Hydro+

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What is hydrodynamics?

- Coarse-grained finite T QFT.

  Physics: evolution towards equilibrium.

  Time-scale hierarchy:

  1) local thermodynamic equilibration – fast;

  2) achieving uniformity – slow.

Hydrodynamics – the description of that slower process.

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- This time-scale hierarchy is controllable (by wavenumber k).

  The hydro degrees of freedom are controllably slow at long wavelengths (small k) because they are conserved.

- Let us focus on this scale hierarchy
  and see what happens when it breaks, controllably.

Parametrically slow local relaxation processes

What if there is a local process (or processes),
whose relaxation rate Γ_s is *finite*
but *parametrically*, i.e., controllably, small
compared to typical local rate Γ?

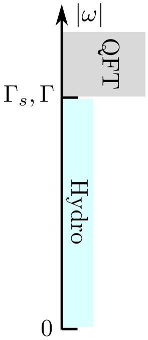
$$\Gamma_s \ll \Gamma$$

Examples:

- slow chemical processes
- slow e/w processes in QGP
- approximately conserved charge (axial, isospin, etc.)
- would-be Goldstone field of spontanesouly broken approximate symmetry
- spin-orbit interaction for nonrelativistic spin
- equilibration at the critical point

Two regimes

- Hydrodynamic regime: $\omega \ll \Gamma_s, \Gamma$.
  Breaks down when $\omega \sim \Gamma_s$.



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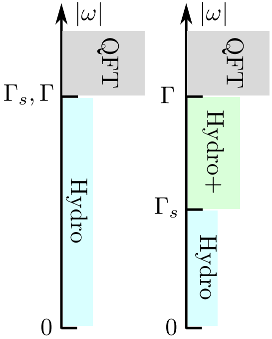
- Hierarchy $\Gamma_s \ll \Gamma$ creates another regime

  $\Gamma_s \ll \omega \ll \Gamma$

  for an effective theory with additional degree(s) of freedom, a.k.a. Hydro+
  (Hydro+ description is also valid
  in hydrodynamic regime, of course.)



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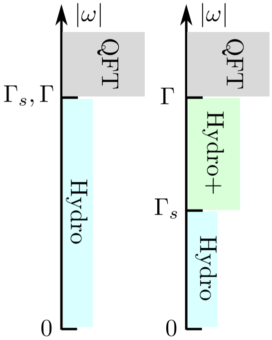

- Unlike hydro variables these additional d.o.f. are not diffusive, i.e., they relax locally. But *very* slowly.
  - E.g., Israel-Stewart hydrodynamics is *not* a Hydro+ theory unless the rate $1/\tau_\Pi$ is *parametrically/controllably* small.

Hydro+ dynamics

MS, Yin, 1712.10305

- Let us introduce generic notation ϕ for such local field.
- On fast time scales $\omega \gg \Gamma_s$: driven by entropy $s_{(+)}(\varepsilon, \phi)$.

  $ds_{(+)} = \beta d\varepsilon - \pi d\phi$, π – thermodynamic “force”.

  $\pi = 0$ in equilibrium and $s_{(+)}(\varepsilon, \phi)|_{\pi=0} = s(\varepsilon)$.

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  $\pi = 0$ in equilibrium and $s_{(+)}(\varepsilon, \phi)|_{\pi=0} = s(\varepsilon)$.
- New equation of motion (Second Law dictates $\gamma_s \geq 0$):
  $(u \cdot \partial)\phi = -\gamma_s \pi +$ gradients
  i.e. ϕ relaxes to equilibrium ($\pi = 0$) at rate $\Gamma_s = \gamma_s/\chi_s$,
  where $\chi_s = (\partial\phi/\partial\pi)_\varepsilon$ is susceptibility.

Bulk viscosity

- However, the equilibrium value of ϕ could depend on ε, which evolves:

  $(u \cdot \partial)\varepsilon = -w(\partial \cdot u)$.

  As a result bulk mode (expansion $\partial \cdot u$) and ϕ mix:

  $$(u \cdot \partial)\pi = -\Gamma_s \pi - \frac{\beta p_\pi}{\chi_s}(\partial \cdot u)$$

  where $p_\pi \equiv (\partial p_{(+)}/\partial \pi)_\varepsilon = -Tw(\partial\phi/\partial\varepsilon)_\pi$ (Maxwell relation).

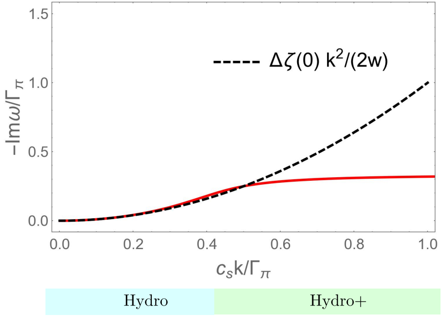
- Therefore, pressure is kept away from equilibrium:

  $$p_{(+)} = p + p_\pi \pi = p - \frac{\beta p_\pi^2}{\chi_s} \frac{1}{\Gamma_s - i\omega} \underbrace{(\partial \cdot u)}_{\text{expansion rate}}$$

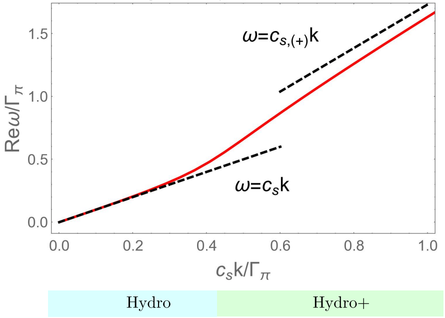
- Bulk viscosity: $\Delta\zeta = \dfrac{\beta p_\pi^2}{\chi_s \Gamma_s}$. In Hydro+: $\Delta\zeta(\omega) = \dfrac{\beta p_\pi^2}{\chi_s} \dfrac{\Gamma_s}{\Gamma_s^2 + \omega^2}$.

Sound attenuation and sound speed

Attenuation rate



Sound frequency



When sound frequency and attenuation rate are both $\sim \Gamma_s$, hydrodynamics breaks down. Crossover to Hydro+ regime.

Attenuation rate $\Delta\zeta(\omega)k^2$ saturates, since $\Delta\zeta(\omega) \sim 1/\omega^2$.

Equation of state "stiffens": $c^2_{s,(+)} = c^2_s + \Delta c^2_s > c^2_s$.

Fluctuations and Hydro+

- Fluctuations at wavenumber q equilibrate at rate $\Gamma_s(q) \sim q^2$.

  They act as Hydro+ variables. Continuous spectrum of them.

- Each q contributes to $\Delta\zeta$ and the total contribution is non-analytic in ω due to small-q fluctuations:

$$\Delta\zeta(\omega) \sim \int d^3q \, \frac{\Gamma_s(q)}{\Gamma_s(q)^2 + \omega^2} \sim \underbrace{\Delta\zeta(0)}_{\xi^3 \text{ near C.P.}} - \underbrace{\mathcal{O}\left(\omega^{1/2}\right)}_{\text{long-time tail}}$$

- Hydrodynamics with fluctuations ("hydrokinetics") is a Hydro+ theory which accounts for these effects.

- Note: these non-analytic contributions to dissipation are of order $\Delta\zeta k^2 \sim \omega^{1/2} k^2 \sim k^{3/2} \gg k^2$.

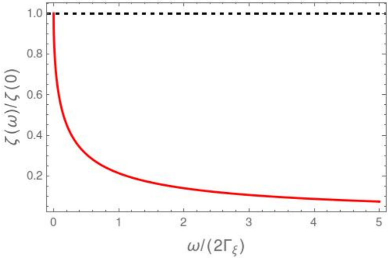
  I.e., fluctuation effects are larger than 2nd order gradients.

Real-time bulk response: hydro vs hydro+

Characteristic Hydro-to-Hydro+ crossover rate $\Gamma_\xi = D\xi^{-2} \sim \xi^{-3}$.

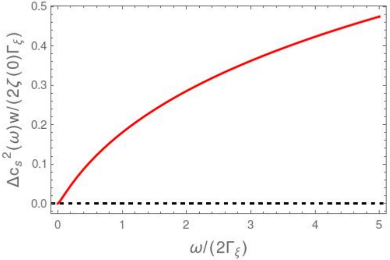
- Dissipation is overestimated in hydro (- - -):

  Only modes with $\omega \ll \Gamma_\xi$ experience large ζ.



- Stiffness of eos (sound speed) is underestimated (- - -):

  Only modes with $\omega \ll \Gamma_\xi$ are critically soft ($c_s \to 0$ at CP).



Approximately conserved axial charge

- Constitutive equations: *MS, Yin, 1712.10305*

$$\partial_t n_V = \underbrace{\mathcal{O}(\nabla^2)}_{\text{diffusion}}; \qquad \partial_t n_A = \underbrace{-\gamma_A \mu_A}_{\text{local relaxation}} + \mathcal{O}(\nabla^2)$$

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- But consider magnetic field $\boldsymbol{B}$

  $$\partial_t n_V = -\boldsymbol{\nabla}(C\mu_A \boldsymbol{B}); \quad \partial_t n_A = -\gamma_A \mu_A - \boldsymbol{\nabla}(C\mu_V \boldsymbol{B});$$

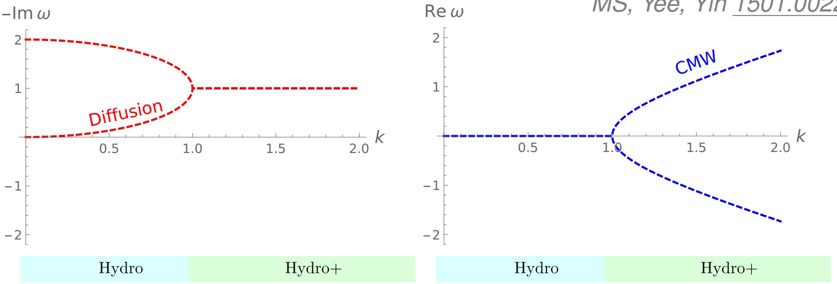
  Now n_A mixes with n_V. Well-known CMW phenomenon.

- The slow non-hydro rate is $\Gamma_s \equiv \Gamma_A = \gamma_A/\chi_A$.

  When $v_{\rm CMW} k_\parallel \gg \Gamma_s$: $\quad \omega_{\rm CMW} = \pm v_{\rm CMW} k_\parallel - i\Gamma_s/2$.

Two regimes

MS, Yee, Yin 1501.00222



- $\omega \ll \Gamma_s$ – hydrodynamic regime (proper).

  Diffusion: $\omega_V = -i \underbrace{\frac{v_{\text{CMW}}^2}{\Gamma_s}}_{\Delta D} k^2$, and non-hydro mode $\omega_A = -i\Gamma_s$.

- Similarity: $\Delta\zeta = w\frac{\Delta c_s^2}{\Gamma_s}$ and $\Delta\lambda = \chi_V \Delta D = \chi_V \frac{\Delta v_{\text{CMW}}^2}{\Gamma_s}$.

Spin hydrodynamics

- σ^i – spin density. EOM: $\partial_t \sigma^i = \epsilon^{ijk}\Theta^{[jk]}$.

  Stress tensor has antisymmetric part:

  $$\Theta^{ij} = p\delta^{ij} - \ldots - \underbrace{\eta_s}_{\text{rotational viscosity}} \left(\underbrace{\partial^{[i}v^{j]}}_{\text{vorticity}} - \underbrace{\mu^{ij}}_{\text{spin chem. potential}} \right)$$

  $\partial\mu^{ij}/\partial\sigma^k = \epsilon^{ijk}/\chi_s$ – where χ_s is spin susceptibility.

- Spin relaxation rate: $\Gamma_s = 2\eta_s/\chi_s$.

  For heavy quarks: $\Gamma_s \sim g^4 \log(1/g) T^3/M^2 \sim (T/M)^2\Gamma \ll \Gamma$,

  so there is Hydro+ regime *Hongo et al, 2201.12390*

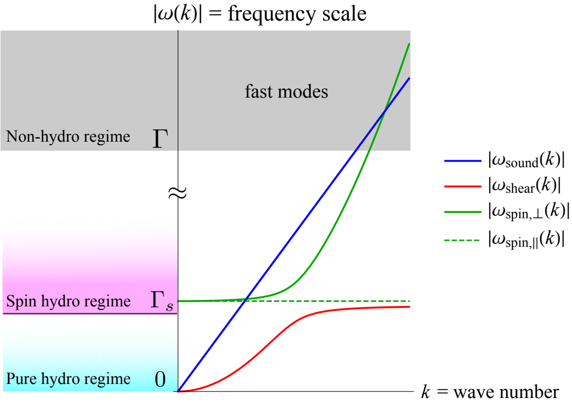
- Linearised e.o.m.:

  $$\left.\begin{aligned} (\partial_t + \Gamma_s)\boldsymbol{\sigma} &= \eta_s \boldsymbol{\nabla} \times \boldsymbol{v}_\perp, \\ (\partial_t + \tfrac{\eta+\eta_s/2}{w}\boldsymbol{\nabla}^2)\boldsymbol{\pi}_\perp &= \eta_s \boldsymbol{\nabla} \times \boldsymbol{\mu}, \end{aligned}\right\}$$

  Spin ($\boldsymbol{\sigma}$) and shear ($\boldsymbol{\pi}_\perp$) modes mix.

Linear response spectrum

Hongo et al, 2107.14231



Mixing is maximal when rates of shear diffusion and spin relaxation are similar, i.e., on the boundary of Hydro regime.

Hydro regime of spin hydrodynamics

- Unlike previous examples, no $1/\Gamma_s$ corrections to η, the limit of $\Gamma_s \to 0$ is smooth. (Mixing $\sim \eta_s \sim \Gamma_s$.)
- In Hydro regime spin d.o.f. can be removed by pseudo-gauge transformation. Stress tensor becomes symmetric.
- Certain 2nd order coefficients in constituitive relations must also change. For example:

  $$\Delta T^{ij} = a_0 \delta^{ij} (\boldsymbol{\nabla} \times \boldsymbol{v})^2 + a_1 \nabla^{[i} v^{k]} \nabla^{[k} v^{j]}$$

  These coefficients are nondissipative. *Li et al, 2011.12318*

Summary

- Hydrodynamics is a low-frequency effective theory and its validity is limited by the local relaxation rate $\omega \ll \Gamma$.
- If controllably slow local relaxation degrees of freedom exist, one can extend hydrodynamics into Hydro+ regime $\Gamma_s \ll \omega \ll \Gamma$.
- Examples: (generic) bulk mode, fluctuations and long-time tails, approximate chiral symmetry, heavy quark spin.
- In the hydro regime ($\omega \ll \Gamma_s$) the presence of slow modes is manifested in contributions to transport coefficients $\Delta\zeta$, $\Delta\lambda$, etc. Typically singular (as $1/\Gamma_s$). But not always (see spin hydro).

  Depends on how hydro modes mix with the slow hydro+ mode.