}?

$$\left\langle A_\mu^a(x)A_\nu^b(y)\right\rangle_0 \sim \delta^{ab}\int_k e^{ik(x-y)}\frac{1}{k^2}\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \xi_\tau\frac{k_\mu k_\nu}{k^2}\right]$$

and no gauge fixing $\rightarrow \xi_\tau = \infty$

GFERG with the gauge fixing?

- Introduce Faddeev-Popov (FP) ghost-anti-ghost and Nakanishi-Lautrup (NL) field
- It is easy to make the diffusion equations invariant under the conventional BRST:

$$
\begin{aligned}
\delta A_\mu^a(x) &= \partial_\mu c^a(x) + g_\tau f^{abc} A_\mu^b(x) c^c(x) \\
\delta c^a(x) &= -\frac{1}{2} g_\tau f^{abc} c^b(x) c^c(x) \\
\delta \bar{c}^a(x) &= B^a(x) \\
\delta B^a(x) &= 0
\end{aligned}
$$

- However, the natural choice

$$
\hat{s} \equiv \exp\left[\int d^D x \, \frac{1}{2} \frac{\delta^2}{\delta A_\mu^a(x) \delta A_\mu^a(x)}\right] \times \exp\left[-\int d^D x \, \frac{\delta}{\delta c^a(x)} \frac{\delta}{\delta \bar{c}^a(x)}\right] \exp\left[-\int d^D x \, \frac{1}{2} \frac{\delta^2}{\delta B^a(x) \delta B^a(x)}\right]
$$

breaks the BRST symmetry $\rightarrow$ (again) modified BRST symmetry...

At least in QED, we can circumvent the difficulty

- We can eliminate NL field
- FP ghost sector completely decouples and solvable
- BRST symmetry reduces to the WT identity

$$ik_\mu \frac{\delta S_\tau}{\delta A_\mu(k)} + \frac{k^2}{\xi_\tau E(e^{-2\tau}k^2)e^{-2k^2}} ik_\mu \left[A_\mu(-k) + \frac{\delta S_\tau}{\delta A_\mu(k)} \right]$$
$$+ ig_\tau \int_p S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(p+k)} \psi(p) - ig_\tau \int_p \bar{\psi}(-p-k) \frac{\delta}{\delta\bar{\psi}(-p)} S_\tau = 0$$

This is linear in the Wilson action.

- WT identity in the conventional ERG is infinite order in the Wilson action
- Perturbative analysis to $O(g_\tau^2)$: The beta function,

$$\beta = -2\gamma g^2 = -\frac{1}{(4\pi)^2}\frac{8}{3}g^4 + \cdots$$

Anomalous dimensions associated with the electron:

$$\beta_m = \frac{6}{(4\pi)^2} g^2 + \cdots \qquad \gamma_F = \frac{3}{(4\pi)^2} g^2 + \cdots$$

The latter coincides with the one for the flowed electron field (Lüscher)

Possible application to the Adler-Bardeen theorem ('t Hooft anomaly is not renormalized)

- $D = 4$, massless fermion
- Assume the global symmetry is $SU(N)_L \times SU(N)_R$
- Introduce external gauge and ghost fields

$$L_\mu^A(x), \quad R_\mu^A(x), \quad \chi_L^A(x), \quad \chi_R^A(x)$$

- Scrambler $\hat{s}$ does not contain these non-dynamical fields
- Generator of the external BRST transf.

$$\begin{aligned}\hat{\delta} \equiv \int d^D x \Bigg\{ & \left[\partial_\mu \chi_L^A(x) + f^{ABC} L_\mu^B(x) \chi_L^C(x)\right] \frac{\delta}{\delta L_\mu^A(x)} \\ & - \frac{1}{2} f^{ABC} \chi_L^B(x) \chi_L^C(x) \frac{\delta}{\delta \chi_L^A(x)} \\ & - \chi_L^A(x) t^A P_L \psi(x) \frac{\delta}{\delta \psi(x)} + \chi_L^A(x) \bar{\psi}(x) P_R t^A \frac{\delta}{\delta \bar{\psi}(x)} \Bigg\} \\ & + \text{(right-handed part)}\end{aligned}$$

- Modified BRST transformation is given by $\tilde{\hat{\delta}} = \hat{s}\hat{\delta}\hat{s}^{-1}$

Possible application to the Adler-Bardeen theorem

- Since the diffusion equations can be BRST invariant, for the 't Hooft anomaly,

$$\tilde{\hat{\delta}} e^{S_\tau} = \hat{s} \int [dA' d\psi' d\bar{\psi}' dL' d\chi'_L dR' d\chi'_R] \,(\text{delta functions})(\hat{s}')^{-1} \tilde{\hat{\delta}}' e^{S_{\tau_0}}$$

- This is identical relation to the Wilson action:

$$e^{S_\tau} = \hat{s} \int [dA' d\psi' d\bar{\psi}' dL' d\chi'_L dR' d\chi'_R] \,(\text{delta functions})(\hat{s}')^{-1} e^{S_{\tau_0}}$$

- Thus, the Wilson action infinitesimally deformed by the anomaly

$$e^{S_\tau} - \eta \tilde{\hat{\delta}} e^{S_\tau} = \exp\left(S_\tau - \eta e^{-S_\tau} \tilde{\hat{\delta}} e^{S_\tau} \right)$$

  is also the solution of the GFERG
- This shows that the anomaly

$$\mathcal{Q}_{\chi_L, \chi_R} \equiv -e^{-S_\tau} \tilde{\hat{\delta}} e^{S_\tau}$$

  is a composite operator with the scaling dimension $D = 4$
- If S_τ is local, then $\mathcal{Q}_{\chi_L, \chi_R}$ is local

Possible application to the Adler-Bardeen theorem

- First assume the anomaly $\mathcal{Q}_{\chi_L,\chi_R}$ is a local functional of external fields
- It can be seen that diffused external fields with $t - t_0 = -1$,

$$L^A_\mu(-1,x) \quad R^A_\mu(-1,x) \quad \chi^A_L(-1,x) \quad \chi^A_R(-1,x)$$

  are and their local products are composite operators
- and they obey the simple BRST transf.

$$\delta L^A_\mu(-1,x) = \partial_\mu \chi^A_L(-1,x) + f^{ABC} L^B_\mu(-1,x)\chi^C_L(-1,x)$$
$$\delta \chi^A_L(-1,x) = -\frac{1}{2} f^{ABC} \chi^B_L(-1,x)\chi^C_L(-1,x)$$
$$\delta R^A_\mu(-1,x) = \partial_\mu \chi^A_R(-1,x) + f^{ABC} R^B_\mu(-1,x)\chi_L R^C(-1,x)$$
$$\delta \chi^A_R(-1,x) = -\frac{1}{2} f^{ABC} \chi^B_R(-1,x)\chi^C_R(-1,x)$$

- Moreover the anomaly obeys the Wess-Zumino consistency

$$\delta \mathcal{Q}_{\chi_L,\chi_R} = 0$$

Possible application to the Adler-Bardeen theorem

- The general solution is

$$\mathcal{Q}_{\chi_L,\chi_R} = c \int d^4x\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\left[\chi_5(-1,x) F_{v,\mu\nu}(-1,x) F_{v,\rho\sigma}(-1,x) + \cdots\right],$$

  where

$$\chi_5(x) \equiv \frac{1}{2}\left[\chi_R(x) - \chi_L(x)\right], \qquad v_\mu(x) \equiv \frac{1}{2}\left[R_\mu(x) + L_\mu(x)\right]$$

- Then, from GFERG,

$$\frac{d}{d\tau} c = 0$$

  't Hooft anomaly does not depend on the renormalization scale

- Thus, cannot depend on the gauge coupling
- Form the lowest order calculation (Y. Miyakawa, arXiv:2201.08181)

$$c = \frac{1}{16\pi^2}$$

Summary

- We formulated GFERG, which keeps a manifest gauge symmetry and the modified chiral symmetry, starting from a connection between ERG and the field diffusion
- We can formulate the corresponding 1PI formalism
- We can argue that GFERG is basically equivalent to the familiar RG flow defined through the gradient flow (Lüscher, Makino-Morikawa-H.S., Carosso-Hasenfratz-Neil, Carosso, Kitazawa-H.S., . . .)

Future issues

- The issue of the gauge fixing term in S_τ
- Presumably, it is necessary at least in perturbative treatment $\Rightarrow$ a manifest BRST symmetry in the FP ghost sector is difficult
- We can circumvent this difficulty at least in QED $\Rightarrow$ reconsideration of non-trivial fixed points in QED (Aoki-Morikawa-Sumi-Terao-Tomoyose, Gies-Jaeckel, Igarashi-Itoh-Pawlowski, Gies-Ziebell, . . .)
- With the gauge fixing but no FP ghost? (cf. stochastic quantization)
- Application in non-Abelian gauge theory?
- Generalization to gravity?

Backup: “Composite operator”

- Composite operator $\mathcal{O}_\tau(x)$ (Wilson, Wegner, Becchi, ...) is a combination that possesses a simple scaling law under ERG ($-y_\tau$ is scaling dimension):

$$\int [d\phi]\, e^{S_\tau[\phi]} \mathcal{O}_\tau(x) \hat{s}^{-1} \left[e^{-\partial^2} \phi(x_1) \cdots e^{-\partial^2} \phi(x_n) \right]$$

$$= e^{-\int_{\tau_0}^{\tau} d\tau'\, y_{\tau'}} Z(\tau, \tau_0)^n$$

$$\times \int [d\phi]\, e^{S_{\tau_0}[\phi]} \mathcal{O}_{\tau_0}(e^{\tau-\tau_0} x) \hat{s}^{-1} \left[e^{-\partial^2} \phi(x_1) \cdots e^{-\partial^2} \phi(x_n) \right]_{x_i \to e^{\tau-\tau_0} x_i}$$

- From the definition, it obeys

$$\left(\partial_\tau - x \cdot \frac{\partial}{\partial x} + y_\tau - \mathcal{D}_\tau \right) \mathcal{O}_\tau(x) = 0$$

$$\mathcal{D}_\tau \mathcal{O}_\tau(x)$$

$$\equiv -e^{-S_\tau} \left[\hat{s} \int d^D x\, \frac{\delta}{\delta \phi(x)} \left(2\partial^2 + \frac{D-2}{2} + \gamma_\tau + x \cdot \frac{\partial}{\partial x} \right) \phi(x) \hat{s}^{-1} e^{S_\tau}, \mathcal{O}_\tau(x) \right]$$

- This can be regarded as an infinitesimal deformation of the Wilson action:

$$S_\tau[\phi] \to S_\tau[\phi] + e^{\int^\tau d\tau'\, y_{\tau'}} \int d^D x\, \epsilon(x) \mathcal{O}_\tau(e^{-\tau} x)$$