Selected topics of the gradient flow in perturbation theory

Robert Harlander

RWTH Aachen University

Gradient Flow Workshop at ECT*

22 March 2023

see also talks by Janosch Borgulat and Fabian Lange

supported by: DFG Deutsche Forschungsgemeinschaft

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

5-dimensional field theory

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

5-dimensional field theory

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

$$\frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

5-dimensional field theory

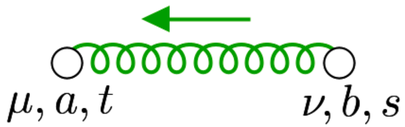
$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

5-dimensional field theory

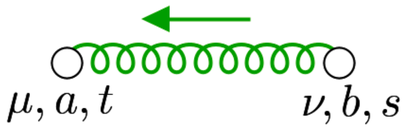
$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

analogously for quarks: Lüscher 2013

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \Delta\right)\chi + \mathrm{h.c.}$$

5-dimensional field theory

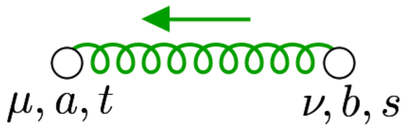
$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

L_μ Lagrange multiplier field

Lüscher, Weisz 2011

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$



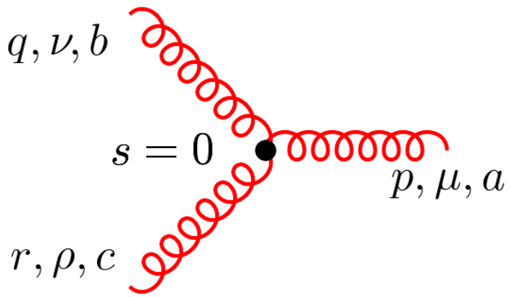
$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

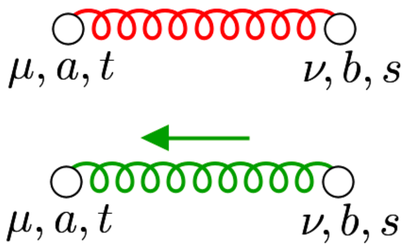
analogously for quarks: Lüscher 2013

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \Delta\right)\chi + \mathrm{h.c.}$$

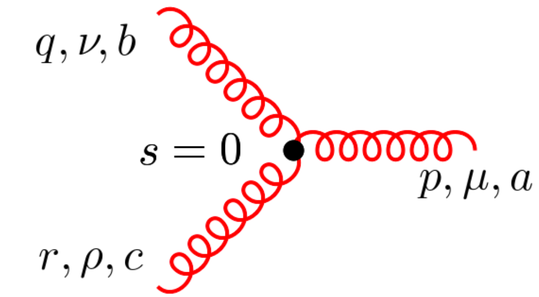
Vertices



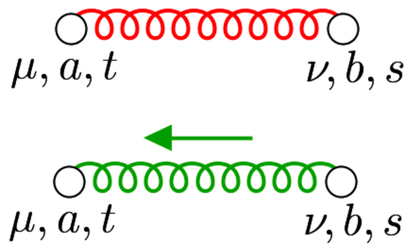
regular 3-gluon vertex

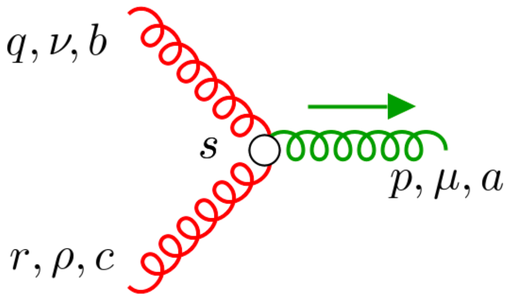


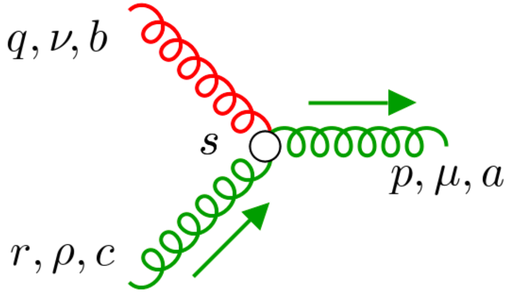
Vertices

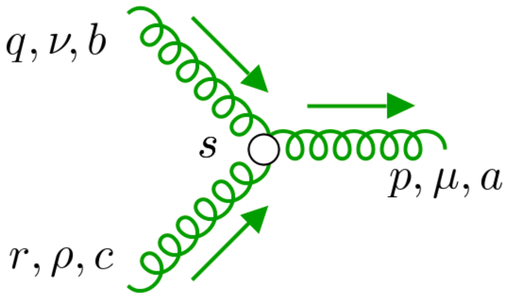


regular 3-gluon vertex



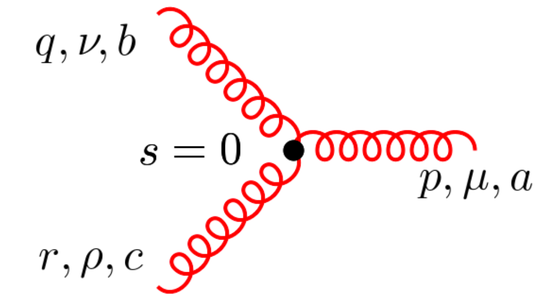




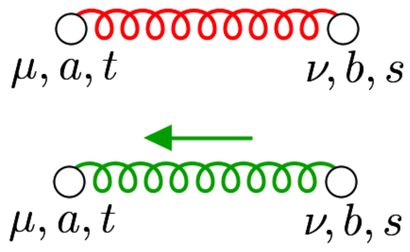


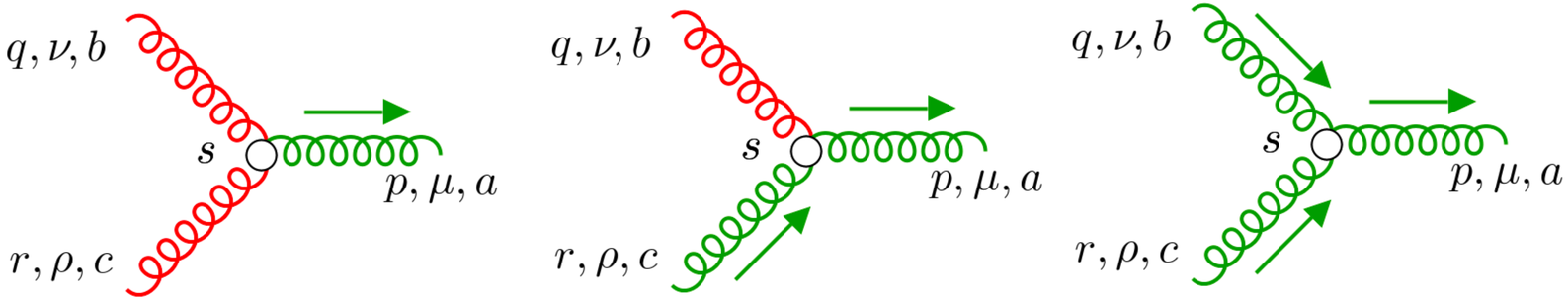
$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

Vertices



regular 3-gluon vertex





$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

analogously for 4-gluon vertex and quarks

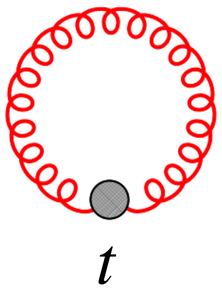
First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

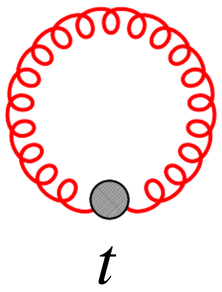
LO:



First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

LO:

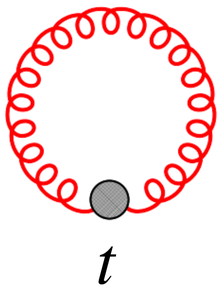


$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

LO:



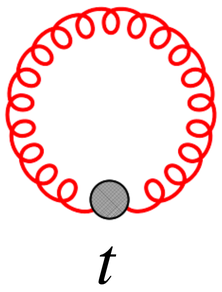
$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

LO:



$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

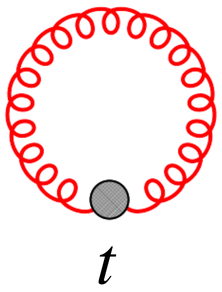
$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

explicitly:

$$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

LO: 

$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

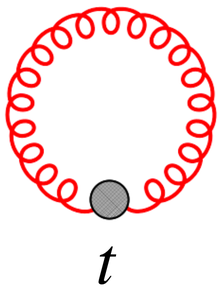
explicitly: $$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

→ measure α_s on the lattice?

First application

$$\langle E(t)\rangle \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

LO:



$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

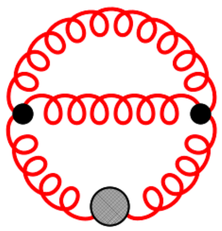
explicitly:

$$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

→ measure α_s on the lattice?

$$\alpha_s = \alpha_s(\mu)$$

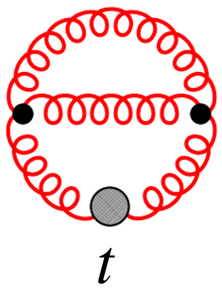
Higher orders



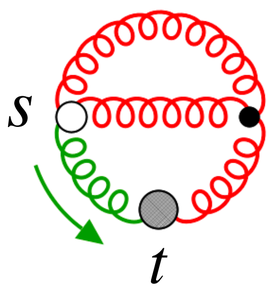
t

$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders

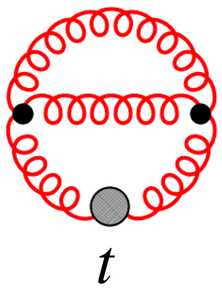


$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

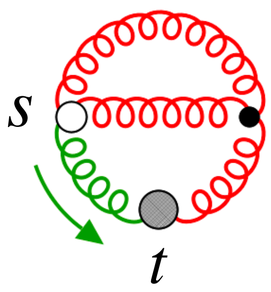


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

Higher orders



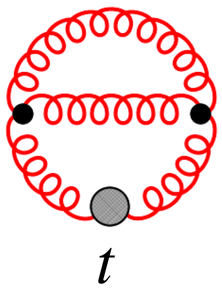
$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$



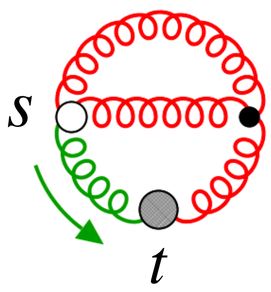
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

- generalized loop integrals

Higher orders



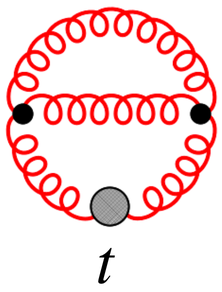
$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$



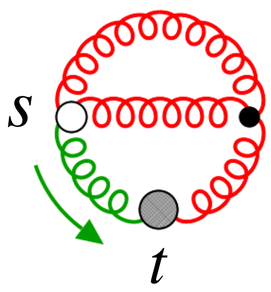
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

Higher orders



$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

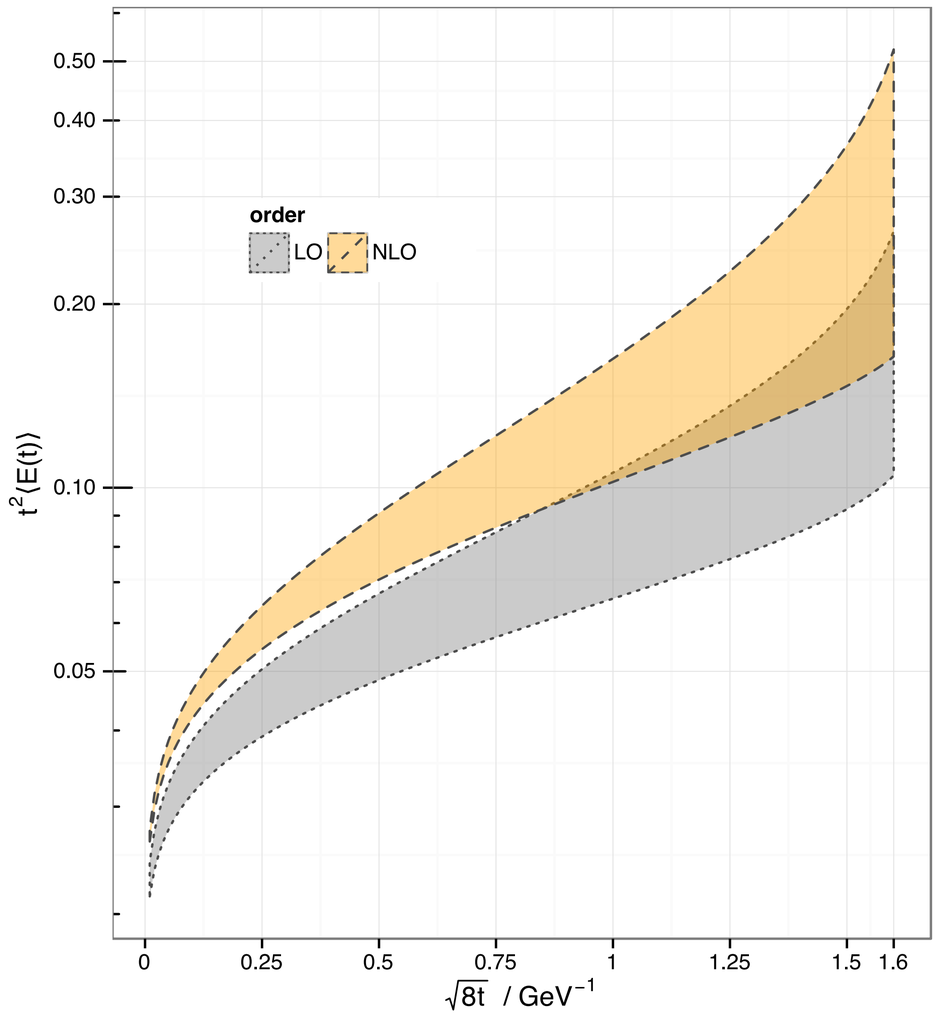


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters
- renormalization: same as fundamental QCD!

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

[Lüscher '10]



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3 - \frac{11}{3}L_{t\mu}\right)C_A - \frac{8}{9}n_f T_R$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on α_s: ± 3-5%

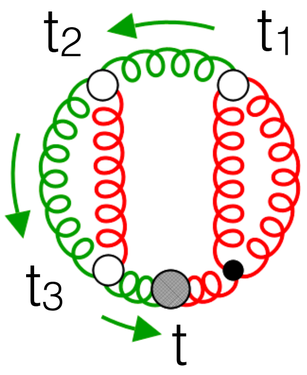
PDG: ± 1%

Three-loop calculation

Three-loop calculation

The usual problems:

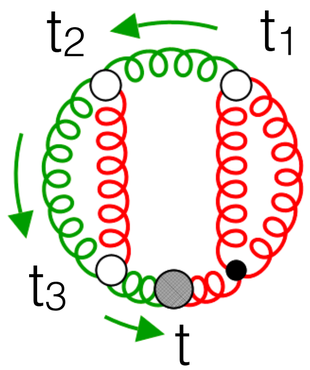
- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

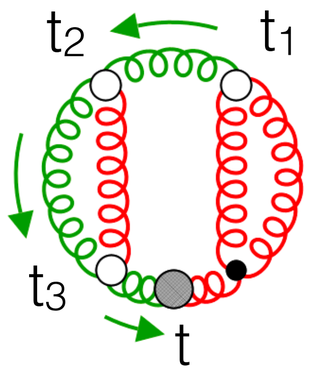


$$I(t,\mathbf{n},\mathbf{a},D) = \left(\prod_{f=1}^{N}\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right)\int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



$$I(t,\mathbf{n},\mathbf{a},D) = \left(\prod_{f=1}^{N}\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right)\int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

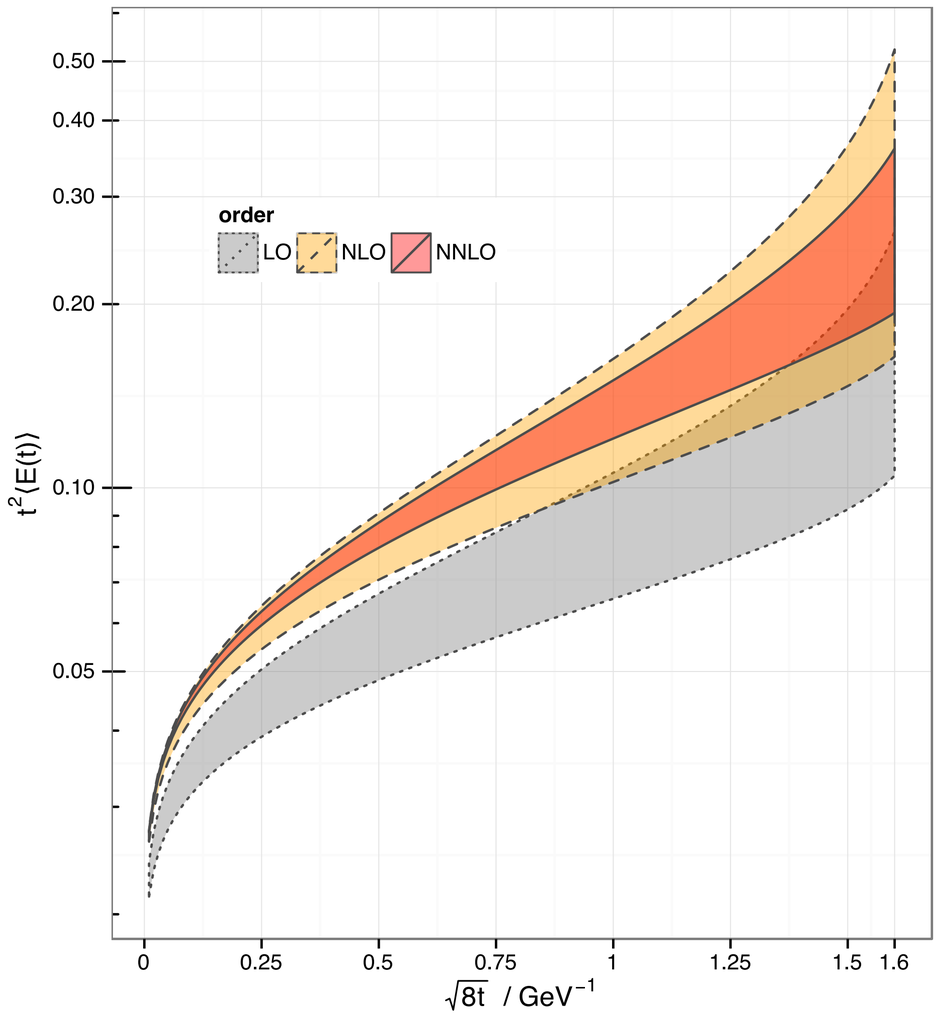
The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa ’19

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann 2016



resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

Derive $\alpha_s(m_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

Gradient-flow coupling

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\, \alpha_s(\mu) + k_2(t,\mu)\, \alpha_s^2(\mu) \right]$$

Gradient-flow coupling

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\, \alpha_s(\mu) + k_2(t,\mu)\, \alpha_s^2(\mu) \right]$$

$$\equiv \frac{3}{4\pi} \hat{\alpha}_s(t) = \frac{3}{4} \hat{a}_s(t)$$

Gradient-flow coupling

$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi} \left[1 + k_1(t,\mu)\, \alpha_s(\mu) + k_2(t,\mu)\, \alpha_s^2(\mu) \right]$$

$$\equiv \frac{3}{4\pi} \hat{\alpha}_s(t) = \frac{3}{4} \hat{a}_s(t)$$

$$\mu^2 \frac{d}{d\mu^2} \hat{a}_s(\mu^2) = \hat{\beta}(\hat{a}_s)$$

Gradient-flow coupling

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

$$\equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$

$$\mu^2\frac{d}{d\mu^2}\hat{a}_s(\mu^2) = \hat{\beta}(\hat{a}_s)$$

$$= -\,\hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

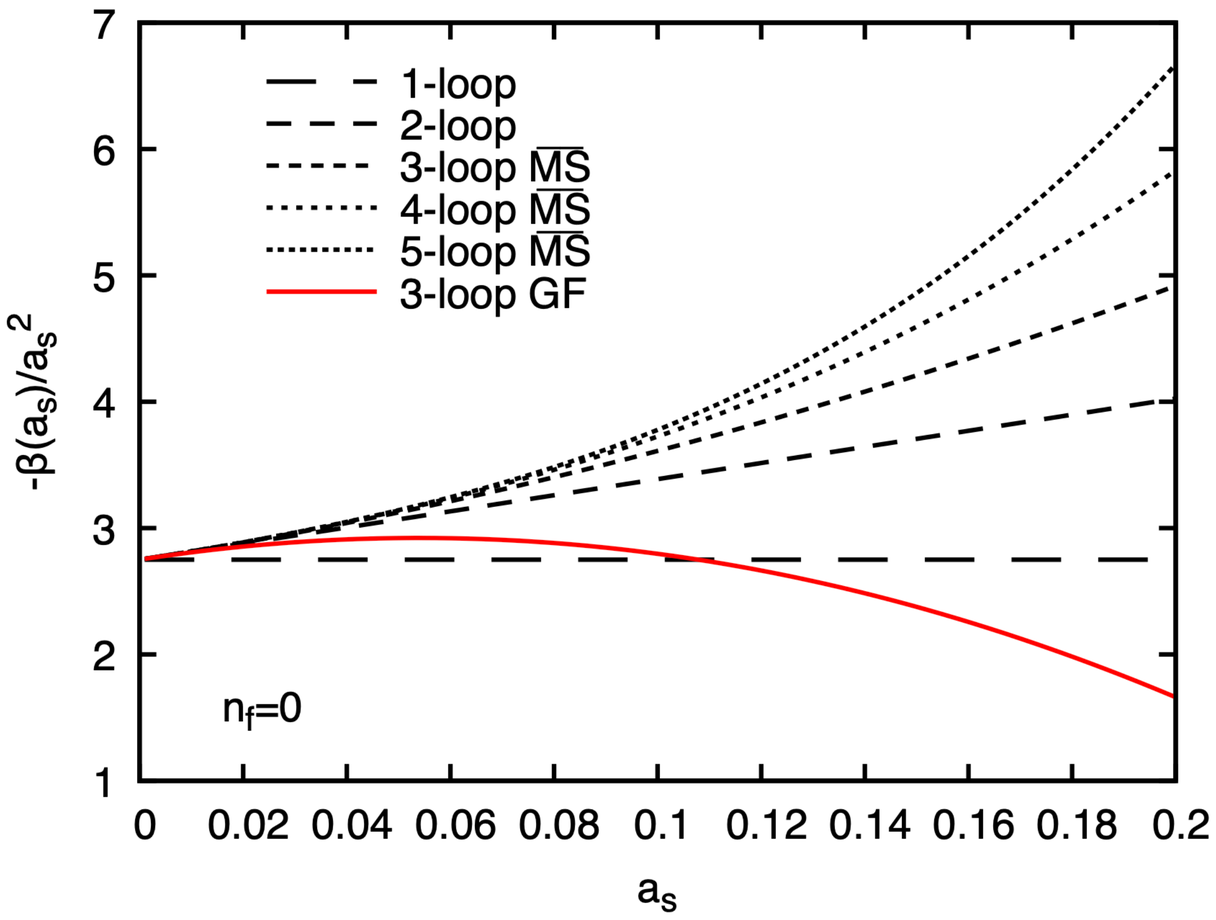
Gradient-flow coupling

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

$$\equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$

$$\mu^2\frac{d}{d\mu^2}\hat{a}_s(\mu^2) = \hat{\beta}(\hat{a}_s)$$

$$= -\,\hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$



Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m$$

perturbative

Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k$$

perturbative

Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k = \sum_m \zeta_{nm}(t)\mathcal{O}^R_m$$

perturbative

Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k \quad = \sum_m \zeta_{nm}(t)\mathcal{O}^R_m$$

perturbative

$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}^R_n \to \sum_{n,m} C_n\, \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t)$$

Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k \quad = \sum_m \zeta_{nm}(t)\mathcal{O}^R_m$$

perturbative

$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}^R_n \to \sum_{n,m} C_n\, \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t)$$

$$\tilde{C}_n(t) = C_m\, \zeta^{-1}_{mn}(t)$$

Small-flow-time expansion [Lüscher, Weisz '13]

$$\tilde{\mathcal{O}}_n(t) \to \sum_m \zeta^B_{nm}(t)\mathcal{O}_m \qquad = \sum_{m,k} \zeta^B_{nm}(t) Z^{-1}_{mk} \mathcal{O}^R_k \quad = \sum_m \zeta_{nm}(t)\mathcal{O}^R_m$$

perturbative

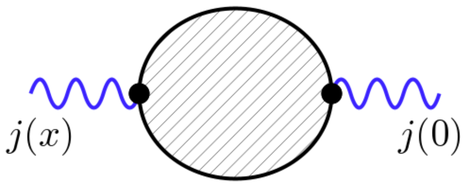
$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}^R_n \to \sum_{n,m} C_n\, \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t)$$

$$\tilde{C}_n(t) = C_m\, \zeta^{-1}_{mn}(t)$$

$$\zeta_{mn}(t) = P_n[\tilde{\mathcal{O}}_m(t)]$$

Hadronic vacuum polarization

$$\int d^4x\, e^{iQx} \langle T\, j(x)\, j(0) \rangle \to \sum_n C_n(Q) \langle \mathcal{O}_n \rangle$$



dim 0: $\mathbf{1}$

dim 2: $m^2\mathbf{1}$

dim 4: $\mathcal{O}_1 = G^a_{\mu\nu} G^a_{\mu\nu}$

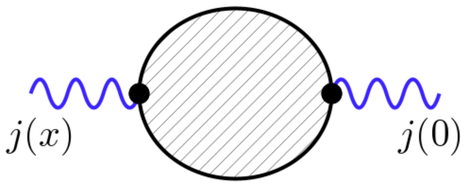
$\mathcal{O}_2 = m\bar{\psi}\psi$

$\mathcal{O}_3 = m^4$

Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T j(x)\, j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n\rangle = \sum_n \tilde{C}_n(Q,t)\langle \tilde{\mathcal{O}}_n(t)\rangle$$

RH, Lange, Neumann '20



dim 0: $\mathbf{1}$

dim 2: $m^2\mathbf{1}$

dim 4: $\mathcal{O}_1 = G^a_{\mu\nu}G^a_{\mu\nu}$

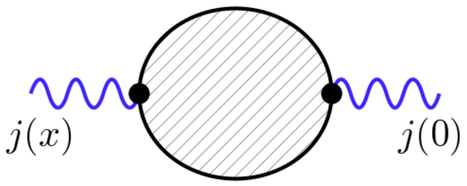
$\mathcal{O}_2 = m\bar{\psi}\psi$

$\mathcal{O}_3 = m^4$

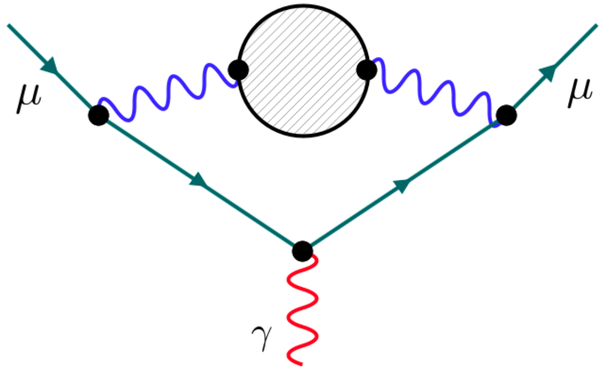
Hadronic vacuum polarization

$$\int d^4x\, e^{iQx}\langle T j(x)\, j(0)\rangle \to \sum_n C_n(Q)\langle \mathcal{O}_n\rangle = \sum_n \tilde{C}_n(Q,t)\langle \tilde{\mathcal{O}}_n(t)\rangle$$

RH, Lange, Neumann '20



contribution to (g-2)mu



dim 0: $\mathbf{1}$

dim 2: $m^2\mathbf{1}$

dim 4: $\mathcal{O}_1 = G^a_{\mu\nu}G^a_{\mu\nu}$

$\mathcal{O}_2 = m\bar{\psi}\psi$

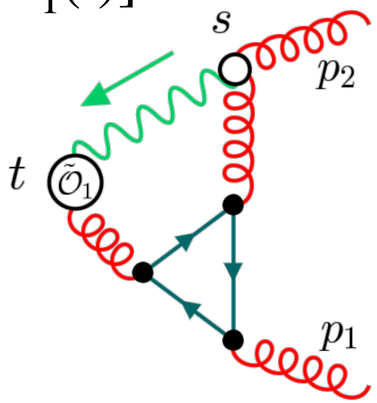
$\mathcal{O}_3 = m^4$

dim 4: $\mathcal{O}_1 = G^a_{\mu\nu} G^a_{\mu\nu}$

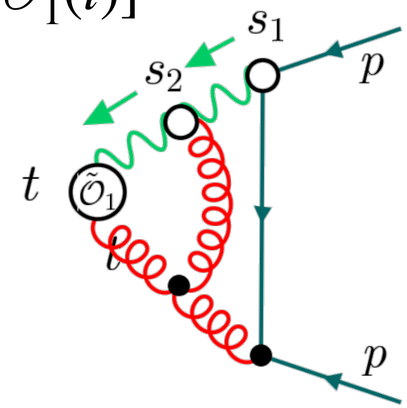
$\mathcal{O}_2 = m\bar{\psi}\psi$

$\mathcal{O}_3 = m^4$

$\zeta_{11}(t) = P_1[\tilde{\mathcal{O}}_1(t)]$



$\zeta_{21}(t) = P_2[\tilde{\mathcal{O}}_1(t)]$



all scales except t can be set to zero [Gorishnii, Larin, Tkachov '83]

QCD energy-momentum tensor

Suzuki, Makino '13, '14

$$T_{\mu\nu} = \sum_n C_n \mathcal{O}_{n,\mu\nu}$$

$$\mathcal{O}_{1,\mu\nu} = \frac{1}{g_0^2} F^a_{\mu\rho} F^a_{\nu\rho} \qquad C_1 \equiv 1$$

$$\mathcal{O}_{2,\mu\nu} = \frac{\delta_{\mu\nu}}{g_0^2} F^a_{\rho\sigma} F^a_{\rho\sigma} \qquad C_2 \equiv -\frac{1}{4}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi \qquad C_3 \equiv \frac{1}{4}$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu} \bar{\psi} \overleftrightarrow{\not{D}} \psi \qquad C_4 \equiv 0$$

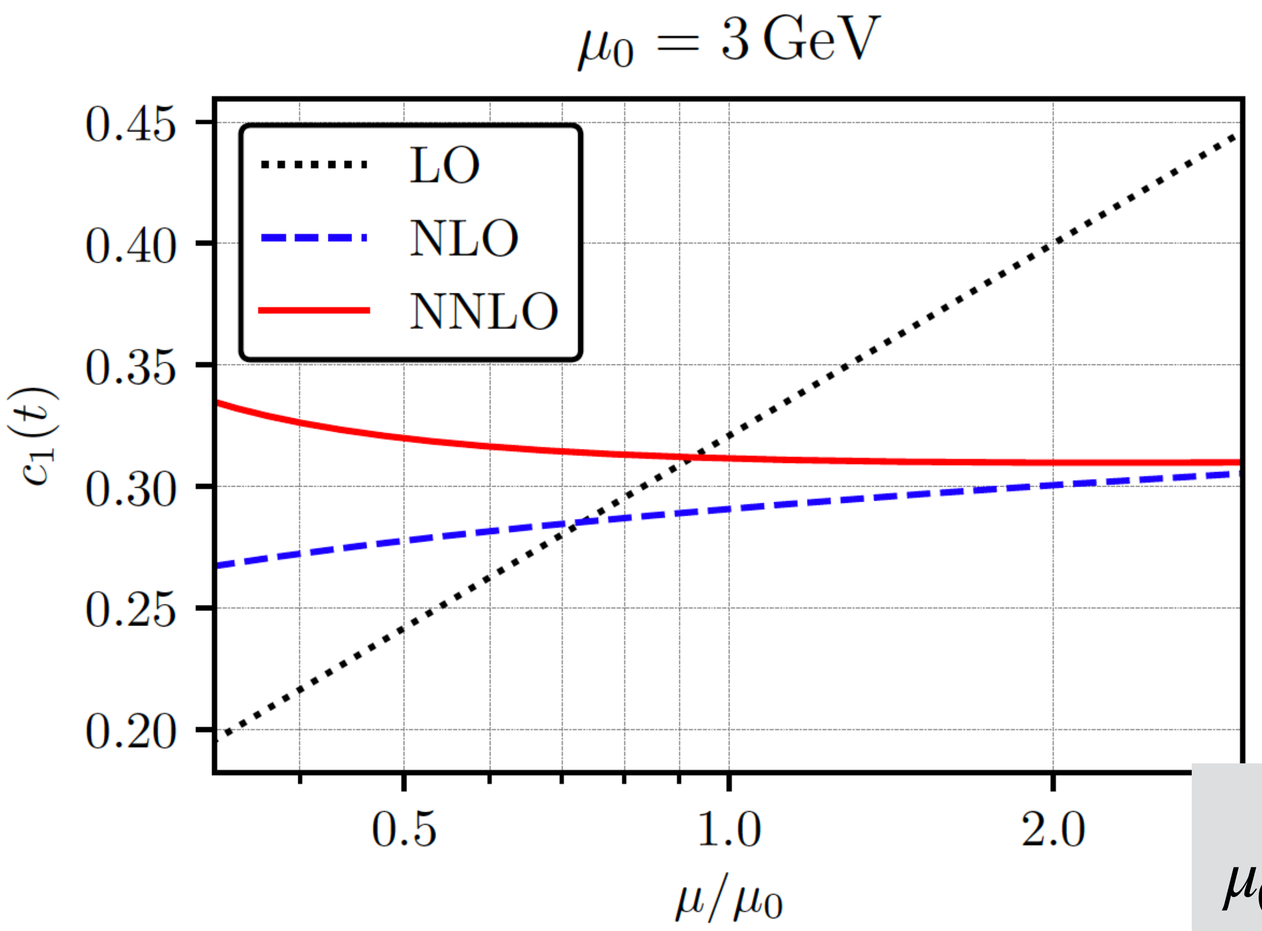
$$T_{\mu\nu} = \sum_n \tilde{C}_n(t) \tilde{\mathcal{O}}_{n,\mu\nu}(t)$$

NNLO result

$$
\begin{aligned}
c_1(t) = \frac{1}{g^2}\Bigg\{ & 1 + \frac{g^2}{(4\pi)^2}\left[-\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0\, L(\mu,t)\right] \\
& + \frac{g^4}{(4\pi)^4}\Bigg[- \beta_1\, L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2 + \frac{1187}{10}\ln 3\right) \\
& + C_A T_F\left(\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2 - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right) \\
& + C_F T_F\left(-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{2587}{108} - \frac{7}{9}\pi^2 - \frac{106}{9}\ln 2 - \frac{161}{18}\ln 3\right)\Bigg] \\
& + \mathcal{O}(g^6)\Bigg\}, \qquad L(\mu,t) \equiv \ln\left(2\mu^2 t\right) + \gamma_{\mathrm{E}}
\end{aligned}
$$

etc.

RH, Kluth, Lange ‘18



$$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$$

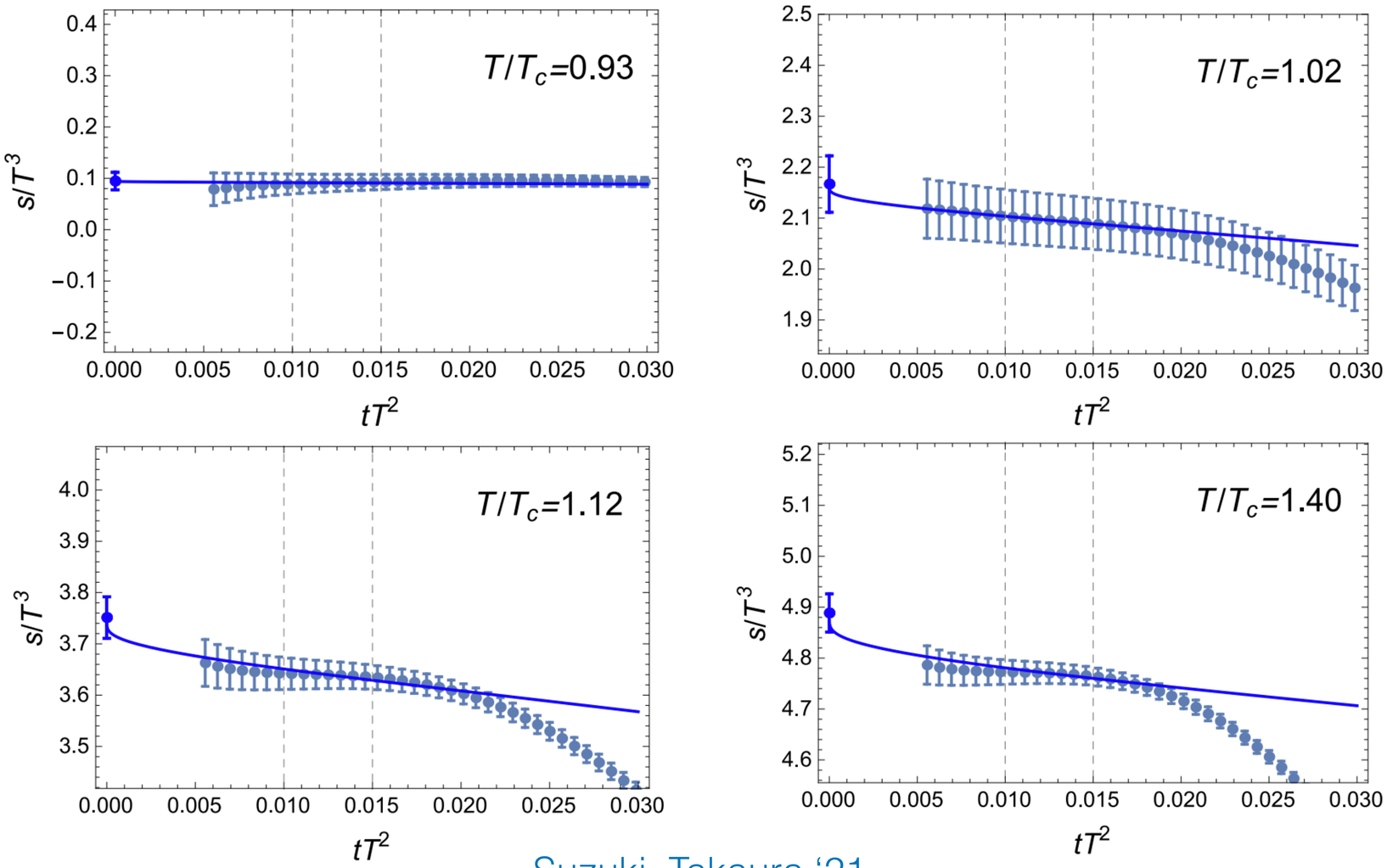
Application

Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$T_{\mu\nu}(x) = \sum_{n=1}^{4} c_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t,x)$$

NLO



Suzuki, Takaura '21

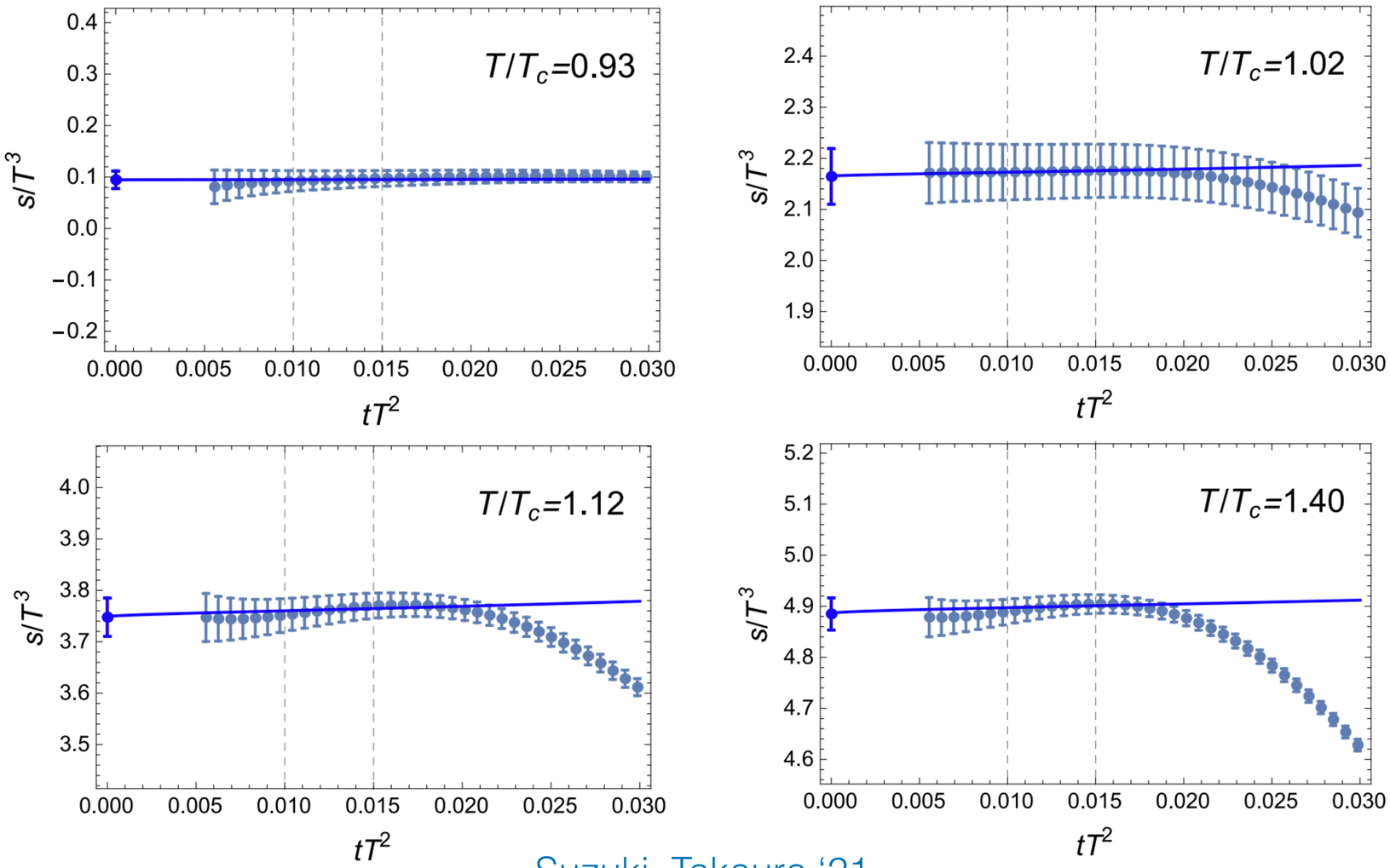
Application

Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

$$T_{\mu\nu}(x) = \sum_{n=1}^{4} c_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t,x)$$

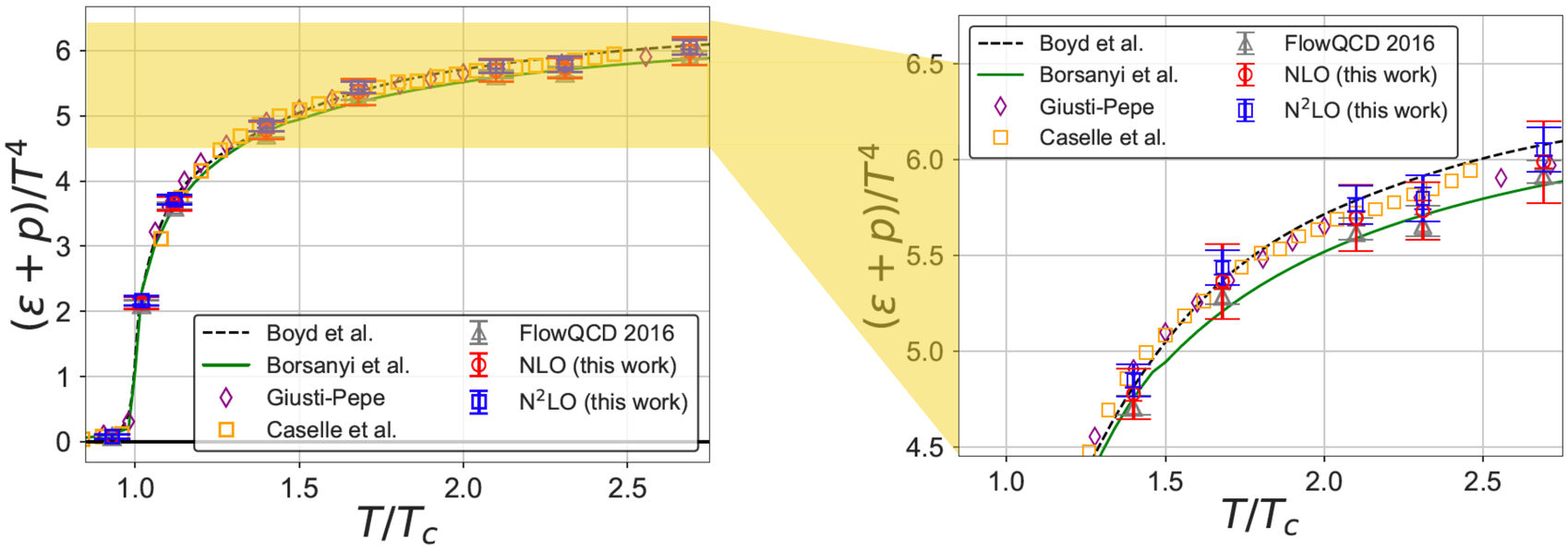
NNLO



Suzuki, Takaura '21

Application

Entropy density: $$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$



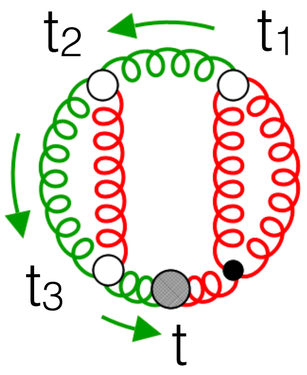
Iritani, Kitazawa, Suzuki, Takaura 2019

Three-loop calculation

Three-loop calculation

The usual problems:

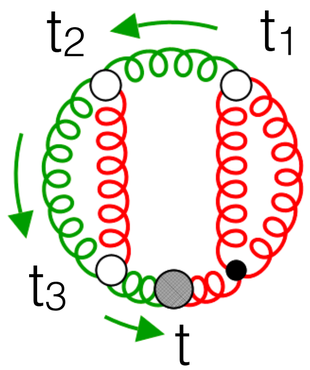
- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

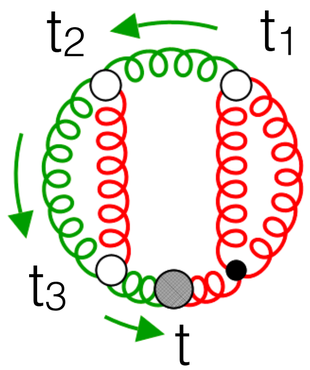


$$I(t,\mathbf{n},\mathbf{a},D) = \left(\prod_{f=1}^{N}\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right)\int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



$$I(t,\mathbf{n},\mathbf{a},D) = \left(\prod_{f=1}^{N}\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right)\int_{p_1,p_2,p_3} \frac{\exp[\sum_{k,i,j} a_{kij} t_k p_i p_j]}{p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} p_4^{2n_4} p_5^{2n_5} p_6^{2n_6}}$$

The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa ’19

Integration-by-parts relations

- After tensor reduction, we end up with many scalar integrals of the form

$$I(\{t_f^{\text{up}}\},\{T_i\},\{a_i\}) = \left(\prod_{f=1}^{F}\int_0^{t_f^{\text{up}}} \mathrm{d}t_f\right)\int_{k_1,\dots,k_L} \frac{\exp[-(T_1 q_1^2+\dots+T_N q_N^2)]}{q_1^{2a_1}\cdots q_N^{2a_N}}$$

  with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$
- Chetyrkin and Tkachov observed [Tkachov 1981; Chetyrkin, Tkachov 1981]

$$\int_{k_1,\dots k_L} \frac{\partial}{\partial k_i^\mu}\left(\tilde{q}_j^\mu \frac{1}{P_1^{a_1}\dots P_N^{a_N}}\right) = 0$$

$\Rightarrow$ Linear relations between Feynman integrals

- Can easily be adopted to gradient-flow integrals
- Additional new relations for gradient-flow integrals: [Artz, RH, Lange, Neumann, Prausa '19]

$$\int_0^{t_f^{\text{up}}} \mathrm{d}t_f \partial_{t_f} F(t_f,\dots) = F(t_f^{\text{up}},\dots) - F(0,\dots)$$

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1) I(a_1, a_2, a_3) + (a_1 - a_2) I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2) I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2) I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$\begin{aligned} 0 &= (d - 1) I(1, 1, 1) + I(2, 1, 0), \\ 0 &= (d - 2) I(2, 1, 1) + I(3, 0, 1) - I(3, 1, 0), \\ &\vdots \end{aligned}$$

  - Solve with Gaussian elimination

$\Rightarrow$ Express integrals through significantly smaller number of master integrals

Institute for Theoretical Particle Physics and
Institute for Astroparticle Physics

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1) I(a_1, a_2, a_3) + (a_1 - a_2) I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2) I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2) I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$0 = (d - 1) I(1, 1, 1) + I(2, 1, 0),$$
$$0 = (d - 2) I(2, 1, 1) + I(3, 0, 1) - I(3, 1, 0),$$

  - Solve with Gaussian elimination

e.g. NNLO chromo-magnetic dipole operator:
O(4000) integrals reduced to 13 master integrals

$\Rightarrow$ Express integrals through significantly smaller number of master integrals

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Schwinger parameters:

$$\frac{1}{p^2} = \int_0^\infty \mathrm{d}x\, e^{-xp^2}$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Schwinger parameters:

$$\frac{1}{p^2} = \int_0^\infty \mathrm{d}x\, e^{-xp^2}$$

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \cdots \int_0^\infty \mathrm{d}x_6 \iint_{p_1,p_2,p_3} \exp\left(-\mathbf{p}^T B(u_1,\ldots,u_f,x_1,\cdots,x_6)\,\mathbf{p}\right)$$

Numerical evaluation of the master integals

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \iint_{p_1,p_2,p_3} \frac{\exp\left(-\mathbf{p}^T A(u_1,\ldots,u_f)\,\mathbf{p}\right)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2}$$

Schwinger parameters:

$$\frac{1}{p^2} = \int_0^\infty \mathrm{d}x\, e^{-xp^2}$$

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \cdots \int_0^\infty \mathrm{d}x_6 \iint_{p_1,p_2,p_3} \exp\left(-\mathbf{p}^T B(u_1,\ldots,u_f,x_1,\cdots,x_6)\,\mathbf{p}\right)$$

$$\int_0^1 \mathrm{d}u_1\, u_1^{c_1} \cdots \int_0^1 \mathrm{d}u_f\, u_f^{c_f} \int_0^\infty \mathrm{d}x_1 \cdots \int_0^\infty \mathrm{d}x_6\, [\det B(u_1,\cdots,u_f,x_1,\cdots,x_6)]^{-D/2}$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\begin{aligned}
\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \quad & u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
& + 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
& + u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
& + 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
& + 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
& + u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
& + u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
& + u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
& + x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
& + 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
& + 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
& + 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
& + x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
& + x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}
\end{aligned}$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \;\; u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 +$$

$$+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 +$$

$$+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 +$$

$$+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 +$$

$$+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 +$$

$$+ 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 +$$

$$+ u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 +$$

$$+ u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 +$$

$$+ u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 +$$

$$+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 +$$

$$+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 +$$

$$+ 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 +$$

$$+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 +$$

$$+ 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 +$$

$$+ x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 +$$

$$+ x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}$$

$$y = \frac{1}{1+x}$$

$$\int_0^\infty \mathrm{d}x\, f(x) = \int_0^1 \frac{\mathrm{d}y}{y^2} f(x(y))$$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \;\; u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 +$$

$$y = \frac{1}{1+x}$$

$$\int_0^\infty \mathrm{d}x\, f(x) = \int_0^1 \frac{\mathrm{d}y}{y^2} f(x(y))$$

$$\begin{aligned}
&+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
&+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
&+ 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
&+ u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
&+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
&+ 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
&+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
&+ 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
&+ x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
&+ x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}
\end{aligned}$$

overlapping singularities
as $x_i, u_j \to 0$

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} \longrightarrow$$

$$\int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3 \int_0^\infty \mathrm{d}x_4 \int_0^\infty \mathrm{d}x_5 \int_0^\infty \mathrm{d}x_6 \;\; u_1 x_1^{-\epsilon} x_2^{-\epsilon} x_3^{-\epsilon} x_4^{-\epsilon} x_5^{-\epsilon} x_6^{-\epsilon} (3\, u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1^2\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 +$$

$$y = \frac{1}{1+x}$$

$$\int_0^\infty \mathrm{d}x\, f(x) = \int_0^1 \frac{\mathrm{d}y}{y^2} f(x(y))$$

$$\begin{aligned}
&+ u_1^2\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1^2\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, u_1\, u_2\, x_1\, x_2\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_4 + \\
&+ u_1\, u_2\, x_1\, x_2\, x_3\, x_5\, x_6 + u_1\, u_2\, x_1\, x_2\, x_3\, x_5 + u_1\, u_2\, x_1\, x_2\, x_3\, x_6 + \\
&+ 3\, u_1\, u_2\, x_1\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_1\, x_3\, x_4\, x_6 + u_1\, u_2\, x_1\, x_3\, x_5\, x_6 + \\
&+ 2\, u_1\, u_2\, x_2\, x_3\, x_4\, x_5\, x_6 + u_1\, u_2\, x_2\, x_3\, x_4\, x_5 + u_1\, u_2\, x_2\, x_3\, x_5\, x_6 + \\
&+ u_1\, u_2\, x_3\, x_4\, x_5\, x_6 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_5 + 3\, u_1\, x_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_3\, x_4 + u_1\, x_1\, x_2\, x_3\, x_5 + u_1\, x_1\, x_2\, x_3\, x_6 + 3\, u_1\, x_1\, x_2\, x_4\, x_5\, x_6 + \\
&+ u_1\, x_1\, x_2\, x_4\, x_6 + u_1\, x_1\, x_2\, x_5\, x_6 + u_1\, x_2\, x_3\, x_4\, x_5 + u_1\, x_2\, x_3\, x_4\, x_6 + \\
&+ u_1\, x_2\, x_4\, x_5\, x_6 + 2\, x_1\, x_2\, x_3\, x_4\, x_5 + 2\, x_1\, x_2\, x_3\, x_4\, x_6 + x_1\, x_2\, x_3\, x_4 + \\
&+ x_1\, x_2\, x_3\, x_5 + x_1\, x_2\, x_3\, x_6 + 2\, x_1\, x_2\, x_4\, x_5\, x_6 + 3\, x_1\, x_2\, x_4\, x_5 + \\
&+ 2\, x_1\, x_2\, x_4\, x_6 + x_1\, x_2\, x_4 + x_1\, x_2\, x_5\, x_6 + x_1\, x_2\, x_5 + x_1\, x_2\, x_6 + \\
&+ 3\, x_1\, x_3\, x_4\, x_5 + 3\, x_1\, x_3\, x_4\, x_6 + x_1\, x_3\, x_4 + x_1\, x_3\, x_5 + x_1\, x_3\, x_6 + \\
&+ 3\, x_1\, x_4\, x_5\, x_6 + x_1\, x_4\, x_6 + x_1\, x_5\, x_6 + 2\, x_2\, x_3\, x_4\, x_5 + 2\, x_2\, x_3\, x_4\, x_6 + \\
&+ x_2\, x_3\, x_4 + x_2\, x_3\, x_5 + x_2\, x_3\, x_6 + 2\, x_2\, x_4\, x_5\, x_6 + \\
&+ x_2\, x_4\, x_5 + x_2\, x_5\, x_6 + x_3\, x_4\, x_5 + x_3\, x_4\, x_6 + x_4\, x_5\, x_6)^{\epsilon-2}
\end{aligned}$$

overlapping singularities as $x_i, u_j \to 0$

→ sector decomposition

example:

[Heinrich '08]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

example: $$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

[Heinrich '08]

singularity as $x \to 0$ and $(x \to 0) \wedge (y \to 0)$

example: $$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

[Heinrich '08]

singularity as $x \to 0$ and $(x \to 0) \wedge (y \to 0)$

sector decomposition: [Binoth, Heinrich '00]

example: [Heinrich '08]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

singularity as $x \to 0$ and $(x \to 0) \wedge (y \to 0)$

sector decomposition: [Binoth, Heinrich '00]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1} [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$

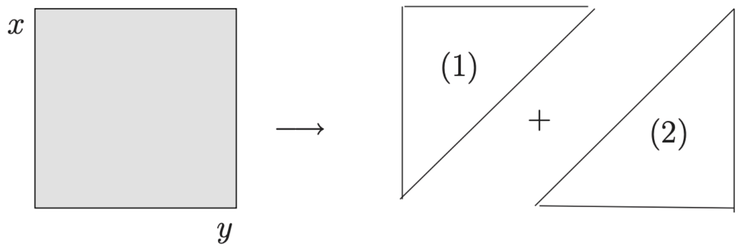
example: [Heinrich '08]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

singularity as $x \to 0$ and $(x \to 0) \wedge (y \to 0)$

sector decomposition: [Binoth, Heinrich '00]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1} [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$



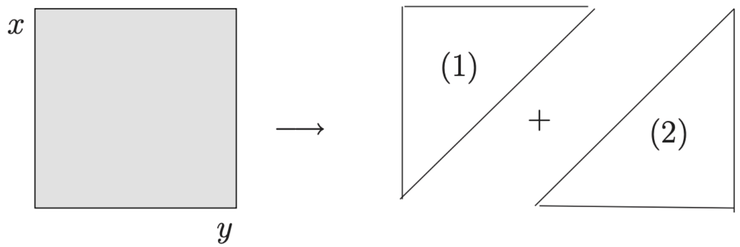
example: [Heinrich '08]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1}$$

singularity as $x \to 0$ and $(x \to 0) \wedge (y \to 0)$

sector decomposition: [Binoth, Heinrich '00]

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\, y\Big)^{-1} [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$



$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

$$= -\frac{1}{(a+b)\,\epsilon} \int_0^1 dt\, \frac{t^{-b\epsilon}}{1+t}$$

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

$$= -\frac{1}{(a+b)\,\epsilon} \int_0^1 dt\, \frac{t^{-b\epsilon}}{1+t} - \frac{1}{(a+b)\,\epsilon} \int_0^1 \frac{dt}{1+t} \left[-\frac{1}{a\,\epsilon}\delta(t) - \left(\frac{1}{t}\right)_+ + a\epsilon \left(\frac{\ln t}{t}\right)_+ + \cdots \right]$$

$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\, t\Big)^{-1} + \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\, t\Big)^{-1}$$

$$= -\frac{1}{(a+b)\,\epsilon} \int_0^1 dt\, \frac{t^{-b\epsilon}}{1+t} - \frac{1}{(a+b)\,\epsilon} \int_0^1 \frac{dt}{1+t} \left[-\frac{1}{a\,\epsilon}\delta(t) - \left(\frac{1}{t}\right)_+ + a\epsilon \left(\frac{\ln t}{t}\right)_+ + \cdots \right]$$

$$\int_0^1 dt \left(\frac{\ln^n t}{t}\right)_+ f(t) = \int_0^1 dt\, \frac{\ln^n t}{t} \left[f(t) - f(0)\right]$$

pySecDec [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke '18]

$$\int_0^1 \mathrm{d}u_1 u_1 \int_0^1 \mathrm{d}u_2 \int_{p_1,p_2,p_3} \frac{\exp(-p_1^2 - u_1 p_2^2 - u_1 u_2 p_3^2 - 2(p_1-p_2)^2)}{p_1^2\, p_2^2\, p_3^2\, (p_1-p_2)^2\, (p_1-p_3)^2\, (p_2-p_3)^2} = \frac{1}{(4\pi)^{3d/2}}$$

```
[ + ep^(-1)*((-1.20205690407937649) +
(6.74709950249940753e-9)*numerr )

 + ep^(0)*((-11.4409624237256917) +
(4.99888756503079786e-8)*numerr ) ]
```

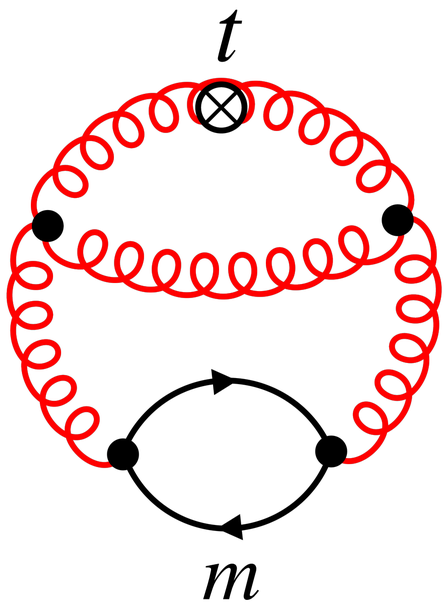
[RH, Nellopoulos '22 (unpublished)]
earlier work: [RH, Neumann '16]

Approximate solution of gradient flow integrals

example:

$$\langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle$$

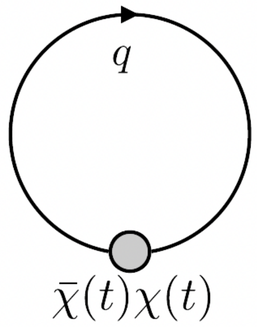
result for $m \neq 0$?



Strategy of regions [Beneke, Smirnov '97]

example:

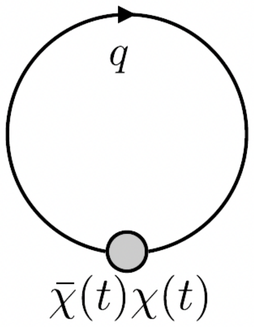
$$S(t) \equiv \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3m}{8\pi^2 t^2} f(m^2, t)$$ [RH '21]



Strategy of regions [Beneke, Smirnov '97]

example:

$$S(t) \equiv \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3m}{8\pi^2 t^2} f(m^2, t)$$ [RH '21]

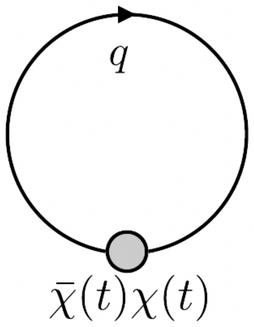


$$f(m^2, t) \equiv t \int_k \frac{e^{-tk^2}}{k^2 + m^2} = 1 - m^2 t e^{m^2 t} \Gamma(0, m^2 t)$$

Strategy of regions [Beneke, Smirnov '97]

example:

$$S(t) \equiv \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3m}{8\pi^2 t^2} f(m^2, t)$$ [RH '21]



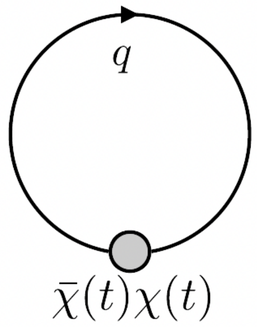
$$f(m^2, t) \equiv t \int_k \frac{e^{-tk^2}}{k^2 + m^2} = 1 - m^2 t e^{m^2 t} \Gamma(0, m^2 t)$$

$$\Gamma(s, x) = \int_x^\infty \mathrm{d}u\, u^{s-1} e^{-u}$$

Strategy of regions [Beneke, Smirnov '97]

example:

$$S(t) \equiv \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3m}{8\pi^2 t^2} f(m^2, t)$$ [RH '21]



$$f(m^2, t) \equiv t \int_k \frac{e^{-tk^2}}{k^2 + m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0, m^2 t) = 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

$$\Gamma(s, x) = \int_x^\infty \mathrm{d}u\, u^{s-1} e^{-u}$$

$$m^2 t \ll 1 \;\Leftrightarrow\; 1/t \gg m^2$$

$$f(m^2,t) \equiv t \int_k \frac{e^{-tk^2}}{k^2+m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0,m^2 t) \; = 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

$$f(m^2,t) \equiv t \int_k \frac{e^{-tk^2}}{k^2+m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0,m^2 t) \; = 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

“naive” expansion:

$$f^{(ii)}(m^2,t) = t\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\int_k \frac{k^{2n}}{k^2+m^2} = m^2 t\left[-\frac{1}{\epsilon} - 1 + \gamma_{\mathrm{E}} + \ln m^2\right] e^{m^2 t}$$

$$f(m^2,t) \equiv t\int_k \frac{e^{-tk^2}}{k^2+m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0,m^2 t) \;= 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

"naive" expansion:

$$f^{(ii)}(m^2,t) = t\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\int_k \frac{k^{2n}}{k^2+m^2} = m^2 t\left[-\frac{1}{\epsilon} - 1 + \gamma_{\mathrm{E}} + \ln m^2\right] e^{m^2 t}$$

assumes

$k^2 t \ll 1$

$k^2 \ll 1/t$

$$f(m^2,t) \equiv t \int_k \frac{e^{-tk^2}}{k^2+m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0,m^2 t) \; = 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

"naive" expansion:

$$f^{(ii)}(m^2,t) = t\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\int_k \frac{k^{2n}}{k^2+m^2} = m^2 t\left[-\frac{1}{\epsilon} - 1 + \gamma_{\mathrm{E}} + \ln m^2\right] e^{m^2 t}$$

assumes
$k^2 t \ll 1$
$k^2 \ll 1/t$

also need to integrate over $k^2 \gtrsim 1/t$

$$f(m^2,t) \equiv t\int_k \frac{e^{-tk^2}}{k^2+m^2} = 1 - m^2 t e^{m^2 t}\Gamma(0,m^2 t) \;= 1 + m^2 t\left(\gamma_E + \log m^2 t\right) + \cdots$$

"naive" expansion:

$$f^{(ii)}(m^2,t) = t\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\int_k \frac{k^{2n}}{k^2+m^2} = m^2 t\left[-\frac{1}{\epsilon} - 1 + \gamma_{\mathrm{E}} + \ln m^2\right] e^{m^2 t}$$

assumes
$k^2 t \ll 1$
$k^2 \ll 1/t$

also need to integrate over $k^2 \gtrsim 1/t$

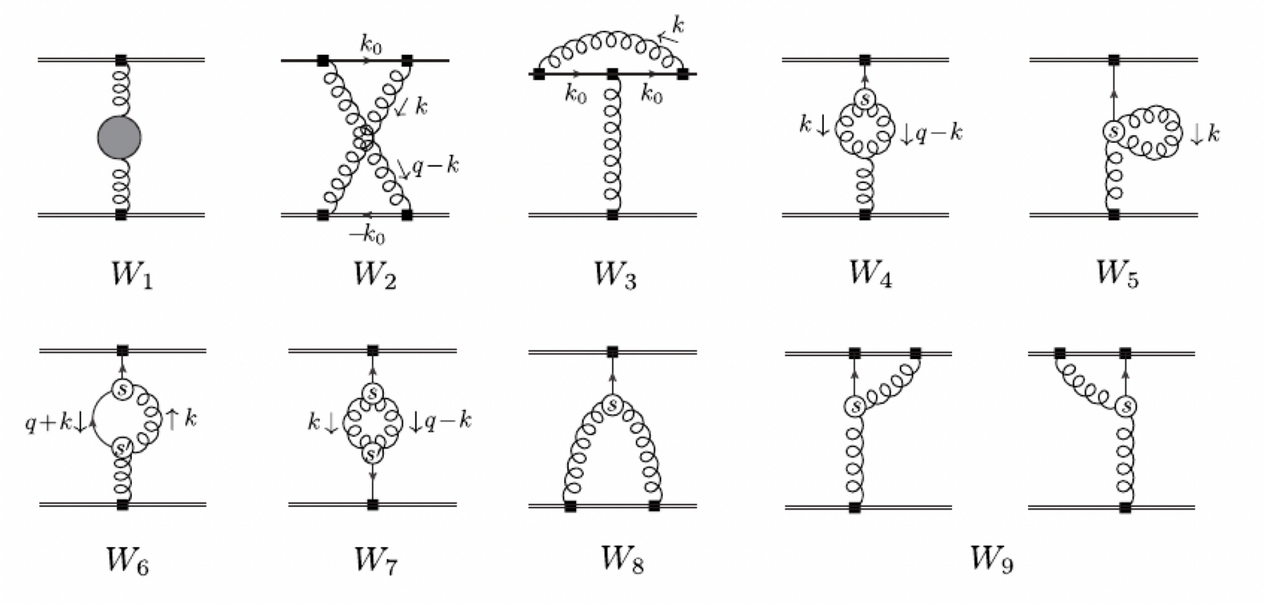
$$f^{(i)}(m^2,t) = t\sum_{n=1}^{\infty}(-m^2)^{n-1}\int_k \frac{e^{-tk^2}}{k^{2n}} = \sum_{n=1}^{\infty}(-m^2)^{n-1} t^{n-1+\epsilon}\frac{\Gamma(D/2-n)}{\Gamma(D/2)}$$

$$= 1 + m^2 t\left(\frac{1}{\epsilon} + \ln t + 1\right) e^{m^2 t} - (m^2 t)^2 - \frac{3}{4}(m^2 t)^2 + \ldots .$$

Application: QCD static force at finite t: [Brambilla, Chung, Vairo, Wang '21]

$$V(r) = -\lim\inf T \frac{\ln\langle W_{r\times T}\rangle}{T} \qquad W_{r\times T} = \mathrm{tr}_{\mathrm{color}} P\exp\left(ig\int_{\mathcal{C}} dz^{\mu} A_{\mu}(z)\right)$$

$$\tilde{V}(\boldsymbol{q};t) = -4\pi\alpha_s(\mu) C_F \frac{e^{-2\boldsymbol{q}^2 t}}{\boldsymbol{q}^2} - \alpha_s^2(\mu) C_F \frac{e^{-2\boldsymbol{q}^2 t}}{\boldsymbol{q}^2}\left[\beta_0 \ln\frac{\mu^2}{\boldsymbol{q}^2} + a_1 + C_A W_{\mathrm{NLO}}^F(\bar{t})\right] + \ldots$$



Application: QCD static force at finite t: [Brambilla, Chung, Vairo, Wang '21]

$$V(r) = -\lim\inf T \frac{\ln\langle W_{r\times T}\rangle}{T} \qquad W_{r\times T} = \mathrm{tr}_{\mathrm{color}} P\exp\left(ig\int_{\mathcal{C}} dz^{\mu} A_{\mu}(z)\right)$$

$$\tilde{V}(\boldsymbol{q};t) = -4\pi\alpha_s(\mu) C_F \frac{e^{-2\boldsymbol{q}^2 t}}{\boldsymbol{q}^2} - \alpha_s^2(\mu) C_F \frac{e^{-2\boldsymbol{q}^2 t}}{\boldsymbol{q}^2}\left[\beta_0 \ln\frac{\mu^2}{\boldsymbol{q}^2} + a_1 + C_A W^F_{\mathrm{NLO}}(\bar{t})\right] + \ldots$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2}\int_0^t \mathrm{d}s\, e^{sq^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-s\left(k^2+(q-k)^2\right)} \frac{2(d-2)k_0^2 + q^2 + 2k^2}{k^2(q-k)^2}$$

$$= g^4 C_A C_F \frac{e^{-2\boldsymbol{q}^2 t}}{16\pi^2 \boldsymbol{q}^2}\left[3\left(\frac{1}{\epsilon_{\mathrm{UV}}} + \log(\mu^2/\boldsymbol{q}^2)\right) + W_4^F(\bar{t}) + O(\epsilon)\right]$$

$$
\begin{aligned}
W_4^F(\bar{t}) = {} & 3\log(2\bar{t}) + 3\gamma_{\mathrm{E}} + \frac{5}{2} \\
& + \frac{1}{2}\int_0^1 dx_1 dx_2 dx_3 \Bigg[\frac{8\exp\left(\frac{x_2(x_1x_2-1)\bar{t}}{x_1(2x_2-1)-2}\right)}{x_2\left(-2x_2x_1+x_1+2\right)^2} + \frac{8e^{\frac{x_1^2(x_2-1)x_2\bar{t}}{2x_1(x_2-1)-1}}}{x_2\left(1-2x_1\left(x_2-1\right)\right)^2} \\
& + \frac{16x_2\left(x_3-1\right)\left(x_2x_3-1\right)\exp\left(\frac{x_2x_3\left(x_1^2x_2(x_3-1)(x_2x_3-1)-1\right)\bar{t}}{2x_1(x_3-1)x_3x_2^2-2x_1(x_3-1)x_2-2x_3x_2+x_2+1}\right)}{x_3\left(2x_1\left(x_3-1\right)x_3x_2^2-2x_1\left(x_3-1\right)x_2-2x_3x_2+x_2+1\right)^3} \\
& + \frac{16x_2\left(x_1x_2x_3-1\right)\left(x_2x_3-1\right)\exp\left(\frac{x_3\left(-x_2x_3+x_1x_2\left(x_2\left(x_3^2-1\right)-x_3\right)+1\right)\bar{t}}{2x_1(x_3-1)x_3x_2^2-(x_1+1)(2x_3-1)x_2+2}\right)}{x_3\left(2x_1\left(x_3-1\right)x_3x_2^2-\left(x_1+1\right)\left(2x_3-1\right)x_2+2\right)^3} \\
& + \frac{16x_2\left(x_3-1\right)\left(x_1x_2x_3-1\right)\exp\left(\frac{x_2x_3\left(-x_3x_2+x_2+x_1\left(x_2^2(x_3-1)x_3-1\right)\right)\bar{t}}{2x_1(x_3-1)x_3x_2^2+(-2x_3x_1+x_1-2x_3+2)x_2+1}\right)}{x_3\left(2x_1\left(x_3-1\right)x_3x_2^2+\left(-2x_3x_1+x_1-2x_3+2\right)x_2+1\right)^3} \\
& - \frac{16x_2}{x_3\left(x_1x_2+x_2+2\right)^3} - \frac{16x_2}{x_3\left(2x_1x_2+x_2+1\right)^3} - \frac{16x_2}{x_3\left(\left(x_1+2\right)x_2+1\right)^3} \\
& + \frac{4\bar{t}\exp\left(\frac{\left((x_1-1)(x_2-1)x_3^2-x_1x_2\right)\bar{t}}{-2x_2x_1+x_1+x_2+2(x_1-1)(x_2-1)x_3}\right)}{\left(-2x_2x_1+x_1+x_2+2\left(x_1-1\right)\left(x_2-1\right)x_3\right)^2} \\
& - \frac{8}{x_2\left(2x_1+1\right)^2} - \frac{8}{x_2\left(x_1+2\right)^2} - 2\log\left(\frac{81}{2}\right)\Bigg],
\end{aligned}
$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$$tq^2 \ll 1$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 1:

$tk^2 \ll 1$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = 1 - s\left(k^2 + (q-k)^2\right) + \cdots$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 1:

$tk^2 \ll 1$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = 1 - s\left(k^2 + (q-k)^2\right) + \cdots$$

$$\int_0^t \mathrm{d}s\, s^n \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^l\,\{1,\, k_0^2,\, k\cdot q, \ldots\}}{k^2(k-q)^2}$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 1:

$tk^2 \ll 1$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = 1 - s\left(k^2 + (q-k)^2\right) + \cdots$$

$$\int_0^t \mathrm{d}s\, s^n \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^l\,\{1,\, k_0^2,\, k\cdot q, \ldots\}}{k^2(k-q)^2}$$

power series in t

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$ $\qquad$ $e^{sq^2} = 1 + sq^2 + \cdots$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$\int_0^t \mathrm{d}s\, s^a \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-2sk^2} \frac{\{1,\, k_0^2,\, (k\cdot q)^2, \ldots\}}{k^{2b}}$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$\int_0^t \mathrm{d}s\, s^a \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-2sk^2} \frac{\{1,\, k_0^2,\, (k\cdot q)^2, \ldots\}}{k^{2b}} \sim \int_0^t \mathrm{d}s\, s^{x+y\epsilon}$$

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$\int_0^t \mathrm{d}s\, s^a \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-2sk^2} \frac{\{1,\, k_0^2,\, (k\cdot q)^2, \ldots\}}{k^{2b}} \sim \int_0^t \mathrm{d}s\, s^{x+y\epsilon}$$

logarithmic in t

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$\int_0^t \mathrm{d}s\, s^a \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-2sk^2} \frac{\{1,\, k_0^2,\, (k\cdot q)^2, \ldots\}}{k^{2b}} \sim \int_0^t \mathrm{d}s\, s^{x+y\epsilon}$$

logarithmic in t

[RH, Koller '22 (unpublished)]

$$W_4 = 2g^4 C_A C_F \frac{e^{-2tq^2}}{\boldsymbol{q}^2} \int_0^t \mathrm{d}s\, e^{sq^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-s\left(k^2+(q-k)^2\right)} \frac{2\,(d-2)\,k_0^2 + q^2 + 2k^2}{k^2\,(q-k)^2}$$

$tq^2 \ll 1$

region 2:

$k^2 \gtrsim 1/t$

$$e^{sq^2} = 1 + sq^2 + \cdots$$

$$e^{-s\left(k^2+(q-k)^2\right)} = e^{-2sk^2} e^{-2k\cdot q + q^2} = e^{-2sk^2}\left(1 - 2sk\cdot q + sq^2 + \cdots\right)$$

$$\int_0^t \mathrm{d}s\, s^a \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-2sk^2} \frac{\{1,\, k_0^2,\, (k\cdot q)^2, \ldots\}}{k^{2b}} \sim \int_0^t \mathrm{d}s\, s^{x+y\epsilon}$$

logarithmic in t

[RH, Koller '22 (unpublished)]

→ two-loop accessible?

Gradient Flow anomalous dimension

$t \to 0$: $$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^{\mathrm{B}}_{nm}(t, \mu, \epsilon)\, \mathcal{O}_m \; + \cdots$$

matrix notation: $$\tilde{\mathcal{O}}(t) = \zeta^{\mathrm{B}}(t, \mu, \epsilon)\, \mathcal{O} \; + \cdots$$

define $$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t, \mu, \epsilon)\, \mathcal{O}$$

Gradient Flow anomalous dimension

$t \to 0:$ $$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^{\mathrm{B}}_{nm}(t,\mu,\epsilon)\,\mathcal{O}_m \;+\cdots$$

matrix notation: $$\tilde{\mathcal{O}}(t) = \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\cdots$$

define $$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O}$$

$$t\frac{\partial}{\partial t}\hat{\mathcal{O}}(t) = \left(t\frac{\partial}{\partial t}\zeta^{\mathrm{B}}(t,\mu,\epsilon)\right)\mathcal{O} = \left(t\frac{\partial}{\partial t}\zeta^{\mathrm{B}}(t,\mu,\epsilon)\right)\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\,\hat{\mathcal{O}}(t)$$

Gradient Flow anomalous dimension

$t \to 0$: $$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta^{\mathrm{B}}_{nm}(t,\mu,\epsilon)\,\mathcal{O}_m \;+\cdots$$

matrix notation: $$\tilde{\mathcal{O}}(t) = \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\cdots$$

define $$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O}$$

$$t\frac{\partial}{\partial t}\hat{\mathcal{O}}(t) = \left(t\frac{\partial}{\partial t}\zeta^{\mathrm{B}}(t,\mu,\epsilon)\right)\mathcal{O} = \left(t\frac{\partial}{\partial t}\zeta^{\mathrm{B}}(t,\mu,\epsilon)\right)\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\,\hat{\mathcal{O}}(t)$$

$$t\frac{\partial}{\partial t}\hat{\mathcal{O}}(t) = \hat{\gamma}(\mu,t)\,\hat{\mathcal{O}}(t) \qquad \hat{\gamma}(\mu,t) = t\frac{\partial}{\partial t}\ln\zeta^{\mathrm{B}}(t,\mu,\epsilon)$$

RH, Lange, Neumann ‘20

Short flow-time expansion

originally:

$$\tilde{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\; t\xi(t,\mu,\epsilon)\,\mathcal{O}^{+2} + \cdots$$

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;= C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) + \cdots \quad = \hat{C}(t)\cdot\hat{\mathcal{O}}(t) + \cdots$$

Short flow-time expansion

originally:

$$\tilde{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\; t\xi(t,\mu,\epsilon)\,\mathcal{O}^{+2} + \cdots$$

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;= C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) + \cdots \quad = \hat{C}(t)\cdot\hat{\mathcal{O}}(t) + \cdots$$

here:

$$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O}$$ exact by definition

Short flow-time expansion

originally:

$$\tilde{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\; t\xi(t,\mu,\epsilon)\,\mathcal{O}^{+2} + \cdots$$

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;= C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) + \cdots \quad = \hat{C}(t)\cdot\hat{\mathcal{O}}(t) + \cdots$$

here:

$$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O}$$ exact by definition

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;\equiv C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) \;\equiv \hat{C}(t)\cdot\hat{\mathcal{O}}(t)$$ exact!

Short flow-time expansion

originally:

$$\tilde{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O} \;+\; t\xi(t,\mu,\epsilon)\,\mathcal{O}^{+2} + \cdots$$

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;= C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) + \cdots \quad = \hat{C}(t)\cdot\hat{\mathcal{O}}(t) + \cdots$$

here:

$$\hat{\mathcal{O}}(t) \equiv \zeta^{\mathrm{B}}(t,\mu,\epsilon)\,\mathcal{O}$$ exact by definition

$$H_{\mathrm{eff}} = C(\epsilon)\cdot\mathcal{O} \;\equiv C(\epsilon)\cdot\zeta^{\mathrm{B},-1}(t,\mu,\epsilon)\cdot\hat{\mathcal{O}}(t) \;\equiv \hat{C}(t)\cdot\hat{\mathcal{O}}(t)$$ exact!

Require a definition of $\hat{\mathcal{O}}(t)$

[Hasenfratz, Monahan, Rizik, Shindler, Witzel '22]

$$G_{\mathcal{O}}(x_4;t/a^2,\beta) = a^3 \sum_{\mathbf{x}} \left\langle \mathcal{O}(\mathbf{x},x_4;t/a^2)\widetilde{\mathcal{O}}(0)\right\rangle_\beta \sim P_{\tilde{\mathcal{O}}}[\mathcal{O}] \;=\; \zeta^{\mathrm{B}}_{\mathcal{O}\tilde{\mathcal{O}}}(t,\epsilon)$$

$$R_{\mathcal{O}}(x_4;t/a^2,\beta) = \frac{G_{\mathcal{O}}(x_4;t/a^2,\beta)}{G_V(x_4;t/a^2,\beta)}$$

$$G_V(x_4;t/a^2,\beta) = \frac{1}{3}\sum_{k=1}^{3} a^3 \sum_{\mathbf{x}} \left\langle V_k(\mathbf{x},x_4;t/a^2)\widetilde{V}_k(0)\right\rangle_\beta$$

$$\gamma_{\mathcal{O}}(t/a^2,\beta) = -2t\frac{\mathrm{d}\log R_{\mathcal{O}}(x_4;t/a^2,\beta)}{\mathrm{d}\,t} = t\partial_t \underbrace{\ln G_{\mathcal{O}}}_{\hat{\gamma}_{\mathcal{O}}} - t\partial_t \ln G_V$$

Conclusions and Outlook

- perturbative approach provides important input for $t \to 0$
- many methods available: learn from decades of experience
- open questions: gradient flow β function vs. MSbar?
- many other applications: gradient flow as UV regulator?