The static force from the lattice with gradient flow

The Gradient Flow in QCD and other Strongly Coupled Field Theories

Julian Frederic Mayer-Steudte

In collaboration with
Nora Brambilla (TUM)
Viljami Leino (HIM)
Antonio Vairo (TUM)

Trento, 22nd of March 2023

Outline

1 Motivation

2 Setup

3 Lattice results

4 Conclusion

Motivation: Static Energy E(r)

- Interested in the QCD static energy of a quark-antiquark pair $E(r)$
- Given by the Wilson loop

$$E(r) = -\lim_{T\to\infty} \frac{\ln\langle \mathrm{Tr}(W_{r\times T})\rangle}{T}, \qquad W_{r\times T} = P\left\{\exp\left(i\oint_{r\times T} dz_\mu g A_\mu\right)\right\}$$

- Can be described by perturbation theory and measured on the lattice
- For $r\Lambda_{\mathrm{QCD}} \ll 1$ both descriptions should agree

➡ can be used for precise α_S-running extraction by comparing PT and lattice

Motivation: Issues with E(r)

- Perturbative form of $E(r)$:

$$E(r) = \Lambda_S - \frac{C_F \alpha_S}{r}\left(1 + \#\alpha_S + \#\alpha_S^2 + \#\alpha_S^3 + \#\alpha_S^3 \ln \alpha_S + ...\right)$$

- $E(r)$ is known up to N^{3}LL
- The perturbative expansion affected by a renormalon ambiguity of order Λ in PT side
- On lattice: Linear UV divergence
- All interesting physics is in the slope

$\Rightarrow$ take a derivative of $E(r)$ for the force $F(r) = \partial_r E(r)$

Outline

1 Motivation

2 Setup

3 Lattice results

4 Conclusion

Setup: Alternative definition of $F(r)$

- Direct measurement of $F(r)$:
  (A. Vairo Mod. Phys. Lett. A 31 (2016) & EPJ Web Conf. 126 (2016), Brambilla et.al.PRD63 (2001))

$$F(r) = -\lim_{T\to\infty} \frac{i}{\langle \mathrm{Tr}(W_{r\times T})\rangle} \left\langle \mathrm{Tr}\left(P\left\{\exp\left(i\oint_{r\times T} dz_\mu g A_\mu\right)\hat{\mathbf{r}}\cdot g\mathbf{E}(\mathbf{r},t^*)\right\}\right)\right\rangle$$
$$= \frac{\langle \mathrm{Tr}\{PW_{r\times T} g E_j(r,t^*)\}\rangle}{\langle \mathrm{Tr}\{PW_{r\times T}\}\rangle}$$

- Chromoelectric field E inserted into Wilson loop
- The insertion location t^* is arbitrary $\rightarrow$ reduce boundary terms and choose $t^* = T/2$
- Can be used to extract α_S without the usual renormalon issues and for scale setting

$\Rightarrow$ On the lattice: modifying Wilson loop with a discretized E-field insertion

Setup: Discretiziation of the E-field insertion

- Clover discretization of E:

$$E_i = \frac{1}{2iga^2}\left(\Pi_{i0} - \Pi_{i0}^\dagger\right) \qquad \Pi_{\mu\nu} = \frac{1}{4}\left(P_{\mu,\nu} + P_{\nu,-\mu} + P_{-\mu,-\nu} + P_{-\nu,\mu}\right)$$

- E has finite size on the lattice
- The self energy contribution of E converges slowly to continuum
  (See e.g. Lepage et.al.PRD48 (1993), G. Bali Phys. Rept. 343 (2001), and many others...)
  $\rightarrow$ need renormalization Z_E

We use **Gradient flow** for targeting the renormalization and the signal to noise ratio problems, new scale: **flowtime** τ_F, **flowradius** $\sqrt{8\tau_F}$, **flowtime ratio** τ_F/r^2

Setup: Gradient flow on the lattice

- Gradient flow is originated in lattice, acts as smearing with flowradius $\sqrt{8\tau_F}$
- Renormalizes lattice field strength components by reducing the self-energy contributions
- Gradient flow equation:
  (Martin Lüscher JHEP 08 (2010))

$$\dot{V}_{\tau_F}(x,\mu) = -g_0^2\{\partial_{x,\mu} S_{W/S}(V_{\tau_F})\}V_{\tau_F}(x,\mu) \qquad V_{\tau_F}(x,\mu)|_{\tau_F=0} = U_\mu(x)$$

- Measuring flowed operators on the lattice:

$$\langle O(\tau_F)\rangle = \frac{1}{Z}\int \mathcal{D}[U]e^{-S_E}O(V_{\tau_F})$$

$\Rightarrow$ Obtain $\lim_{\tau_F\to 0}\langle O(\tau_F)\rangle$ for the physical quantity

Setup: Specific extra motivations

- The statice force is complementary to the static energy extraction from Wilson loops
- We measured the force directly with the E-field insertion in Wilson loops and Polyakov loops first, with multilevel so far
  (Brambilla et. al. Phys.Rev.D 105 (2022))
  which introduces an additional factor Z_E
- One to one comparison to $\partial_r E(r)$ is possible
- Here we address first time the force measurement with gradient flow
- This study is a preparation for similar objects with field insertions needed it NREFTs

$\Rightarrow$ Use the implications of the force measurement with gradient flow

Setup: Continuum results

- One-loop calculation of the flowed force is known:
  (Hee Sok Chung et. al. JHEP01(2022)184 (2022)), Xiangpeng Wang's talk
- Relevant scale: $\mu_b = \frac{1}{\sqrt{r^2+8b\tau_F}}$, $\mu_0 = 1/r$, $\mu_1 = \frac{1}{\sqrt{r^2+8\tau_F}}$
- We focus on the $n_f = 0$ result
- Small τ_F exansion:

$$r^2 F(r;\tau_F) \approx r^2 F(r;\tau_F = 0) + \frac{\alpha_S^2 C_F}{4\pi} \underbrace{\left[-12\beta_0 - 6C_A c_L\right]}_{8n_f} \frac{\tau_F}{r^2} \qquad c_L = -\frac{22}{3}$$

➡ At small flowtime the force is constant in pure gauge ($n_f = 0$)

Outline

1 Motivation

2 Setup

3 Lattice results

4 Conclusion

Lattice results: setup and parameters

- Parameters:

N_S	N_T	β	a [fm]	N_{conf}	Label
20	40	6.284	0.060	6000	L20
26	52	6.481	0.046	6000	L26
30	60	6.594	0.040	6000	L30
40	80	6.816	0.030	2700	L40

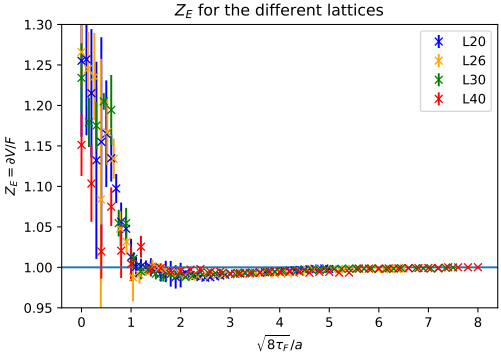
- Pure gauge configuration produced with overrelaxation and heatbath
- Scale setting with ($r_0 = 0.5$ fm)
  $\ln(a/r_0) = -1.6804 - 1.7331(\beta - 6) + 0.7849(\beta - 6)^2 - 0.4428(\beta - 6)^3$
  (1S. Necco & R. Sommer. Nucl. Phys. B622 (2002))
- Gradient flow with **fixed** and **adaptive** solver, with **Symanzik** action
  (Bazavov and Chuna 2101.05320 (2021)

Lattice results: Discretization effects

- Nonperturbative determination of Z_E:

$$Z_E(r) = \frac{\partial_r E(r)}{F(r)}$$

- Z_E has low r-dependence
  (Brambilla et. al. Phys.Rev.D 105 (2022))
- Examine the flowed Z_E
- $Z_E \to 1$ for flowradius $\sqrt{8\tau_F} > a$



Gradient flow reduces discretization effects of field insertions for $\sqrt{8\tau_F} > a$

Lattice results: r_0 scale

- Sommer scale $r_0(\tau_F)$ defined as:

$$r^2 F(r, \tau_F)|_{r=r_0(\tau_F)} = 1.65$$

- $r_0(\tau_F)$ has strong τ_F-dependence
- Approaches Necco & Sommer scale setting for $\tau_F/r^2 \approx 0.02$
- Plateau extraction within a suitable flowtime range could be use to extract r_0

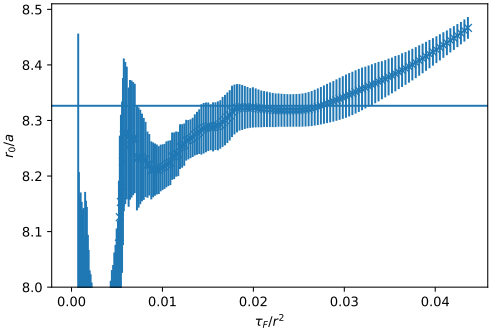


Figure 1 Lattice L20

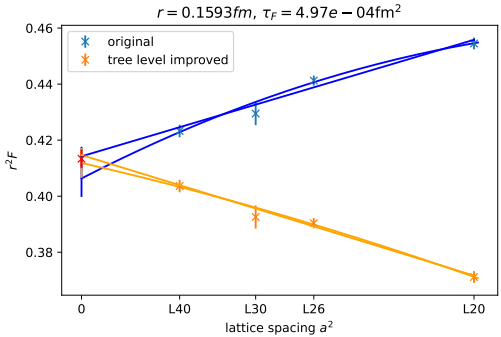
➡ Direct force measurement with gradient flow can be used to set the scale

Lattice results: Continuum limit

- Tree level improvement:

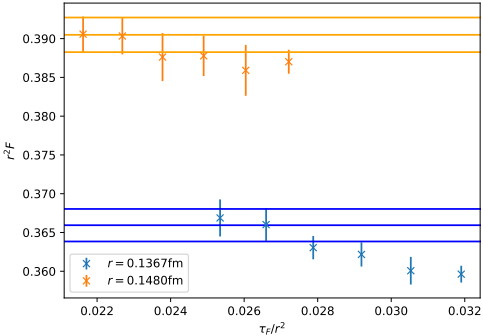
$$F_{\text{Latt}}^{\text{Impr}}(r,\tau_F) = \frac{F_{\text{Cont}}^{\text{Tree}}}{F_{\text{Latt}}^{\text{Tree}}} \cdot F_{\text{Latt}}^{\text{Meas}}$$

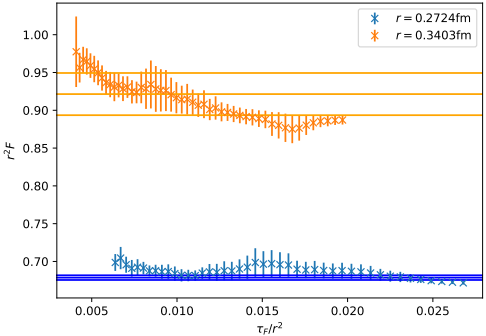
- Continuum limit at fixed r and fixed τ_F
- Akaike average of linear and quadratic in a^2 limit
- restrict to $\chi^2/dof < 3.6$



Direct force measurement with gradient flow leads to reliable continuum limits

Lattice results: Continuum results at large r



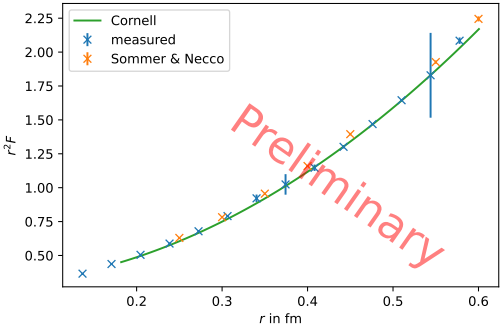


- Constant zero flowtime limits for $r > 0.137$ fm

Lattice results: Continuum results at large r

- Close to force from Necco & Sommer
- Cornell fit $r^2 F(r) = A + \sigma r^2$
- $\sigma = 5.25(3)$ fm^{-2}, literature: 5.5 fm^{-2}
- $A = 0.277(3)$
  Koma & Koma: $A = 0.2808(5)$

➡ Direct force measurement with gradient flow gives reliable large r results



Lattice results: Continuum results at small r

- Perturbative results are valid at small r
- Perturbative force known up to N^{3}LL at zero flow time, at NLO at finite flow time
- We combine the perturbative knowledge at zero flow time with 1-loop flow time behavior:

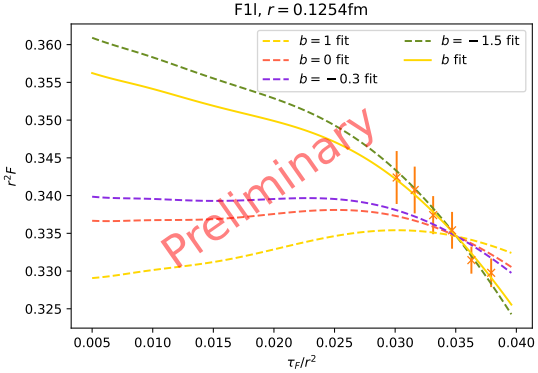
$$r^2 F^{\text{order}}(r, \tau_F) \equiv r^2 F^{\text{order}}(r, \tau_F = 0) + f^{1-\text{loop}}(r, \tau_F)$$
$$f^{1-\text{loop}}(r, \tau_F) \equiv r^2 F^{1-\text{loop}}(r, \tau_F) - r^2 F^{1-\text{loop}}(r, \tau_F = 0)$$

- The perturbative result depends on $\Lambda_0 \equiv \Lambda_{\overline{\text{MS}}}^{n_f=0}$ which is a flow time independent quantity and defines the physics at zero flow time
- Extracting Λ_0 gives the zero flow time limit
- Compare two ways:
  - at fixed r and fit along flow time
  - at fixed τ_F and fit along r

Lattice results: Continuum results at small r, fixed r

- Select valid points ($Z_E \approx 1$, $\chi^2/dof < 3.6$))
- Various scale options will catch the flow time in different shapes
- Fit parameter for pure 1-loop:

$$\Lambda_0 = 0.300...0.360\text{GeV}$$



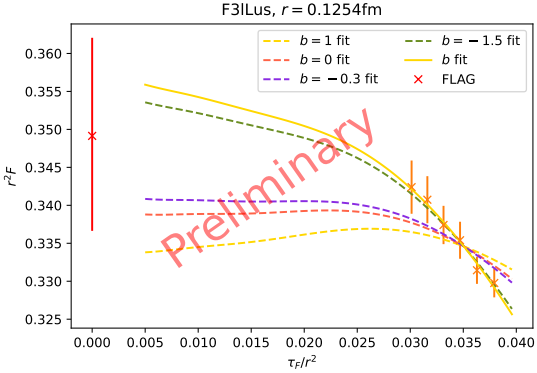
Lattice results: Continuum results at small r, fixed r

- Select valid points ($Z_E \approx 1$, $\chi^2/dof < 3.6$))
- Different choices for the scale catch the flow time shape different well
- Fit parameter for pure 1-loop:

$$\Lambda_0 = 0.300...0.360\text{GeV}$$

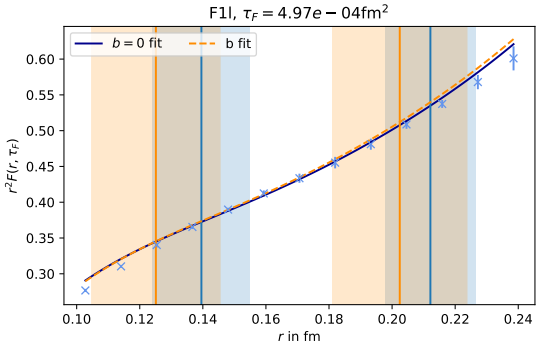
- Fit parameter for 3-loop + u.s.:

$$\Lambda_0 = 0.24...0.27\text{GeV}$$



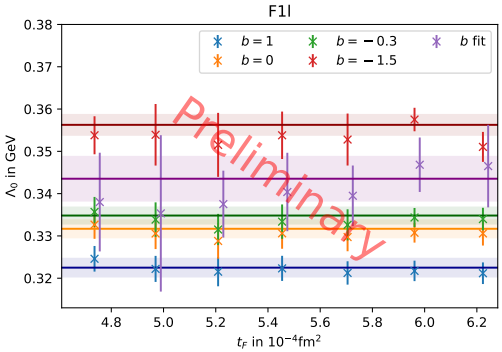
Lattice results: Continuum results at small r, fixed τ_F

- Select valid points ($Z_E \approx 1$, $\chi^2/dof < 3.6$))
- Perform Aikaike averaging for different r fit windows
- Vertical lines corresponds to the effective lower and upper fit bounds
- Multiple values for Λ_0 at different flowtimes



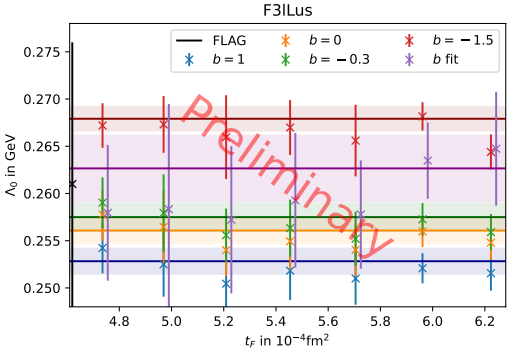
Lattice results: Continuum results at small r, fixed τ_F

- $\Lambda_0(\tau_F)$ is constant within the errors
- The value depends on the scale choice



Lattice results: Continuum results at small r, fixed τ_F

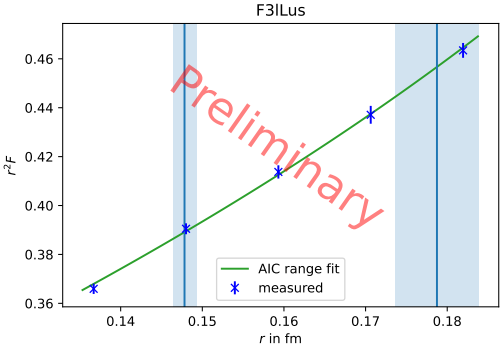
- $\Lambda_0(\tau_F)$ is constant within the errors
- The value depends on the scale choice
- For combined 3-loop + u.s., all values are within the FLAG error



Fixed τ_F fit for Λ_0 gives good result

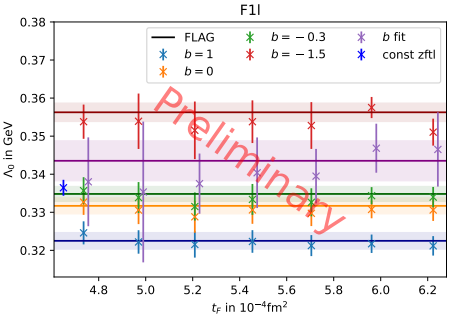
Lattice results: Constant zero flow time limit at small r

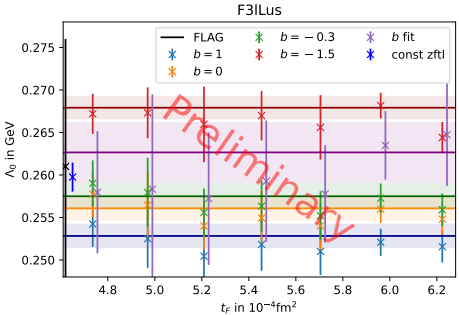
- Perform constant zero flow time limit at smallest r
- Fit perturbative result:
  - 1-loop: $0.336(2)$ GeV
  - 2-loop: $0.273(2)$ GeV
  - 2-loop+u.s.: $0.286(2)$ GeV
  - 3-loop: $0.253(2)$ GeV
  - 3-loop+u.s.: $0.260(2)$ GeV
  - FLAG: $0.262(15)$ GeV



Constant zero flow time limit at small r can be used to compare with perturbative results

Lattice results: Fixed τ_F and constant zero flow time limit at small r comparison





Outline

1 Motivation

2 Setup

3 Lattice results

4 Conclusion

Conclusion

- Summary ond observations:
  - Gradient flow reduces effectively discretization effects
  - Gradient flow improves qualitatively the signal to noise ratio
  - Direct force measurement with gradient flow can be used for scale setting
  - Good preparation for future applications in NREFTs
- For the future:
  - Get better scale dependence for Λ extraction
  - Go to finer lattices
  - Other operators with field insertions
  - Extend to dynamical fermions

Thank you for your attention!

fixed r

b	1	0	-0.3	-1.5	b_{fitted}
	F1l				-0.65(10)
Λ_0^{fit} in GeV	0.284(2)	0.302(2)	0.313(1)	0.384(6)	0.326(5)
χ^2/dof	1.76	0.60	0.28	5.6	0.13
	F2l				-1.00(10)
Λ_0^{fit} in GeV	0.229(2)	0.240(2)	0.245(1)	0.288(2)	0.262(4)
χ^2/dof	1.8	0.92	0.62	1.4	0.14
	F3lLus				-1.03(10)
Λ_0^{fit} in GeV	0.226(2)	0.237(2)	0.241(1)	0.276(2)	0.257(3)
χ^2/dof	1.8	0.94	0.64	1.2	0.1

Table 1 The fit results at fixed $r = 0.102$ fm for three different order choices.

fixed r

b	1	0	-0.3	-1.5	b_{fitted}
	F1I				-1.31(10)
Λ_0^{fit} in GeV	0.304(3)	0.317(2)	0.323(2)	0.358(1)	0.351(4)
χ^2/dof	1.93	1.04	0.75	0.12	0.07
	F2I				-1.61(12)
Λ_0^{fit} in GeV	0.247(2)	0.255(2)	0.258(1)	0.277(1)	0.279(3)
χ^2/dof	1.46	0.91	0.73	0.07	0.07
	F3ILus				-1.65(12)
Λ_0^{fit} in GeV	0.242(2)	0.249(2)	0.251(2)	0.268(1)	0.271(3)
χ^2/dof	1.42	0.90	0.73	0.07	0.07

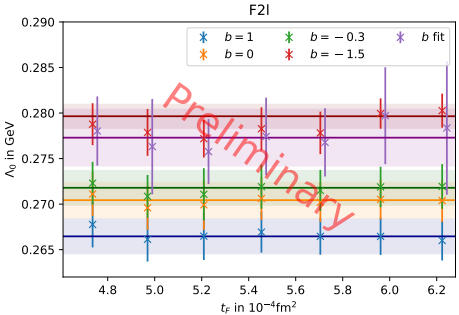
Table 2 The fit results at fixed $r = 0.1254$ fm for three different order choices.

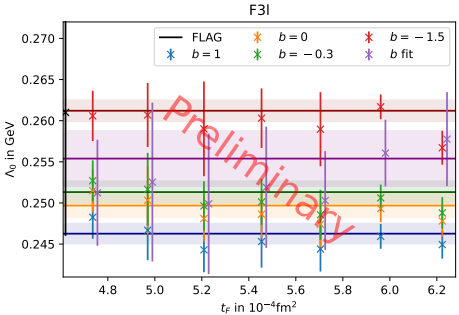
fixed r

b	1	0	-0.3	-1.5	b_{fitted}
	F1l				-1.18(16)
Λ_0^{fit} in GeV	0.321(1)	0.331(1)	0.334(1)	0.353(1)	0.347(3)
χ^2/dof	1.56	0.75	0.52	0.23	0.16
	F2l				-1.61(23)
Λ_0^{fit} in GeV	0.265(1)	0.270(1)	0.272(1)	0.281(1)	0.282(2)
χ^2/dof	0.89	0.54	0.44	0.14	0.15
	F3lLus				-1.74(25)
Λ_0^{fit} in GeV	0.252(1)	0.255(1)	0.257(1)	0.263(1)	0.264(2)
χ^2/dof	0.80	0.51	0.42	0.15	0.15

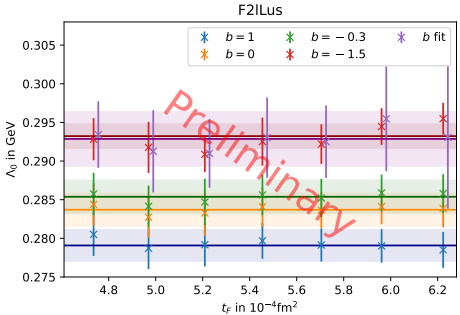
Table 3 The fit results at fixed $r = 0.1819$ fm for three different order choices.

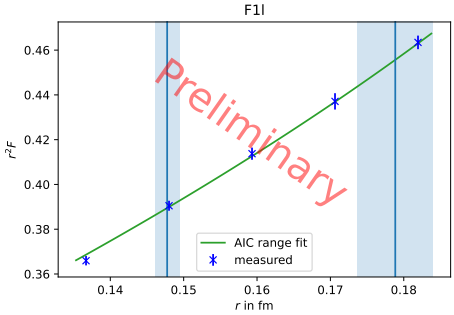
Fixed τ_F


F2l
$b = 1$
$b = 0$
$b = -0.3$
$b = -1.5$
b fit
Preliminary
Λ_0 in GeV
0.290
0.285
0.280
0.275
0.270
0.265
4.8
5.0
5.2
5.4
5.6
5.8
6.0
6.2
t_F in 10^{-4}fm^2


F3l
FLAG
$b = 1$
$b = 0$
$b = -0.3$
$b = -1.5$
b fit
Preliminary
Λ_0 in GeV
0.270
0.265
0.260
0.255
0.250
0.245
4.8
5.0
5.2
5.4
5.6
5.8
6.0
6.2
t_F in 10^{-4}fm^2

Fixed τ_F, small constant r zero flow time limit





Effective distance

- Solve for r_{Eff}:

$$
\begin{aligned}
& r^2 F(r, \tau_F, \mu = 1/r) \\
= & r_{\mathrm{Eff}}^2 F(r_{\mathrm{Eff}}, 0, \mu = 1/r_{\mathrm{Eff}})
\end{aligned}
$$