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The gradient flow formulation of the electroweak Hamiltonian

The Gradient Flow in QCD and Other Strongly-Coupled Field Theories

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in collaboration with Robert V. Harlander | March 20, 2023

The effective electroweak Hamiltonian

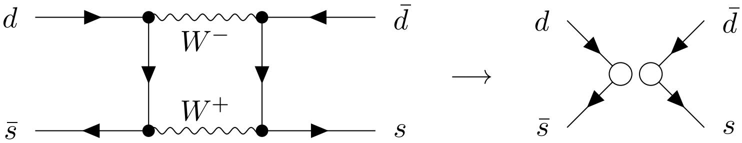
- Observables in flavor physics often computed with effective Hamiltonian of electroweak interactions

$$\mathcal{H}_{\mathrm{eff}} = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^{x} V_{\mathrm{CKM}} \sum_i C_i \mathcal{O}_i$$

  with four-fermion operators like

$$\mathcal{O}^{|\Delta S|=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1-\gamma_5)d)$$

  for $K^0 - \bar{K}^0$ mixing:



- Wilson coefficients $C_i(\mu)$ obtained from perturbative matching to Standard Model at $\mu = \mu_W \sim M_W$
- V_{CKM}: relevant entries of the CKM matrix, e.g. $V_{is}^* V_{id} V_{js}^* V_{jd}$ with $i, j = c, t$

Computing observables

- Flavor observables mostly at low energies

$\Rightarrow$ Use renormalization group equations to evolve down to appropriate scale to avoid large logarithms

- Schematically for Kaon mixing:

$$\langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{|\Delta S|=2} | K^0 \rangle \approx C(\mu_W) U(\mu_W, \mu_K) \langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu_K) | K^0 \rangle$$

- Running with $U(\mu_W, \mu_K)$ determined by anomalous dimension γ of $\mathcal{O}^{|\Delta S|=2}$
- Matrix element $\langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu_K) | K^0 \rangle$ nonperturbative

$\Rightarrow$ Compute on lattice

Complications

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

- While $\mathcal{H}_{\text{eff}}$ is scheme independent, C_i and $\mathcal{O}_i$ are not:

  Perturbative C_i:

  - Dimensional regularization with $D = 4 - 2\epsilon$
  - Operators mix through renormalization, also with evanescent operators (vanish in $D = 4$):

    $$\mathcal{O}^{\text{R}} = Z_{\mathcal{O}\mathcal{O}}\mathcal{O} + Z_{\mathcal{O}\text{E}}E$$

  - C_i scheme dependent:
    1. Explicit dependence on μ
    2. Scheme for γ_5
    3. Choice of evanescent operators

  Lattice $\langle\mathcal{O}_i\rangle$:

  - Lattice spacing a as UV regulator
  - Have to take continuum limit $a \to 0$ in the end
  - Operators mix through renormalization:

    $$\mathcal{O}^{\text{R}} = Z_{11}\mathcal{O}_1 + Z_{12}\mathcal{O}_2$$

  - $\langle\mathcal{O}_i\rangle$ scheme dependent

$\Rightarrow$ Scheme matching between lattice and perturbative results additional source of uncertainty

Gradient flow

- Introduce parameter *flow time* $t \geq 0$ [Narayanan, Neuberger 2006; Lüscher 2009; Lüscher 2010]
- *Flowed fields* in $D+1$ dimensions obey differential *flow equations*:

Gluon flow equation [Narayanan, Neuberger 2006; Lüscher 2010]

$$\partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu} \quad \text{with} \quad B^a_\mu(t,x)\big|_{t=0} = A^a_\mu(x)$$

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - f^{abc}B^c_\mu, \qquad G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc}B^b_\mu B^c_\nu$$

Quark flow equation [Lüscher 2013]

$$\partial_t \chi = \Delta\chi \quad \text{with} \quad \chi(t,x)|_{t=0} = \psi(x)\,,$$

$$\partial_t \bar{\chi} = \bar{\chi}\overleftarrow{\Delta} \quad \text{with} \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x)$$

$$\Delta = (\partial_\mu + B^a_\mu T^a)(\partial_\mu + B^b_\mu T^b), \qquad \overleftarrow{\Delta} = (\overleftarrow{\partial}_\mu - B^a_\mu T^a)(\overleftarrow{\partial}_\mu - B^b_\mu T^b)$$

Flowed operator product expansion

- Flowed composite operators $\tilde{\mathcal{O}}_i(t, x)$ finite [Lüscher, Weisz 2011]
- Small flow-time expansion [Lüscher, Weisz 2011]:

$$\tilde{\mathcal{O}}_i(t, x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

Flowed OPE

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) \equiv \sum_j \tilde{C}_j(t)\tilde{\mathcal{O}}_j(t)$$

- T defined in regular QCD expressed through finite flowed operators $\tilde{\mathcal{O}}_j(t)$
- Gradient-flow definition of T valid both on the lattice and perturbatively

Flowed OPE for the electroweak Hamiltonian

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- Write electroweak Hamiltonian as

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_F}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_F}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_F}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Gradient-flow scheme valid both on the lattice and perturbatively:

Perturbative $\tilde{C}_j$:

- Dimensional regularization with $D = 4 - 2\epsilon$
- Finite and scheme independent:
  1. No explicit dependence on μ
  2. No dependence on scheme for γ_5
  3. Independent of evanescent operators

Lattice $\langle \tilde{\mathcal{O}}_j \rangle$:

- Lattice spacing a as UV regulator
- Finite for $a \to 0$
- No operator mixing

Flowed OPE for the electroweak Hamiltonian

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Lattice $\langle \tilde{\mathcal{O}}_j \rangle$:

- Lattice spacing a as UV regulator
- Finite for $a \to 0$
- No operator mixing

- Three ingredients:
  - C_i known perturbatively through (N)NLO (depending on process)
  - ζ_{ij}^{-1} has to be computed, some first results in [Suzuki, Taniguchi, Suzuki, Kanaya 2020; Harlander, **FL** 2022] $\Rightarrow$ this talk
  - $\langle \tilde{\mathcal{O}}_j \rangle$ to be computed on the lattice $\Rightarrow$ Matthew Black's talk

Lagrangian

- Write Lagrangian for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi ,$$

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f (\not{D}^{\mathrm{F}} + m_f) \psi_f + \dots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t,x)$ and $\lambda_f(t,x)$:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr}\left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right] , \qquad \partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}$$

$$\mathcal{L}_\chi = \sum_{f=1}^{n_\mathrm{f}} \int_0^\infty \mathrm{d}t \left(\bar{\lambda}_f \left(\partial_t - \Delta \right) \chi_f + \bar{\chi}_f \left(\overleftarrow{\partial_t} - \overleftarrow{\Delta} \right) \lambda_f \right), \qquad \partial_t \chi = \Delta \chi, \;\; \partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta}$$

$\Rightarrow$ Flow equations automatically fulfilled

$\Rightarrow$ QCD Feynman rules + gradient-flow Feynman rules (complete list in [Artz, Harlander, **FL**, Neumann, Prausa 2019])

Solving the flow equations

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- Split flow equation into linear part and remainder [Lüscher 2010]

$$\partial_t B_\mu^a = \partial_\nu \partial_\nu B_\mu^a + R_\mu^a \quad \text{with} \quad B_\mu^a(t,x)\big|_{t=0} = A_\mu^a(x)$$

- Solved by

$$B_\mu^a(t,x) = \int_y K_{\mu\nu}(t, x-y) A_\nu^a(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s, x-y) R_\nu^a(s,y)$$

with integration kernel

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

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Propagators

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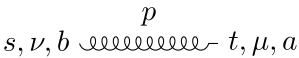
$$B^a_\mu(t,x) = \int_y K_{\mu\nu}(t, x-y) A^a_\nu(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s, x-y) R^a_\nu(s,y)$$

$$K_{\mu\nu}(t,x) = \int_p e^{ip\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{ip\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

- Flowed gluon propagator contains fundamental gluon propagator:

$$\left\langle \widetilde{B}^a_\mu(t,p) \widetilde{B}^b_\nu(s,q) \right\rangle \Big|_{\mathrm{LO}} = \widetilde{K}_{\mu\rho}(t,p) \widetilde{K}_{\nu\sigma}(s,q) \left\langle \widetilde{A}^a_\rho(p) \widetilde{A}^b_\sigma(q) \right\rangle$$

$\Rightarrow$ Can express both by same Feynman rule



$$= \delta^{ab} \frac{1}{p^2} \delta_{\mu\nu}\, e^{-(t+s)p^2}$$

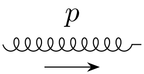
Flow lines

- Flowed gluon Lagrangian:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b \right) \right]$$

$\Rightarrow$ No squared L_μ^a in $\mathcal{L}_B$ $\Rightarrow$ no propagator

- Instead mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p) \widetilde{L}_\nu^b(s,q) \right\rangle$ called *flow line*:

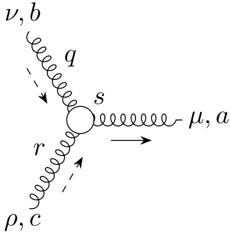
s, ν, b  t, μ, a $= \delta^{ab} \, \theta(t-s) \delta_{\mu\nu} \, e^{-(t-s)p^2}$

- Directed towards increasing flow time

Flow vertices

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\, \mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Example:



$$= -\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$

- Integral restricted by $\theta(t-s)$ from outgoing flow line
- Incoming lines can be both flow lines and flowed propagators

Renormalization

- QCD renormalization of QCD parameters like α_s and quark masses
- Flowed gluon fields do not require renormalization [Lüscher 2010; Lüscher, Weisz 2011]
- Flowed quark fields have to be renormalized: $\chi^\mathrm{R} = Z_\chi^{1/2} \chi^\mathrm{B}$ [Lüscher 2013]

$\Rightarrow$ χ acquire anomalous dimension and not scheme independent

- "Physical" scheme: Ringed fermions $\mathring{\chi} = \mathring{Z}_\chi^{1/2} \chi^\mathrm{B}$ [Makino, Suzuki 2014]:

$$\mathring{Z}_\chi = -\frac{2N_\mathrm{c}}{(4\pi t)^2 \left.\langle \bar{\chi}^\mathrm{B} \overleftrightarrow{\not{D}} \chi^\mathrm{B} \rangle\right|_{m=0}}$$

$\Rightarrow$ $\mathring{\chi}$ formally independent of renormalization scale μ

- $\mathring{Z}_\chi$ available through NNLO [Artz, Harlander, **FL**, Neumann, Prausa 2019]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

$\Rightarrow$ Gradient-flow scheme without operator mixing

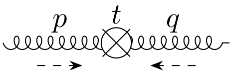
Gluon action density

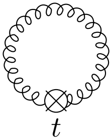
- Simple first observable: vacuum expectation value of gluon action density

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x),$$

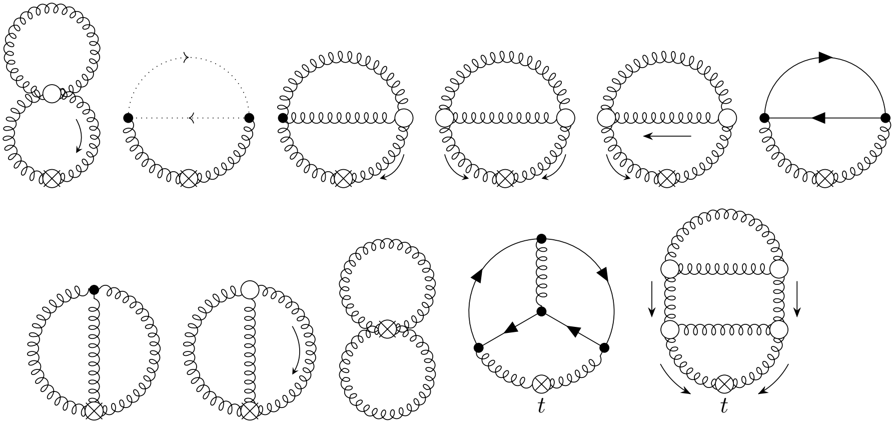
$$G^a_{\mu\nu}(t,x) = \partial_\mu B^a_\nu(t,x) - \partial_\nu B^a_\mu(t,x) + f^{abc} B^b_\mu(t,x) B^c_\nu(t,x)$$

- Feynman rules like

μ, a  ν, b $= -g^2 \delta^{ab} \left(\delta_{\mu\nu} p \cdot q - p_\mu q_\nu \right)$

$\langle E(t) \rangle|_{\mathrm{LO}} =$  $= \dfrac{3\alpha_s}{4\pi t^2} \dfrac{N_A}{8}$

Sample Feynman diagrams for $\langle E(t) \rangle$ at higher orders



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Integration-by-parts relations

- After tensor reduction, we end up with many scalar integrals of the form

$$I(\{t_f^{\mathrm{up}}\}, \{T_i\}, \{a_i\}) = \left(\prod_{f=1}^{F} \int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right) \int_{k_1,\ldots,k_L} \frac{\exp[-(T_1 q_1^2 + \cdots + T_N q_N^2)]}{q_1^{2a_1} \cdots q_N^{2a_N}}$$

with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$

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  with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$
- Chetyrkin and Tkachov observed [Tkachov 1981; Chetyrkin, Tkachov 1981]

$$\int_{k_1,\ldots k_L} \frac{\partial}{\partial k_i^\mu} \left(\tilde{q}_j^\mu \frac{1}{P_1^{a_1} \ldots P_N^{a_N}}\right) = 0$$

⇒ Linear relations between Feynman integrals
- Can easily be adopted to gradient-flow integrals
- Additional new relations for gradient-flow integrals:

$$\int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f \partial_{t_f} F(t_f, \ldots) = F(t_f^{\mathrm{up}}, \ldots) - F(0, \ldots)$$

Laporta algorithm

- Schematically integration-by-parts read

$$0 = (d - a_1)I(a_1, a_2, a_3) + (a_1 - a_2)I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2)I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2)I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
    - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$\begin{aligned} 0 &= (d-1)I(1,1,1) + I(2,1,0), \\ 0 &= (d-2)I(2,1,1) + I(3,0,1) - I(3,1,0), \\ &\vdots \end{aligned}$$

    - Solve with Gaussian elimination

$\Rightarrow$ Express integrals through significantly smaller number of master integrals

Automatized calculation

- `qgraf` [Nogueira 1991]: Generate Feynman diagrams
- `q2e` and `exp` [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]: Assign diagrams to topologies and prepare `FORM` code
- `FORM` [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013]: Insert Feynman rules, perform tensor reduction, Dirac traces, color algebra, and expansions
- Generate system of equations employing integration-by-parts-like relations [Tkachov 1981; Chetyrkin, Tkachov 1981] with in-house `Mathematica` code
- `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ `FireFly` [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020]: Solve system to express all integrals through master integrals with Laporta algorithm [Laporta 2000]
- Calculation of master integrals:
  - Direct integration with Mathematica
  - Expansion employing `HyperInt` [Panzer 2014]
  - Numerical integration with following sector decomposition strategy [Binoth, Heinrich 2000 + 2003] with `FIESTA` [Smirnov, Tentyukov 2008; Smirnov, Smirnov, Tentyukov 2009; Smirnov 2013] and in-house integration routines [Harlander, Neumann 2016] or `pySecDec` [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017; Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk 2018; Heinrich, Jahn, Jones, Kerner, Langer, Magerya, Poldaru, Schlenk, Villa 2021]

$\langle E(t)\rangle$ through NNLO (I)

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$$
\begin{aligned}
\langle E(t)\rangle &= \frac{3\alpha_s}{4\pi t^2}\frac{N_A}{8}\left[1+\frac{\alpha_s}{4\pi}e_1+\left(\frac{\alpha_s}{4\pi}\right)^2 e_2+O(\alpha_s^3)\right]+O(m^2)\\
e_1 &= e_{1,0}+\beta_0\,L(\mu^2 t)\\
e_2 &= e_{2,0}+(2\beta_0\,e_{1,0}+\beta_1)\,L(\mu^2 t)+\beta_0^2\,L^2(\mu^2 t)\\
L(z) &\equiv \ln(2z)+\gamma_E
\end{aligned}
$$

- $\langle E(t)\rangle$ finite after QCD renormalization of α_s only!
- NLO [Lüscher 2010] :

$$
e_{1,0}=\left(\frac{52}{9}+\frac{22}{3}\ln 2-3\ln 3\right)C_A-\frac{8}{9}n_f T_R
$$

- NNLO [Harlander, Neumann 2016; Artz, Harlander, **FL**, Neumann, Prausa 2019] :

$$
e_{2,0}=27.9786\,C_A^2-(31.5652\ldots)\,n_f T_R C_A+\left(16\zeta(3)-\frac{43}{3}\right)n_f T_R C_F+\left(\frac{8\pi^2}{27}-\frac{80}{81}\right)n_f^2 T_R^2
$$

$\langle E(t) \rangle$ through NNLO (II)

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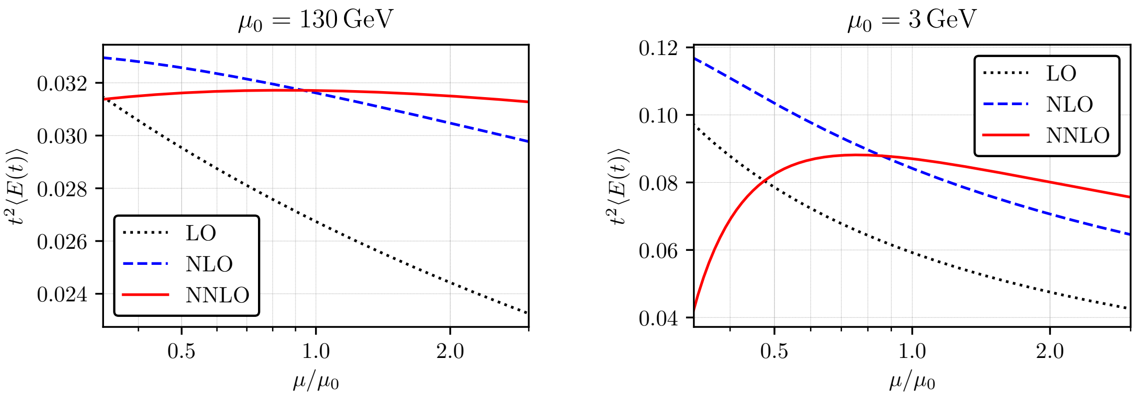


- Uncertainty through scale variation reduces from 3.3 % to 0.29 % at 130 GeV and from 19 % to 3.4 % at 3 GeV

Operator basis

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^{x} V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^{x} V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^{x} V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Operator basis depends on process
- We focus on the current-current operators relevant for $|\Delta F| = 2$ processes
- Operator basis not unique even for the same process, but different bases related by basis transformations
- CMM basis [Chetyrkin, Misiak, Münz 1997]:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\text{L}}\gamma_\mu T^a \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu T^a \psi_{4,\text{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\text{L}}\gamma_\mu \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu \psi_{4,\text{L}}\right)$$

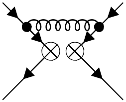
with

$$\psi_{\text{R/L}} = P_\pm \psi = \tfrac{1}{2}(1 \pm \gamma_5)\psi$$

Evanescent operators

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$

- In dimensional regularization, loop corrections produce additional non-reducible γ structures:



$$\Rightarrow \qquad (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}) \otimes (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3})$$

- These contributions have to be attributed to *evanescent* operators like [Buras, Weisz 1990]

$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2 \qquad \text{with} \qquad \gamma_{\mu_1\cdots\mu_n} \equiv \gamma_{\mu_1}\cdots\gamma_{\mu_n}$$

- Algebraically they are of $O(\epsilon)$ and vanish for $D \to 4$
- Nonetheless required to renormalize the physical operators
- Renormalization has to take care of finite pieces from $\frac{1}{\epsilon}$ (poles) $\times\, \epsilon$ (operators)
- Every loop order introduces more evanescent operators

Complete operator basis

- Physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

- Evanescent operators through NNLO:

$$E_1^{(1)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_1,$$
$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2,$$
$$E_1^{(2)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{4,\mathrm{L}}\right) - 20E_1^{(1)} - 256\mathcal{O}_1,$$
$$E_2^{(2)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{4,\mathrm{L}}\right) - 20E_2^{(1)} - 256\mathcal{O}_2$$

Flowed operator basis

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- Flowed physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a\psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a\chi_{4,\mathrm{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu\chi_{4,\mathrm{L}}\right)$$

- Flowed evanescent operators:

$$\begin{aligned}
\tilde{E}_1^{(1)} &= -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}T^a\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}T^a\chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(1)} &= \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_2\,,\\
\tilde{E}_1^{(2)} &= -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a\chi_{4,\mathrm{L}}\right) - 20\tilde{E}_1^{(1)} - 256\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(2)} &= \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}\chi_{4,\mathrm{L}}\right) - 20\tilde{E}_2^{(1)} - 256\tilde{\mathcal{O}}_2
\end{aligned}$$

- Since flowed operators do not have to be renormalized, the flowed evanescent operators actually vanish and could be dropped
- Keeping them allows us to check our results

Renormalization (I)

- Small-flow-time expansion for operators of electroweak Hamiltonian:

$$\begin{pmatrix}\tilde{\mathcal{O}}(t)\\ \tilde{E}(t)\end{pmatrix} \asymp \zeta^{\mathrm{B}}(t)\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}$$

with $\quad \mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)^{\mathrm{T}}, \qquad E = (E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)})^{\mathrm{T}}$

- Since regular operators divergent, $\zeta^{\mathrm{B}}(t)$ divergent as well
- Regular operators renormalized through

$$\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}^{\mathrm{R}} = Z\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix} \equiv \begin{pmatrix}Z_{\mathrm{PP}} & Z_{\mathrm{PE}}\\ Z_{\mathrm{EP}} & Z_{\mathrm{EE}}\end{pmatrix}\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}$$

- Renormalized $\zeta(t)$:

$$\begin{pmatrix}\tilde{\mathcal{O}}(t)\\ \tilde{E}(t)\end{pmatrix} \asymp \zeta^{\mathrm{B}}(t)Z^{-1}\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}^{\mathrm{R}} \equiv \zeta(t)\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}^{\mathrm{R}} \equiv \begin{pmatrix}\zeta_{\mathrm{PP}}(t) & \zeta_{\mathrm{PE}}(t)\\ \zeta_{\mathrm{EP}}(t) & \zeta_{\mathrm{EE}}(t)\end{pmatrix}\begin{pmatrix}\mathcal{O}\\ E\end{pmatrix}^{\mathrm{R}}$$

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Renormalization (II)

- Renormalization matrix Z includes finite renormalization:

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)} \quad \text{with} \quad Z_{ij}^{(k)} = \sum_{l=0}^{k} \frac{1}{\epsilon^l} Z_{ij}^{(k,l)}$$

- Related to anomalous dimension of operators and Wilson coefficients:

$$\mu \frac{\mathrm{d}\mathcal{O}_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ij} \mathcal{O}_j(\mu) \qquad \text{and} \qquad \mu \frac{\mathrm{d}C_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ji} C_j(\mu) \qquad \Rightarrow \qquad \gamma_{ij} = 2\alpha_s \beta_\epsilon Z_{ik} \frac{\partial Z_{kj}^{-1}}{\partial \alpha_s}$$

- Block form [Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich, Nierste 1995]:

$$\gamma^{(k)} = \begin{pmatrix} \gamma_{\mathrm{PP}}^{(k)} & \gamma_{\mathrm{PE}}^{(k)} \\ 0 & \gamma_{\mathrm{EE}}^{(k)} \end{pmatrix} \qquad \text{and} \qquad Z^{(k,0)} = \begin{pmatrix} 0 & 0 \\ Z_{\mathrm{EP}}^{(k,0)} & 0 \end{pmatrix}$$

- Ensures that matrix elements of renormalized evanescent operators vanish:

$$\langle E^{\mathrm{R}} \rangle = Z_{\mathrm{EP}} \langle \mathcal{O} \rangle + Z_{\mathrm{EE}} \langle E \rangle \stackrel{!}{=} O(\epsilon)$$

Method of projectors

- Define projectors [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

$$P_k[\mathcal{O}_i] \equiv D_k \langle 0|\mathcal{O}_i|k\rangle \stackrel{!}{=} \delta_{ik} + O(\alpha_s)$$

- Apply to small flow-time expansion:

$$P_k[\tilde{\mathcal{O}}_i(t)] = \sum_j \zeta_{ij}(t) P_k[\mathcal{O}_j]$$

- $\zeta_{ij}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j]$, because all loop integrals scaleless

“Master formula”

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t)]\Big|_{p=m=0}$$

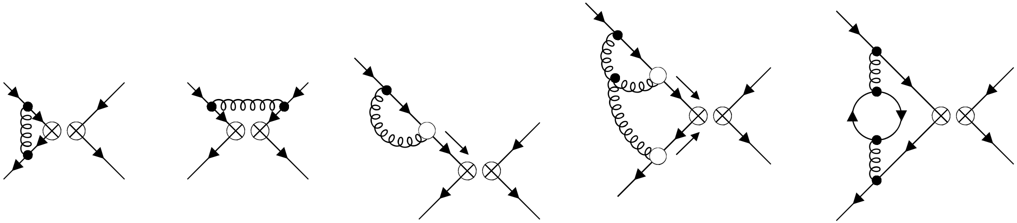
Projectors and example diagrams

- Schematic projector for $\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$:

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\mathrm{line}\,1}\mathrm{Tr}_{\mathrm{line}\,2}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

$\Rightarrow$ more detailed construction of projectors in Janosch Borgulat's talk

- Sample diagrams:



Results in CMM basis

- Physical matching matrix $(\zeta^{-1})_{\mathrm{PP}}$:

$$(\zeta^{-1})_{11}(t) = 1 + a_\mathrm{s}\left(4.212 + \frac{1}{2}L_{\mu t}\right) + a_\mathrm{s}^2\left[22.72 - 0.7218\,n_\mathrm{f} + L_{\mu t}\,(16.45 - 0.7576\,n_\mathrm{f}) + L_{\mu t}^2\left(\frac{17}{16} - \frac{1}{24}\,n_\mathrm{f}\right)\right],$$

$$(\zeta^{-1})_{12}(t) = a_\mathrm{s}\left(-\frac{5}{6} - \frac{1}{3}L_{\mu t}\right) + a_\mathrm{s}^2\left[-4.531 + 0.1576\,n_\mathrm{f} + L_{\mu t}\left(-3.133 + \frac{5}{54}\,n_\mathrm{f}\right) + L_{\mu t}^2\left(-\frac{13}{24} + \frac{1}{36}n_\mathrm{f}\right)\right],$$

$$(\zeta^{-1})_{21}(t) = a_\mathrm{s}\left(-\frac{15}{4} - \frac{3}{2}\,L_{\mu t}\right) + a_\mathrm{s}^2\left[-23.20 + 0.7091\,n_\mathrm{f} + L_{\mu t}\left(-15.22 + \frac{5}{12}\,n_\mathrm{f}\right) + L_{\mu t}^2\left(-\frac{39}{16} + \frac{1}{8}\,n_\mathrm{f}\right)\right],$$

$$(\zeta^{-1})_{22}(t) = 1 + a_\mathrm{s}\,3.712 + a_\mathrm{s}^2\left[19.47 - 0.4334\,n_\mathrm{f} + L_{\mu t}\,(11.75 - 0.6187\,n_\mathrm{f}) + \frac{1}{4}\,L_{\mu t}^2\right]$$

- $a_\mathrm{s} = \alpha_\mathrm{s}(\mu)/\pi$ renormalized in $\overline{\mathrm{MS}}$ scheme and $L_{\mu t} = \ln 2\mu^2 t + \gamma_\mathrm{E}$
- Set $N_\mathrm{c} = 3$, $T_\mathrm{R} = \frac{1}{2}$, and transcendental coefficients replaced by floating-point numbers

Checks

$$\zeta^{-1} = Z(\zeta^{\mathrm{B}})^{-1} = \begin{pmatrix} (\zeta^{-1})_{\mathrm{PP}} & (\zeta^{-1})_{\mathrm{PE}} \\ (\zeta^{-1})_{\mathrm{EP}} & (\zeta^{-1})_{\mathrm{EE}} \end{pmatrix}$$

- Finite after α_{s} + field renormalization and with Z from [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]
- $(\zeta^{-1})_{\mathrm{EP}} = O(\epsilon)$
- Independent of QCD gauge parameter

Basis transformations

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- Different operator bases related by

$$\vec{\mathcal{O}}' = R(\vec{\mathcal{O}} + W\vec{E}) \qquad \text{and} \qquad \vec{E}' = M(\epsilon U\vec{\mathcal{O}} + [1 + \epsilon V]\vec{E})$$

- Not sufficient to simply rotate the physical submatrix with R: $\zeta'_{\text{PP}} \neq R\zeta_{\text{PP}}R^{-1}$
- 1. possibility:
  - Transform whole ζ^{B}
  - Perform renormalization in the same way as before with a different Z
- 2. possibility:
  - Rotate renormalized ζ_{PP}
  - But: basis transformation also changes the scheme of Z!

$\Rightarrow$ Restore the scheme by an additional finite renormalization [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]:

$$\zeta'_{\text{PP}} = R\zeta_{\text{PP}}R^{-1}(1 + Z_{\text{fin}}^{-1})$$

Transformation to non-mixing basis

- Physical operators:

$$\mathcal{O}_\pm = \frac{1}{2}\left[\left(\bar{\psi}_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\alpha\right)\left(\bar{\psi}_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\beta\right) \pm \left(\bar{\psi}_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\beta\right)\left(\bar{\psi}_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\alpha\right)\right]$$

- Evanescent operators and transformation matrices through NNLO defined in [Buras, Gorbahn, Haisch, Nierste 2006]
- Anomalous dimension diagonal, i.e. operators do not mix under RGE running
- We did the transformation in both ways and find agreement as well as diagonal form:

$$\zeta_{++}^{-1} = 1 + a_s\left(2.796 - \frac{1}{2}L_{\mu t}\right) + a_s^2\left[14.15 - 0.1739\, n_f + L_{\mu t}\,(6.509 - 0.4798\, n_f) + L_{\mu t}^2\left(-\frac{9}{16} + \frac{1}{24}\, n_f\right)\right],$$

$$\zeta_{--}^{-1} = 1 + a_s\,(5.546 + L_{\mu t}) + a_s^2\left[32.01 - 0.9524\, n_f + L_{\mu t}(21.23 - 0.8965\, n_f) + L_{\mu t}^2\left(\frac{15}{8} - \frac{1}{12}\, n_f\right)\right]$$

Summary

- Discussed automatized setup for perturbative calculations in gradient-flow formalism $\Rightarrow$ further discussions and applications in talks of Janosch Borgulat and Robert Harlander
- Constructed gradient-flow version of electroweak Hamiltonian:

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^x V_{\mathrm{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^x V_{\mathrm{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^x V_{\mathrm{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Valid both on the lattice and perturbatively

$\Rightarrow$ $\tilde{C}_i$ and $\langle \tilde{\mathcal{O}}_j \rangle$ can be computed in different regularization schemes, e.g. perturbatively and on the lattice

- Perturbative ingredients C_i and ζ_{ij}^{-1} required in exactly same scheme (operator basis, evanescent operators, scheme for γ_5), but no major problem

Status and outlook

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$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^{x} V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j :$$

- Kaon mixing ($|\Delta S| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein] with two of three contributions known through NNLO [Brod, Gorbahn 2010 + 2012]
  - ζ_{ij}^{-1}: NNLO [Harlander, **FL** 2022] (NLO in different basis and scheme in [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
  - $\langle\tilde{\mathcal{O}}_j\rangle$: ?
- Non-leptonic $|\Delta F| = 1$ decays:
  - C_i: NNLO [Bobeth, Misiak, Urban 2000; Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1}: NNLO, but without penguin operators [Harlander, **FL** 2022], extension to QCD penguin operators planned
  - $\langle\tilde{\mathcal{O}}_j\rangle$: ?
- Neutral B-meson mixing ($|\Delta B| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein; Kirk, Lenz, Rauh 2017]
  - ζ_{ij}^{-1}: NNLO for mass difference [Harlander, **FL** 2022], calculation of remaining matching matrix planned
  - $\langle\tilde{\mathcal{O}}_j\rangle$ $\Rightarrow$ Matthew Black's talk

Treatment of γ_5 (I)

- In dimensional regularization,

$$\{\gamma_\mu, \gamma_5\} = 0$$

  is incompatible with the trace requirement

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) \neq 0 \xrightarrow[D \to 4]{} 4i\epsilon_{\mu\nu\rho\sigma}$$

- Different prescriptions for γ_5 (NDR, 't Hooft-Veltmann, DREG) lead to different results for scheme-dependent quantities like Wilson coefficients!

Treatment of γ_5 (II)

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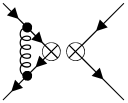
$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\text{line 1}}\mathrm{Tr}_{\text{line 2}}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$



- The quarks in our operators cannot annihilate due to different flavors

$\Rightarrow$ No γ_5 in traces produced by loop corrections

- Define external quarks in projectors to be left-handed, anticommute γ_5 from operator, and use $P_L^2 = P_L = \frac{1}{2}(1-\gamma_5)$

$\Rightarrow$ No traces with γ_5, simply use naively anticommuting γ_5

- Note: CMM basis avoids γ_5 in traces also for penguin operators ($|\Delta F| = 1$) [Chetyrkin, Misiak, Münz 1997]