Gradient Flow and Exact Renormalization Group

ECT*workshop - The Gradient Flow in QCD and other Strongly Coupled Field Theories

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Outline

Let's explore some connections between gradient flow and RG.

1. GF-RG transformation by analogy with spin blocking
   - Ratio formulas
   - Numerical results
2. Relation to Wilson's "Exact RG" from 1973
   - Stochastic RG
   - Perturbative WFFP
3. Summary and Future possibilities...

Gradient flow vs. blocking

Consider the free gradient flow of a scalar field theory:

$$\partial_t \phi_t(x) = \partial^2 \phi_t(x) \quad \text{has solution} \quad \phi_t(x) = \frac{1}{(4\pi t)^{d/2}} \int \mathrm{d}^d y \; \mathrm{e}^{-\frac{(x-y)^2}{4t}} \varphi(y)$$

and compare with a traditional blocking transforrmation

$$\phi_b(x_b) = \frac{b^{\Delta_1}}{b^d} \sum_{\varepsilon} \varphi(x+\varepsilon), \quad x_b = x/b$$

and $a' = ba$. Similar under the identification $b_t \propto \sqrt{t}$.

But the GF solution does not have a rescaling factor of b^{Δ_1}. Thus we can try augmenting the GF solution and define

$$\Phi_t(x_t) := b_t^{\Delta_1} \phi_t(x)$$

where Δ_1 is not necessarily known beforehand, and x_t are sites on a fictitious blocked lattice.

Here we'll explore the consequences... but first some review.

RG fixed points: review

Iterated blocking transformations generate a sequence of actions in "theory space." A fixed point is characterized by $S_* \to S_*$ under the blocking.

Perturbations near the FP transform like

$$S_* + \sum_a u_a R_a \longrightarrow S_* + \sum_a b^{y_a} u_a R_a$$

The R_a are called *scaling operators*. The relative importance of these operators near a FP is determined by the "RG eigenvalues" y_a, e.g. relevant, marginal, or irrelevant.

Scaling ops are typically linear combinations of familiar operators:

$$R_a(\phi) = \sum_i c_{ai} \mathcal{O}_i(\phi), \quad \{\mathcal{O}_i\} = \{\phi^2, \phi^4, \phi\partial^2\phi, ...\}$$

Near a FP, correlations of the scaling ops are related as

$$\langle R_a(z/b) R_a(0) \rangle_{S_b} \sim b^{2\Delta_a} \langle R_a(z) R_a(0) \rangle_{S_1} \quad \text{(large } z\text{)}$$

where $\Delta_a = d - y_a$.

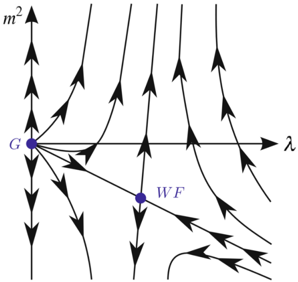


Figure: Adapted from Kopietz [1].

GF ratio formulas

The MCRG equivalence for blocking lets one calculate observables in the blocked theory:

$$\langle \mathcal{O}(\Phi) \rangle_{S_b} = \langle \mathcal{O}(\phi_b) \rangle_{S_0} \quad \Rightarrow \quad \langle \mathcal{O}(\Phi) \rangle_{S_t} = \langle \mathcal{O}(b_t^{\Delta_1} \phi_t) \rangle_{S_0}$$

The ratio formula then reads (near a FP)

$$\langle R_a(\Phi_{t'}; z) R_a(\Phi_{t'}; 0) \rangle_{S_0} \sim (b_{t'}/b_t)^{2\Delta_a} \langle R_a(\Phi_t; z) R_a(\Phi_t; 0) \rangle_{S_0}$$

where $\Phi_t = b_t^{\Delta_1} \phi_t$, $z_t = z/b_t$.

Scaling operators are not known a priori. However, since

$$\mathcal{O}_i(\phi) = \sum_a c_{ia} R_a(\phi),$$

the operators with smallest Δ_a dominate correlations of arbitrary operators:

$$\frac{\langle \mathcal{O}_i(\phi_{t'}; z) \mathcal{O}_j(\phi_{t'}; 0) \rangle_{S_0}}{\langle \mathcal{O}_i(\phi_t; z) \mathcal{O}_j(\phi_t; 0) \rangle_{S_0}} \sim (b_{t'}/b_t)^{2\Delta_a - (n_i + n_j)\Delta_1}$$

The LHS can be measured on the lattice!

From any pair of operators i, j, we expect to extract an estimate for

$$\delta_{ij} := 2\Delta_a - (n_i + n_j)\Delta_1$$

Testbed: ϕ^4 theory

We performed a lattice simulation with action

$$S(\varphi) = \sum_x \Big[-\beta \sum_\mu \varphi_x \varphi_{x+\mu} + \varphi_x^2 + \lambda(\varphi_x^2 - 1)^2 \Big]$$

tuned close to criticality (we want the RG flow to skim the critical surface closely)

$$3d: \quad \lambda = 1.1, \quad \beta_c \approx 0.3750966 \quad (\textit{Hasenbusch}, \ 1999\ [2])$$

$$2d: \quad \lambda = 1.0, \quad \beta_c \approx 0.6806048 \quad (\textit{Kaupuzs}, \ 2016\ [3])$$

Criticality was checked by analyzing the Binder cumulant.

We hope to measure

Δ	$d=2$	$d=3$
Δ_1	0.125	0.51790(20)
Δ_2	1	1.41169(76)
Δ_3	2.125	~ 2.5
Δ_4	2	3.845(11)

Figure: Leading scaling dims for 2d, 3d Ising universality class. (3d values from Hasenbusch '99 and Rychkov 2015 [4] (for Δ_3)).

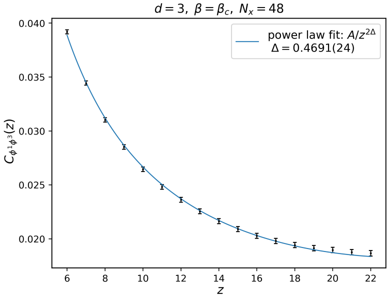
Correlators

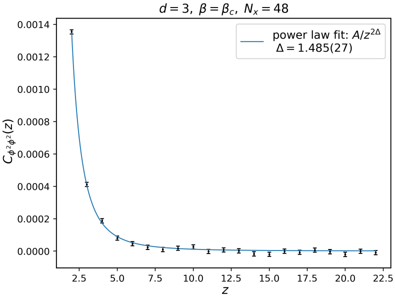
Operators in ϕ^4 theory are either even or odd under $\mathbb{Z}_2$ reflection $\phi \to -\phi$.

We measured mixed point-to-point correlators in each set

$$\mathit{odd} \;:\; \{\phi, \phi^3\}, \quad \mathit{even} \;:\; \{\phi^2, \phi^4\}$$

So close to criticality, the correlators are power-law-like, $C(z) = A/z^{2\Delta}$, where Δ is in the ballpark of the leading scaling dim of that subspace:





Correlator ratios

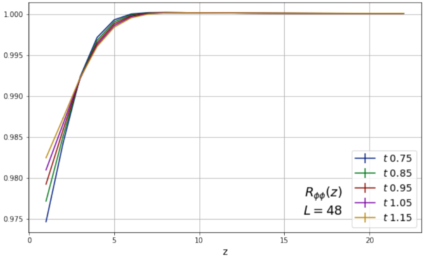
We considered ratios of correlators at pairs of flowtimes $t, t+\epsilon$, with $\epsilon = 0.05$.

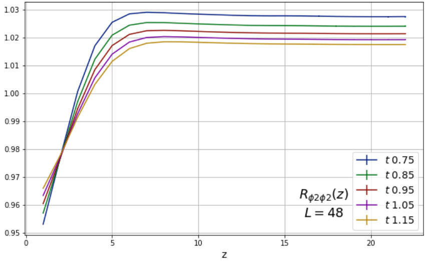
The ratio formula predicts a plateau at large z:

$$\mathscr{R}_{ij}(z;t) = \frac{\langle \mathcal{O}_i(\phi_{t+\epsilon};z)\mathcal{O}_j(\phi_{t+\epsilon};0)\rangle_{S_0}}{\langle \mathcal{O}_i(\phi_t;z)\mathcal{O}_j(\phi_t;0)\rangle_{S_0}} \sim b_\epsilon(t)^{2\Delta_a-(n_i+n_j)\Delta_1}$$

where $b_\epsilon(t) \sim \sqrt{1+\epsilon/t}$ at large t.

(The $\phi\phi$ ratio is independent of t, consistent with the formula above.)





Extracting scaling dimensions

The ratio formula is expected to be valid for $t \gg 0$ and $z \gg \sqrt{2dt}$.

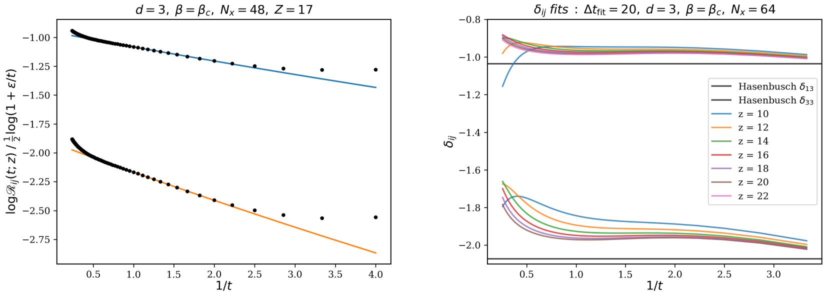
Ideally, the following should have a finite limit as $t \to \infty$:

$$\frac{\log \mathscr{R}_{ij}(z;t)}{\log b_\epsilon(t)} \longrightarrow \delta_{ij} + \text{subleading in } t$$

but large smearing radii intrude; one can't use too-large flow times.

Assuming $b_t \sim \sqrt{t}$, we fit ratios to the ansatz

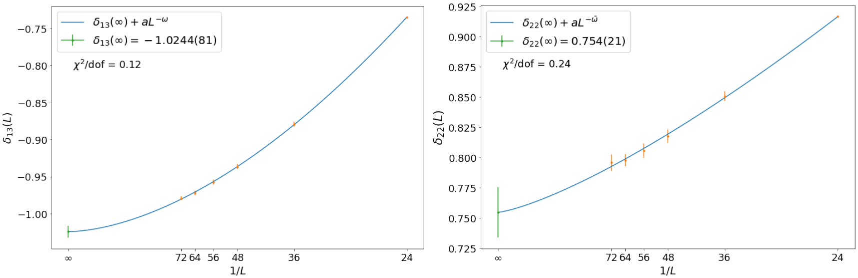
$$f(t) = \left(1 + \epsilon/t\right)^{\delta_{ij}/2 + a/t}$$



Infinite volume extrapolations

We assumed a corrections-to-scaling-like ansatz for extrapolating to infinite volume:

$$\delta_{ij}(L) = \delta_{ij}(\infty) + \frac{a}{L^{\omega}}$$



Note: Hasenbusch '99 values correspond to

$$\delta_{13} = -1.03580(40)$$
$$\delta_{22} = 0.7518(17)$$

Tabulated results for $\delta_{ij} := 2\Delta_a - (n_i + n_j)\Delta_1$

(i,j)	δ_{ij}(table 2.2)	δ_{ij}	$\bar{\omega}$	a	χ^2/dof
$(1,3)$	-1.03580(40)	-1.0283(65)	1.667(80)	59(14)	0.16
$(3,3)$	-2.07160(80)	-2.055(18)	1.73(11)	154(51)	0.30
$(2,2)$	0.7518(17)	0.743(17)	1.17(18)	7.2(3.5)	0.17
$(2,4)$	-0.2840(19)	-0.308(38)	0.96(20)	5.8(2.9)	0.11
$(4,4)$	-1.3198(22)	-1.310(29)	1.56(16)	84(38)	0.20

Figure: Infinite volume extrapolations of the δ_{ij} in $d = 3$ dimensions.

This analysis was repeated in two dimensions:

(i,j)	δ_{ij}(table 2.2)	δ_{ij}	$\bar{\omega}$	a	χ^2/dof
$(1,3)$	-0.25	-0.2616(14)	2.48(21)	127(83)	0.21
$(3,3)$	-0.50	-0.5279(28)	2.35(18)	161(91)	0.55
$(2,2)$	1.50	1.538(20)	1.92(35)	90(93)	0.35
$(2,4)$	1.25	1.299(31)	1.79(31)	103(93)	0.30
$(4,4)$	1.00	1.061(60)	1.62(25)	129(93)	0.39

Figure: Infinite volume extrapolations of the δ_{ij} in $d = 2$ dimensions.

GF-RG was also applied to a 12-flavor SU(3) gauge theory in 4d: arXiv:1806.01385 [5]

GF effective action and Wilson's exact RG

For a blocking transformation, one can define an effective (blocked) action via

$$e^{-S_b(\phi)} := \int_\varphi \delta(\phi - \varphi_b)\, e^{-S_0(\varphi)} \quad \text{where} \quad \varphi_b(x_b) = b^{\Delta_1 - d} \sum_\varepsilon \varphi(x+\varepsilon)$$

Can this be done for the GF-RG transformation?

- × Integrating against $\delta(\phi - \phi_t)$ where $\phi_t = f_t\varphi$ does not yield a satisfactory effective action:

$$S_t(\phi) = N_t + S_0(f_t^{-1}\varphi)$$

(where f(t) is the heat kernel).

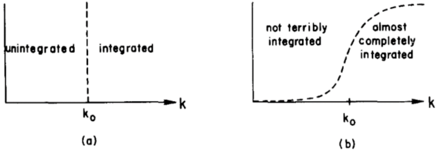
Enter "exact RG": Wilson-Kogut 1973, Wegner-Houghton 1973.

Wilson & Kogut defined a low-mode Boltzmann factor by

$$\rho_t(\phi) = \int_\varphi P_t(\phi;\varphi)\, \rho_0(\varphi)$$

where P_t is a "constraint functional":

$$P_t(\phi;\varphi) = \mathcal{N}\, \exp\left[-\frac{1}{2}\int_p \frac{2\omega_p}{K_0(p)}\, \frac{(\phi_p - f_t(p)\varphi_p)^2}{1 - f_t^2(p)}\right]$$



Adapted from *Wilson & Kogut (1973)*

Exact RG and Stochastic RG

Wilson & Kogut's constraint functional satisfies

$$\partial_t P_t(\phi) = \frac{1}{2} \int_p \left(K_0(p) \frac{\delta^2 P_t}{\delta\phi_p \delta\phi_{-p}} + 2\omega_p \phi_p \frac{\delta P_t}{\delta \phi_p} \right)$$

where $K_0(p)$ is a smooth cutoff function (e.g. $K_0 = \mathrm{e}^{-p^2}$).

This is a functional Fokker-Planck equation! $P_t(\phi, \varphi) = P_t(\phi, t; \varphi, 0)$ is a transition function;
$\Rightarrow$ perhaps exact RG can be viewed as a diffusion in *field space*.

The stochastic process which generates this distribution is

$$\partial_t \phi_t(p) = -\omega_p \phi_t(p) + \eta_t(p)$$

where $\eta_t(p)$ is gaussian noise with variance $K_0(p)$: *Looks like a stochastic GF equation.*

MCRG: observables of the effective theory $\rho_t(\phi) = \mathrm{e}^{-S_t(\phi)}$ are related to the bare theory by

$$\langle \mathcal{O}(\phi) \rangle_{S_t} = \langle \mathbb{E}_\eta \left[\mathcal{O}(\phi_t[\varphi, \eta]) \right] \rangle_{S_0}$$

The effective low-mode action $S_t(\phi)$ can be written in terms of bare-theory connected (time-dependent) Green functions

$$S_t(\phi) = F_t + \frac{1}{2}(\phi, A_t \phi) - W_0^{(t)}[B_t f_t \phi]$$

Wilson-Fisher FP and relation to GF

When the field is rescaled as

$$\phi(p) = \Lambda_t^{d_\phi} \zeta_t \Phi(p/\Lambda_t), \quad b_t = \Lambda_0/\Lambda_t = \sqrt{1+2t}$$

the effective action $\bar{S}_t$ for Φ has an interacting fixed point at first order in perturbation theory ([8]: arxiv:1904.13057; see also arxiv:2006.07481).

Gradient flow:

It can be demonstrated that effective theory correlations reduce to the GF correlations at large distances:

$$\langle \phi(x)\phi(y) \rangle_{S_t} = A_t(x,y) + \langle f_t\varphi(x) f_t\varphi(y) \rangle_{S_0}$$

since A_t decays fast with $x-y$. Similar formulae hold for higher composite operators.

(⇒ Wilson-Kogut '73 [6], Sonoda-Suzuki 2019 [7], Carosso 2019 [8])

So

$$\langle \Phi(x/b_t)\Phi(y/b_t) \rangle_{\bar{S}_t} \sim b_t^{2\Delta_1} \langle f_t\varphi(x) f_t\varphi(y) \rangle_{S_0}$$

which suggests the relation we found from the spin-blocking analogy, $\Phi_t = b_t^{\Delta_1} f_t\varphi$.

Summary and Future

We have seen that the block-spin analogy implies that GF can be used to extract leading (and perhaps subleading) scaling dimensions.

We have also seen a possibility for properly defining an *effective action* associated with the GFRG transformation, which implies that GFRG is closely related to Wilson's "exact RG."

Future directions:

- Can we apply GF-RG scaling laws in the case of local expectation values? Perhaps a *continuous* Swendsen equation?
- What is the effect of interacting flows? Wilson & Bell '74 suggested that nonlinear RG's may have some advantages over linear RG's.

Thank you for listening!

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