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Exact form of currents at global equilibrium with rotation and acceleration for free massless fermions

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OUTLINE

- Introduction : global thermodynamic equilibrium
- Iterative solution for the two-point function
- Analytic continuation and analytic distillation
- Results and discussion

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Global thermodynamic equilibrium

Density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

The vector is constant and the thermal vorticity ϖ is a constant antisymmetric tensor.
The four-temperature β is a Killing vector

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu \equiv \frac{u^\mu}{T}$$

Acceleration and vorticity (only at global equilibrium)

$$\frac{A^\mu}{T} = \varpi^{\mu\nu} u_\nu = \alpha^\mu \qquad \frac{\omega^\mu}{T} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho} u_\sigma = w^\mu$$

GOAL: calculate

$$\langle \widehat{O} \rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

The method

Factorization of the density operator

The generators of the Poincaré group appear in the density operator.

Analytic continuation to imaginary thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J$\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (\underbrace{\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\ldots\phi^{\alpha_{k-1}}{}_{\alpha_k}}_{k \text{ times}}) b^{\alpha_k} \qquad \widehat{\Lambda} \equiv \mathrm{e}^{-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}}$$

by using known **group theory** relations

Calculation of the two-momentum function

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\tilde{b}_\mu(\phi)\widehat{P}^\mu]\widehat{\Lambda}\ \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\right)$$

$$[\widehat{a}_s^\dagger(p),\widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin S):

$$\begin{aligned}\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle =& (-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle + \\ &+ 2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')\end{aligned}$$

$\mathrm{D(W)}$ is the “Wigner rotation” in the S-spin representation.

Solution by iteration

We find a solution by iteration:

I

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

II

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\,(-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\left(\Lambda p+\Lambda^2 p\right)}\delta^3(\Lambda^2\boldsymbol{p}-\boldsymbol{p}')+ + 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

∞

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\boldsymbol{p}-\boldsymbol{p}')D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

For vanishing vorticity (i.e. Λ=I):

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}\,\mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\,\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}}{\mathrm{e}^{b\cdot p}+(-1)^{2S+1}}$$

Wigner function

The Wigner for free fermions:

$$W(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\ \mathrm{e}^{-ik\cdot y}\langle :\Psi(x-y/2)\overline{\Psi}(x+y/2):\rangle$$

Wigner equation, a constraint:

$$\left(m - \not{k} - \frac{i\hbar\not{\partial}}{2}\right) W(x,k) = 0$$

Holds regardless of the density operator

Solution (to be continued analytically):

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$

$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-(\Lambda^n p+p)/2\right) + (m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+(\Lambda^n p+p)/2\right)\right]$$

Currents

Currents (vector, axial, stress-energy tensor) can be expressed as *k* integrals of *W(x,k)* for instance:

$$j^\mu(x) = \int \mathrm{d}^4k \; \mathrm{tr}\left(\gamma^\mu W(x,k)\right)$$

$$T_C^{\mu\nu} = \int \mathrm{d}^4k \, k^\nu \mathrm{tr}\left(\gamma^\mu W(x,k)\right)$$

They can be decomposed onto a suitable basis:

$$u^\mu = \frac{\beta^\mu}{\sqrt{\beta^2}}, \qquad \alpha^\mu = \varpi^{\mu\nu} u_\nu, \qquad w^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma, \qquad l^\mu = \epsilon^{\mu\nu\rho\sigma} w_\nu \alpha_\rho u_\sigma,$$

Decomposition of the stress-energy tensor:

$$T_B^{\mu\nu}(x) = \rho\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + W\, w^\mu w^\nu + A\,\alpha^\mu\alpha^\nu + G^l\, l^\mu l^\nu + G\,(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}\,(\alpha^\mu u^\nu + \alpha^\nu u^\mu)$$
$$+ G^\alpha\,(l^\mu\alpha^\nu + l^\nu\alpha^\mu) + \mathbb{W}\,(w^\mu u^\nu + w^\nu u^\mu) + A^w\,(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w\,(l^\mu w^\nu + l^\nu w^\mu)\,.$$

Analytic continuation and distillation

Analytic results can be obtained in the *m=0* case, but an analytic continuation is necessary

Example (scalar field) $$\langle : \widehat{\psi}^2(x) : \rangle_I = \frac{T_0^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{\phi^2}{\sinh^2(n\phi/2)}$$

The series cannot be resummed for the physical thermal vorticity -iϕ; it is not an analytic function in zero. Does this mean that analytic continuation fails?

It can be shown that the solution obtained by iteration may contain unwanted non-analytic terms (which solve the associated homogenous equation):

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = (-1)^{2S} \sum_r D^S(W(\Lambda,p))_{rs} \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} \langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle +$$

$$+ 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts} \delta^3(\Lambda \boldsymbol{p} - \boldsymbol{p}')$$

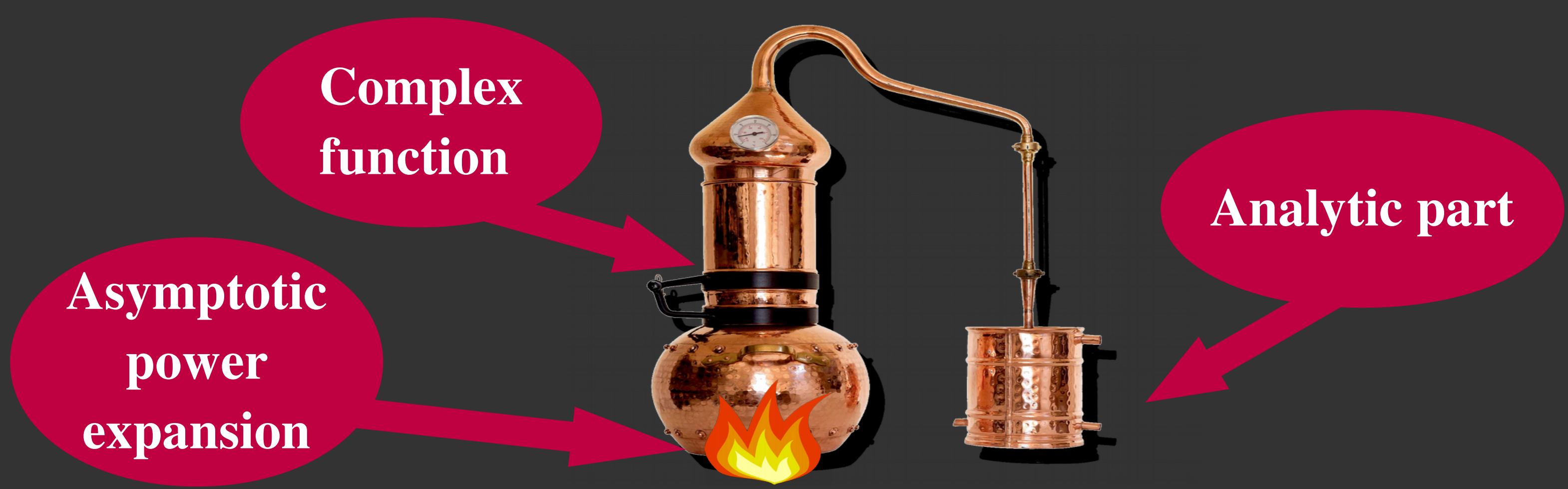
Analytic distillation

Goal: to define and extract the analytic part of a function at some point

Definition. *Let $f(z)$ be a function on a domain D of the complex plane and $z_0 \in \bar{D}$ a point where the function may not be analytic. Suppose that asymptotic*[3] *power series of $f(z)$ in $z - z_0$ exist in subsets $D_i \subset D$ such that $\cup_i D_i = D$:*

$$f(z) \sim \sum_n a_n^{(i)} (z - z_0)^n$$

where n can take integer negative values. If the series formed with the common coefficients in the various subsets restricted to $n \geq 0$ has a positive radius of convergence, the analytic function defined by this power series is called analytic distillate of $f(z)$ in z_0 and it is denoted by $\mathrm{dist}_{z_0} f(z)$.



An example

Suitable theorem for series:

D. Zagier, The Mellin transform and related analytic techniques, in Quantum Field Theory I: Basics in Mathematics and Physics. A Bridge Between Mathematicians and Physicists, Springer (2006), pp. 305–323

$$f(x) \equiv \frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx} .$$

$$\mathrm{dist}_0 f(x) = ?$$

just write down the Laurent series of *f(z)* and remove negative powers

There is another way, showing Zagier's theorem basic idea: expand the exponential and invert the summations:

$$\to \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} n^k \right) \frac{(-x)^k}{k!} ,$$

Continue analytically to the Riemann ζ and resum:

$$\to \sum_{k=0}^{\infty} \zeta(-k) \frac{(-x)^k}{k!} = \frac{1}{e^x - 1} - \frac{1}{x} ,$$

$$= \mathrm{dist}_0 f(x)$$

$$\langle : \widehat{\psi}^2(x) : \rangle_I = \frac{T_0^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{\phi^2}{\sinh^2(n\phi/2)}$$

$$S_2(\phi) = \sum_{n=1}^{\infty} \frac{\phi^2}{\sinh^2(n\phi/2)} = \phi^2 G_2(\phi).$$

Zagier's theorem

$$\mathrm{dist}_0 S_2(\phi) = \mathrm{dist}_0 \phi^2 G_2(\phi) = \frac{2\pi^2}{3} + \frac{\phi^2}{6}.$$

$$\langle : \widehat{\psi}^2(x) : \rangle = \frac{1}{\beta(x)^2} \left(\frac{1}{12} + \frac{\alpha^2(x)}{48\pi^2} \right).$$

This is the mean value of the massless field squared for constant acceleration obtained by solving Klein-Gordon equation in Rindler coordinates.It vanishes at T=A/2π, that is at the Unruh temperature.

Results – Stress energy tensor

$$T_B^{\mu\nu}(x) = \rho\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + W\, w^\mu w^\nu + A\,\alpha^\mu\alpha^\nu + G^l\, l^\mu l^\nu + G\,(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}\,(\alpha^\mu u^\nu + \alpha^\nu u^\mu)$$
$$+ G^\alpha\,(l^\mu\alpha^\nu + l^\nu\alpha^\mu) + \mathbb{W}\,(w^\mu u^\nu + w^\nu u^\mu) + A^w\,(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w\,(l^\mu w^\nu + l^\nu w^\mu)\,.$$

Full exact expression of the stress-energy tensor at global equilibrium for massless fermions

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{w^2}{8\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} + \frac{w^4}{64\pi^2\beta^4} + \frac{23\alpha^2 w^2}{1440\pi^2\beta^4} + \frac{11(\alpha\cdot w)^2}{720\pi^2\beta^4},$$
$$p = \frac{7\pi^2}{180\beta^4} - \frac{\alpha^2}{72\beta^4} - \frac{w^2}{24\beta^4} - \frac{17\alpha^4}{2880\pi^2\beta^4} + \frac{w^4}{192\pi^2\beta^4} + \frac{(\alpha\cdot w)^2}{96\pi^2\beta^4},$$
$$G^l = -\frac{11}{160\pi^2\beta^4},$$
$$G = \frac{1}{18\beta^4} - \frac{31\alpha^2}{360\pi^2\beta^4} - \frac{13w^2}{120\pi^2\beta^4},$$
$$W = -\frac{61\alpha^2}{1440\pi^2\beta^4},$$
$$A = -\frac{61w^2}{1440\pi^2\beta^4},$$
$$A^w = \frac{61\alpha\cdot w}{1440\pi^2\beta^4},$$
$$\mathbb{A} = \mathbb{W} = G^\alpha = G^w = 0.$$

$$\beta^2 = \frac{1}{T}$$
$$\alpha^2 = -\frac{A^2}{T^2}$$
$$w^2 = -\frac{\omega^2}{T^2}$$

For the pure acceleration case (w=0), the stress-energy tensor vanishes for finite T=A/2π

Axial current

$$j_{\rm A}^{\mu} = \frac{1}{\beta^2}\left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2}\right)\frac{w^\mu}{\sqrt{\beta^2}}.$$

These results are in agreement with exact solutions obtained solving Dirac equation
in cylindrical and Rindler coordinates, for the special case of rotation and pure acceleration ($A \neq 0, \omega = 0$)

A. Vilenkin, Phys. Rev. D 21 (1980) 2260 [AXIAL CURRENT].
V.E. Ambruş and E. Winstanley, Phys. Lett. B 734 (2014) 296.
V.E. Ambrus and E. Winstanley, arXiv:arXiv:1908.10244.
V.E. Ambrus, *Dirac fermions on rotating space-times*, Ph.D. Thesis, Sheffield University, Sheffield, U.K. (2014).

G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, JHEP 03 (2020) 137.
G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, Phys. Rev. D 99 (2019) 071901.
G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, Particles 3 (2020) 1 [ALL OF THEM PERTURBATIVE TO 4th ORDER].

This method does not require to solve field equations in special coordinates and it shows that solutions in different equilibria and geometries are deeply linked. New terms involving $\alpha \cdot w$

Entropy current

Entropy current can be derived from first principles of quantum statistical mechanics and has a unique expression at global equilibrium (F.B., D. Rindori, Phys.Rev.D 99 (2019) 12, 125011)

$$S = \log Z_{\rm LE} + \int_\Sigma {\rm d}\Sigma_\mu \left(\langle \hat{T}^{\mu\nu} \rangle_{\rm LE} \beta_\nu - \zeta \langle \hat{j}^\mu \rangle_{\rm LE} \right),$$

$$\log Z_{\rm LE} = \int_\Sigma {\rm d}\Sigma_\mu \phi^\mu$$

$$\phi^\mu = \int_1^{+\infty} {\rm d}\lambda [(\langle \hat{T}^{\mu\nu} \rangle_{\rm LE}(\lambda) - \langle 0 | \hat{T}^{\mu\nu} | 0 \rangle) \beta_\nu - \zeta (\langle \hat{j}^\mu \rangle_{\rm LE}(\lambda) - \langle 0 | \hat{j}^\mu | 0 \rangle)].$$

$$\hat{\rho}_{\rm LE}(\lambda) = \frac{1}{Z_{\rm LE}(\lambda)} \exp\left[-\lambda \int_\Sigma {\rm d}\Sigma_\mu (\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu) \right]$$

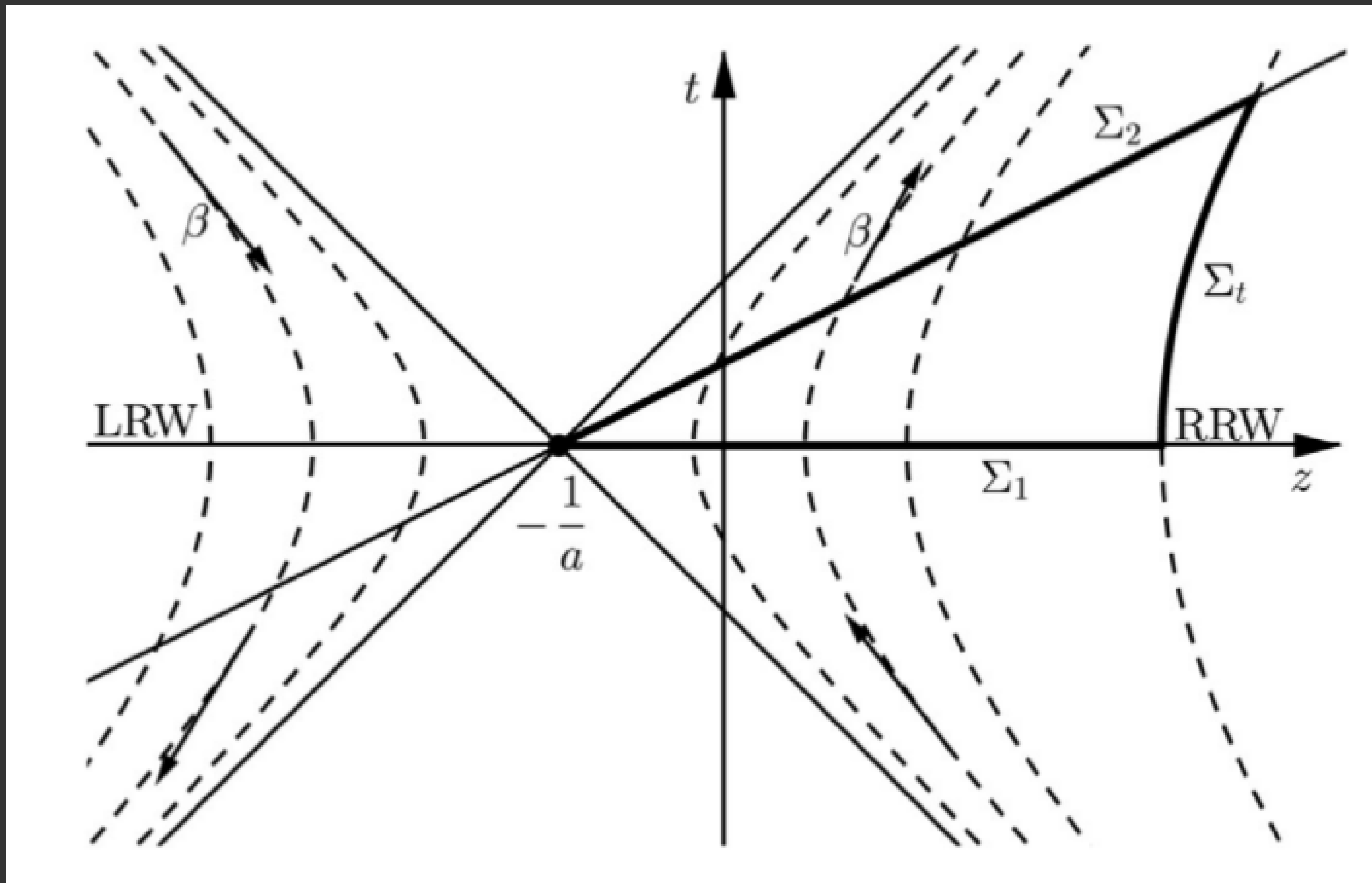
$$s^\mu = \phi^\mu + (\langle \hat{T}^{\mu\nu} \rangle_{\rm LE} - \langle 0 | \hat{T}^{\mu\nu} | 0 \rangle) \beta_\nu - \zeta (\langle \hat{j}^\mu \rangle_{\rm LE} - \langle 0 | \hat{j}^\mu | 0 \rangle).$$

If we apply it to our last obtained expression we find an interesting expression:

$$s^\mu = \left(\frac{7}{45} \pi^2 T^3 + \frac{1}{12} A^2 T + \frac{1}{4} \omega^2 T \right) u^\mu + \frac{1}{9} \epsilon^{\mu\nu\rho\sigma} T \omega_\nu A_\rho u_\sigma$$

Conclusions

- A method to obtain exact expressions of currents (and more) at global thermodynamic equilibrium with rotation and acceleration for free fields
- Analytic continuation requires the definition of an analytic part of a function and a method to extract it: analytic distillation
- The stress-energy tensor features corrections to the familiar form which are of quantum origin (proportional to A^2T^2, ω^2T^2, A^4, ω^4)
- Unruh temperature as an outcome
- Entropy current shows a non-longitudinal component (along the Killing vector) even at global equilibrium
- This method can be extended to include the axial chemical potential



t
β
β
Σ2
Σt
LRW
RRW
z
Σ1
$-\frac{1}{a}$