FONDAZIONE BRUNO KESSLER

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EUROPEAN CENTRE
FOR THEORETICAL STUDIES
IN NUCLEAR PHYSICS AND RELATED AREAS

Λ-parameter of SU(3) Yang-Mills theory from the continuous β-function

Curtis Taylor Peterson
In collaboration with Anna Hasenfratz, Jake van Sickle, and Oliver Witzel

arXiv:2303.00704

The renormalization group (RG) β-function

- The RG beta function describes the dependence of renormalized coupling $g^2(\mu)$ on some energy scale μ^2

$$\beta(g^2) \equiv \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} g^2(\mu)$$

- Solution yields the famous Λ-parameter from dimensional transmutation

$$\Lambda^2/\mu^2 \sim \exp\left[- \int^{g^2(\mu)} \mathrm{d}x \frac{1}{\beta(x)} \right]$$

Requires connection to low-energy hadronic scale

$\implies$ Non-perturbative determination of β-function is essential

RG β-function on the lattice

- **Gradient flow** (GF) is a continuous smearing operation
  - Describes a real-space RG transformation in infinite volume when combined with appropriately-defined coarse graining step

[Lüscher, M., *JHEP* **08** (2010) 71]

$$\frac{\mathrm{d}}{\mathrm{d}t} V_t(x,\mu) = -g_0^2 (\partial_{x,\mu} \mathcal{S}_F) V_t(x,\mu)$$
$$V_0(x,\mu) = U(x,\mu)$$

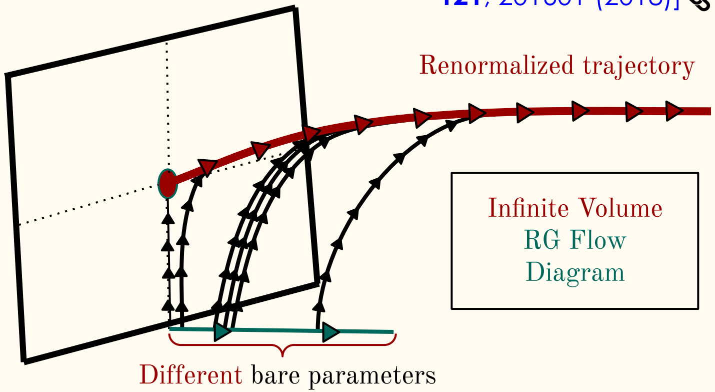
- Define a renormalized running coupling ($\mu^2 \propto 1/8t$)*

$$g^2_{\rm GF}(t; g_0^2) \equiv \mathcal{N} \langle t^2 E(t) \rangle$$

  - Describes flow along renormalized trajectory with corresponding β-function

$$\beta_{\rm GF}(t; g_0^2) = -t \frac{\mathrm{d}}{\mathrm{d}t} g^2_{\rm GF}(t; g_0^2)$$

[Carosso, A., Hasenfratz, A., Neil, E. *PRL* **121**, 201601 (2018)]



*$\mathcal{N} = 128\pi^2/3(N^2-1)$ chosen such that the GF coupling matches $\overline{\rm MS}$ at tree level

Λ parameter of the SU(3) Yang-Mills theory from the continuous β function

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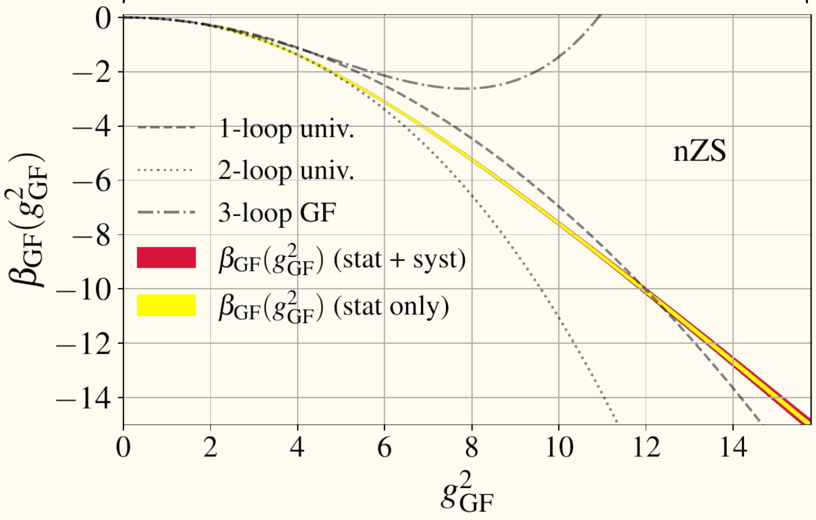
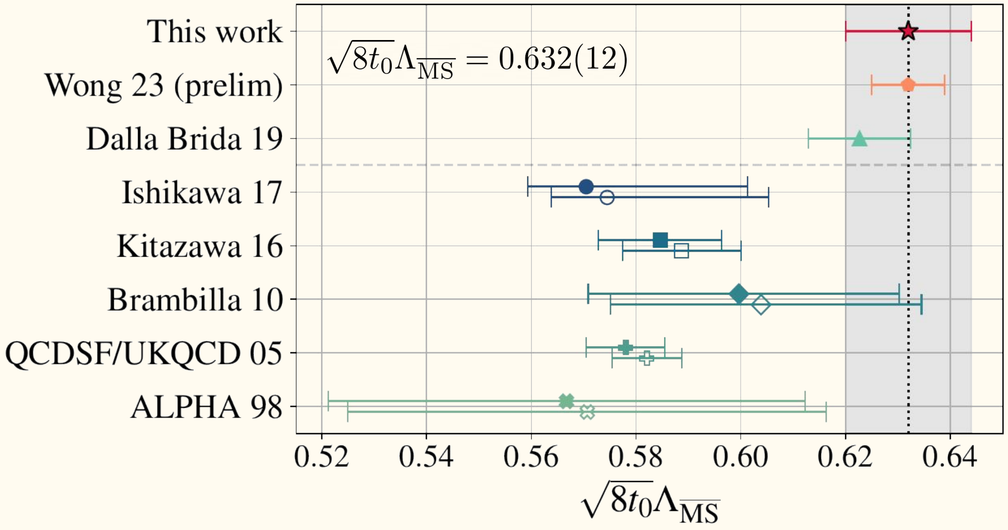
[Hasenfratz, A., Peterson, C.T., Witzel, O., van Sickle, J., arXiv:2303.00704, *sub. to PRD*]

Continuous β-function from UV to IR in continuum



Direct determination of the Yang-Mills Λ-parameter



The continuous β-function method (CBFM)

1. Calculate GF coupling & β-function
   a. Tree-level normalization (tln) factor corrects for gauge zero-modes and tree-level cutoff effects

$$g^2_{\rm GF}(t; L, g_0^2) = \mathcal{N}\,\frac{1}{C(t,L)}\,\langle t^2 E(t)\rangle$$

   b. $\mathrm{d}/\mathrm{d}t$ discretized with 5-point stencil

$$\beta_{\rm GF}(t; L, g_0^2) = -t\frac{\mathrm{d}}{\mathrm{d}t} g^2_{\rm GF}(t; L, g_0^2)$$

2. Infinite volume extrapolation $(a/L)^4 \to 0$ at fixed $\beta_b \equiv 6/g_0^2$ and t/a^2
3. Interpolate $\beta_{\rm GF}(t; g_0^2)$ in $g^2_{\rm GF}(t; g_0^2)$ at fixed t/a^2
4. Continuum extrapolation $a^2/t \to 0$ at fixed $g^2_{\rm GF}$

$$C(t,L) = \frac{2\eta t^2}{L^4} + \frac{\eta t^2}{L^4}\sum_{n_\mu=0, n^2\neq 0}^{L/a-1} \mathrm{Tr}\left[\underbrace{e^{-t(\mathcal{S}_F+\mathcal{G})}}_{\text{Flow action}} \underbrace{(\mathcal{S}_G+\mathcal{G})^{-1}}_{\text{Gauge action}} \underbrace{e^{-t(\mathcal{S}_F+\mathcal{G})}}_{\text{Flow action}} \underbrace{\mathcal{S}_O}_{E(t)\text{ observable}} \right]$$

[Fodor, Z., Holland, K., Kuti, J., Mondal, S., Nogradi, D., *JHEP* (2014) 018]

[Hasenfratz, A., Witzel, O., *PRD* **101**, 034514 (2019)]

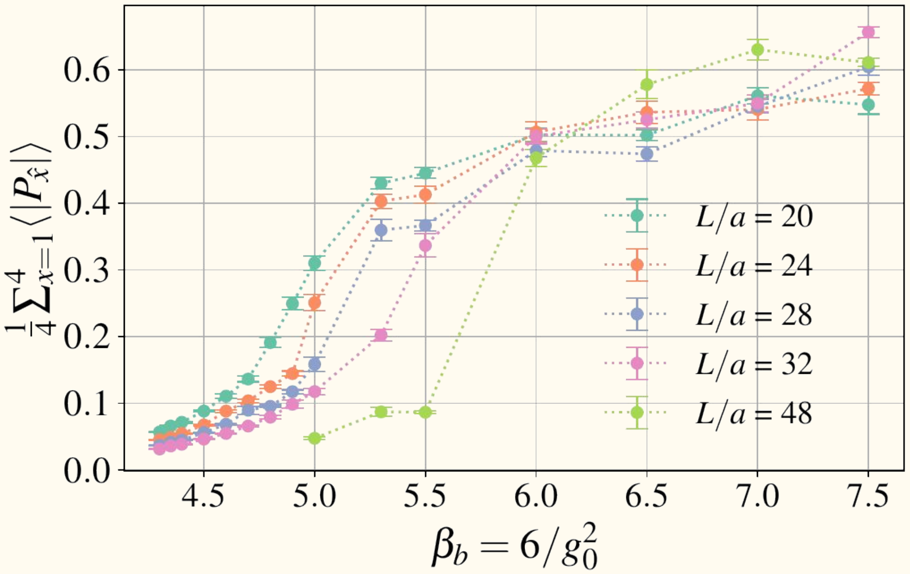
[Kuti, J., Fodor, Z., Holland, K., Wong, K. H. *PoS*, LATTICE2021 (**2021**) 321]

[Hasenfratz, A., Peterson, C.T., Witzel, O., van Sickle, J., arXiv:2303.00704, *sub. to PRD*]

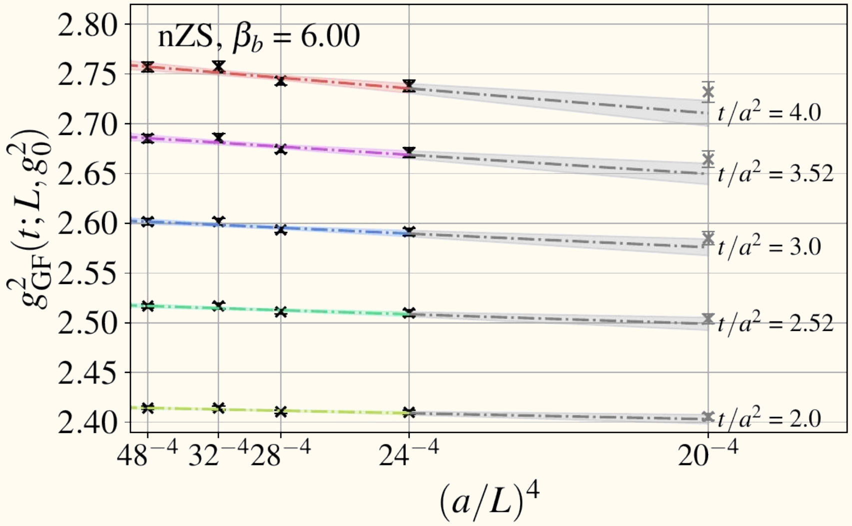
Simulation details

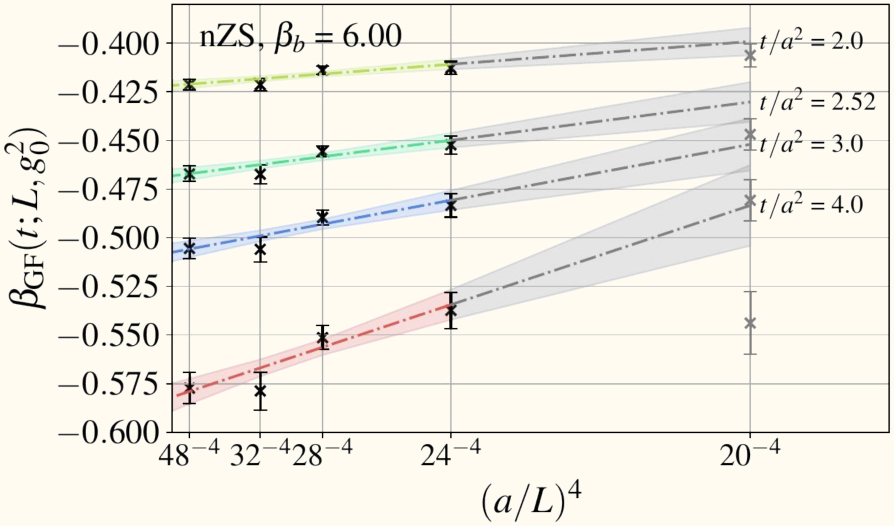
- Simulations are performed with a pure gauge Symanzik action using GRID
  - $4.3 \leq \beta_b \leq 7.5$ (15 overall)
  - Symmetric volumes with periodic boundary conditions in all four directions
    - $L/a = 20, 24, 28, 32, 48$
- Gradient flow performed with Wilson flow and Zeuthen flow using QLUA
  - Flow (F) and operator (O) combinations abbreviated
    - FO for no tln (e.g. ZS)
    - nFO with tln (e.g. nZS)

Flowed Polyakov loop



Infinite volume extrapolation (weak coupling)

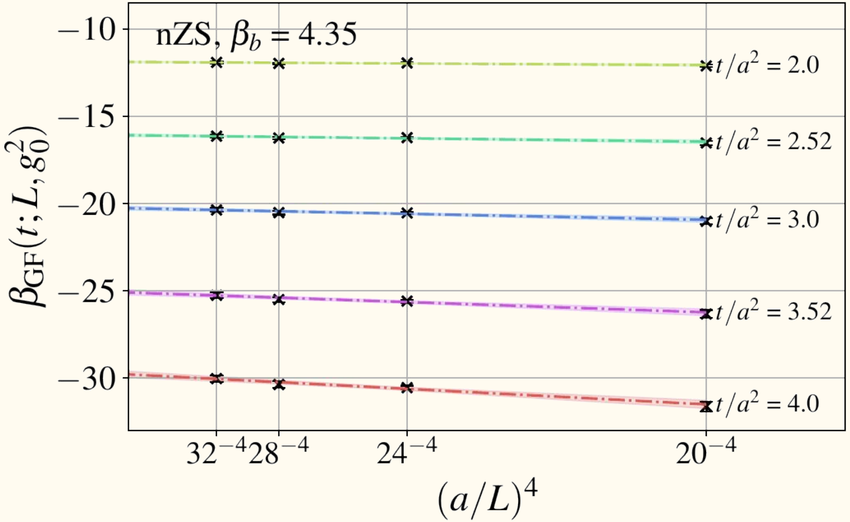


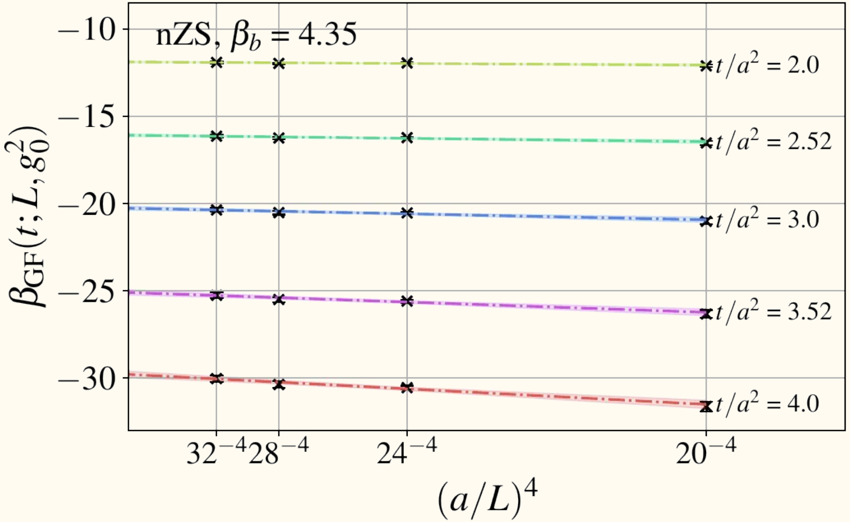


Extrapolate both $g^2_{\mathrm{GF}}(t; L, g_0^2)$ and $\beta_{\mathrm{GF}}(t; L, g_0^2)$ linearly in $(a/L)^4 \to 0$ at fixed β_b and t/a^2

$24 \leq L/a \leq 48$ for $\beta_b \geq 5.5$

Infinite volume extrapolation (strong coupling)





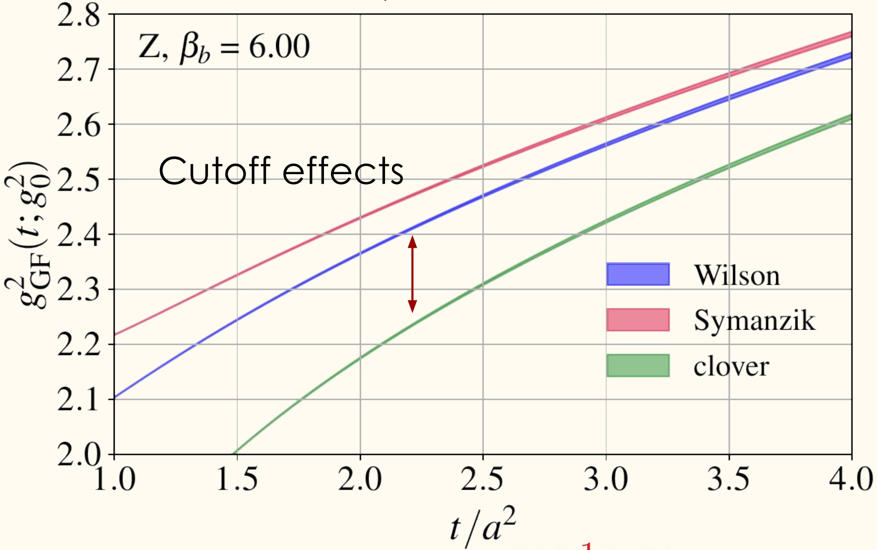
Confinement introduces infrared scale μ_{conf}

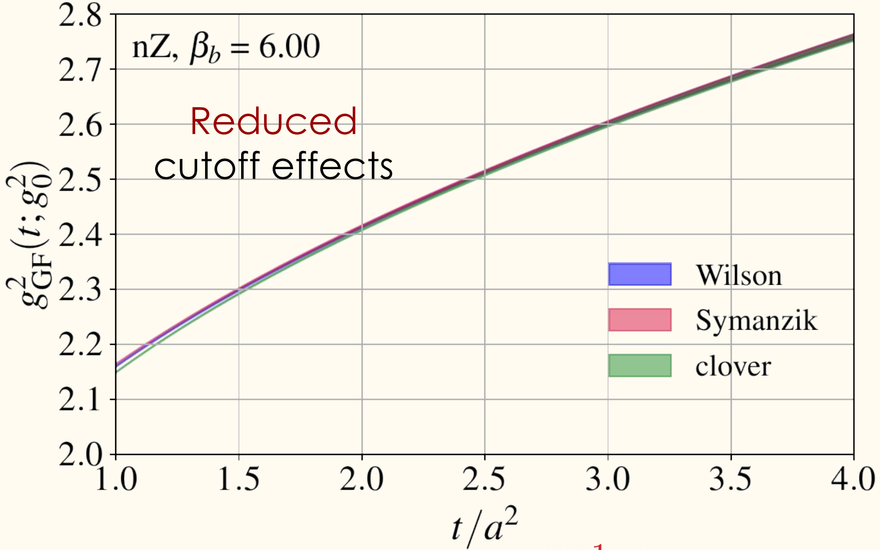
$\implies$ Finite-volume effects suppressed

$20 \leq L/a \leq 32$ for $\beta_b \leq 4.9$

Tree-level normalization (weak coupling)

Infinite volume extrapolated*





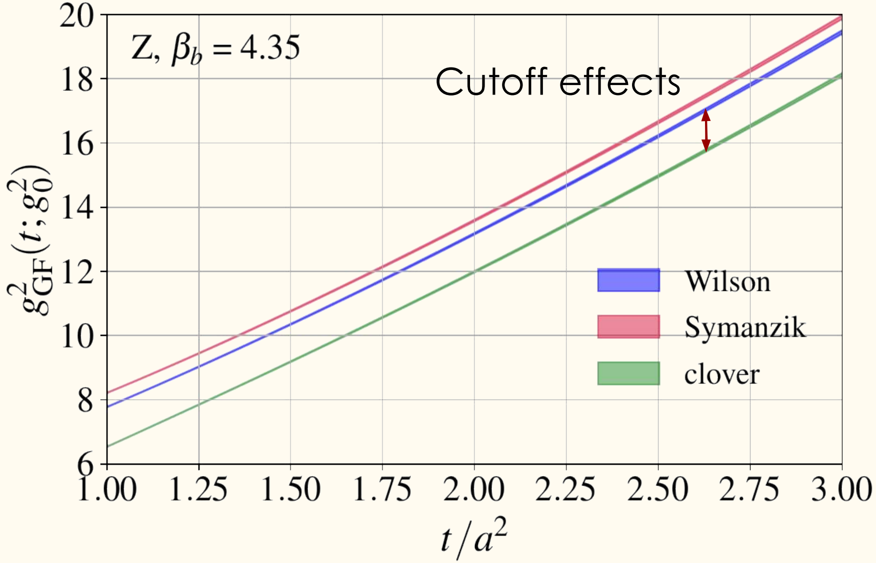
$$g^2_{\mathrm{GF}}(t; L, g_0^2) = \mathcal{N} \underbrace{\frac{1}{1+\delta(t,L)}}_{\text{Gauge zero-mode correction}} \langle t^2 E(t) \rangle$$

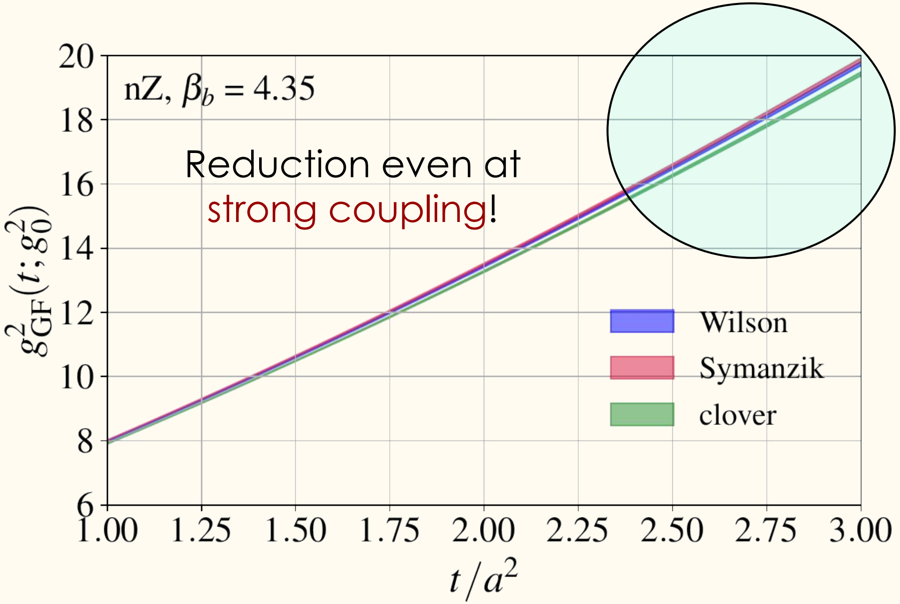
$$g^2_{\mathrm{GF}}(t; L, g_0^2) = \mathcal{N} \underbrace{\frac{1}{C(t,L)}}_{\text{Tree-level normalization}} \langle t^2 E(t) \rangle$$

[Fodor, Z., Holland, K., Kuti, J., Mondal, S., Nogradi, D., *JHEP* (2014) 018]

Tree-level normalization (strong coupling)

Infinite volume extrapolated*





See also:
[Schneider, C., Hasenfratz, A., Witzel, O., *arXiv:2211.12406*]

We use nZS for our central analysis

[Fodor, Z., Holland, K., Kuti, J., Mondal, S., Nogradi, D., *JHEP* (2014) 018]

Interpolation

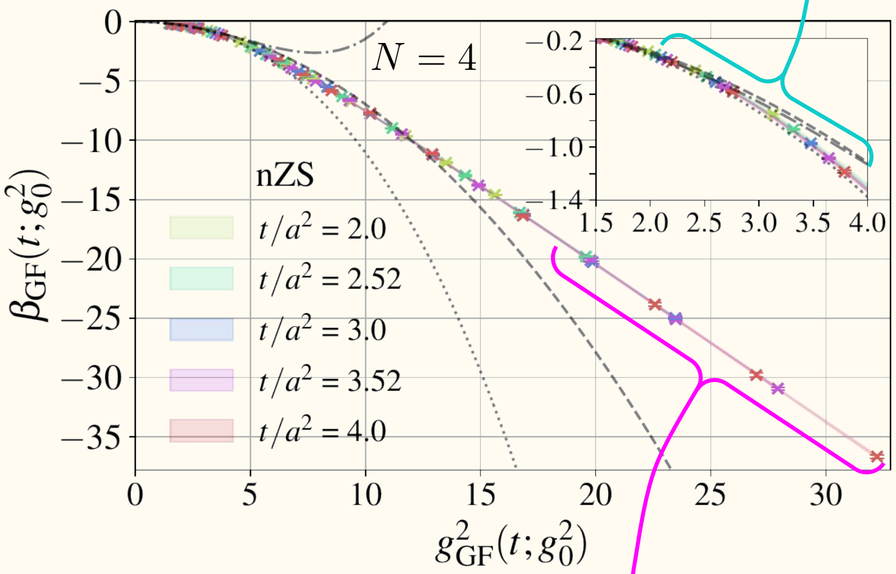
- Must interpolate $\beta_{\mathrm{GF}}(t; g_0^2)$ in $g^2_{\mathrm{GF}}(t; g_0^2)$ at fixed t/a^2 before performing continuum extrapolation.
  - Interpolation should accommodate weak/strong coupling asymptotics.

$$\mathcal{I}_N(g^2_{\mathrm{GF}}) = \frac{-p_0 g^4_{\mathrm{GF}}(1 + p_1 g^2_{\mathrm{GF}} + \ldots + p_N g^{2N}_{\mathrm{GF}})}{1 + q_1 g^2_{\mathrm{GF}} + \ldots + q_{N+1} g^{2N+2}_{\mathrm{GF}}}$$

$$\propto g^4_{\mathrm{GF}} \qquad \propto g^2_{\mathrm{GF}}$$

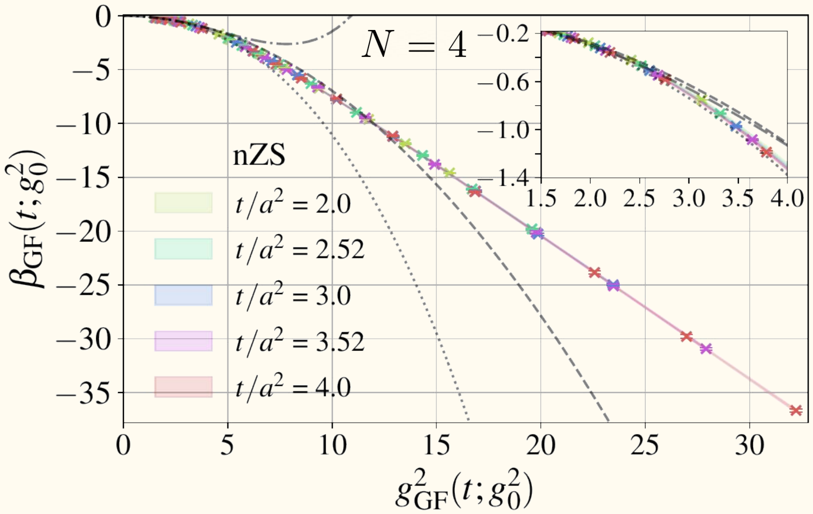
  - Interpolation stable for $N \geq 4$

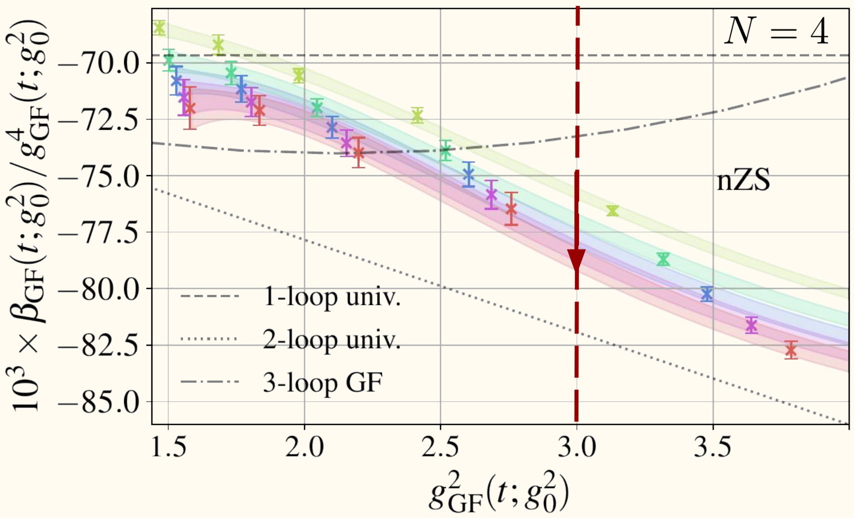
$$\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}}) \sim b_0 g^4_{\mathrm{GF}} + b_1 g^6_{\mathrm{GF}} + \mathcal{O}(g^8_{\mathrm{GF}})$$



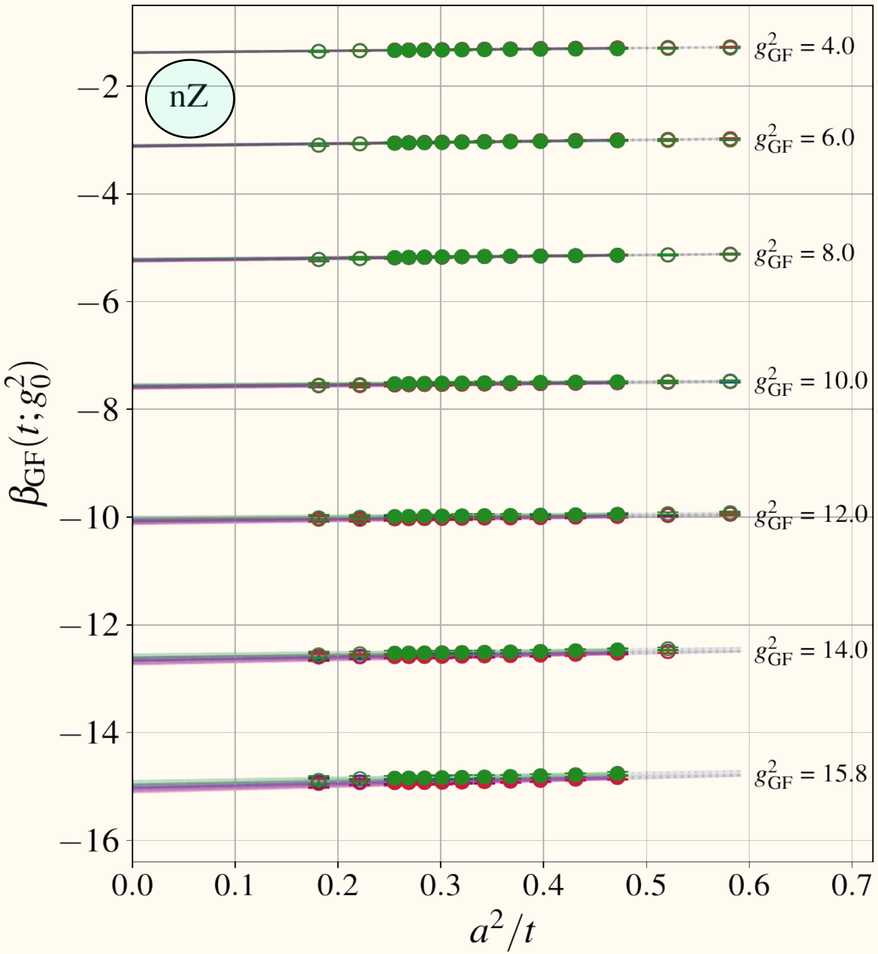
$$\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}}) \approx c_0 + c_1 g^2_{\mathrm{GF}}$$

Interpolation

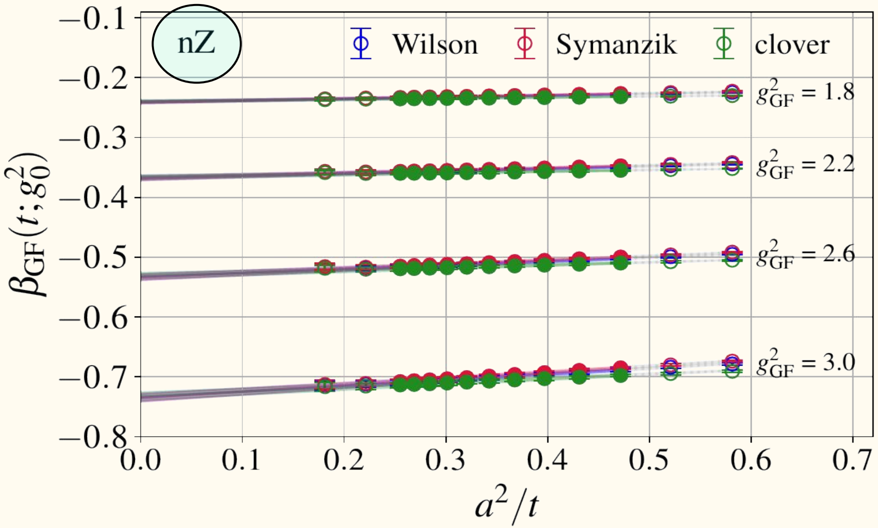




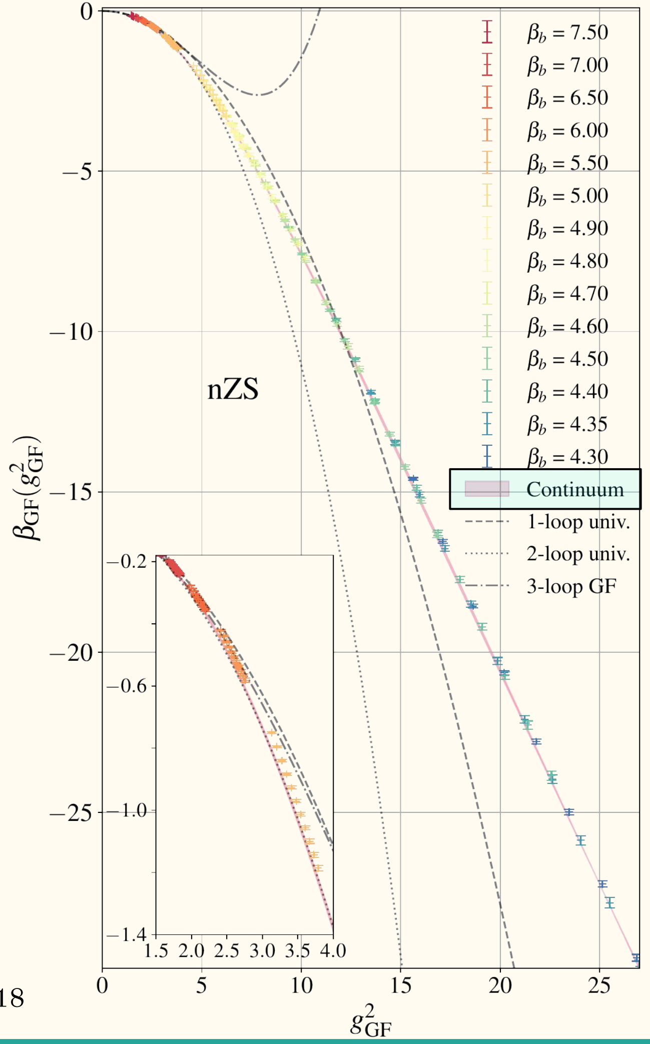
Continuum limit can now be taken at fixed $g^2_{\rm GF}$



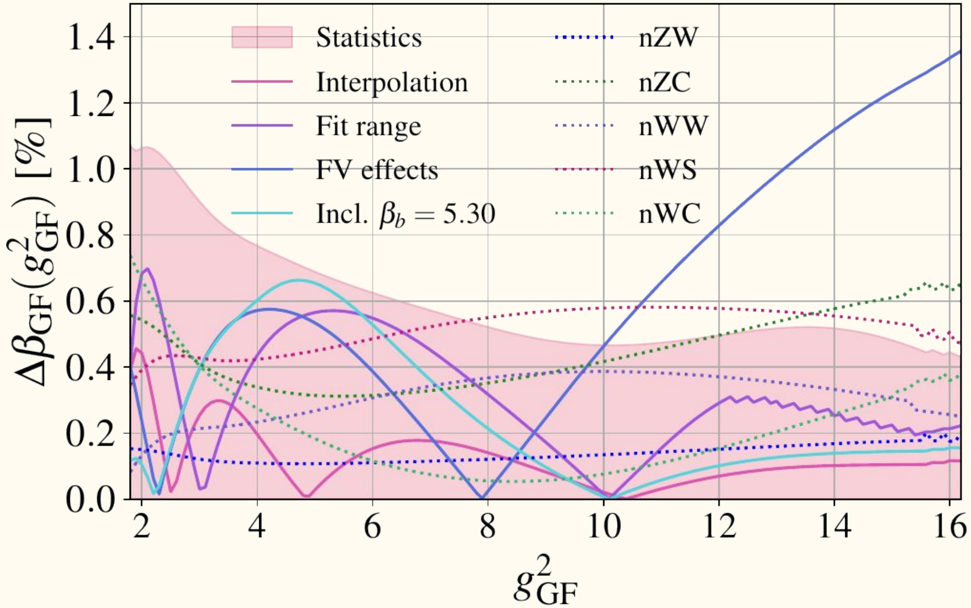
Continuum limit (extrapolation)



- Linear extrapolation in a^2/t at fixed $g^2_{\rm GF}$
  - $t_{\rm min}/a^2 = 2.0, t_{\rm max}/a^2 = 4.0$
  - Not sensitive to $t_{\rm min}/a^2, t_{\rm max}/a^2$
- tln flattens continuum extrapolation
  - Also little operator dependence



Continuum limit (errors)



- Dominant sources of systematic error
  - Variations in continuum fit range
  - Finite-volume effects
  - Exclusion of $\beta_b \sim 5.30$
  - Other flow/operator combinations

Continuum limit (result)

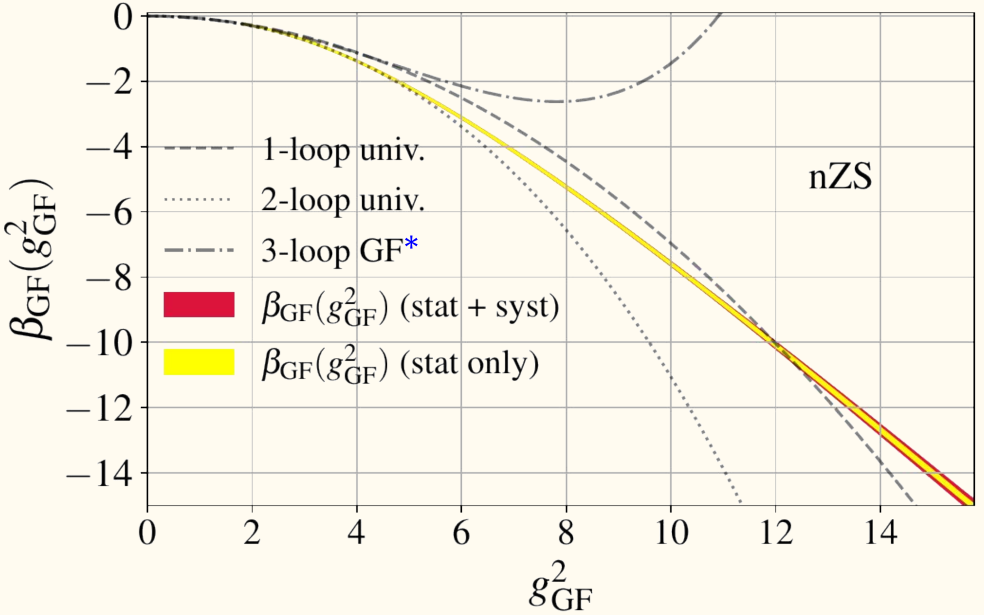
- Systematic error estimated from largest systematic effect at each g^2_{GF}
- Total error estimated by adding systematic and statistical errors in quadrature
- Linear behavior at strong coupling

$$\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}}) \approx c_0 + c_1 g^2_{\mathrm{GF}}$$

$$c_0 = 5.91(30) \quad c_1 = -1.326(12)$$

- Implies scaling relation between $\langle t^2 E(t)\rangle$ and GF flow time

$$\langle t^2 E(t)\rangle \sim a_0 + a_1 (t/\tilde{t})^{-c_1}$$



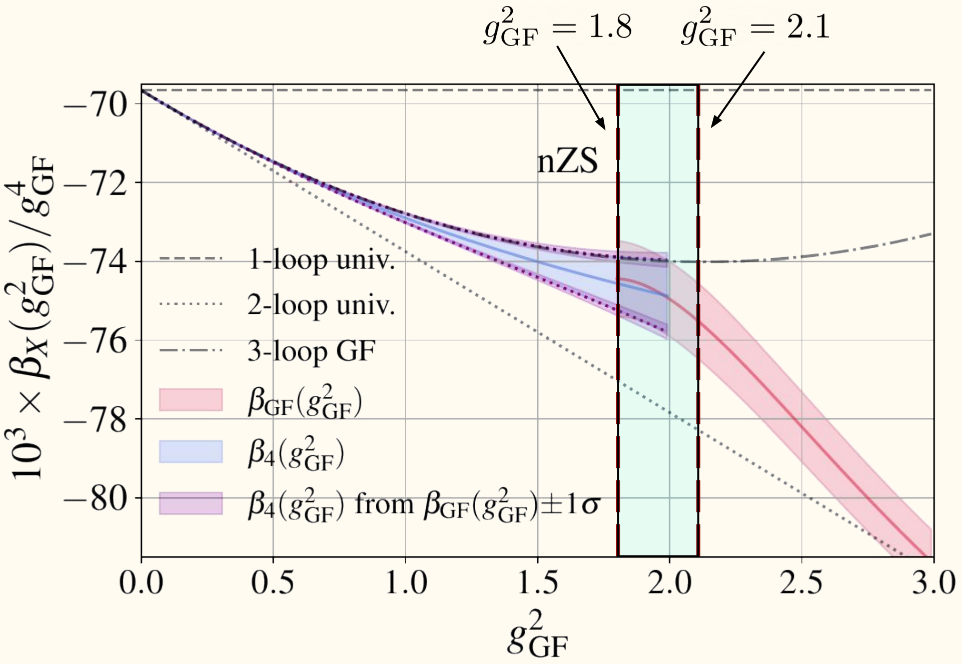
*[Artz, Harlander, Lange, Neumann, Prausa *JHEP* **06** (2019) 121]

Effective β-function

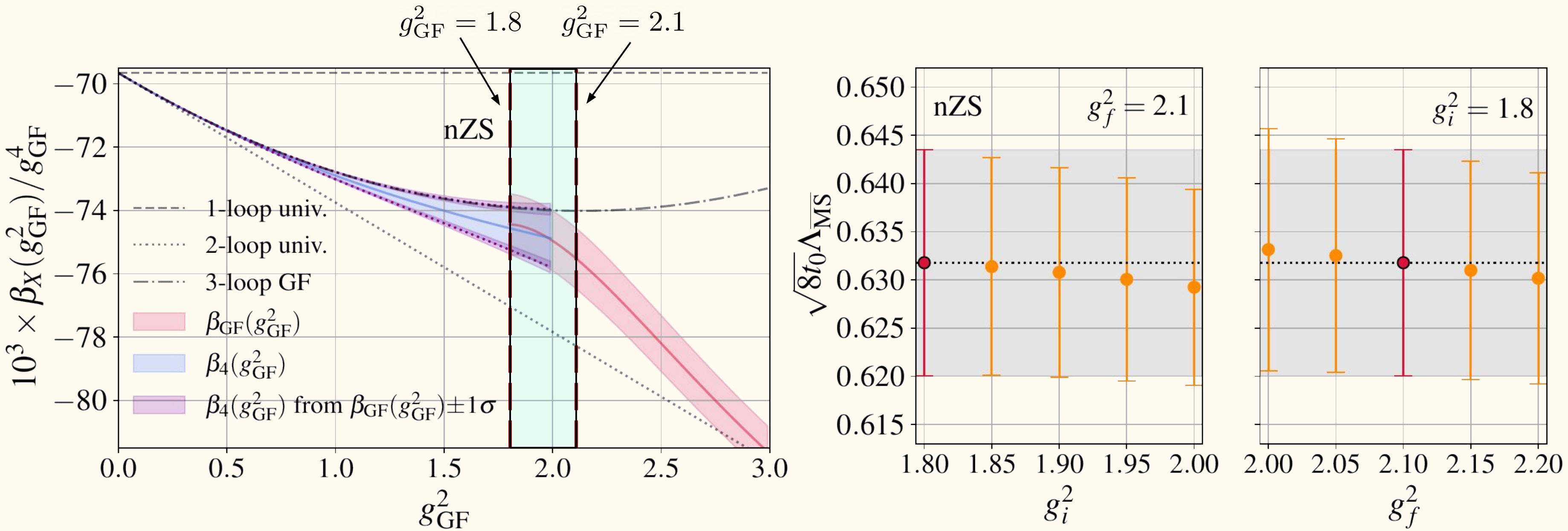
- Need $\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}})$ down to $g^2_{\mathrm{GF}} = 0$ for Λ-parameter
- Extend our β-function below weakest coupling by parameterization

$$\beta_4(g^2_{\mathrm{GF}}) \equiv -g^4_{\mathrm{GF}}\left(b_0 + b_1 g^2_{\mathrm{GF}} + b_2 g^4_{\mathrm{GF}} + b_p g^6_{\mathrm{GF}}\right)$$

  - b_p determined by matching β_{GF} to β_4 between $g_i^2 = 1.8, g_f^2 = 2.1$



Effective β-function



Systematic uncertainties from g_i^2, g_f^2 are under control

Final result for $\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}}$ from pure Yang-Mills

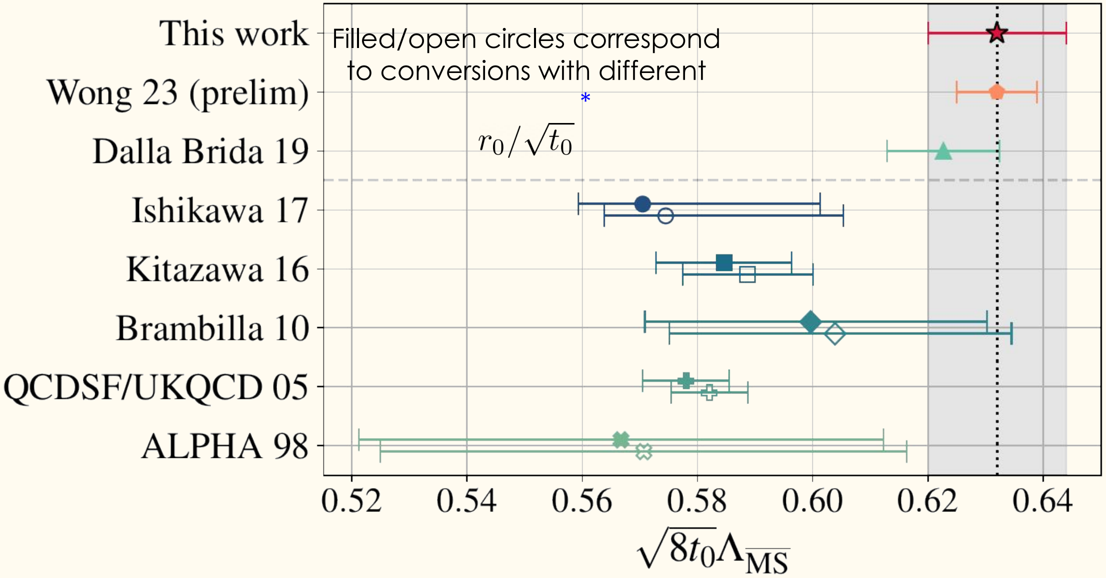
- We obtain for our final result

$$\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}} = 0.632(12)$$

  - Given t_0, predicts $\Lambda_{\overline{\mathrm{MS}}}$ (and vice-versa)

- Agrees with recent GF determinations

- Considerable spread amongst different determinations
  - Warrants further scrutiny
  - Is the culprit $r_0/\sqrt{t_0}$?



[Wong, C.H., Borsanyi, S., Fodor, Z., Holland, K., Kuti, J., arXiv:2301.06611 (2023)]
*[Dalla Brida, M., Ramos, A. *EPJC* **79**, 720 (2019)]
[FLAG 2021, arXiv:2111.09849 (2021)]
*[Lüscher, M., *JHEP* **08** (2010) 71]

Conclusions

[Hasenfratz, A., Peterson, C.T., Witzel, O., van Sickle, J., arXiv:2303.00704, *sub. to PRD*]

Λ parameter of the SU(3) Yang-Mills theory from the continuous β function

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- ❖ The continuous β-function method (CBFM) is well-suited to study RG properties in confining regime
  - ➢ Infinite volume extrapolation is controlled
  - ➢ Should choose interpolating function that reproduces β-function asymptotics
  - ➢ Small cutoff effects with tln

- ❖ CBFM allows for a direct determination of the Λ-parameter from the t_0 scale
  - ➢ In agreement with recent GF determinations

Please refer to our recent arXiv submission for more details

Acknowledgements

U. Colorado: RMACC Summit

USQCD: BNL SDCC

U. Siegen: OMNI

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Supplemental Slides

Λ-parameter calculation

$$\frac{\Lambda^2}{\mu^2} = \left(b_0 g^2(\mu)\right)^{-\frac{b_1}{b_0^2}} \exp\left(-\frac{1}{b_0 g(\mu)^2}\right)$$

$$\times \exp\left[-\int_0^{g^2(\mu)} \mathrm{d}x \left(\frac{1}{\beta(x)} + \frac{1}{b_0 x^2} - \frac{b_1}{b_0^2 x}\right)\right]$$

$$g_Y^2(\mu) = g_X^2(\mu) + c_1 g_X^4(\mu) + \mathcal{O}\left(g_X^6(\mu)\right)$$

$$\Longrightarrow \Lambda_Y^2/\Lambda_X^2 = \exp\left[c_1/b_0\right]$$

A proposal for non-perturbative matching

β-functions in different schemes "X" and "Y" are related by…

$$\beta_Y\left(g_Y^2\right) = \frac{\partial g_Y^2}{\partial g_X^2}\beta_X\left(g_X^2\right)$$

Parameterize…

$$g_Y^2 = \phi(g_X^2) = g_X^2 + g_X^2 \sum_{i=1}^{N} k_i g_X^{2i}$$

Fit!

$$\beta_Y\left(\phi(g_X^2)\right)/\phi'\left(g_X^2\right) = \beta_X\left(g_X^2\right)$$