Renormalization group for fractons

Piotr Surówka
Wrocław University of Science and Technology

Collaborators: Kevin Grosvenor, Ruben Lier

Trento, 17.03.2023

Motivation

- Understand new phases of matter. Fractons appear to be elusive new excitations that exhibit mobility restrictions
- Elucidate kinematically constrained systems that have been studied for a long time in statistical physics
- Shed light on dynamics dislocations and disclinations that appear in elasticity and whose macroscopic description is rather crude from a modern effective field theory perspective
- Set a foundation for a new paradigm of field theories, in which low-energy physics is coupled to high-energy physics

Vortices in superfluids

When weakly-interacting bosons condense they form a superfluid, spontaneously breaking global internal U(1)-symmetry. The resulting Goldstone mode is the zero-sound mode of the superfluid, and it is a single free massless mode described by a scalar field. The partition function for the relativistic zero-sound mode is

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-\int_0^\beta d\tau \int d^D x \; \mathcal{L}}$$

$$\mathcal{L} = \frac{1}{2g}\Big(\tfrac{1}{c_{\text{ph}}^2}(\partial_\tau \varphi)^2 + (\partial_m \varphi)^2\Big) \equiv \frac{1}{2g}(\partial_\mu \varphi)^2$$

c_{ph} is the phase velocity of the superfluid condensate

Phase is in fact a compact variable originating from and a change by 2π brings it back to its original value. Such windings of the phase variable correspond to vortices.

To incorporate the vortices, we must treat the phase, having both smooth and singular contributions

$$\varphi(x) = \varphi_{\rm smooth}(x) + \varphi_{\rm sing}(x).$$

$$\oint_{\partial\mathfrak{S}} dx^\mu \partial_\mu \varphi_{\rm sing}(x) = 2\pi N,$$

where $\partial\mathfrak{S}$ is the contour of integration that encloses the singularity, and N is the winding number of the quantized vorticity.

For a vortex of winding number N we have by Stokes' theorem

$$2\pi N = \oint_{\partial\mathfrak{S}} dx^\mu \partial_\mu \varphi = \int_{\mathfrak{S}} dS^\lambda \epsilon_{\lambda\nu\mu} \partial_\nu \partial_\mu (\varphi_{\rm smooth} + \varphi_{\rm sing}) \quad = \int_{\mathfrak{S}} dS^\lambda \epsilon_{\lambda\nu\mu} \partial_\nu \partial_\mu \varphi_{\rm sing} \equiv \int_{\mathfrak{S}} dS^\lambda J^{\rm V}_\lambda.$$

We have defined the vortex current.

To see how vortices interact in the superfluid, we perform the duality operation, which is in fact a Legendre transformation to the relativistic canonical momentum associated with the field-theoretic velocity field

$$\xi_\mu = -i\frac{\partial \mathcal{S}}{\partial(\partial_\mu \varphi)} = -i\frac{1}{g}\partial_\mu \varphi.$$

Now we can go from a Lagrangian in terms of the velocity to a Hamiltonian density in terms of the momentum

$$\mathcal{H} = -i\xi_\mu \partial_\mu \varphi + \mathcal{L} = \frac{g}{2}\xi_\mu^2.$$

We can formally obtain a Lagrangian, but keep the momentum as the principal field

$$\mathcal{L}_{\text{dual}} = \mathcal{H} + i\xi_\mu \partial_\mu \varphi = \frac{g}{2}\xi_\mu^2 + i\xi_\mu \partial_\mu \varphi.$$

Now comes the important step in the duality construction. We again separate the phase field in smooth and singular parts

$$\mathcal{Z} = \int \mathcal{D}\xi_\mu \mathcal{D}\varphi_{\text{smooth}} \mathcal{D}\varphi_{\text{sing}} \; e^{-\int \mathcal{L}_{\text{dual}}}$$

On the smooth part, one is allowed to perform integration by parts

$$\xi_\mu \partial_\mu \varphi_{\text{smooth}} \to -(\partial_\mu \xi_\mu)\varphi_{\text{smooth}}$$

then integrate out the smooth part in the path integral as a Lagrange multiplier field, producing the constraint

$$\partial_\mu \xi_\mu = 0.$$

We obtain the conserved current of the superfluid. The constraint can be enforced explicitly by expressing it as the curl of another vector field

$$\xi_\mu(x) = \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda(x).$$

Particle-vortex duality

We substitute the condition to obtain the final form of the dual Lagrangian

$$\mathcal{L}_{\text{dual}} = \frac{g}{2}(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)^2 + i(\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda)\partial_\mu\varphi_{\text{sing}}$$
$$= \frac{g}{4}(\partial_\nu b_\lambda - \partial_\lambda b_\nu)^2 + i b_\lambda J^{\text{V}}_\lambda .$$

We introduced a gauge redundancy since the Lagrangian is invariant under the addition of the gradient of any smooth scalar field

$$b_\lambda(x) \to b_\lambda(x) + \partial_\lambda \alpha(x).$$

XY- sine-Gordon duality

The phenomenon of different systems exhibiting the same critical behavior is called the universality, and its explanation requires the renormalization group framework.

$$\mathcal{Z} = \prod_i \int_0^{2\pi} \exp\left(K \sum_i \cos(\theta_i - \theta_{i+1}) \right) \qquad K \equiv \frac{J}{T}$$

Now various technical steps follow based on mathematical identities and approximations

$$\exp(\beta \cos\theta) \approx \exp(\beta) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{2}\beta(\theta - 2\pi n)^2 \right)$$

We use Hubbard-Stratonovich transformation

$$\sum_{n=-\infty}^{\infty} \exp\left[-K(\theta - 2\pi n)^2/(2\pi)\right] = \frac{1}{\sqrt{2K}} \sum_{p=-\infty}^{\infty} \exp(-\pi p^2/(2K) - ip\theta)$$

In this representation integrals factorize and we get a constraint

$$\Delta_\mu p_{im} = 0 \to p_{i\mu} = \epsilon_{\mu\nu}\Delta_\nu m_i$$

We pass to the continuous variables, apply Poisson formula and introduce a chemical potential

$$\sum_m e^{m^2} = \int_{-\infty}^{\infty} d\phi e^{\phi^2} \sum_m \delta(\phi - m) = \sum_m \int_{-\infty}^{\infty} d\phi e^{\phi^2} e^{m^2 \ln y + im\phi} \Big|_{y=1}$$

Now we cheat a bit and expand in a small chemical potential, arguing that the mistake should not be large. The result is up to $\mathcal{O}(y^2)$

$$\mathcal{Z}_{sG} = \int_{-\infty}^{\infty} \prod_i d\phi_i \exp\left(-\frac{1}{2K} \sum_{i,\mu} (\Delta_\mu \phi_i)^2 - 2y \sum_i \cos(2\phi_i) \right)$$

Renormalisation and flow equations

The critical behavior of the continuum sine-Gordon theory may be understood by using renormalization group framework applying momentum-shell transformations

$$\phi(x) = \phi_<(x) + \phi_>(x),$$

where as usual $\phi_<(x)$ contains only the Fourier components with $k < \Lambda/b$, and $\phi_>(x)$ only with $\Lambda/b < k < b$.

$$g_>(x) = (2\pi)^2 \langle \phi_>(x)\phi_>(0) \rangle = \frac{1}{T} \int_{\Lambda/b}^{b} \frac{dk}{k} \int_0^{2\pi} d\alpha e^{ikx\cos(\alpha)}$$

Still divergent and we need a soft cut-off $\int_0^{\Lambda} dk \to \int_0^{\infty} dk \frac{\Lambda^2}{k^2+\Lambda^2}$

RG equations

$$\frac{d\hat{y}}{d\ln(b)} = \left(2 - \frac{\pi}{T}\right)\hat{y} + \mathcal{O}(\hat{y}^3),$$

$$\frac{dT}{d\ln(b)} = \frac{\hat{y}^2}{2T} + \mathcal{O}(\hat{y}^4).$$

$$\hat{y} = (4\pi)^2 y/\Lambda$$

Berezinskii-Kosterlitz-Thouless transition

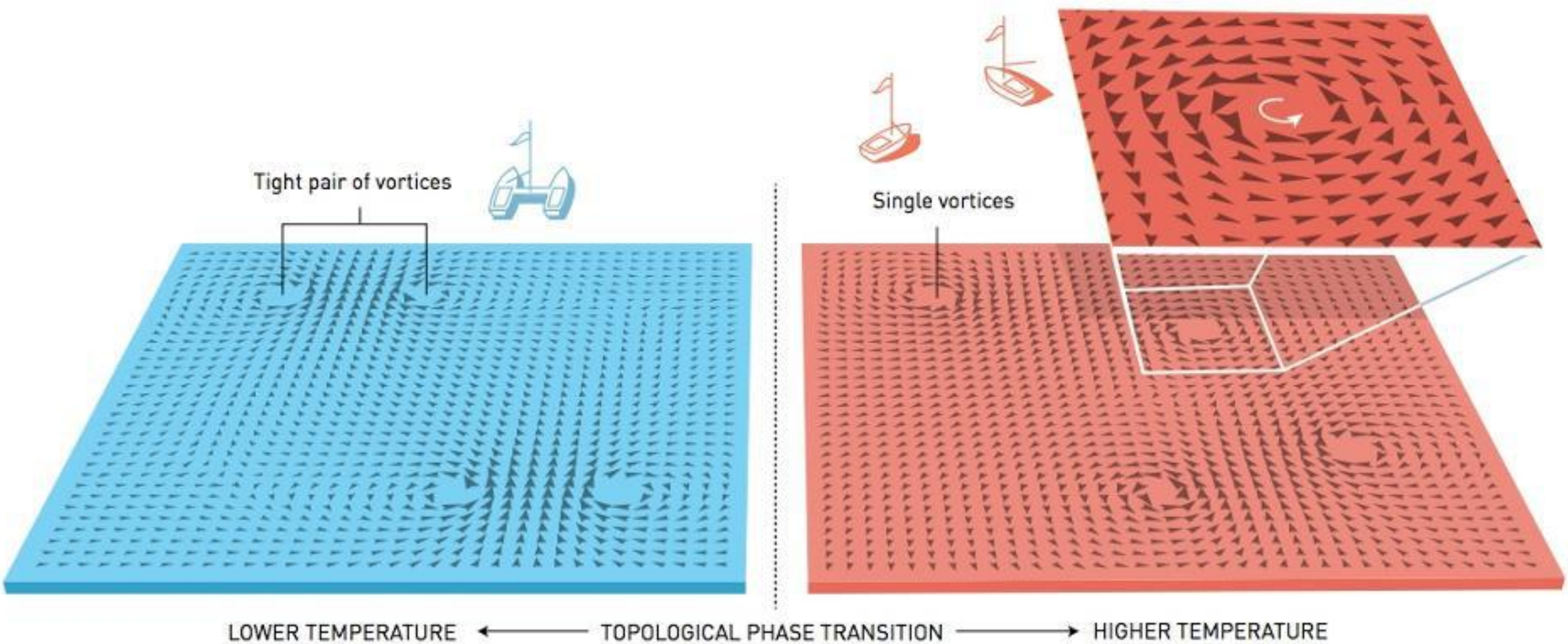
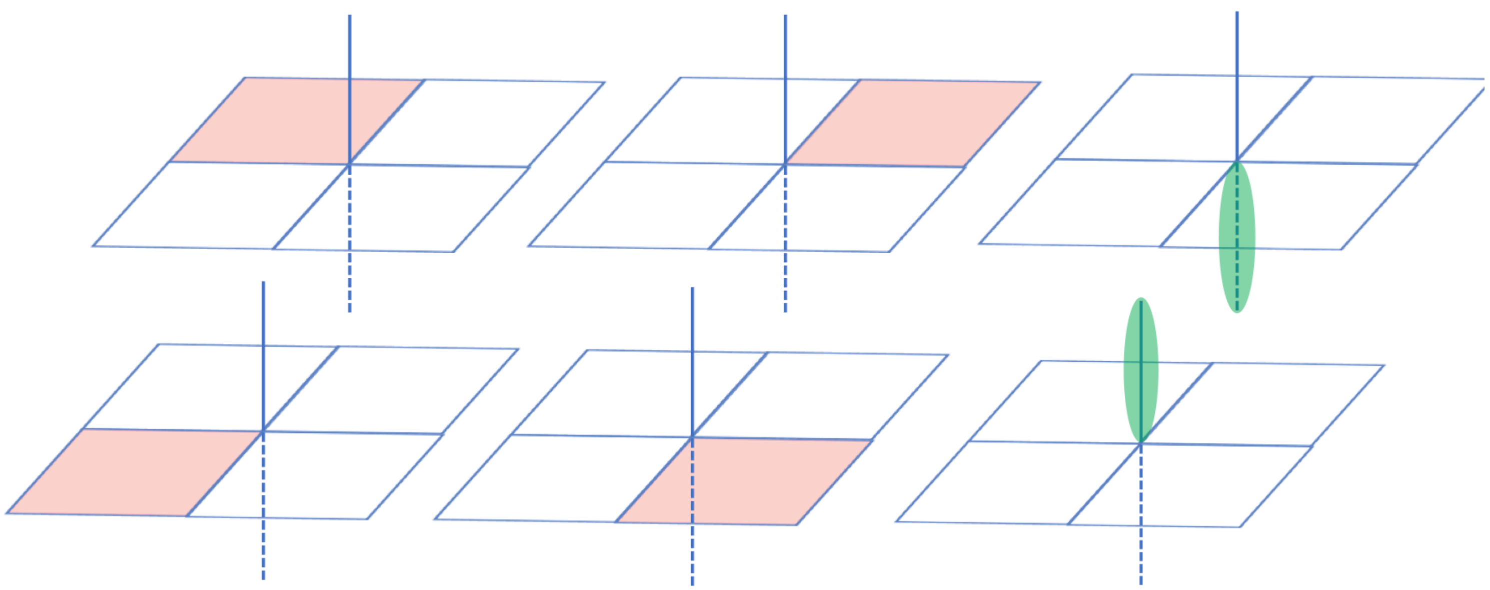


Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

Fractons in plaquette models

A natural question to ask is what are the macroscopic signatures of fractons. One recent suggestion is that fractons are realised in close-packed tilings models. This is motivated by previous works on dimer models where constraints on the dimers can be interpreted as the Gauss law.



Close-packed tilings of the cubic lattice. Each site is either part of a plaquette (red) in the x-y plane or a z-dimer (green).

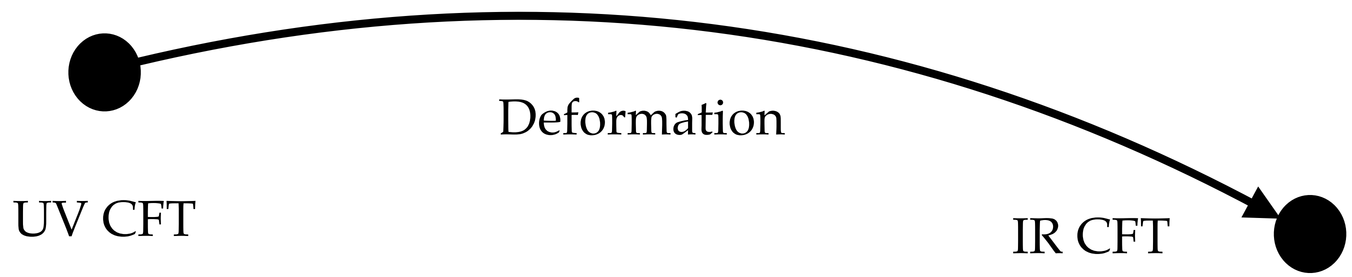
You, Moessner

$$\mathcal{Z} = \int \mathcal{D}E_z \, \mathcal{D}E_{xy} \, e^{-\kappa(E_z^2 + E_{xy}^2)}$$

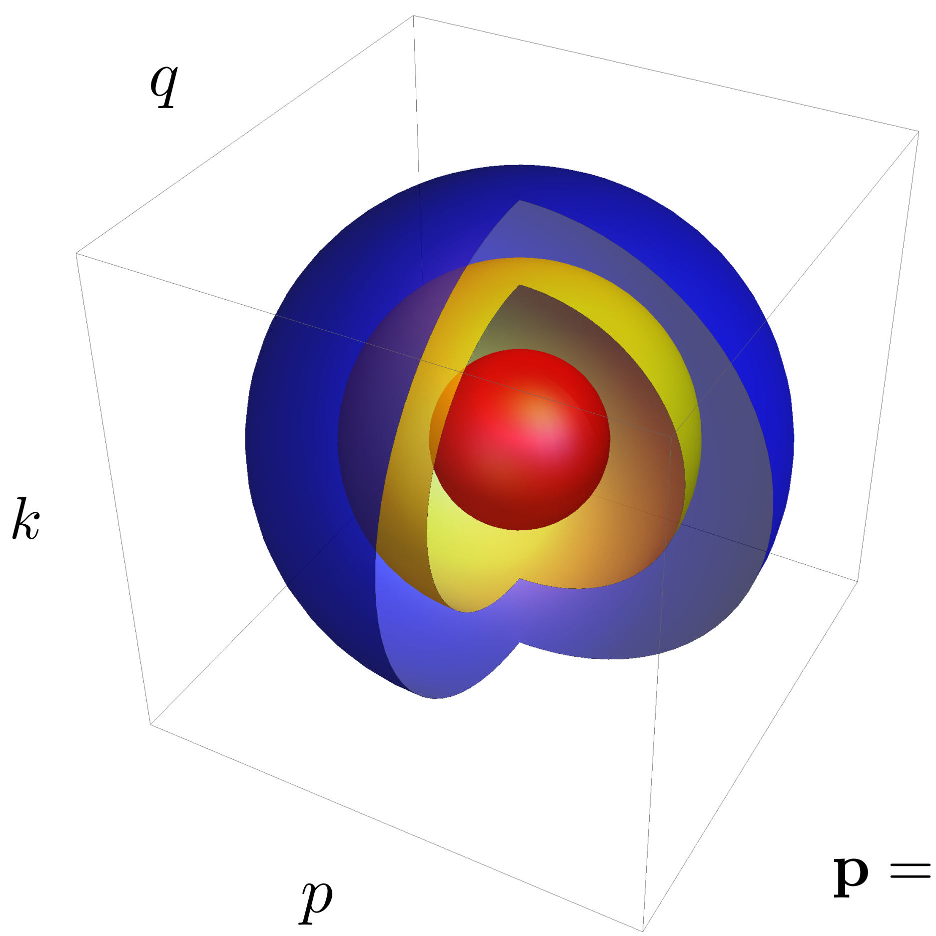
$$= \int \mathcal{D}h \, e^{\frac{-\kappa}{2}[(\partial_x \partial_y h)^2 + (\partial_z h)^2] + 2\alpha_0 \cos(2\pi h) + 2\alpha_x \cos(2a_x \partial_x h) + 2\alpha_y \cos(2\pi a_y \partial_y h)}$$

h is a discrete integer-valued field living on the dual lattice at the center of each cube which characterizes the local fluctuation of the dimer-plaquette pattern.

How to do RG for fractons



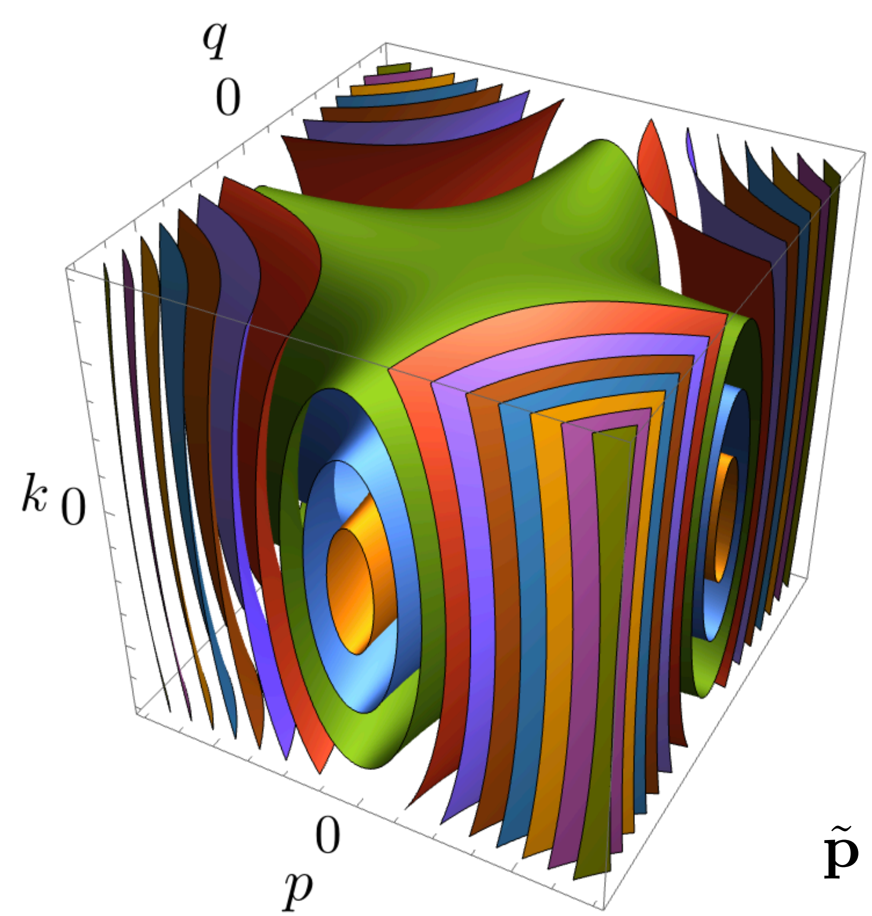
Momentum shells for ordinary EFTs



$$\mathbf{p} = (p,\, q,\, k)^{\intercal}$$

$$D = \mathbf{p} \cdot \nabla_{\mathbf{p}}$$

Momentum shells for fraction EFTs



$$[p] = \frac{q^2}{p^2+q^2}\,, \quad [q] = \frac{p^2}{p^2+q^2}\,, \quad [k] = 1$$

$$\tilde{\mathbf{p}} = \left([p]\, p,\, [q]\, q,\, [k] k\right)^{\intercal}$$

$$\tilde{D} = \tilde{\mathbf{p}} \cdot \nabla_{\mathbf{p}}$$

Lake

Grosvenor, Lier, Surówka

Modified dilatation operator and the corresponding dimensional analysis

For the simple cosine term we have

$$\lim_{x'\to x}(2\pi)^2\big\langle h(x)h(x')\big\rangle = (2\pi)^2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\kappa\big[(pq)^2+k^2\big]} \quad ,$$

where $\mathbf{p}=(p,q,k)$ is the momentum vector in three dimensions. Let us pass to dimensionless variables by defining

$$\tilde{p}=\frac{a_x}{\pi}\,p \quad , \qquad \tilde{q}=\frac{a_y}{\pi}\,q \quad , \qquad \tilde{k}=\frac{a_x a_y}{\pi^2}\,k \quad .$$

In this way, the Brillouin zone in the pq-plane is translated into the square $|\tilde{p}|,|\tilde{q}|\le 1$. The expression of the integral actually remains the same, just with tildes on the variables $\frac{1}{2\pi\kappa}\int\frac{d^3\tilde{\mathbf{p}}}{(\tilde{p}\tilde{q})^2+\tilde{k}^2}$.

The shell goes from $\sqrt{(pq)^2+k^2}=\frac{\Lambda}{b}$ to Λ, which, in these dimensionless variables corresponds to

$$\frac{\tilde{\Lambda}}{b}\le\sqrt{(\tilde{p}\tilde{q})^2+\tilde{k}^2}\le\tilde{\Lambda} \quad , \qquad \tilde{\Lambda}=\frac{\Lambda a_x a_y}{\pi^2} \quad .$$

After the integration, which again requires a smooth cut-off

$$\left.\frac{d\alpha_0(b)}{d\log(b)}\right|_{b=1} = \left[2 - \frac{1}{\kappa}\log\left(\frac{2}{\tilde{\Lambda}}\right)\right]\alpha_0 \quad .$$

Since $\tilde{\Lambda} \ll 1$, α_0 decays very quickly under the RG flow towards the infrared. Thus, around the fractonic fixed point, this operator is irrelevant. This operator does not destabilize the fractonic phase.

For the deformation by the cosine gradient term $\alpha_x \cos(2\pi a_x \partial_x h)$ we have

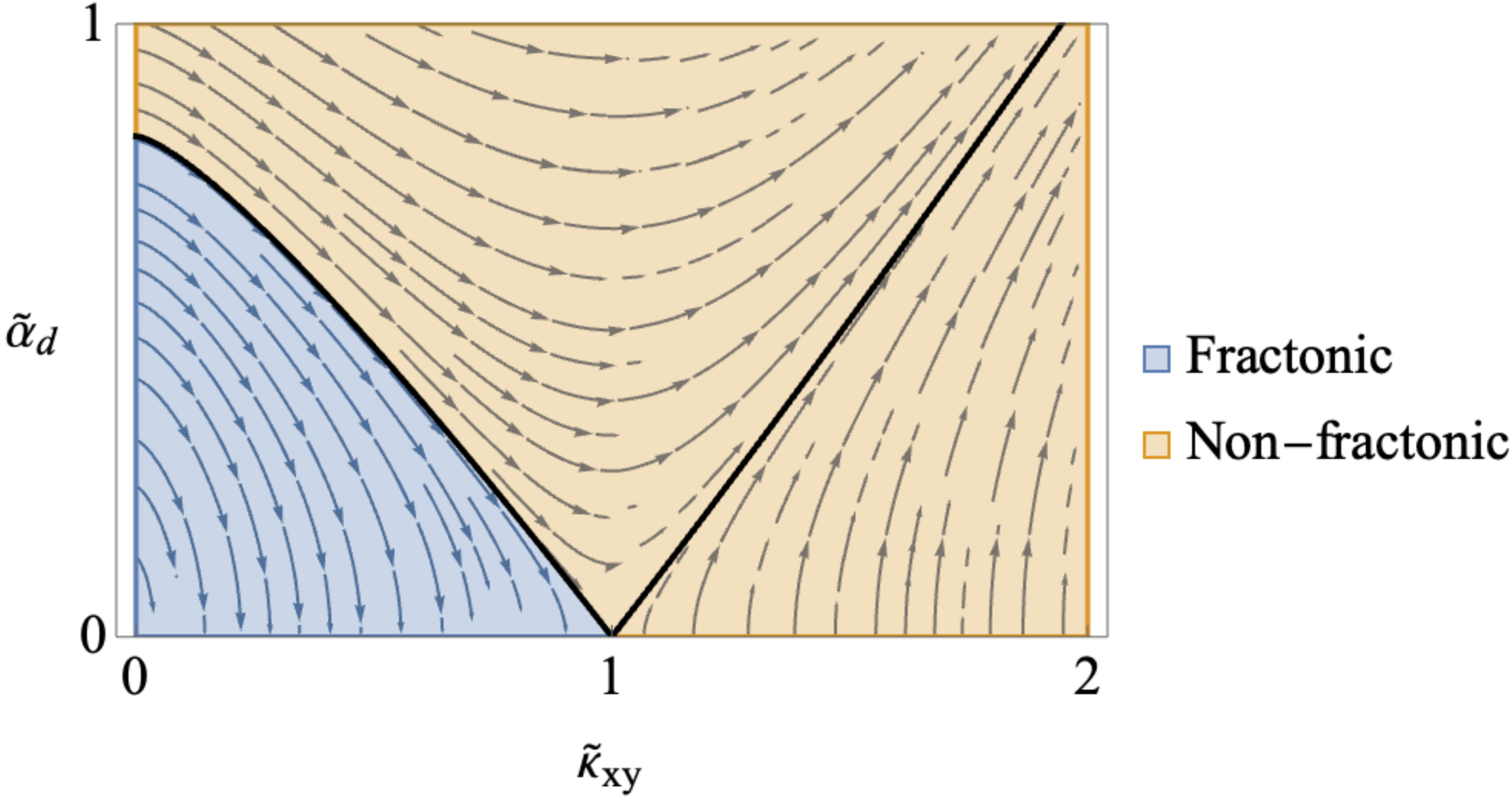
$$\lim_{x'\to x}(2\pi)^2\big\langle a_x\partial_x h(x)\, a_x\partial_{x'} h(x')\big\rangle = (2\pi)^2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{(a_x p)^2}{\kappa\big[(pq)^2 + k^2\big]} \quad .$$

This leads to the following RG equation

$$\left.\frac{d\alpha_x(b)}{d\log(b)}\right|_{b=1} \approx \left(2 - \frac{\pi^2}{2\kappa}\right)\alpha_x \quad ,$$

and a critical value for κ. In other words, this operator does destabilize the fractonic phase when $\kappa > \kappa_c$. This is precisely the process of dipole proliferation and a fractonic analog of the BKT transition.

Fractonic Berezinskii-Kosterlitz-Thouless transition



Conclusions

- Dualities are a convenient tool to study fracton theories
- New universally class and fractonic phase transition
- Fractons require a new approach to RG
- Playground for numerical methods in dimer models as we now have quantitative predictions

Thank you!