

# Gradient flow and topological properties of (hot) Quantum Chromodynamics

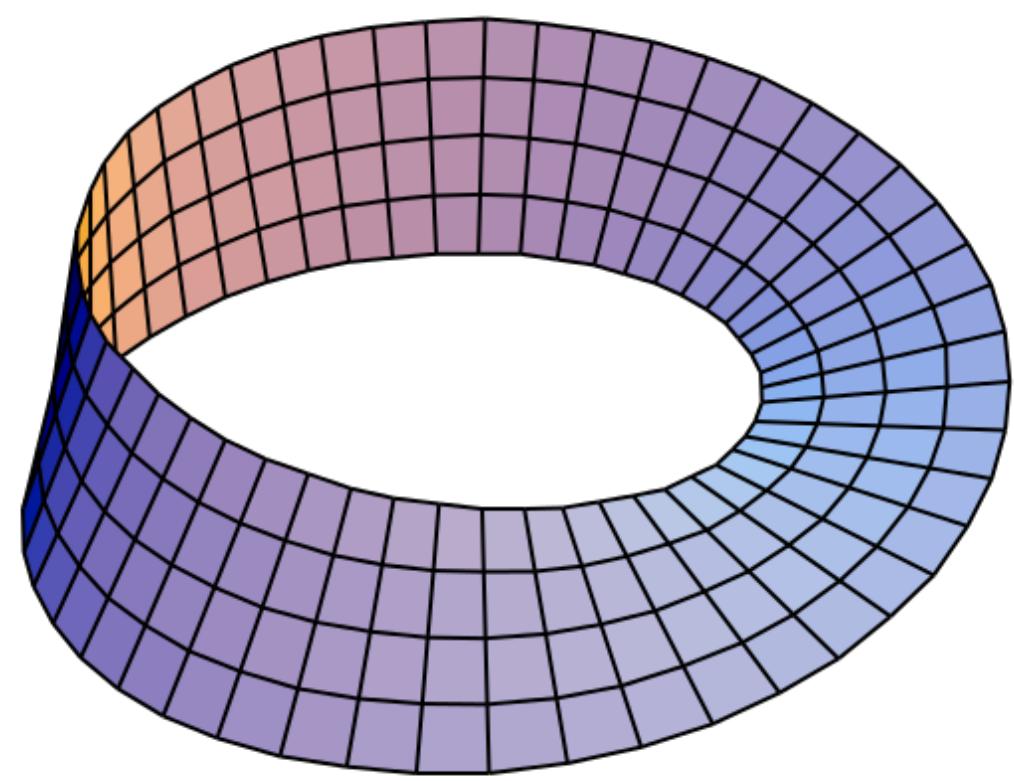
A. Yu. Kotov



[AYuK, JETP Letters, 2018]  
[AYuK, M.P. Lombardo, A. Trunin, Phys. Lett. B 794, 2019]  
[AYuK, M.P. Lombardo, A. Trunin, PoS Lattice, 2021]  
[AYuK, M.P. Lombardo, A. Trunin, in progress]

# **Background and motivation**

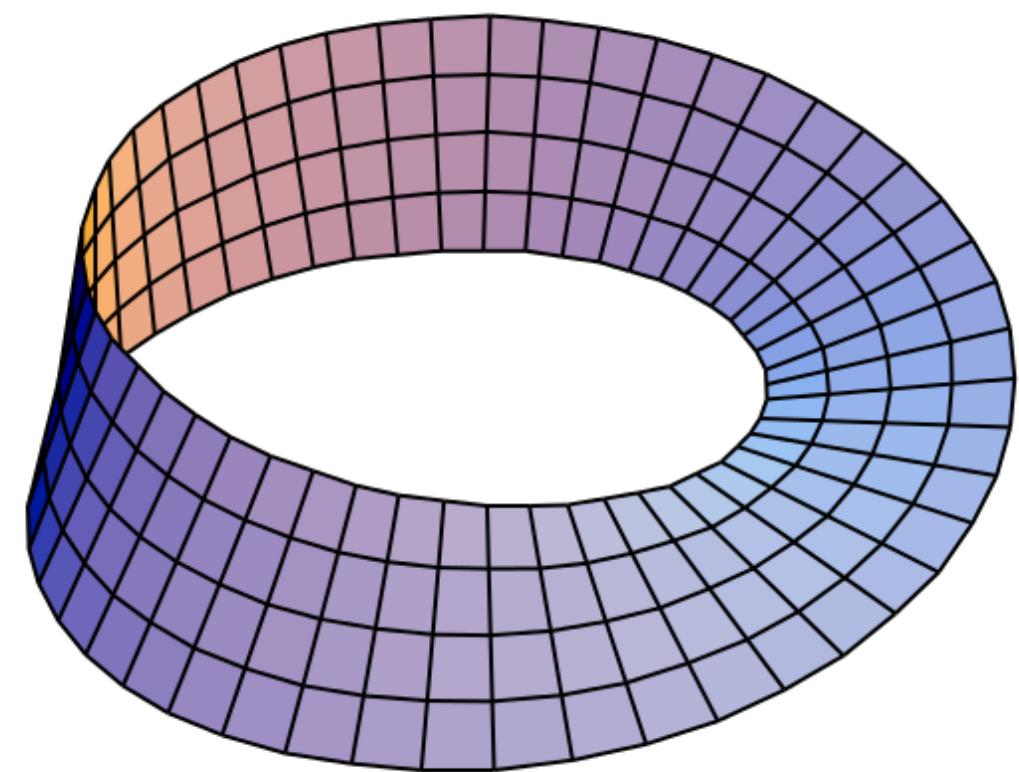
## **QCD and topology**



# Background and motivation

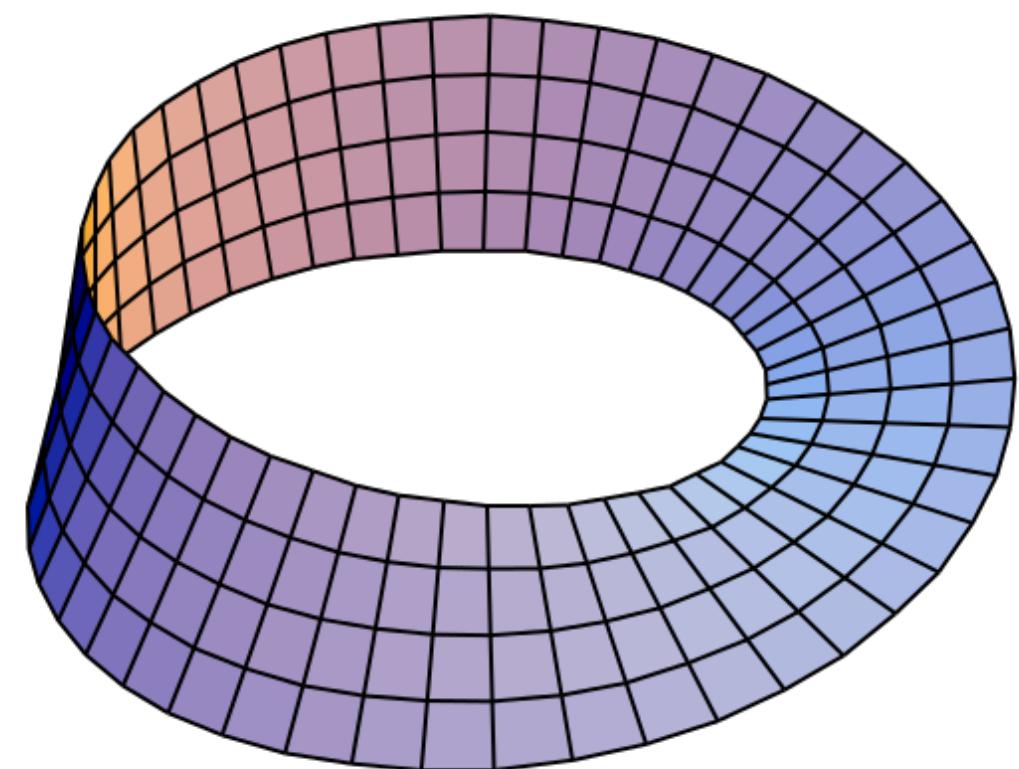
## QCD and topology

- Topological charge  $Q = \int d^4x q(x)$ ,  $q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}$  of the gluon configuration
- Topological susceptibility  $\chi = \frac{\langle Q^2 \rangle}{V}$ , correlator of the topological charge density  $\langle q(x)q(0) \rangle$



# Background and motivation

## QCD and topology

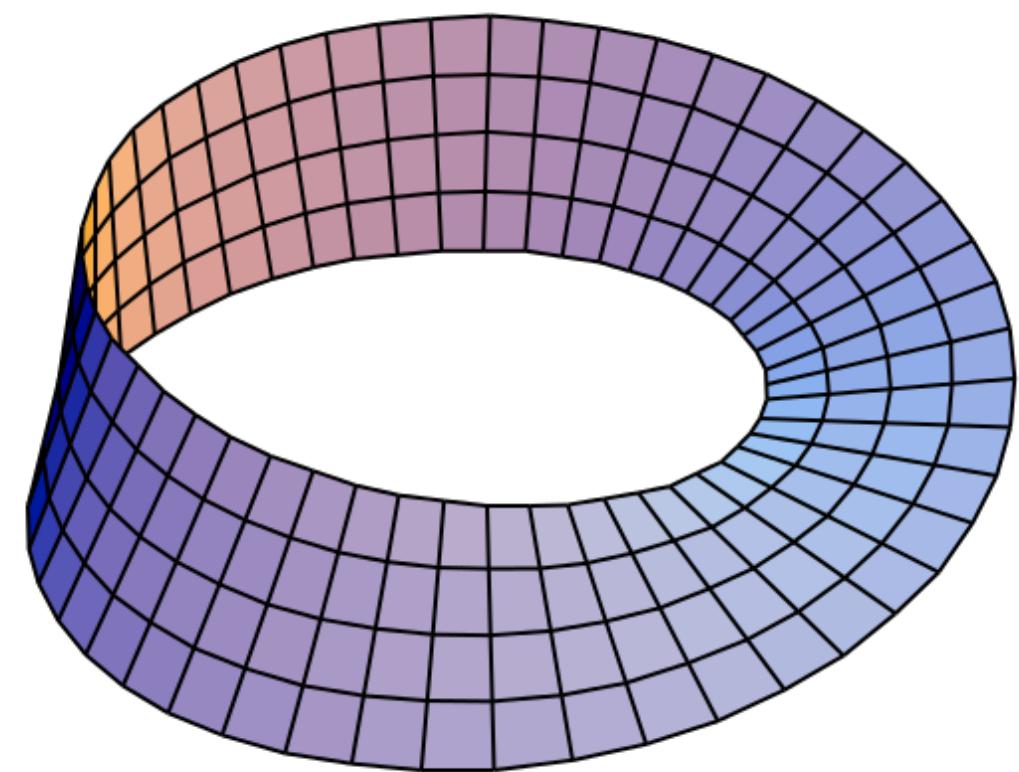


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- QCD symmetries and symmetry breaking pattern (and restoration @ finite T)

$$L_{\text{QCD}} = \sum_f \bar{q}_L^f \not{D} q_L^f + \bar{q}_R^f \not{D} q_R^f - m \left( \bar{q}_L^f q_R^f + \bar{q}_R^f q_L^f \right) + L_{\text{gauge}}$$

# Background and motivation

## QCD and topology



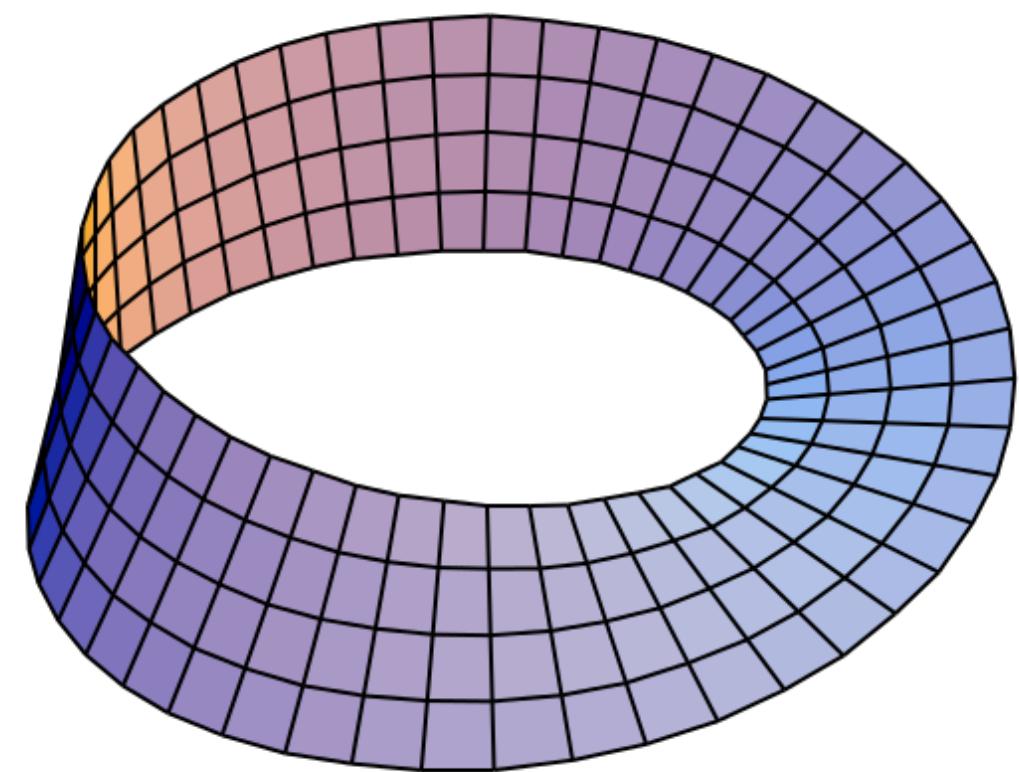
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- Global symmetry:  $U_L(n) \times U_R(n) \simeq SU_L(n) \times SU_R(n) \times U_B(1) \times U_A(1)$

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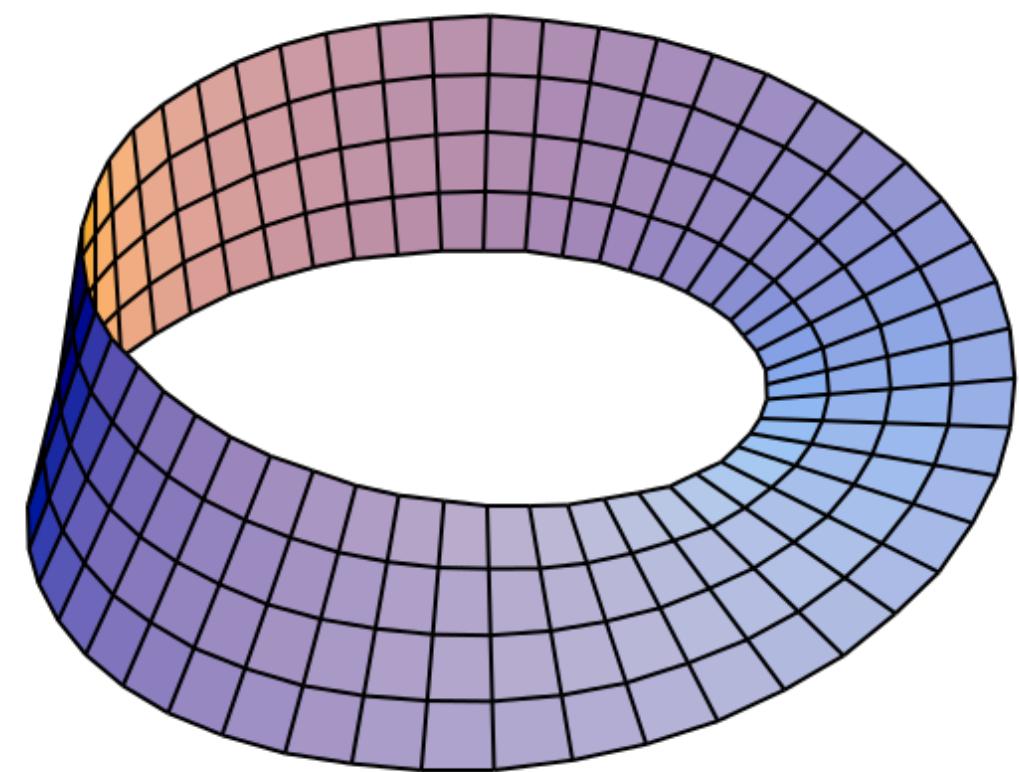
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  - Spontaneously broken  $\downarrow$   $SU_V(n)$
  - Baryon number
  - Explicitly broken  
(by topological fluctuations)

# Background and motivation

## QCD and topology



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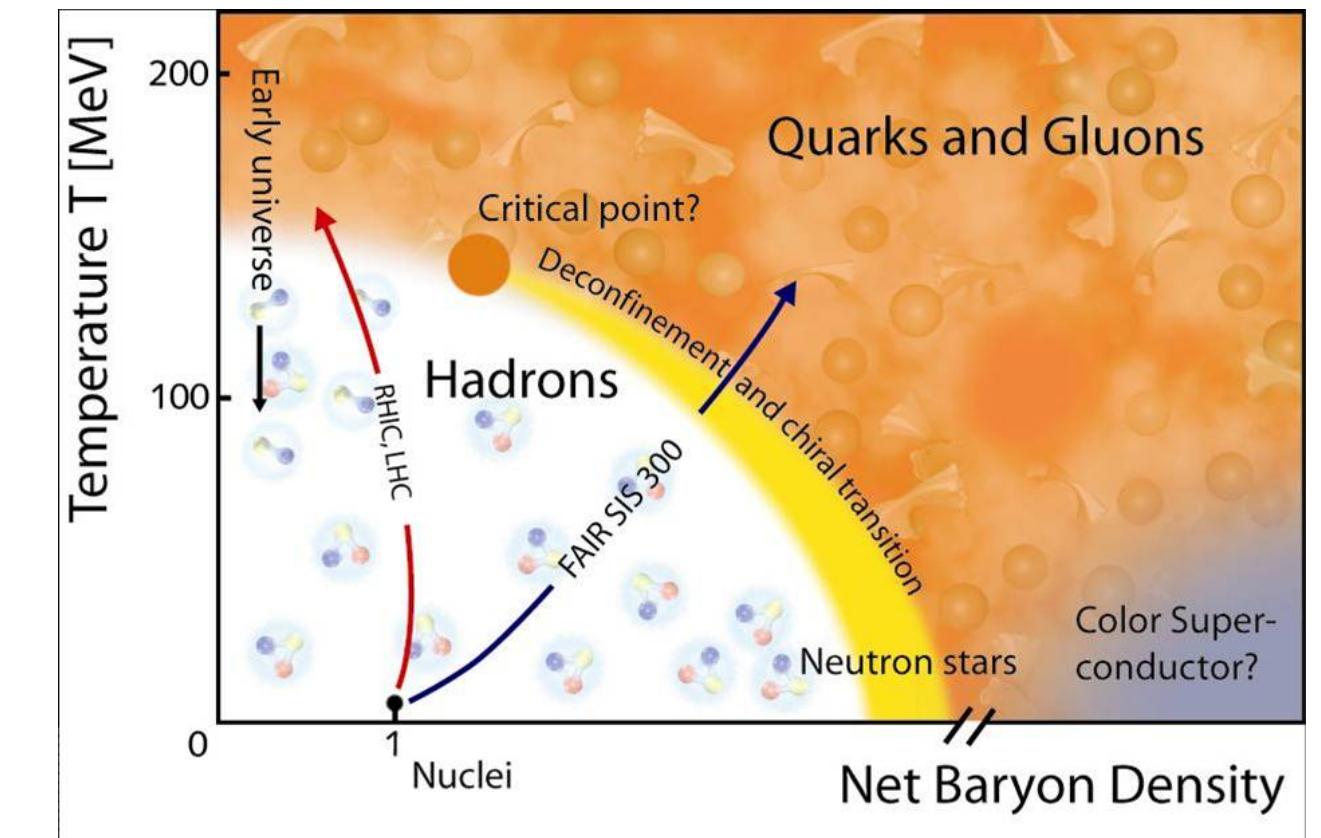
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  - Spontaneously broken  $\downarrow$  Baryon number (by topological fluctuations)
  - $SU_V(n)$
- Consequence:  $\eta'$  is not a (pseudo) Goldstone boson:  $\frac{f_0^2}{4N_f} \left( M_\eta^2 + M_{\eta'}^2 - 2M_K^2 \right) = \chi_\infty$  [Witten, 1979][Veneziano, 1979]

# Background and motivation

## QCD and topology, finite temperature

- *Which symmetry is restored at finite temperature?*



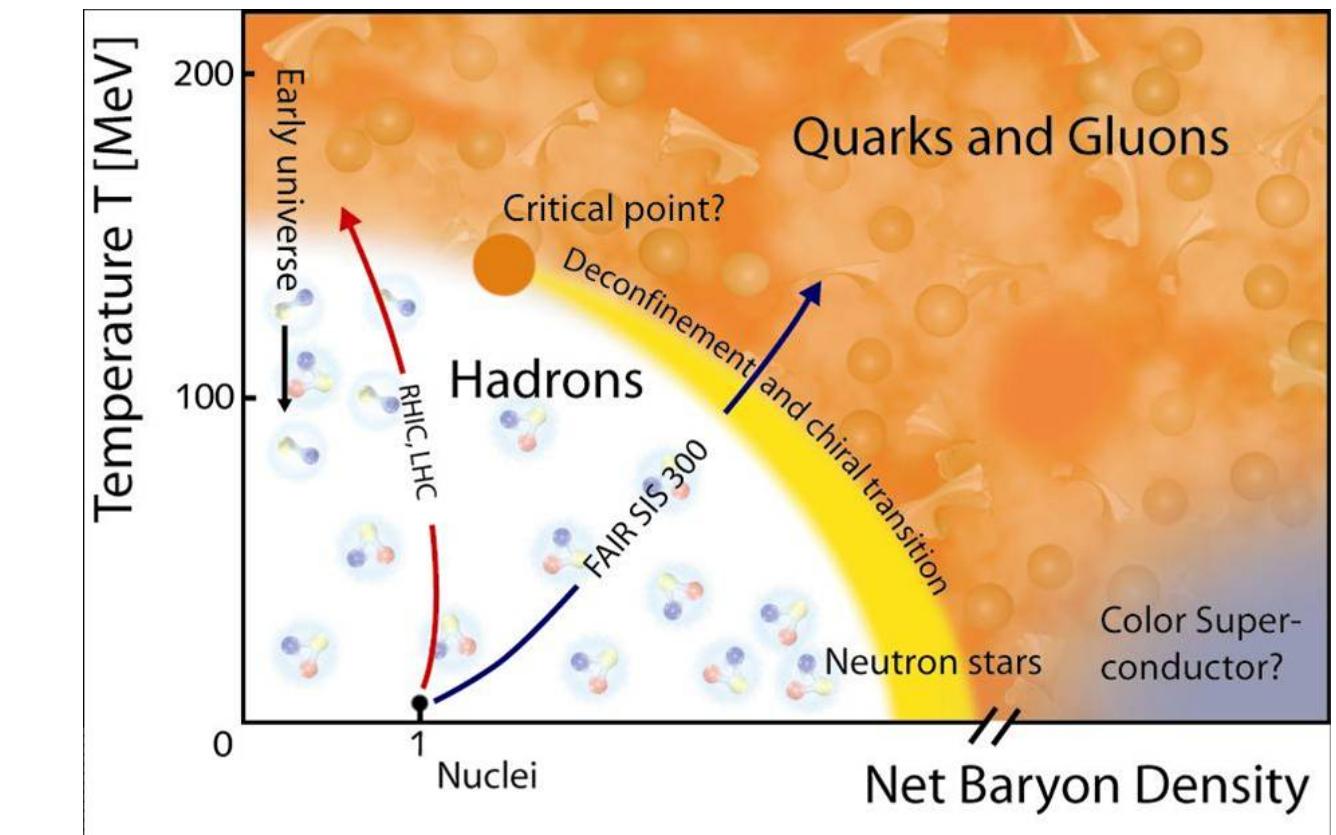
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?

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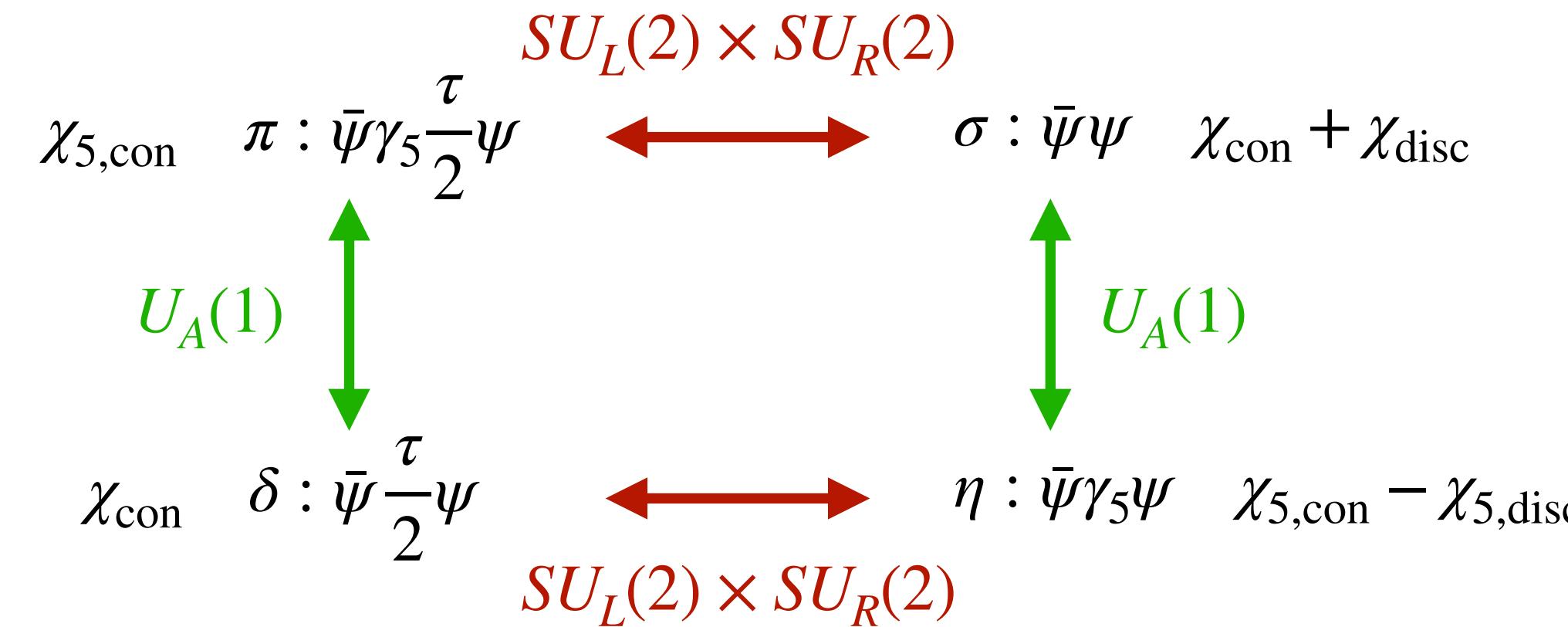


$$U_L(n) \times U_R(n) \simeq SU_L(n) \times SU_R(n) \times U_B(1) \times U_A(1)$$

$$\chi_{\text{top}} = \langle Q_{\text{top}} \rangle^2 / V = m_l^2 \chi_{5,\text{disc}}$$

$$m \int d^4x \bar{\psi} \gamma_5 \psi = Q_{\text{top}}$$

?



Not fully settled

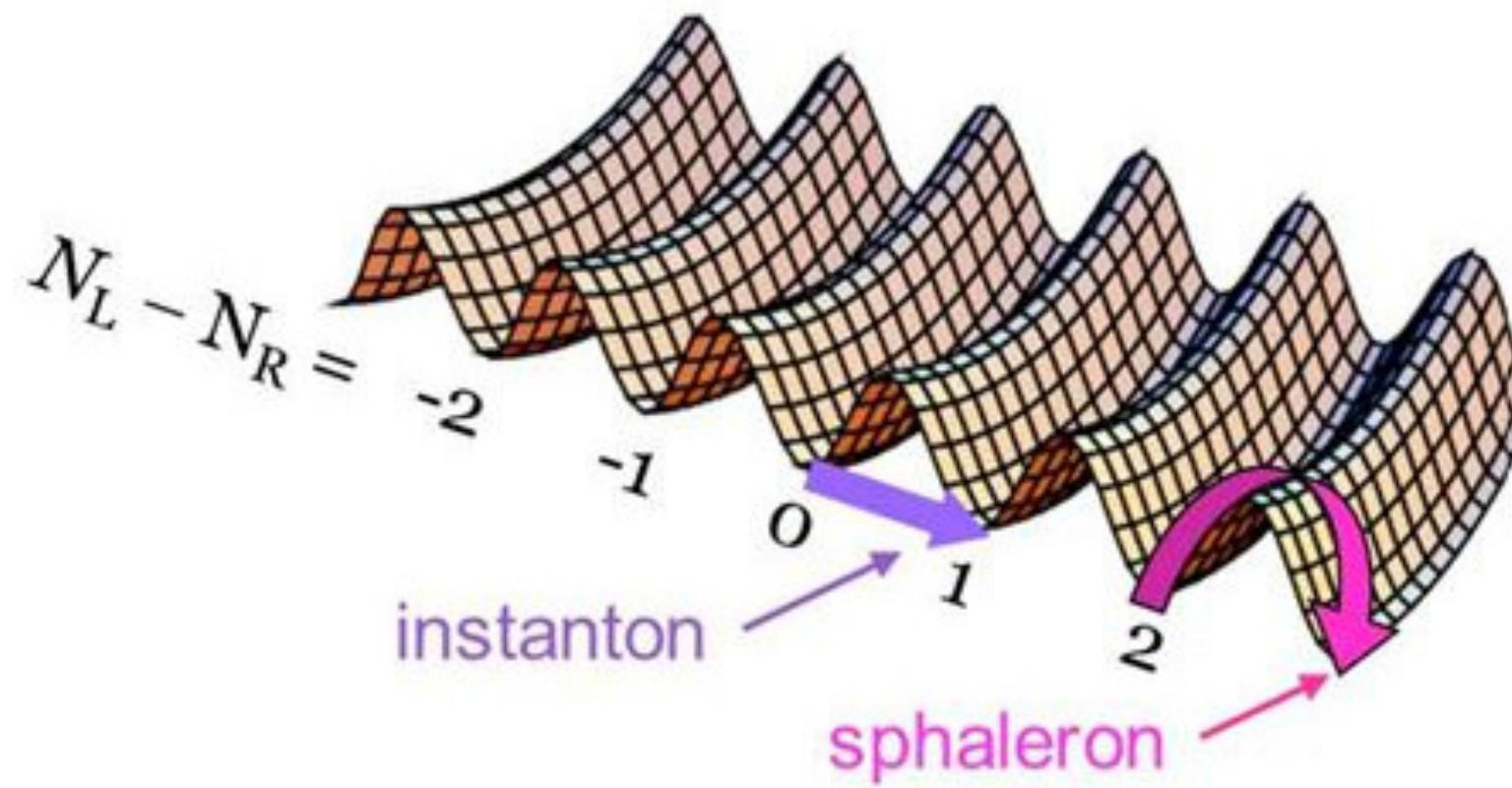
$$\chi_\pi - \chi_\delta = \chi_{\text{disc}} = \chi_{5,\text{disc}}, \quad \text{for } T \geq T_C, m_l \rightarrow 0 \implies \chi_{\text{top}} = \langle Q_{\text{top}} \rangle^2 / V = m_l^2 \chi_{\text{disc}}$$

Topology is closely related to the symmetry breaking pattern and particle spectrum in finite temperature QCD

# Background and motivation

## QCD, chiral dynamics

Quenched theory

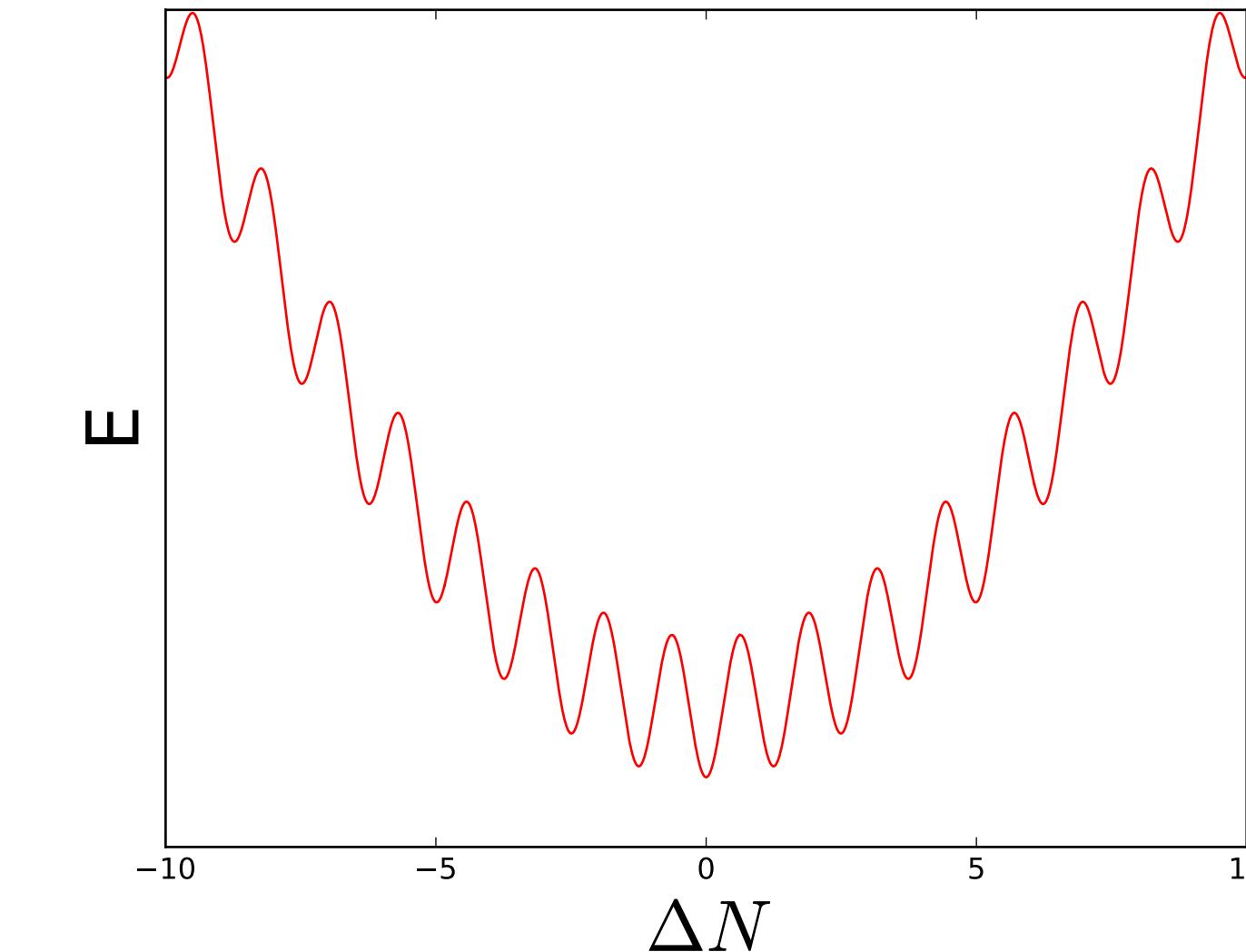


$$\Gamma_{CS} = \lim_{t \rightarrow \infty} \frac{\langle \Delta N_{CS}^2 \rangle}{Vt}$$

$$\Delta N_{CS} = Q_4 = \int d^4x q(x)$$

$$q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$$

Theory with fermions



$$\frac{dQ_5}{dt} = -CQ_5 \frac{\Gamma_{CS}}{2T}$$

$$\Gamma_{CS} \sim \lim_{\omega \rightarrow 0} \langle \bar{q}q(\omega, \vec{p} = 0) \rangle$$

Governs the dynamics of the chiral density

# Background and motivation

## Strong CP problem & Axions

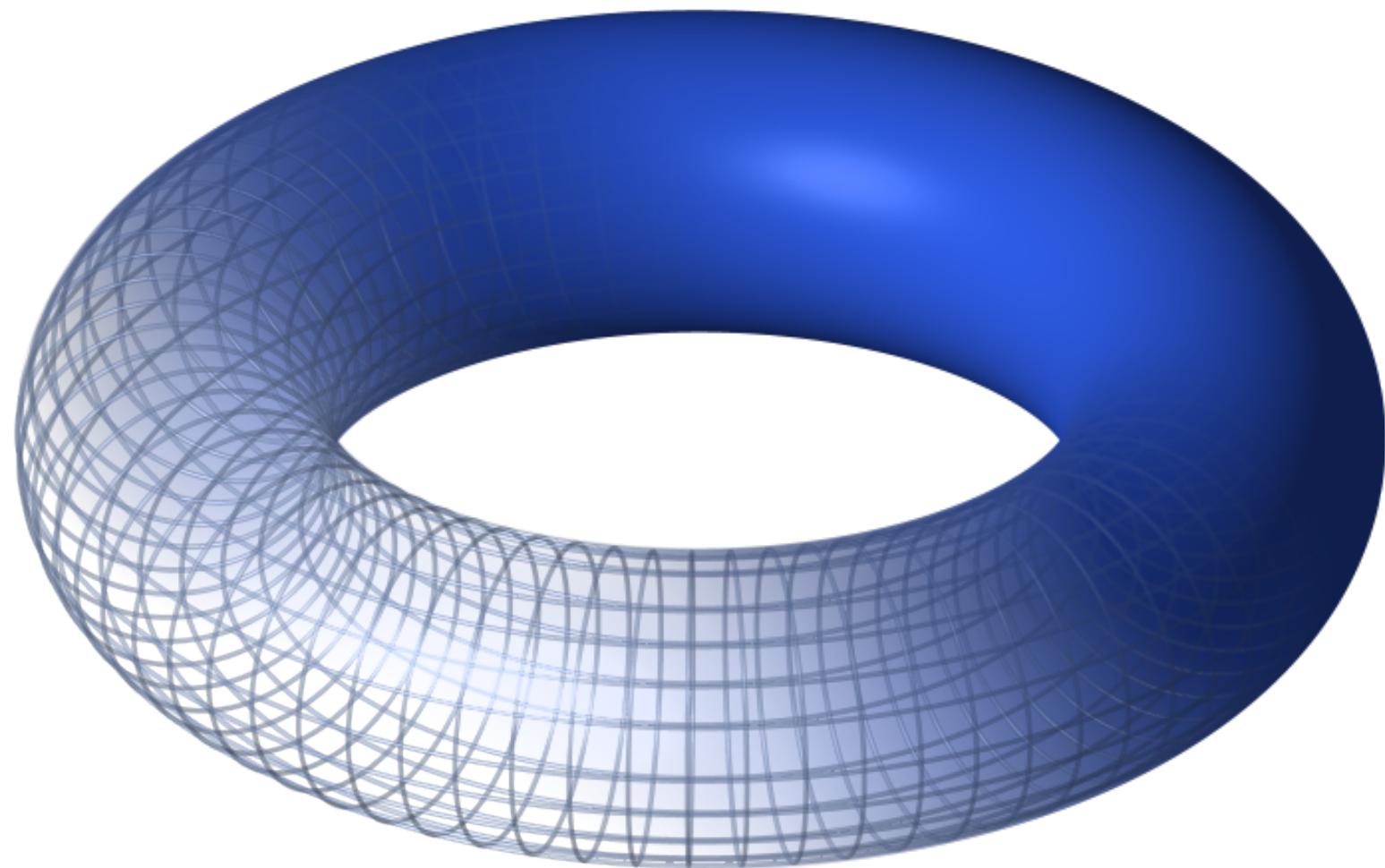
- $L_{\text{QCD}} = \sum_f \bar{q}_L^f D q_L^f + \bar{q}_R^f D q_R^f - m(\bar{q}_L^f q_R^f + \bar{q}_R^f q_L^f) + \theta \sum_a \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} + L_{\text{gauge}}$
- Experiments on Neutron electric dipole moment:  $\theta < 0.5 \times 10^{-10}$  Talk by [A. Shindler, Mon]
- Solution: the axion, the minimum is effectively at  $\theta = 0$
- $L = L_{\text{QCD}} + (\partial_\mu a)^2 + \left( \theta + \frac{a}{f_A} \right) \sum_a \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}$
- In general:  $m_A^2 f_A^2 = \chi_{\text{top}}$
- Axions are possible candidates of dark matter

Talk by [G. Schierholz, Tue]

# Problem

**How to measure topological charge and density on the lattice?**

- Simulations on the torus: no topological invariant
- Integer-valued
- Discretization effects

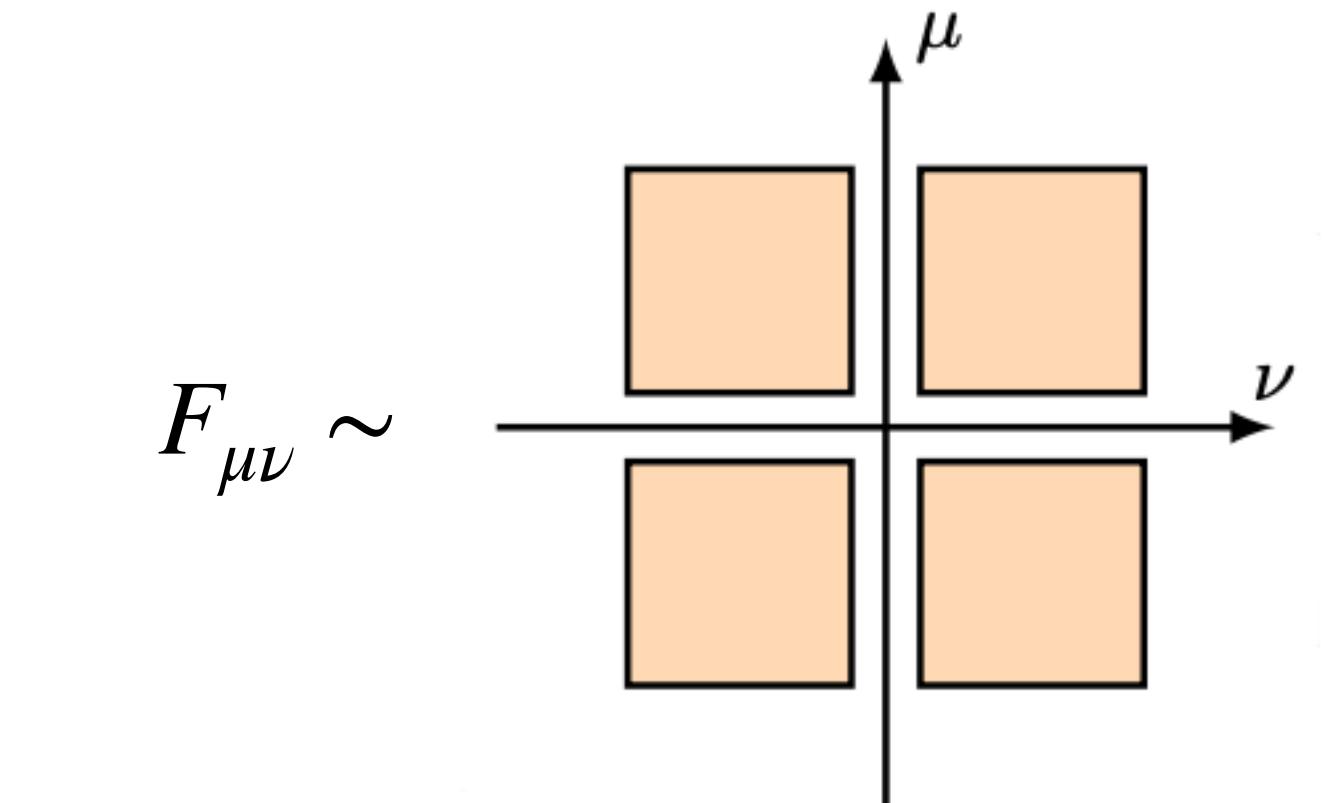


# Definitions of topological charge

- Index of the overlap Dirac operator:  $Q = n_- - n_+$ ,  $D = (1 + \gamma_5 \text{sgn}(\gamma_5 D_W(-m_W)))$
  - Wilson-Dirac spectral flow Numerically expensive [ETMC, 2020]
  - Spectral projectors
  - Fermionic definition:  $\chi_{\text{top}} = \langle Q_{\text{top}} \rangle^2 / V = m_l^2 \chi_{\text{disc}}$ , in the chirally symmetric phase
  - Gluonic definitions:  $q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu}$ , gluonic definition of  $F_{\mu\nu}^a$  (clover, e.g.)
  - ...
    - Gradient Flow and other smearing routines (removing UV fluctuations)  
What about continuum limit? Talk by [M. Lüscher, Tue]
- No definition is definitely preferable over others

# Gradient Flow and topological susceptibility

- GF evolution:  $\partial_\tau A_\mu(\tau, t, x) = - \frac{\partial S}{\partial A_\mu}$
- $\partial_\tau q^\tau = \partial_\rho \omega_\rho^\tau, \partial_\tau Q^\tau = 0$
- $\langle q^\tau(x)q^\tau(0) \rangle = \langle q^{\tau=0}(x)q^{\tau=0}(0) \rangle + O_\tau(a^2) + O(\tau)$
- One can use any definition of  $q(x)$ , we use gluonic clover discretization



[M.Ce et al., 2015]

# Questions to discuss

## QCD @ finite T

- Topological susceptibility
- Topological charge correlator
  - Thermal (sphaleron) rate (gluodynamics)
  - $\eta'$  mass

# Lattice setup: TWEXT collaboration

## Wilson twisted mass fermions

$N_f = 2 + 1 + 1$  Wilson twisted mass fermions at maximal twist

Iwasaki gauge action

$m_\pi \in [m_\pi^{\text{phys}}, 370 \text{ MeV}]$

Physical strange and charm quarks

Fixed scale approach:  $a = \text{fixed}$ ,  $T \leftrightarrow N_t$

Based on ETMC  $T = 0$  parameters

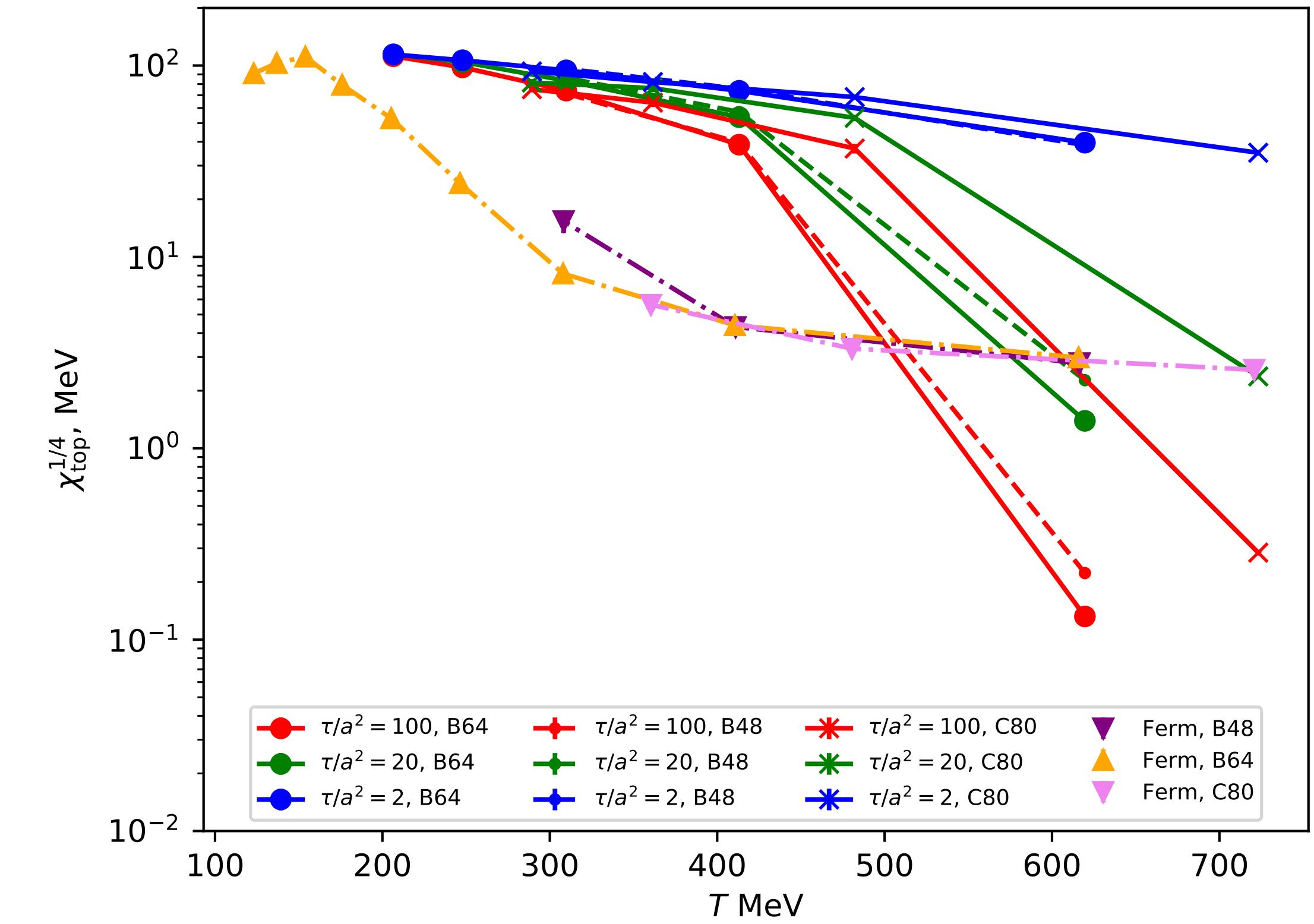
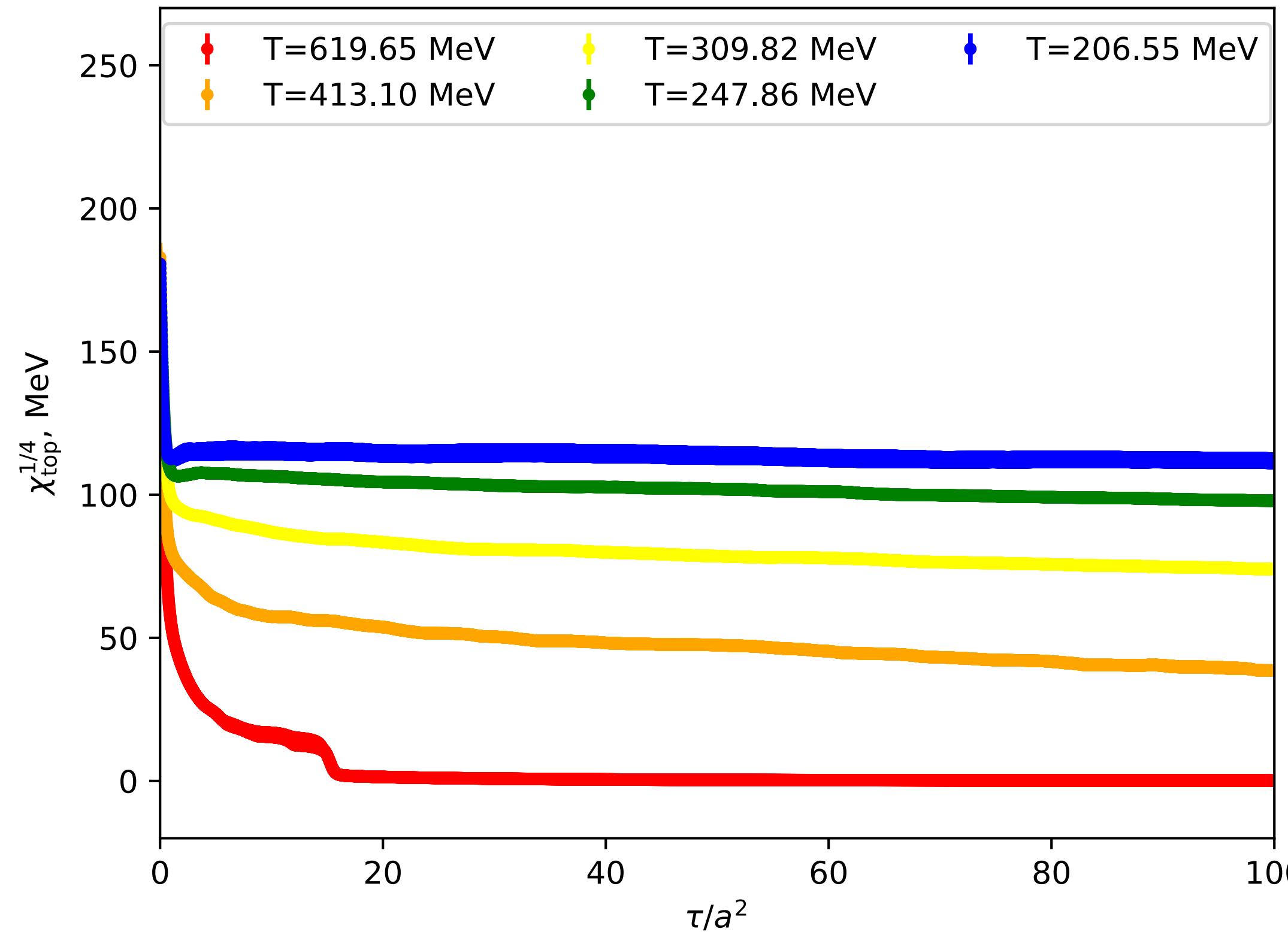
[C. Alexandrou et al., 2018][C. Alexandrou et al., 2021]

**Quenched simulations with Wilson action**

$m_\pi \text{ [MeV]}$	$a \text{ [fm]}$
136.2(3)	0.0684(3)
139.7(3)	0.0801(4)
225(5)	0.0619(18)
383(11)	0.0619(18)
376(14)	0.0815(30)

# Systematics from twisted mass fermions

## Gradient flow definition vs Fermion definition

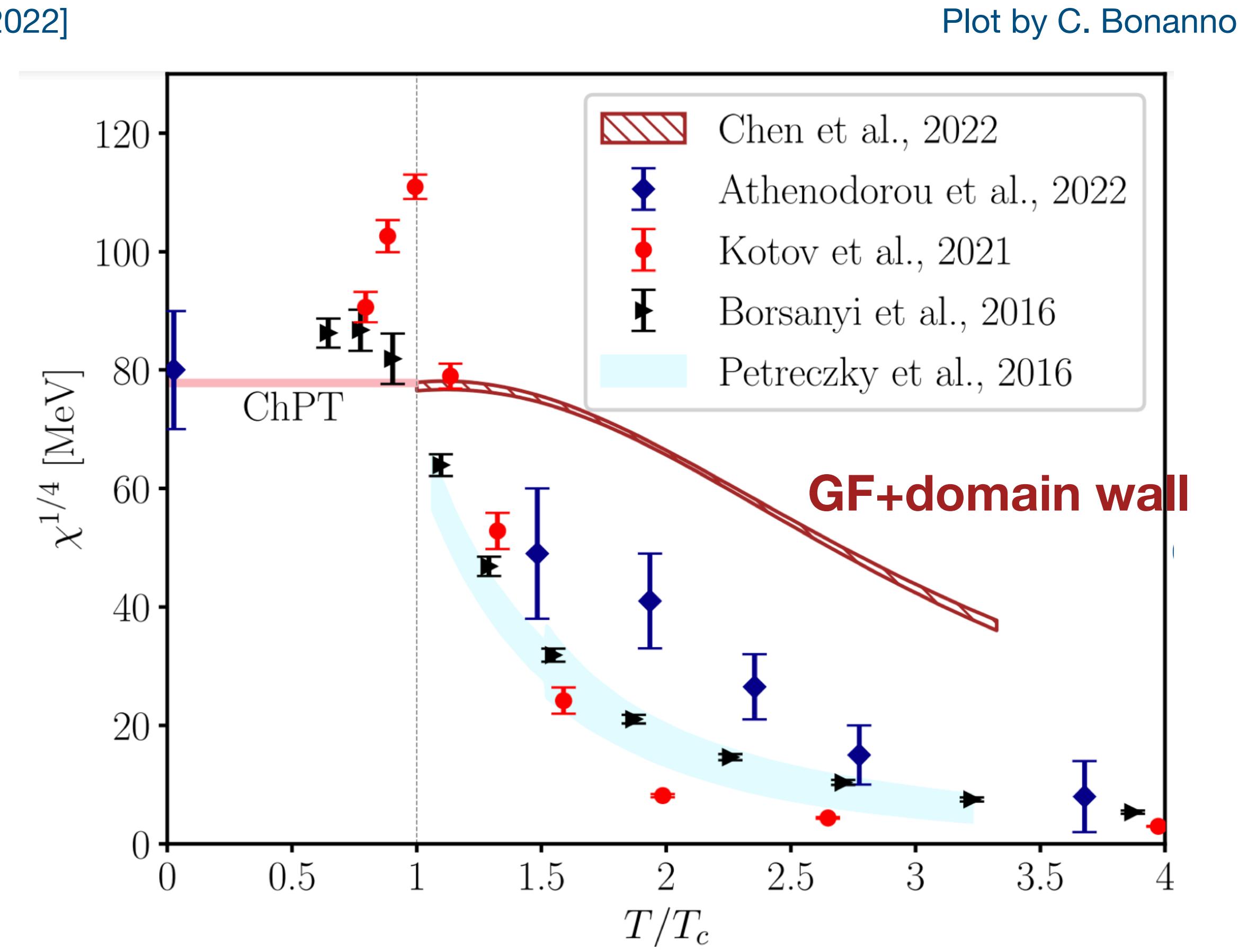
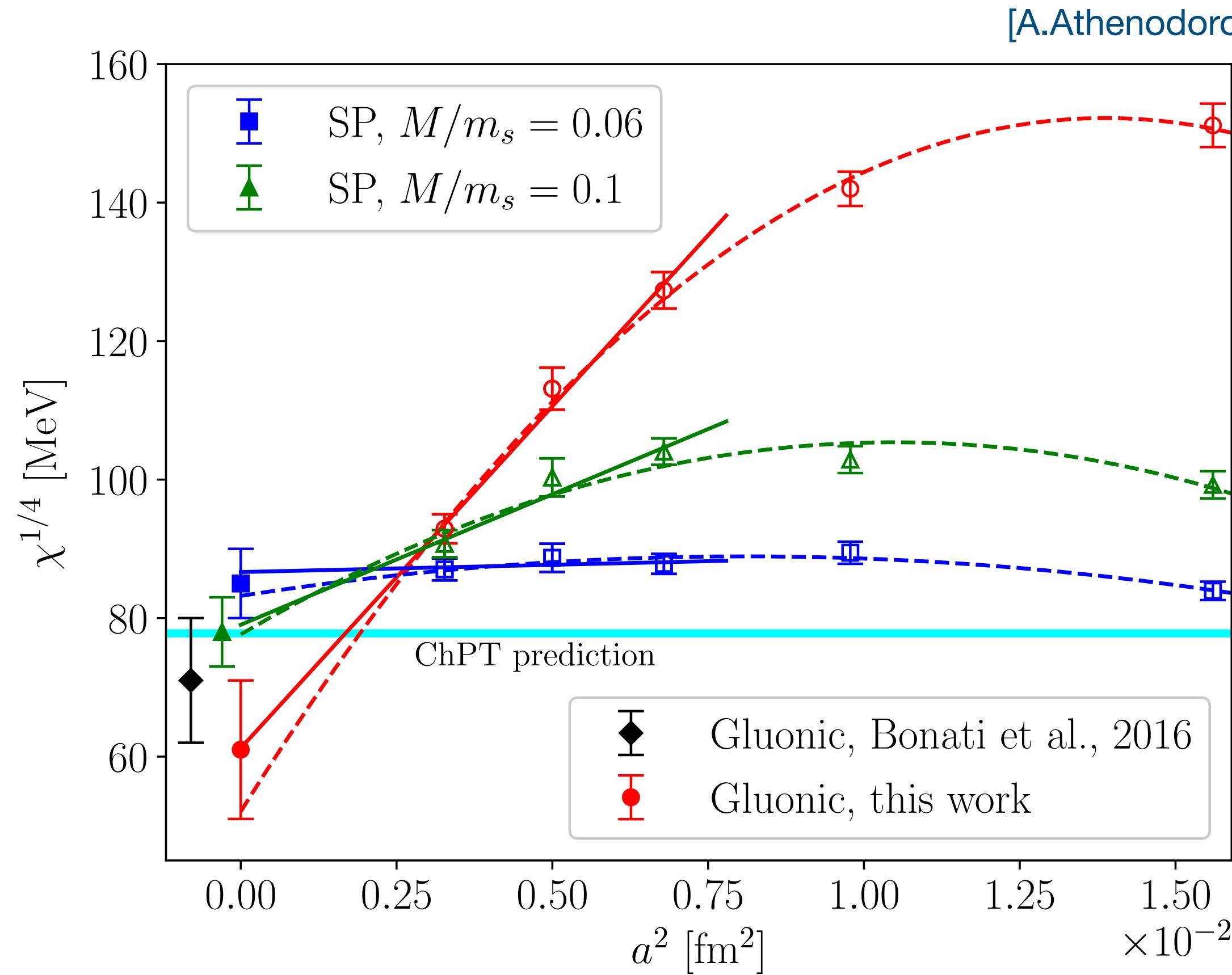


PRELIMINARY

[AYuK, M.P.Lombardo, A.Trunin, 2021 + in preparation]

# Systematics from other fermions

## Spectral projector method vs Gradient flow definition



GF in general gives larger result - can they converge in the continuum limit?

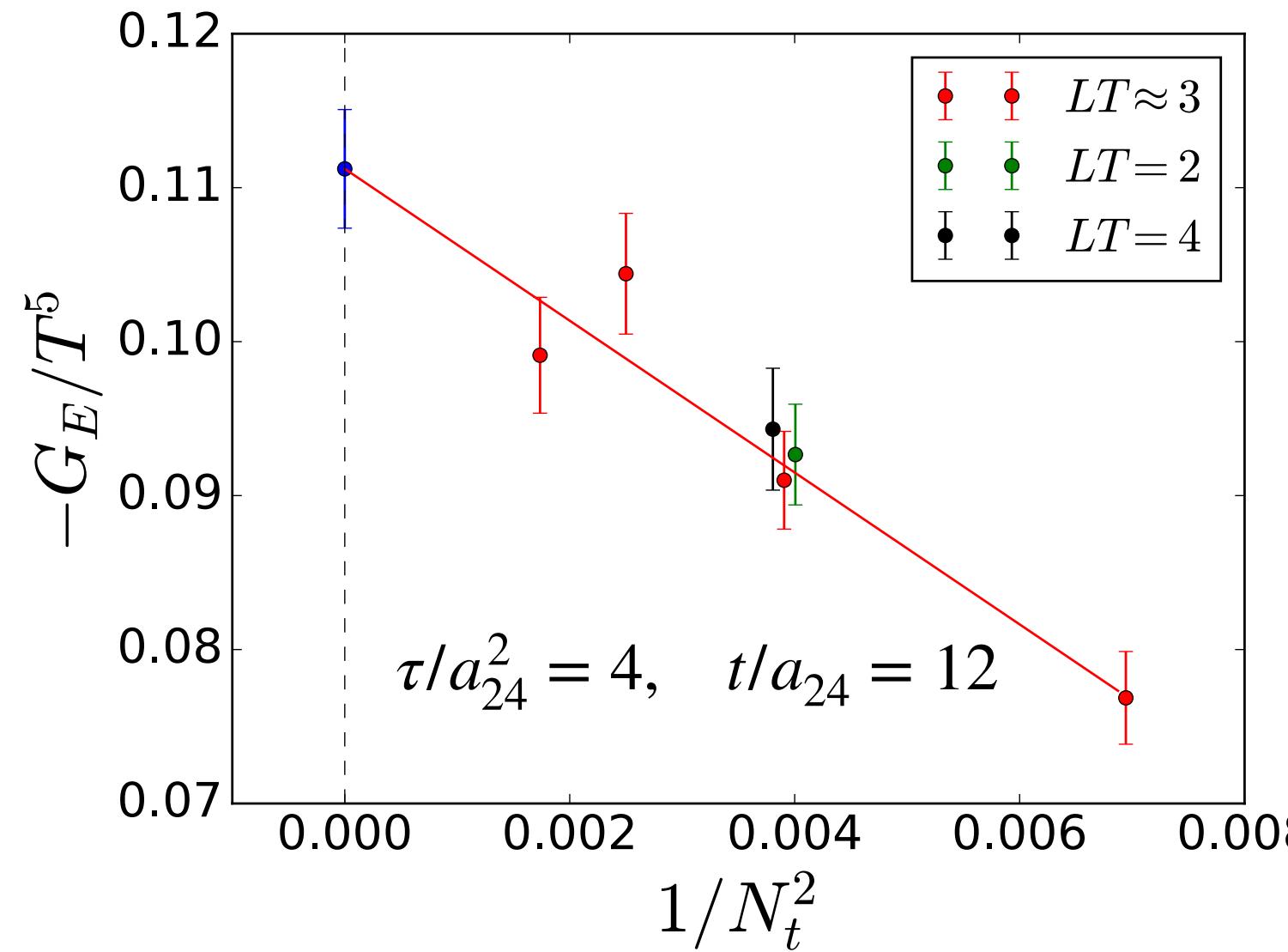
# Topological charge correlator

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- $C_q(x) = \lim_{\tau \rightarrow 0} \lim_{a \rightarrow 0} \langle q_L^\tau(x) q_L^\tau(0) \rangle$  [M.Ce et al., 2015]
- $\langle q_L^\tau(x) q_L^\tau(0) \rangle = C_q(x) + O_\tau(a^2) + O(\tau)$

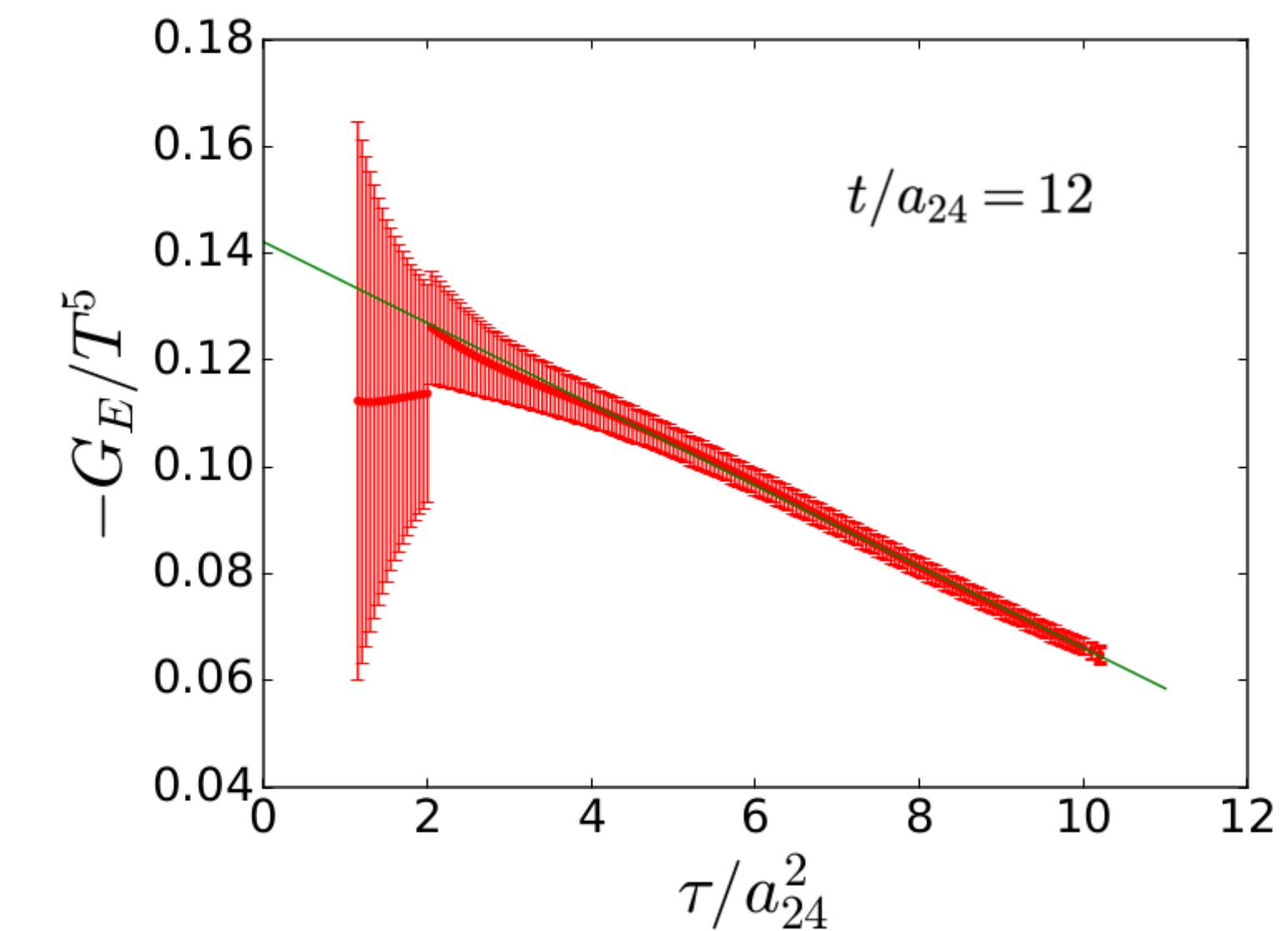
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- $\langle q_L^\tau(x) q_L^\tau(0) \rangle = C_q(x) + O_\tau(a^2) + O(\tau)$
- $t^2/8 \gtrsim \tau \gtrsim a^2$
- Clover discretization of  $q_L^\tau(x)$  Quenched theory



$T/T_c = 1.24$

[AYuK, 2018]



# Sphaleron (thermal) rate

$$G(t) = \sum_x \langle q(t, x)q(0, 0) \rangle$$

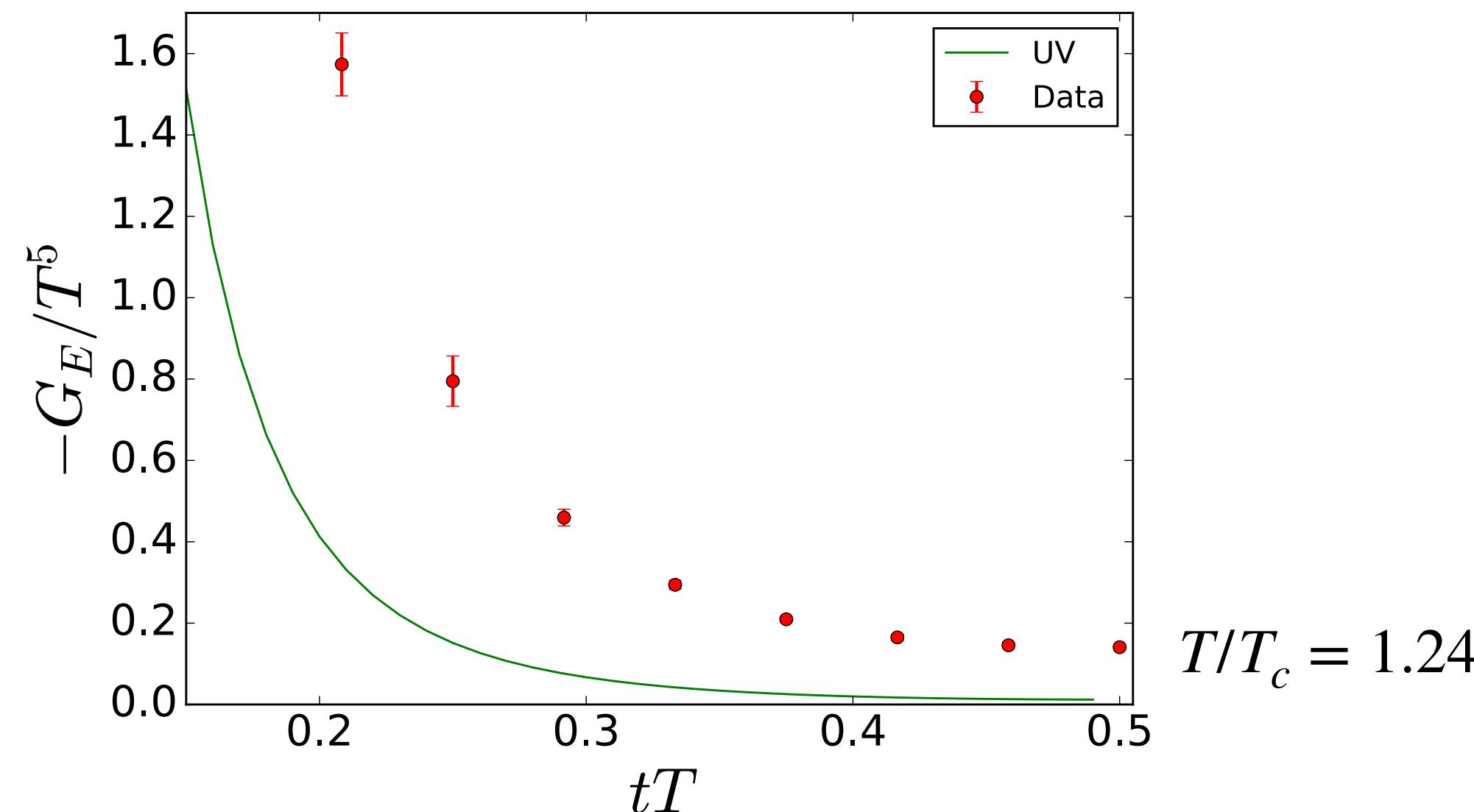
$$G_E(t) = \int \rho(\omega) \frac{\cosh \omega(\beta/2 - t)}{\sinh \beta\omega/2} d\omega$$

$$\Gamma_{CS} = -2\pi T \lim_{\omega \rightarrow 0} \frac{\rho(\omega)}{\omega}$$

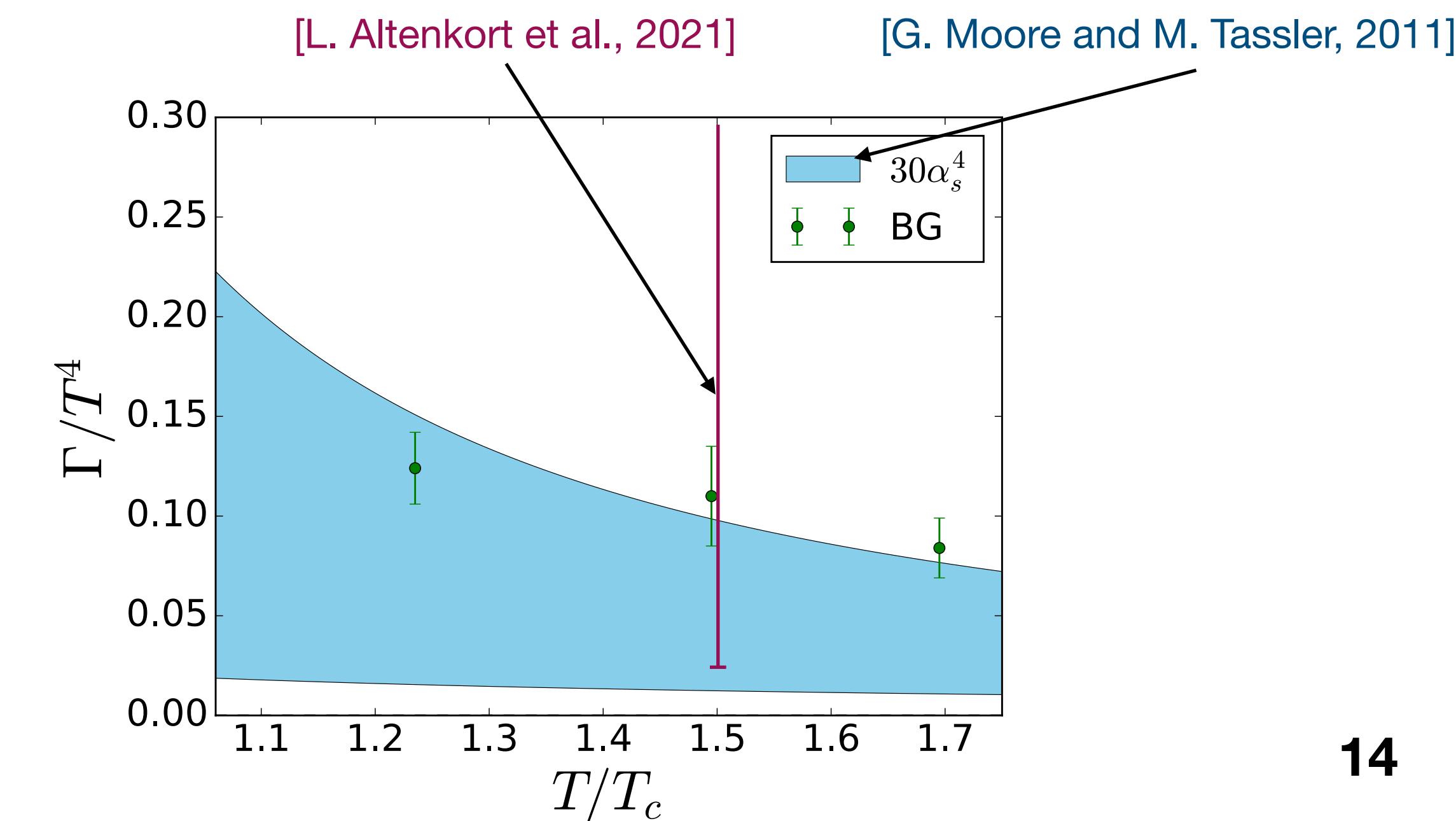
$$\rho(\omega) = \frac{1}{\pi} \text{Im} G^R(\omega, \vec{k} = 0)$$

UV part:  $\rho_q(\omega) = -\frac{d_A \alpha_s^2}{256\pi^4} \omega^4$  : seems to be small

Inversion with Backus-Gilbert method



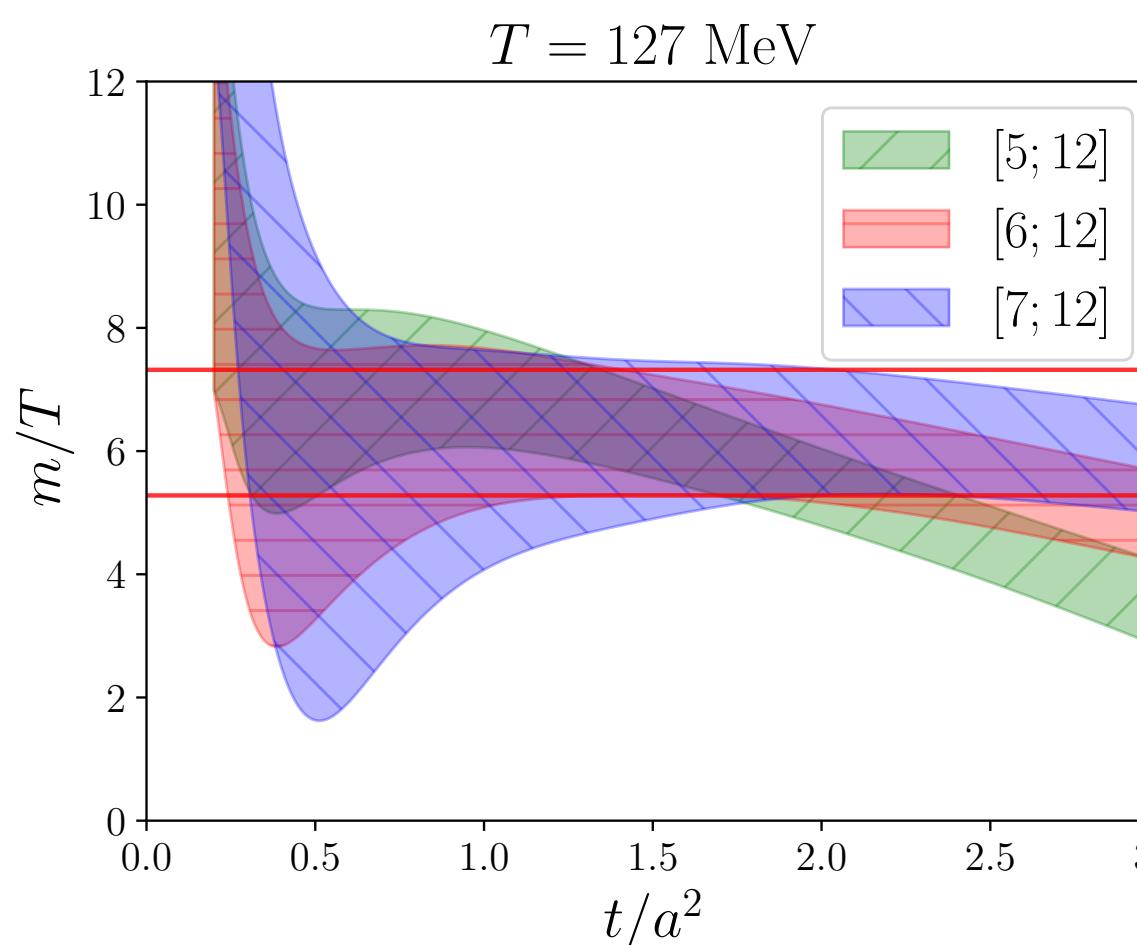
[AYuK, 2018]



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# Topological charge correlator in theory with fermions

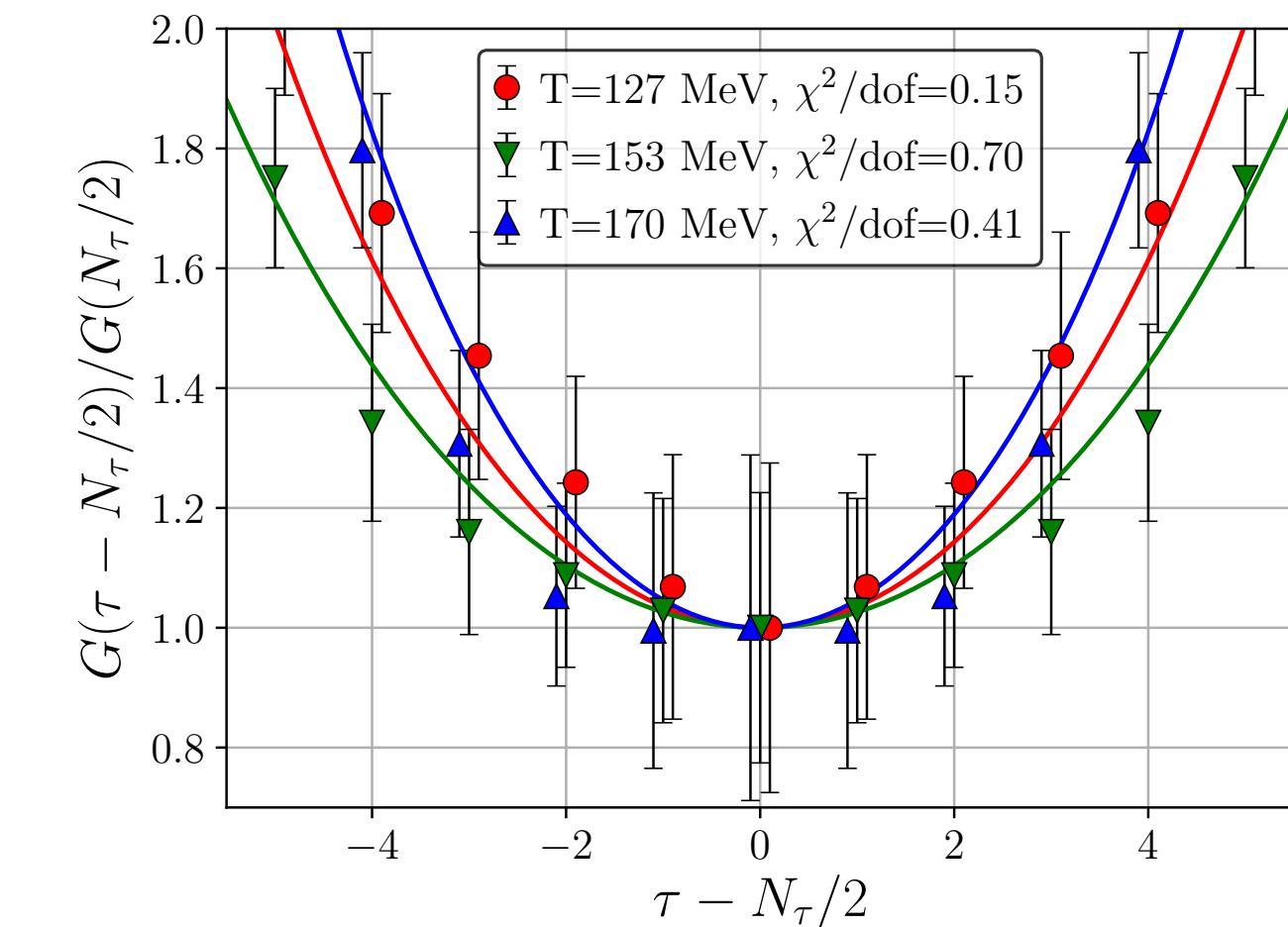
- Noisy of for the extraction of  $\Gamma$
- Assuming that the major part of the spectral function  $\rho(\omega)$  is given by  $\eta'$ , we can estimate  $m_{\eta'}(T)$
- Results suggest that anomalous contribution to  $m_{\eta'}$  disappears at the transition



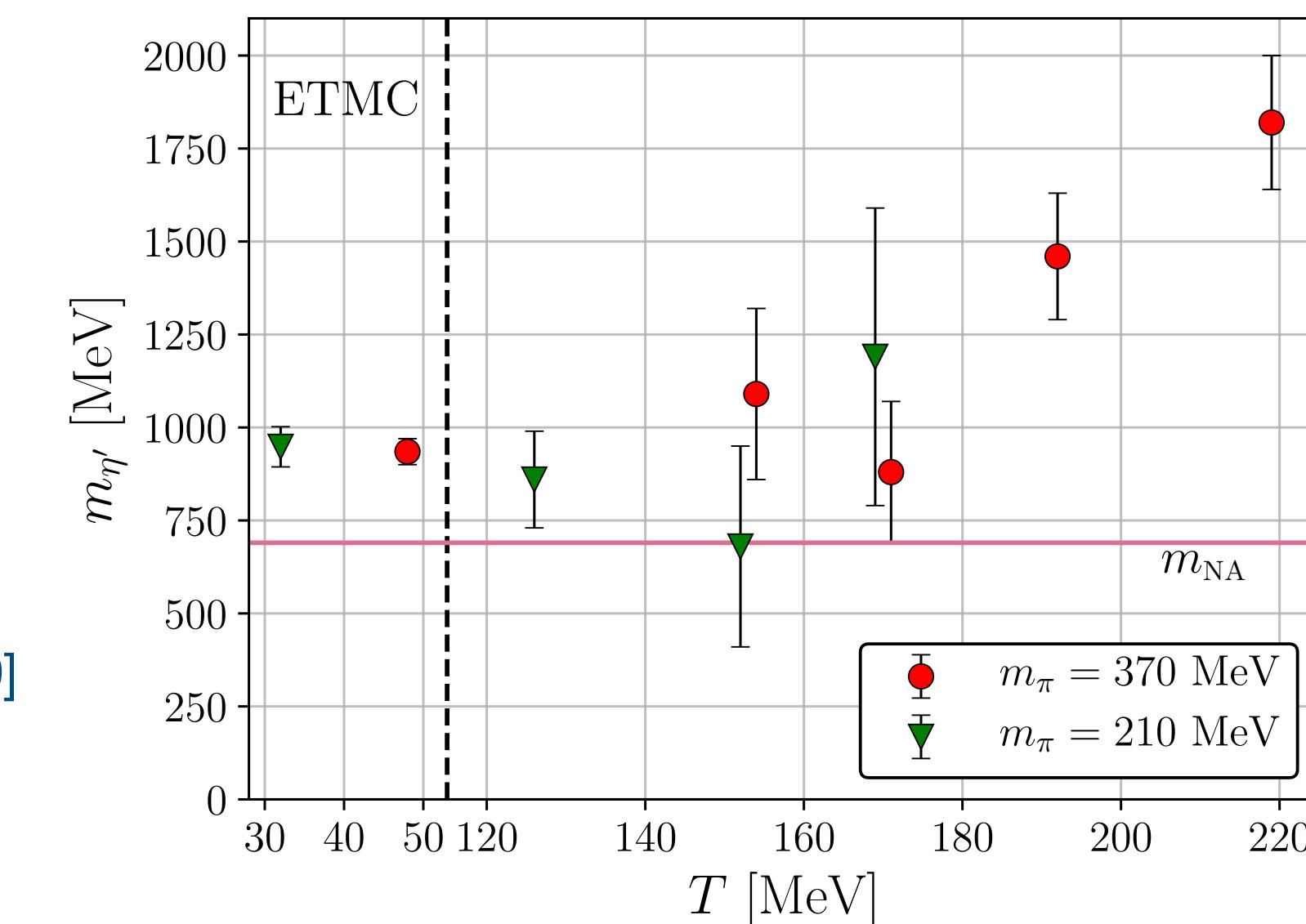
Effective mass vs GF time

$m_\pi = 210 \text{ MeV}$

[AYuK, M.P. Lombardo, A. Trunin, 2019]



$m_\pi = 210 \text{ MeV}, t/a^2 = 1.5, a = 0.065 \text{ fm}$



# Summary

- Applications of the Gradient Flow to topological properties in QCD and QCD-like theories (in hot phase as well)
- GF works for the topological charge correlator and related quantities
  - Sphaleron (thermal) rate in gluodynamics
  - $\eta'$  mass
- GF and topological susceptibility: difficult to make a continuum extrapolation, especially, at high temperatures

