Computation of Relativistic Corrections to the Static Potential

from Generalized Wilson Loops at Finite Flow Time

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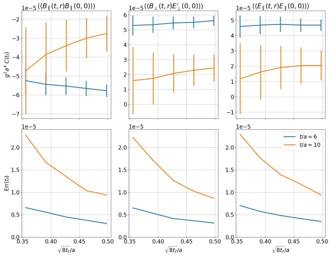
Trento, 24.03.2023

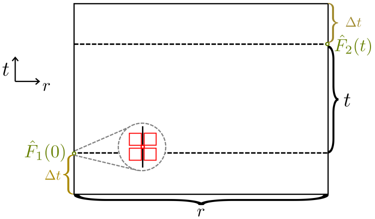
Introduction

- Non-relativistic quarkonia can be described via static potentials + rel. corrections [2212.11107].
- Well understood for $t_f = 0$ to high order in $\bar{ms}$, but only *few results* from lattice QCD.
- Correlators strongly affected by *UV fluctuations* and *need renormalization* [hep-lat/9703019].
- Gradient flow expected to effectively solve these problems (cf. talk by Julian Mayer-Steudte).
- The types of Wilson loops used here are great to study *properties of colour-fields* in gradient flow, e.g. t_f-dependence and matching (cf. talks by Viljami Leino, Nora Brambilla).

Outline

- Overview of the potentials.
- How to compute Correlators.
- Preliminary results for correlators.
- Where to improve and what can be learned.





Potentials

$$V(r) = V^{(0)}(r) + \frac{1}{m} V^{(1)}(r) + \frac{1}{m^2}\left(V_{\mathrm{SD}}(r) + V_{\mathrm{SI}}(r)\right) + \mathcal{O}(1/m^3)$$

Pineda, Vairo, 2001

- $V^{(0)}$: Static $\bar{q}q$-potential. Rest: Relativistic correction terms.
- $V^{(0)}$ and corrections well understood in *perturbation theory*, but results from lattice QCD *incomplete*. Bali, 1997, Koma, Koma, 2007
- **Precise** long-range expressions relevant for phenomenology, e.g. determination of quarkonia masses and related areas.
- **Properties** of colour-field insertions in gradient flow (flow dependence, matching, ...) can be studied here independently $\Rightarrow$ Relevant for e.g. computation of HQ diffusion coefficients.

Potentials

Potentials are computed using correlators $\langle\langle\ldots\rangle\rangle = \langle\ldots\rangle_W / \langle 1\rangle_W \Rightarrow V_0$ (in particular $V_{\mathrm{self}}(\mu)$) does not appear in resulting spectrum.

$$
\begin{aligned}
V_{\mathbf{p}^2} =& \frac{1}{2}\left\{\mathbf{p}^2, \left(\mathcal{I}(E_{\|}(t,0)E_{\|}(0,0)) + \mathcal{I}(E_{\|}(t,r)E_{\|}(0,0))\right)\right\},\\
V_{\mathbf{LS}} =& \epsilon_{ij\|}\frac{c_F(\mu)}{r}\left(\mathcal{I}(B_i(t,0)E_j(0,0)) + \mathcal{I}(B_i(t,r)E_j(0,0))\right)\hat{\mathbf{L}}\hat{\mathbf{S}},\\
V_{\mathbf{S}^2} =& \frac{2c_F^2(\mu)}{3}\sum_i \left(\mathcal{I}(B_i(t,r)B_i(0,0))\right)(\hat{\mathbf{S}}_1\hat{\mathbf{S}}_2),\\
& \ldots
\end{aligned}
$$

with $\mathcal{I}(F_i^1(t_f,r_2)F_j^2(t_i,r_1)) = \lim_{T\to\infty}\int_0^T dt\ t^s\ \langle\langle g^2 F_i^1(t_f,r_2)F_j^2(t_i,r_1)\rangle\rangle_{(c)}$

$c_F(\mu)$: Matching coefficient for B-insertions.
Indices $\|$, $\perp$, $\perp'$: Paralell or orthogonal directions, relative to $\bar{q}q$-separation axis.

Differences for gradient flow

- Insertions without gradient flow:
  - Small signal-to-noise ratio due to UV-fluctuations.
  - Sizeable discretization errors with poor convergence .
  - B-insertions log-divergent in μ, lattice artefacts difficult to remove.
- In gradient flow:
  - \+ UV-fluctuations effectively suppressed.
  - \+ Additional renormalization of insertions not neccessary $r_f/a > 1$.
  - \+ Divergencies regulated by t_f.
  - − Effect of r_f larger than loop separations unknown and may be complicated to handle.
- Find the best compromise: Use flow times *large enough* to make precise measurements, while keeping r_f *small compared to separations*.

Correlators

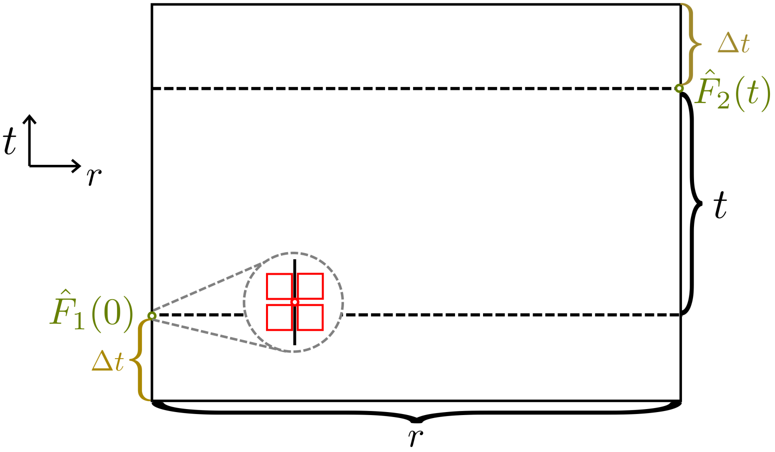
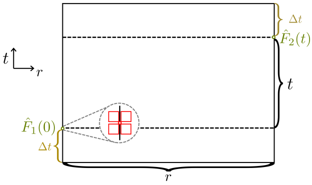


Figure: Schematics of a Wilson loop which ranges from $-\Delta t$ to $t + \Delta t$ in temporal, and 0 to r in spatial direction with insertions at 0 and t.

Correlators

- $\mathbf{e}_{\parallel}$: Axis parallel to $\bar{q}q$-separation. $\mathbf{e}_{\perp}$, $\mathbf{e}'_{\perp}$: Orthogonal axises.
- Insertions $F_{\mu\nu}$ using clover definition.
- Composition: Insertions on temporal lines separated by t from each other and Δt from upper and lower bounds.
- Choose $\sqrt{8t_f} < d/2$, where d is the *smallest relevant distance* between two operators in the loop (e.g. r, t, $\sqrt{r^2+t^2}$,...).



Correlators

- Integration of correlation functions: Fit ansatz and integrate analytically.
- Motivate ansatz from spectral decomposition: $\langle\langle F^2(t_2, r_2)F^1(t_1, r_1)\rangle\rangle = \sum_m D_m^{12} e^{-\Delta V_m t}(1 + \ldots)$ with $\Delta V_m = V_m - V_0$.
- $(1 + \ldots) \to 1$ for $\Delta t \to \infty$, or $|\langle 0|n, r\rangle|^2/|\langle 0|0, r\rangle|^2 \Rightarrow$ use additional *ground state enhancement*.
- Here: APE-smearing[1] with N_{APE} adjusted such that $\ln(W(t_W, r)/W(t_W + a, r))$ forms plateau for small t_W.
- ΔV_m's can be extracted from combined fits for multiple correlators at fixed t_f, r.
- Number of exponentials $m_{\max}$ and range for integration must be chosen carefully!

[1] Spatial HYP- or gradient-flow-smearing planned for future computations.

Noise reduction for colour fields

Preliminary

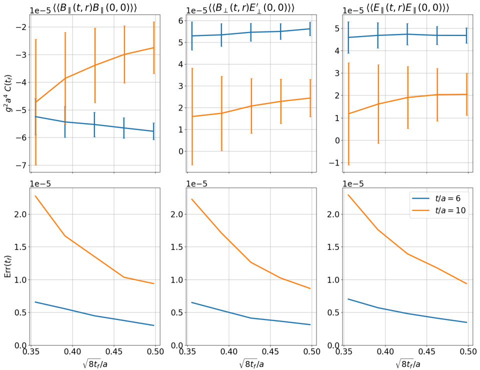


Figure: **Noise reduction** for different correlation functions for $a = 0.06$ fm at $r/a = 8$ as function of t_f. **Top** row: Data points with error bars. **Bottom** row: Statistical errors. The effect is most noticeable, if t_f is fixed in physical units when taking the continuum limits, as this goes as function of t_f/a^2.

Gradient flow vs. smearing

Treat correlators with gradient flow, or smearing? (talk by Rainer Sommer)

- Smearing possible, if correlator contains E-field insertions only. B-field insertions are log-divergent and correlators cannot be continuum extrapolated.
- t_f/a^2 =const. for smearing, amount of noise reduction remains similar for ensembles at different lattice spacings.
- For physical gradient flow, larger t_f/a^2 are accessed for decreasing lattice spacings, giving more noise reduction.
- Continuum extrapolated result at finite t_f has to be related to $t_f = 0$, either by extrapolation, or matching (e.g. with B-fields).

  $\rightarrow$ Matching coefficients (without GF) known in msbar, computation in gradient flow might be similar.

$\Rightarrow$ Gradient flow seems to be the most sensible choice here, but matching to $t_f = 0$ has to be worked out.

Lattice ensembles

- Simulation: $SU(3)$ heatbath with CL2QCD.
- Wilson flow with adaptive step size.
- Whole Wilson loop, including insertions, flowed.
- Statistical errors propagated via bootstrap samples.

Table: Lattice ensembles shown in the following (black), used, but not shown (dark grey), or planned (light grey).

β	T/a	L/a	a [fm]	# confs.
6.091	36	18	0.08	20000
6.284	48	24	0.06	10000
6.451	60	30	0.048	–
6.594	36	18	0.04	–

Static potential

Preliminary

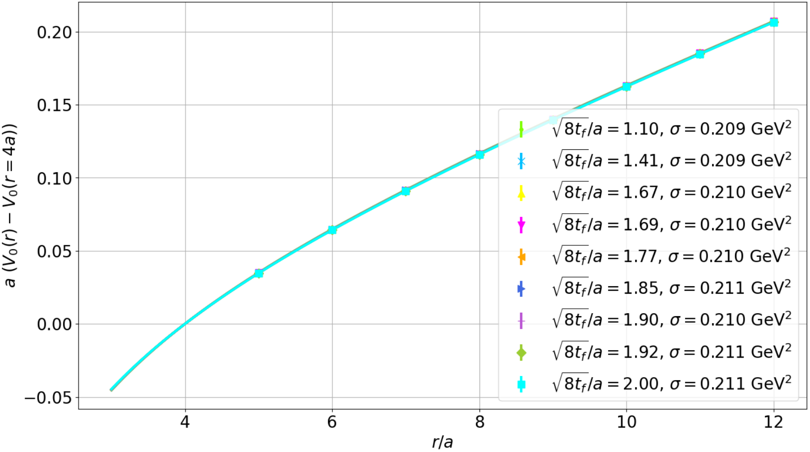


Figure: Static potential for $a = 0.06$ fm, normalized with $V_0(r/a = 4)$, at different flow times. The physical scale has been fixed with $r_0 = 0.5$ fm.

Noise reduction for colour fields

Preliminary

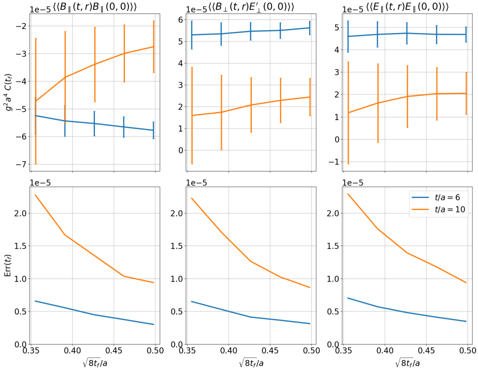


Figure: **Noise reduction** for different correlation functions for $a = 0.06$ fm at $r/a = 8$ as function of t_f. **Top** row: Data points with error bars. **Bottom** row: Statistical errors.

Fitting of correlation functions I

Preliminary

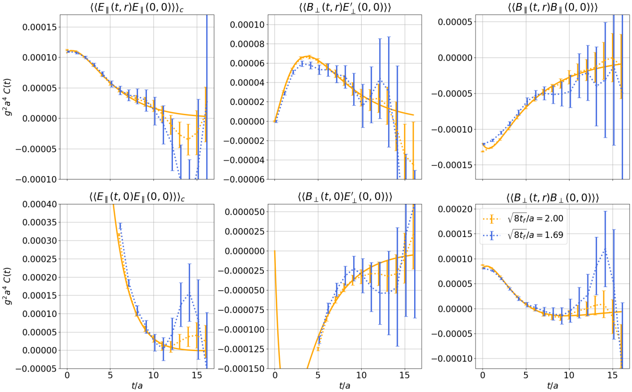


Figure: (Unweighted) correlators at $r/a = 8$ for $a = 0.06$ fm as function of euclidean time t. For $\sqrt{8t_f}/a = 2$ (orange), the fit result is drawn. The first data point was left out of the fitting procedure, for demonstration purposes.

Fitting of correlation functions II

Preliminary

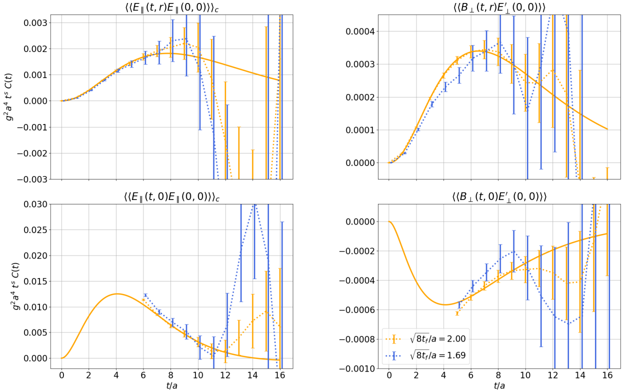


Figure: Weighted correlators at $r/a = 8$ for $a = 0.06$ fm as function of euclidean time t. For $\sqrt{8t_f}/a = 2$ (orange), the fit result is drawn. The first data point was left out of the fitting procedure, for demonstration purposes.

Spin-orbit potential

Preliminary

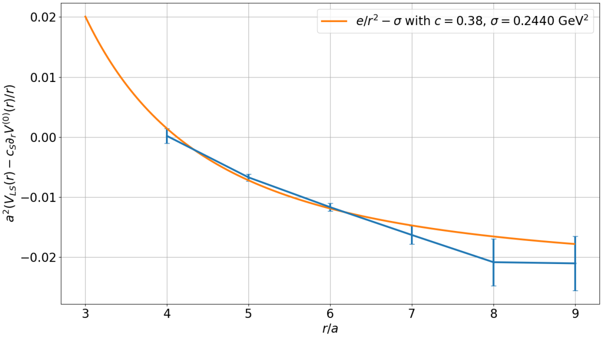


Figure: Spin-orbit potential for $a = 0.06$ fm at $\sqrt{8t_f}/a = 0.5$. A fit motivated from non-GF models [Bali, 2001] is shown in orange. In the model, the parameter c corresponds to the parameter from the $1/r$-term in the Cornell potential and σ is the string-tension. This is not meant to be physically meaningful.

Conclusions

Summary

- $\mathcal{O}(1/m^2)$-corrections to the static $\bar{q}q$-potential have been computed at $a = 0.06$ fm and several values of t_f for distances up to ~ 0.7 fm.
- *Effective reduction of noise* in colour-field correlators.
- Good ground to *test effect of gradient flow* on colour-fields.
- Data at small separations not accessible due to $r_f/a > 1$, *difficult to get robust fits*.

Outlook

- Do continuum extrapolation and relate it to $t_f = 0$, e.g. via matching (perturbative calculation required).
- Improve robustness of correlator fits, e.g. *combined fitting* for multiple correlators.
- Access *smaller* min. separations < 0.2 fm.