On conductivities in anomalous hydrodynamics

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Introduction: anomalous conductivities

- We have seen that hydrodynamic frame can be more subtle than previously thought[1].
- Consider the $U(1)_A$ model

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \ , \qquad \nabla_\mu J^\mu = c E_\mu B^\mu \ , \tag{1}$$

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma \ . \tag{2}$$

- Work with a fluid at finite T and axial chemical potential μ.
- We take $E^\mu, B^\mu \sim \mathcal{O}(\partial^1)$ in arbitrary hydrodynamic frame and compute conductivities.
- Physical observables must be frame independent.

[1] Many papers by authors in this room and their collaborators e.g. arXiv: 2210.15605, Phys. Rev. D. 106.066023, Phys. Rev. Lett. 126, 222301 etc. Sorry if I missed yours.

A class of hydrodynamic frames

- If $B \sim \mathcal{O}(\partial^1)$ frame transformation

$$u^\mu \to u^\mu + f_B(\mu, T)B^\mu + f_\Omega(\mu, T)\Omega^\mu \tag{3}$$

- Constitutive relations covariant under this frame transformation

$$\begin{aligned} T^{\mu\nu} =& \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \xi_B^\epsilon (u^\mu B^\nu + u^\nu B^\mu) \\ &+ \xi_\Omega^\epsilon (u^\mu \Omega^\nu + u^\nu \Omega^\mu) - \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\alpha\beta} \\ &- \zeta \Delta^{\mu\nu}\partial_\alpha u^\alpha \ , \\ J^\mu =& n u^\mu + \sigma_0 \Delta^{\mu\nu}\left(E_\nu - T\partial_\nu \frac{\mu}{T}\right) + \xi_\Omega \Omega^\mu + \xi_B B^\mu \ . \end{aligned}$$

- For these frames

$$\begin{aligned} T^{\mu\nu}u_\nu &= -(\epsilon u^\mu + \xi_B^\epsilon B^\mu + \xi_\Omega^\epsilon \Omega^\mu) \ , & \text{(5a)} \\ J^\mu u_\mu &= -n \, . & \text{(5b)} \end{aligned}$$

Entropy positivity

- Positivity gives four constraints[2], but we have six anomalous transport coeffs. ξ

$$\partial_\mu S^\mu = \left(\frac{\partial \xi_B^s}{\partial T} + \frac{\mu}{T}\frac{\partial \xi_B^s}{\partial \mu} - \frac{\xi_B^\epsilon}{T^2}\right) B^\mu \left(\partial_\mu^\perp T + T a_\mu\right) + \ldots \geq 0 \,. \tag{6}$$

- We have employed zeroth order equations to simplify.
- Landau frame supplies additional condition and we find

$$\xi_B^\epsilon = 0 \qquad \xi_B = c\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + p}\right) \,, \tag{7a}$$

$$\xi_\Omega^\epsilon = 0 \qquad \xi_\Omega = c\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + p}\right) \,. \tag{7b}$$

[2] Just follow original calculation of Son & Surówka paper (Phys. Rev. Lett. 103, 191601) with more coefficients.

Linearised hydrodynamics

- Linearised hydrodynamics with first order constitutive relations

$$\left(D + \mathcal{O}(\partial^3)\right) X = S + \mathcal{O}(\partial^3) \tag{8}$$

where $X = (\delta T, \delta\mu, \delta v^i)$.

- Then we have something like

$$\delta J = \left(M + \mathcal{O}(\partial^2)\right) \delta X \; . \tag{9}$$

- Typical Green's function

$$G_{\mathrm{R}} \sim \frac{a(\omega, k) + \mathcal{O}(\varepsilon^2)}{b(\omega, k) + \mathcal{O}(\varepsilon^3)} \tag{10}$$

where ε is a small counting parameter $(\omega, k, B \sim \varepsilon)$.

Conductivities

- Set $B^z = Be^z$ then $\sigma(\omega)$

$$\sigma_0 + \frac{in^2}{\omega w} + \frac{iB^2}{\omega w^2(\frac{\partial\epsilon}{\partial T}\frac{\partial n}{\partial\mu} - \frac{\partial\epsilon}{\partial\mu}\frac{\partial n}{\partial T})} \times$$
$$\left[\left(w\frac{\partial\xi_B}{\partial\mu} - n\frac{\partial\xi_B^\epsilon}{\partial\mu}\right)\left(w\left(c\frac{\partial\epsilon}{\partial T} - \frac{\partial n}{\partial T}\xi_B\right) - \frac{\partial\epsilon}{\partial T}n\xi_B + 2\frac{\partial n}{\partial T}n\xi_B^\epsilon\right)\right.$$
$$\left. - \left(w\frac{\partial\xi_B}{\partial T} - n\frac{\partial\xi_B^\epsilon}{\partial T}\right)\left(w\left(c\frac{\partial\epsilon}{\partial\mu} - \frac{\partial n}{\partial\mu}\xi_B\right) - \frac{\partial\epsilon}{\partial\mu}n\xi_B + 2\frac{\partial n}{\partial\mu}n\xi_B^\epsilon\right)\right] .$$

- Unsurprisingly some of the result is frame dependent.
- Perform $\omega, B \sim \varepsilon$:

$$\sigma_{\mathrm{AC}}(\omega) = \sigma_0 + \frac{in^2}{(\epsilon + P)\omega} + (\text{untrustworthy stuff}) . \qquad (11)$$

Equilibrium generating functional

- If we want anomalous contributions to σ_{AC} then... $B \sim \mathcal{O}(\partial^0)$.
- Generating functional

$$W[g, A, F] = \int d^{3+1}x \sqrt{-g}\, p(T, \mu, B^2)\ . \qquad (12)$$

- Canonical definition of quantities

$$T = \frac{T_0}{\sqrt{-V^2}}\ , \qquad \mu = \frac{A_\mu V^\mu + \Lambda_V}{\sqrt{-V^2}}\ . \qquad (13)$$

- No frame redefinitions by B and we find that the following anomalous transport coefficients

$$\xi_B^\epsilon = \frac{1}{2} c\mu^2\ , \qquad \xi_B = c\mu\ . \qquad (14)$$

Conductivities

- At small frequency[3] we find

$$\sigma(\omega) = \frac{i}{\omega}\left[\frac{n^2}{(p+\epsilon)} + \Xi B^2\right] + \mathcal{O}(\omega^0) \ , \tag{15a}$$

$$\alpha(\omega) = \frac{i}{\omega}\left[\frac{ns}{(p+\epsilon)} - \mu\Xi B^2\right] + \mathcal{O}(\omega^0) \ , \tag{15b}$$

$$\kappa(\omega) = \frac{i}{\omega}\left[\frac{s^2 T}{(p+\epsilon)} + \frac{\mu^2\Xi B^2}{T}\right] + \mathcal{O}(\omega^0) \ , \tag{15c}$$

  where $\Xi = \Xi(T, \mu, B^2)$.
- Dependence of conductivities on B is not purely quadratic.
- There is no transformation that gives the pure Landau frame.

[3] Thanks for the help guys! (JHEP 78 (2021))

Step change: energy and charge relaxation

- DC conductivities are not finite due to unbroken translation invariance - add relaxation!
- Momentum relaxation is not enough for anomalous hydro.
- Let's try a deeper study of charge, energy and momentum relaxation in a relativistic theory.
- Caveat - extra transport coefficients can appear as we are breaking Lorentz (+more) - for future work[4].
- Moreover, anomaly is tangential to what we find.

[4] Shameless advertisement for one of my papers which tackles these issues in one specific way- arXiv: 2211.05791.

Relaxed hydro equations; Onsager/entropy

- The relaxed equations (minus momentum)

$$\partial_t \delta\varepsilon + \ldots = -\left(\frac{1}{\tau_{\varepsilon\varepsilon}}\delta\varepsilon + \frac{1}{\tau_{\varepsilon n}}\delta n\right) , \quad (16a)$$

$$\partial_t \delta n + \ldots = -\left(\frac{1}{\tau_{n\varepsilon}}\delta\varepsilon + \frac{1}{\tau_{nn}}\delta n\right) . \quad (16b)$$

- Onsager[5] implies

$$\frac{\chi_{\varepsilon\varepsilon}}{\tau_{n\varepsilon}} - \frac{\chi_{\varepsilon n}}{\tau_{\varepsilon\varepsilon}} + \frac{\chi_{n\varepsilon}}{\tau_{nn}} - \frac{\chi_{nn}}{\tau_{\varepsilon n}} = 0 . \quad (17)$$

- Entropy positivity

$$T_{(0)}\partial_\mu \delta s^\mu = \delta\varepsilon\left(\frac{\mu}{\tau_{n\varepsilon}} - \frac{1}{\tau_{\varepsilon\varepsilon}}\right) + \delta n\left(\frac{\mu}{\tau_{nn}} - \frac{1}{\tau_{\varepsilon n}}\right) \geq 0 . \quad (18)$$

[5] Partial results were already known e.g. JHEP 05 (2019) 206.

Relaxed hydro equations; Linearised stability and integrability

- Onsager+entropy positivity

$$\frac{\partial \varepsilon}{\partial T}\frac{1}{\tau_{\varepsilon\varepsilon}} + \frac{\partial n}{\partial T}\frac{\mu}{\tau_{nn}} = 0 \,. \tag{19}$$

- Linearised modes

$$\omega = 0\,, \qquad \omega = -\frac{i}{\tau_{nn}}\left(\frac{\frac{\partial \varepsilon}{\partial T} - \mu\frac{\partial n}{\partial T}}{\frac{\partial \varepsilon}{\partial T}}\right)\,. \tag{20}$$

- Integrability

$$\frac{1}{\tau_{nn}(T,\mu)} = \frac{f(\mu)\frac{\partial \varepsilon}{\partial T}}{\frac{\partial n}{\partial \mu}\frac{\partial \varepsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial \varepsilon}{\partial \mu}}\,. \tag{21}$$

Fixing variational method

- Naive covariantisation + background field method

$$\langle T^{tt} T^{xx} \rangle - \langle T^{xx} T^{tt} \rangle \big|_{k=0} \neq 0 \,. \tag{22}$$

- Background independence of theory is broken.
- Solution:

$$\text{energy:} \quad -\left(\frac{1}{\tau_{\varepsilon\varepsilon}} \delta\varepsilon + \frac{1}{\tau_{\varepsilon n}} \delta n \right) - c_{\varepsilon,1} \delta h_{tt} - r_{\varepsilon,1} \delta A_t \tag{23a}$$

$$\text{charge:} \quad -\left(\frac{1}{\tau_{n\varepsilon}} \delta\varepsilon + \frac{1}{\tau_{nn}} \delta n \right) - c_{n,1} \delta h_{tt} - r_{n,1} \delta A_t \tag{23b}$$

- No additional constraints on Martin-Kadanoff.

Summary

Part I:

- For derivative order magnetic fields $B \sim \mathcal{O}(\partial^1)$ anomalous terms drop from the conductivity.
- For $B \sim \mathcal{O}(\partial^0)$ the anomalous transport coefficients take the thermodynamic frame values.
- Reported conductivities.

Part II:

- Charge relaxation + constraints $\Rightarrow$ energy relaxation + mixed.
- Unique linearised, curved space equations of motion consistent with time reversal covariance.

Thanks for listening!

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