From the chiral anomaly to the gradient-flow expression for the topological susceptibility

Martin Lüscher, Theoretical Physics Department, CERN

Based in part on work done with Peter Weisz

ECT* Workshop on "The gradient flow in QCD and other strongly coupled field theories"

Trento, 20–24 March 2023

Topological charge beyond the semi-classical level

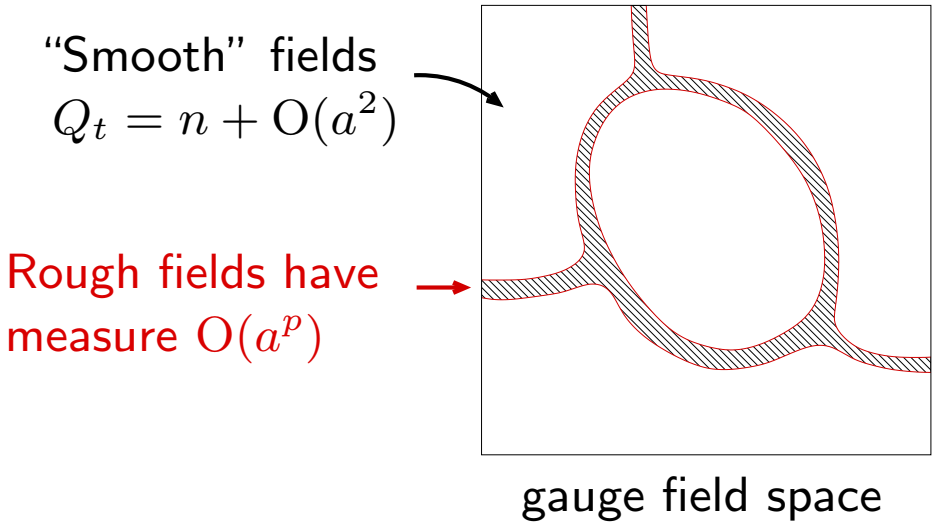
Conceptually challenging, since

- Fields are typically nowhere continuous
- $\langle q(x)q(0)\rangle \underset{x\to 0}{\sim} |x|^{-8} \quad \Rightarrow \quad \langle Q^n\rangle$ not obviously well defined
- Field space in LQCD is connected

Using the Yang–Mills gradient flow . . .

ML '10

Lattice gauge fields at flow time $t > 0$ become increasingly "smooth" as $a \to 0$



$\Rightarrow$ In the continuum limit, the topological sectors emerge dynamically!

But how exactly are the moments $\langle Q_t^n \rangle$ related to the chiral anomaly?

Renormalization of $q(x)$

Dimensional regularization

Avoid Levi-Civita symbol and γ_5 by replacing

Larin & Vermaseren '91
Larin '93

$$\epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \quad \to \quad F^a_{[\mu\nu} F^a_{\rho\sigma]}$$

$$\overline{\psi}_r \gamma_\mu \gamma_5 \psi_s \quad \to \quad \overline{\psi}_r \gamma_{[\mu} \gamma_\nu \gamma_{\rho]} \psi_s$$

$\Rightarrow$ To all orders

Breitenlohner et al. '84
Larin '93
Zoller '13
Ahmed et al. '15, '21
ML & Weisz '21

$$(F^a_{[\mu\nu} F^a_{\rho\sigma]})_{\mathrm{R}} = F^a_{[\mu\nu} F^a_{\rho\sigma]} + \frac{1}{3} X_A \partial_{[\mu} A_{\nu\rho\sigma]}$$

$$(A_{\mu\nu\rho})_{\mathrm{R}} = Z_A A_{\mu\nu\rho}$$

Proof is based on the "descent equations" Stora '77, '84; Zumino '84

$$F^a_{\mu\nu}F^a_{\rho\sigma}\mathrm{d}x_\mu\mathrm{d}x_\nu\mathrm{d}x_\rho\mathrm{d}x_\sigma = \text{2nd Chern character} = \mathrm{d}\phi_3$$

$$\delta_{\mathrm{BRS}}\phi_3 = \mathrm{d}\phi_2$$

$$\delta_{\mathrm{BRS}}\phi_2 = \mathrm{d}\phi_1$$

$$\delta_{\mathrm{BRS}}\phi_1 = \mathrm{d}\phi_0$$

$$\delta_{\mathrm{BRS}}\phi_0 = 0$$

i.e. of the algebraic (cohomology) properties of the charge density

Lattice regularization

Choose a formulation preserving chiral symmetry

Ginsparg & Wilson '82
Kaplan '92
...

$\Rightarrow$ There are lattice versions of

$$A_\mu^{rs} = \overline{\psi}_r \gamma_\mu \gamma_5 \psi_s, \qquad P_{rs} = \overline{\psi}_r \gamma_5 \psi_s,$$

Kikukawa & Yamada '99
Hasenfratz et al. '02

and the charge density q such that

$$\partial_\mu A_\mu^{rs} + (m_r + m_s) P_{rs} = -2\delta_{rs} q$$

holds exactly in on-shell correlation functions

Using theses symmetries and standard arguments (power-counting etc.)

$$q_{\mathrm{R}} = q + X_A \partial_\mu A_\mu, \qquad X_A = \mathrm{O}(g_0^4)$$

$$(A_\mu)_{\mathrm{R}} = Z_A A_\mu$$

as in the case of dimensional regularization

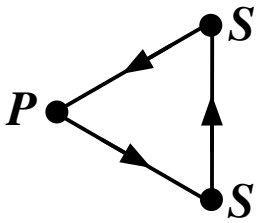
Giusti et al. '02

Density-chain formulae for $\langle Q^n \rangle$

Giusti, Rossi & Testa '04
ML '04

Example

$$m_1 m_2 m_3 \underbrace{\int_{x,y,z} [P_{12}(x) S_{23}(y) S_{31}(z)]_{\mathrm{Wick}}}$$



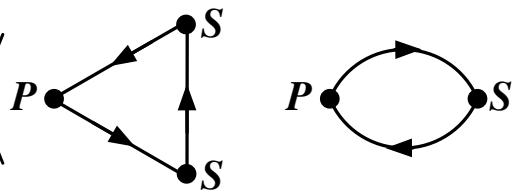
$$= m_1 m_2 m_3 \, \mathrm{Tr}\{\gamma_5 (\not{D} + m_1)^{-1} (\not{D} + m_2)^{-1} (\not{D} + m_3)^{-1}\}$$

$$= \mathsf{index}(\not{D}) = Q$$

Holds exactly, for all gauge fields, in lattice QCD with exact chiral symmetry!

Assuming 5 or more quark flavours

$$\langle Q^2 \rangle = m_1 \ldots m_5 \times \Big\langle$$



$$\Big\rangle$$

- Short-distance singularities are integrable
- Does not require renormalization

Well-defined <u>universal</u> formula for the topological susceptibility!

Flowed topological charge

ML '10
Weisz & ML '11
Hieda, Makino & Suzuki '17

At gradient-flow time $t > 0$

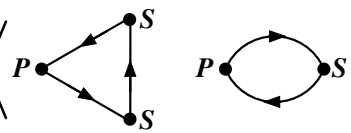
- No short-distance singularities
- The topological charge Q_t does not require renormalization
- In the continuum limit, $\partial_t \langle Q_t \ \ldots \rangle = 0$

In particular, the moments $\langle Q_t^n \rangle$ are well defined and independent of t

Gradient flow vs density chains

Does the equality

$$\langle Q_t^2 \rangle = m_1 \ldots m_5 \times \Big\langle$$



$$\Big\rangle$$

hold beyond the semi-classical level?

Topology of field space $\leftrightarrow$ chiral anomaly

Cè, Consonni, Engel & Giusti '15 (YM theory)
ML '21 (full QCD)

Small flow-time expansion

As $t \to 0$

$$q_t(x) \sim c_1(t)\phi_1(x) + c_2(t)\phi_2(x) + \mathrm{O}(t)$$

where

$$\phi_1(x) = q(x) + X_A \partial_\mu A_\mu(x)$$

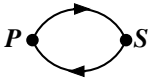
$$\phi_2(x) = Z_A \partial_\mu A_\mu(x)$$

Moreover, using perturbation theory and the RG

$$\lim_{t\to 0} c_1(t) = 1, \qquad \lim_{t\to 0} c_2(t) = 0$$

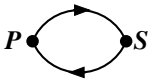
Hieda & Suzuki '16; ML & Weisz '21

As a consequence

$$\langle Q_t^2 \rangle = \langle Q_t Q_s \rangle = \langle Q_t Q \rangle = m_4 m_5 \times \Big\langle Q_t$$ 

$\Big\rangle$

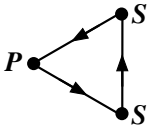
The correlation function

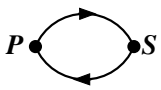
$$\Big\langle Q_t$$ 

$$\Big\rangle = \int_{x,y,z} \langle q_t(x) P_{12}(y) S_{21}(z) \rangle$$

however develops additional short-distance singularities as $t \to 0$

Must control these to be able to establish the identity

$$\langle Q_t^2 \rangle = m_1 \ldots m_5 \times \Big\langle$$ 



$\Big\rangle$

Generating function for density chains

Consider a complex mass matrix M

$$S_F = \int_x \overline{\psi}(x)\{\not{D} + MP_- + M^\dagger P_+\}\psi(x), \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$\partial^S_{rs} S_F = \int_x S_{rs}(x), \qquad \partial^P_{rs} S_F = \int_x P_{rs}(x)$$

$\Rightarrow$ The free energy

$$F(M) = -\ln\{Z(M)\}$$

generates density chains, e.g.

$$\partial^P_{12}\partial^S_{23}\partial^S_{31} F(M) = \int_{x,y,z} \langle P_{12}(x) S_{23}(y) S_{31}(z)\rangle_{\rm c}$$

Gasser & Leutwyler '83; Leutwyler & Smilga '92

Exact flavour symmetries

For any anti-Hermitian flavour matrix λ

$$\delta_\lambda^V M = [\lambda, M] \quad \Rightarrow \quad \delta_\lambda^V F(M) = 0$$

$$\delta_\lambda^A M = \{\lambda, M\} \quad \Rightarrow \quad \delta_\lambda^A F(M) = -2\,\mathrm{tr}\{\lambda\}\langle Q\rangle$$

Renormalization ($N_\mathrm{f} \geq 5$)

$F(M)$ requires additive renormalization

$$\Delta F(M) = V\left[\frac{z_0}{a^4} + \frac{z_1}{a^2}\mathrm{tr}\{M^\dagger M\} + z_2\mathrm{tr}\{M^\dagger M\}^2 + z_3\mathrm{tr}\{(M^\dagger M)^2\}\right]$$

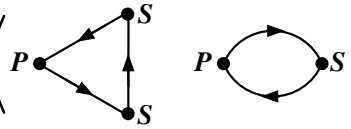
$\Rightarrow$ *$F + \Delta F$ has the same symmetry properties as F*

Final steps

Free energy $\leftrightarrow$ density-chain formula at $a > 0$

$$\left[\delta_\lambda^A \delta_\eta^A \{F(M) + \Delta F(M)\}\right]_{\text{diagonal } M} = 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q^2\rangle$$

$$= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\, m_1 \ldots m_5 \times \Big\langle$$



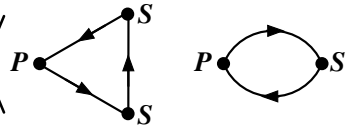
$\Big\rangle$

Final steps

Free energy $\leftrightarrow$ density-chain formula at $\cancel{a \gg 0}$ $a = 0$

$$\left[\delta_\lambda^A \delta_\eta^A \{F(M) + \Delta F(M)\}\right]_{\text{diagonal } M} \cancel{= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q^2\rangle}$$

$$= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\, m_1 \ldots m_5 \times \Big\langle$$



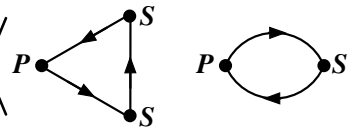
$$\Big\rangle$$

Final steps

Free energy $\leftrightarrow$ density-chain formula at $\cancel{a>0}$ $a=0$

$$\left[\delta^A_\lambda \delta^A_\eta \{F(M)+\Delta F(M)\}\right]_{\text{diagonal } M} \cancel{= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q^2\rangle}$$

$$= 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\, m_1 \ldots m_5 \times \Big\langle$$



$$\Big\rangle$$

Free energy $\leftrightarrow$ gradient-flow formula at $a=0$ and any M

(1) $\quad 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_t^2\rangle_{\mathrm{c}} = 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_t Q\rangle_{\mathrm{c}} = -2\,\mathrm{tr}\{\eta\}\delta^A_\lambda\langle Q_t\rangle$

(2) $\quad -2\,\mathrm{tr}\{\eta\}\langle Q_t\rangle = -2\,\mathrm{tr}\{\eta\}\langle Q\rangle = \delta^A_\eta\{F(M)+\Delta F(M)\}$

$\Rightarrow \quad 4\,\mathrm{tr}\{\lambda\}\mathrm{tr}\{\eta\}\langle Q_t^2\rangle_{\mathrm{c}} = \delta^A_\lambda\delta^A_\eta\{F(M)+\Delta F(M)\}$

Conclusions

The equality of the density-chain and gradient-flow formulae

- ⋆ *Relates the chiral anomaly to the topology of field space at the fully non-perturbative level*

- ⋆ *Provides the definitive justification for using the flow-formula for the topological susceptibility*

Would be difficult to show w/o formulations of LQCD preserving chiral symmetry!