Chiral Hydrodynamics

in Strong Magnetic Fields

based on [1703.08757], [2012.09183]
and w.i.p. with M. Ammon, S. Grieninger and M. Kaminski

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Outline

1. Thermodynamics
   - Strong magnetic field
2. Hydrodynamics
3. Chiral hydrodynamics
   - Broken $U(1)_A$
   - $U(1)_V \times U(1)_A$ hydrodynamics

Outline

Thermodynamics

Generating functional and hydrodynamic equations

System with diffeomorphism and $U(1)$ symmetry coupled to external sources $g_{\mu\nu}$,A_μ. Partition function given by

$$Z[g,A] = \mathrm{Tr} e^{-\beta H[g,A]} = \int \mathcal{D}\phi e^{iS[\phi;g,A]} \, .$$

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Generating functional $W[g, A] = -i \ln Z[g, A]$ for n-pt functions

$$\delta W[g, A] = \int d^d x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu \right] .$$

where $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$ and $J^\mu = \langle \hat{J}^\mu \rangle$.

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Diffeomorphism and gauge invariance of W ensures these obey

Conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \,, \quad \nabla_\mu J^\mu = 0 \,.$$

Thermodynamics

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Thermodynamics

Equilibrium constraints and derivative expansion [1203.3544] & [1203.3556]

Equilibrium is given by a time-like killing vector V^μ

$$\mathcal{L}_V A_\mu = 0\,, \quad \mathcal{L}_V g_{\mu\nu} = 0\,.$$

Thermodynamics

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Can expand W_s order by order in derivatives

$$W_s = \int d^d x \sqrt{-g}\left[p(T,\mu) + \sum_{i,n_i} M_{n_i}^{(i)}(T,\mu) s_{n_i}^{(i)}\right]\,,$$

where

$$T = \frac{T_0}{\sqrt{-V^2}}\,, \quad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}}\,, \quad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}\,,$$

and $s_{n_i}^{(i)}$ are $\mathcal{O}(\partial^i)$ equilibrium scalars.

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Strong magnetic field and derivative expansion [1703.08757]

For $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} \sim \mathcal{O}(1)$, the generating functional is

$$W_s = \int d^4x\sqrt{-g}\left[\tilde{p}(T,\mu,B^2) + \sum_{i,n_i}\tilde{M}^{(i)}_{n_i}(T,\mu,B^2)\tilde{s}^{(i)}_{n_i}\right],$$

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where, for example

$$\tilde{p}(T, \mu, B^2) = p(T, \mu) + M^{(2)}_{B^2}(T, \mu) B^2 + M^{(4)}_{B^4}(T, \mu) B^4 + \cdots ,$$

and

$$\tilde{s}^{(i)}_{n_i} = s^{(j)}_{n_j} ,$$

for some n_j with $j \geq i$.

Thermodynamics

Constitutive relations, strong magnetic fields

For $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma} \sim \mathcal{O}(1)$, five $\mathcal{O}(\partial)$ equilibrium scalars s_n.

n	1	2	3	4	5
s_n	$B^\mu \partial_\mu (\frac{B^2}{T^4})$	$\epsilon^{\mu\nu\rho\sigma} u_\mu B_\nu \partial_\rho B_\sigma$	$B{\cdot}a$	$B{\cdot}E$	$B{\cdot}\Omega$
$(C,P,T)_{\text{vector}}$	$-/-/-$	$+/-/+$	$-/-/-$	$+/-/-$	$-/+/+$
$(C,P,T)_{\text{axial}}$	$+/+/-$	$+/-/+$	$+/+/-$	$+/-/-$	$+/-/+$
W	3	5	n/a	4	3

where $a^\mu = u^\nu \nabla_\nu u^\mu \,, \quad E^\mu = F^{\mu\nu} u_\nu \,, \quad \Omega^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \,.$

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Equilibrium constitutive relations

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}\,,$$

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu\,.$$

Thermodynamics

Example, Nersnt effect and conformal anomaly

Consider the leading term in the generating functional

$$W_s \sim \int d^d x \sqrt{-g}\, p(T, \mu, B^2)\,.$$

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$$W_s \sim \int d^d x \sqrt{-g}\, p(T, \mu, B^2)\,.$$

We can find the conformal anomaly coefficient and the Nernst coefficient

$$T^{\mu}_{\mu} \sim c_A F_{\mu\nu} F^{\mu\nu} \approx 2 c_A B^2\,, \quad \mathcal{J}^{\mu} \sim N_{Nernst} \epsilon^{\mu\nu\rho\sigma} u_{\nu} B_{\rho} \partial_{\sigma} T\,,$$

Thermodynamics

Example, Nersnt effect and conformal anomaly

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where

$$N_{Nernst} = \frac{2c_A}{T} = -\chi_{B,T} - \mu \frac{\chi_{B,\mu}}{T} \,,$$

and $\chi_B = 2p_{,B^2}$ is the magnetic susceptibility.

Thermodynamics

Example, magneto-vortical susceptibility M_5

$$W_s \sim \int d^4x\sqrt{-g}M_5 B\cdot\Omega$$

Einstein-de Haas, Barnett effects

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho B_\sigma\,, \quad m^\mu \sim \Omega^\mu$$

Momentum Nernst effect

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma T$$

Response to Poynting vetor

$$\mathcal{Q}^\mu \sim \epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma$$

Thermodynamics

Example, perpendicular magnetic vorticity susceptibility M_2

$$W_s \sim \int d^4x\sqrt{-g}\,M_2\epsilon^{\mu\nu\rho\sigma}u_\mu B_\nu \partial_\rho B_\sigma$$

Magnetic Nernst effect

$$m^\mu \sim \epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma T$$

Response to magnetic vorticity

$$m^\mu \sim \epsilon^{\mu\nu\rho\sigma}u_\nu \partial_\rho B_\sigma$$

Response to Poynting vetor

$$\mathcal{T}^{\mu\nu} \sim B^{<\mu}\epsilon^{\nu>\rho\sigma\alpha}u_\rho E_\sigma B_\alpha$$

Thermodynamics

Thermodynamic Kubo formulas

Second order variations of W_s give thermodynamic two point functions ($\omega = 0, k \ll T$)

$$\delta W_s = \int d^d x \sqrt{-g} \left[G_{JJ} \delta A \delta A + \frac{1}{2} G_{TJ} \delta g \delta A + \frac{1}{4} G_{TT} \delta g \delta g \right] .$$

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This allows us to derive

Thermodynamic Kubo formulas $(g_{\mu\nu} = \eta_{\mu\nu}\,,\ B^\mu = B_0 \delta^\mu_z)$

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{T^{xz} T^{yz}}(\omega = 0, k\hat{z}) = -2\, B_0^2\, M_2\,,$$

$$\lim_{k\to 0} \frac{1}{k} \mathrm{Im}\, G_{T^{tx} T^{yz}}(\omega = 0, k\hat{z}) = -B_0\, M_5\,.$$

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Thermodynamic Kubo formulas

$$\lim_{k\to 0}\frac{1}{k}\mathrm{Im}\, G_{J^t T^{xx}}(\omega = 0, k\hat{z}) = -\frac{2\,B_0^3}{T_0^4}\,\frac{\partial M_1}{\partial \mu} ,$$

$$\lim_{k\to 0}\frac{1}{k}\mathrm{Im}\, G_{J^t J^t}(\omega = 0, k\hat{z}) = B_0\frac{\partial M_4}{\partial \mu} ,$$

$$\lim_{k\to 0}\frac{1}{k}\mathrm{Im}\, G_{J^t T^{tt}}(\omega = 0, k\hat{z}) = -B_0\left(\frac{\partial M_3}{\partial \mu} + \frac{4B_0^2}{T_0^4}\frac{\partial M_1}{\partial \mu}\right) .$$

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Hydrodynamics

Non-equilibrium constitutive relations

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p - \zeta \nabla \cdot u)\, \Delta^{\mu\nu} - \eta \sigma^{\mu\nu}\,,$$

$$J^\mu = n u^\mu + \sigma \left(E^\mu - T \Delta^{\mu\nu} \partial_\nu \frac{\mu}{T} \right)\,.$$

where $\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1} \Delta_{\alpha\beta} \nabla \cdot u \right)$.

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Hydrodynamic transport coefficients

$$\zeta\,,\ \eta\,,\ \sigma\,.$$

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Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \ \text{s.t.}\ \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \zeta \geqslant 0\,,\ \eta \geqslant 0\,,\ \sigma \geqslant 0\,.$$

Hydrodynamics

Non-equilibrium constitutive relations, $B^\mu \sim \mathcal{O}(1)$

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots \,,$$

$$J^\mu = \cdots \,.$$

Hydrodynamic transport coefficients

$$\eta_\perp \,,\; \eta_\parallel \,,\; \eta_1 \,,\; \eta_2 \,,\; \zeta_1 \,,\; \zeta_2 \,,\; \sigma_\parallel \,,\; \sigma_\perp \,,\; \tilde{\eta}_\perp \,,\; \tilde{\eta}_\parallel \,,\; \tilde{\sigma}_\perp \,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic}$$

Hydrodynamics

Non-equilibrium constitutive relations, $B^\mu \sim \mathcal{O}(1)$, parity violating

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots \,, \quad J^\mu = \cdots \,.$$

Hydrodynamic transport coefficients

$$\eta_\perp \,,\ \eta_\parallel \,,\ \eta_1 \,,\ \eta_2 \,,\ \zeta_1 \,,\ \zeta_2 \,,\ \sigma_\parallel \,,\ \sigma_\perp \,,\ \tilde{\eta}_\perp \,,\ \tilde{\eta}_\parallel \,,\ \tilde{\sigma}_\perp \,,$$
$$\sigma_1^V \,,\ \sigma_2^V \,,\ \sigma_3^V \,,\ \sigma_4^V \,,\ \zeta^V \,,\ \eta^V \,,\ \eta_\parallel^V \,,\ \tilde{\eta}_\parallel^V \,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic} + \text{qubic}$$

Hydrodynamics

Eigenmodes and Kubo formulas

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \,, \quad \nabla_\mu J^\mu = 0 \,.$$

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Linearize hydrodynamic equations for $\delta A_\mu \,, \delta g_{\mu\nu} \propto \exp(-i\,(\omega t - \mathrm{k}\cdot\mathrm{x}))$

Eigenmodes

$$\omega = \omega(k) \,.$$

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Eigenmodes

$$\omega = \omega(k) \,.$$

Varying on-shell $T^{\mu\nu}[g, A]$ and $J^\mu[g, A]$ (limit $k \to 0$ first, $\omega \to 0$ second)

Hydrodynamic Kubo formulas

$$\lim_{\omega \to 0} \mathrm{Im}\, G_{\mathcal{O}\mathcal{O}}(\omega, k = 0) \sim \zeta_i \,, \; \eta_i \,, \; \sigma_i \,, \; c_i \,.$$

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Chiral hydrodynamics

Consistent generating functional [2012.09183]

Generating functional W not $U(1)$ invariant

$$\delta_\alpha W = \frac{C}{24} \int d^4x \sqrt{-g}\epsilon^{\mu\nu\rho\sigma} \alpha F_{\mu\nu} F_{\rho\sigma} \, .$$

Chiral hydrodynamics

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$$\delta_\alpha W = \frac{C}{24}\int d^4x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\alpha F_{\mu\nu}F_{\rho\sigma}\,.$$

Consistent generating functional

$$W_{cons} = W_s + \frac{C}{6}\int d^4x\sqrt{-g}\mu(\mu\Omega\cdot A + 2B\cdot A)\,,$$

$$\delta W_{cons} = \int d^4x\sqrt{-g}(J^\mu_{cons}\delta A_\mu + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu})\,.$$

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$$\delta W_{cons} = \int d^4x\sqrt{-g}(J^\mu_{cons}\delta A_\mu + \frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu})\,.$$

Because $\delta_\alpha W_{cons} \neq 0$

$$\delta_\alpha J^\mu_{cons} \neq 0\,.$$

But since $\delta_\alpha W_{cons}$ is independent of the metric

$$\delta_\alpha T^{\mu\nu} = 0\,.$$

Chiral hydrodynamics

Hydrodynamic equations, constitutive relations and Kubo formulas

From W_{cons} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cons} - A^\nu \nabla_\mu J^\mu_{cons}\,, \quad \nabla_\mu J^\mu_{cons} = \frac{C}{3} E\cdot B\,.$$

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)}$$

$$J^\mu_{cons} = J_s^\mu + \tfrac{1}{3} CB\cdot A u^\mu + \xi\,\Omega^\mu + \left(\xi_B - \tfrac{1}{3} C\mu\right) B^\mu + \tfrac{1}{3} C\epsilon^{\mu\nu\rho\sigma} A_\nu u_\rho E_\sigma\,.$$

Chiral conductivities

$$\xi = \tfrac{1}{2} C\mu^2\,, \quad \xi_B = C\mu\,,$$

$$\xi_T = \tfrac{1}{3} C\mu^3\,, \quad \xi_{TB} = \tfrac{1}{2} C\mu^2\,.$$

Chiral hydrodynamics

Anomaly inflow and covariant generating functional

Consider an auxiliary manifold $\mathcal{M}$ whose boundary $\partial\mathcal{M}$ is where the chiral fluid lives.

Covariant generating functional

$$W_{cov} = W_{cons} - \frac{C}{24}\int_{\mathcal{M}} d^5x\sqrt{-G}\epsilon^{mnopq}A_m F_{no}F_{pq}\,,$$

$$\delta W_{cov} = \int_{\partial\mathcal{M}} d^4x\sqrt{-g}(J^{\mu}_{cov}\delta A_{\mu} + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}) + \int_{\mathcal{M}} d^5x\sqrt{-G}J^m_H\delta A_m\,.$$

Where $J^{\mu}_{cov} = J^{\mu}_{cons} - \frac{C}{6}\epsilon^{\mu\nu\rho\sigma}A_{\nu}F_{\rho\sigma}$, and $J^m_H = -\frac{C}{8}\epsilon^{mnopq}F_{no}F_{pq}$
Since $\delta_{\alpha}W_{cov} = 0$

$$\delta_{\alpha}J^{\mu}_{cov} = 0\,.$$

Chiral hydrodynamics

Constitutive relations and Kubo formulas

From W_{cov} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cov} \,, \quad \nabla_\mu J^\mu_{cov} = J^\rho_H = CE\cdot B \,.$$

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi_T u^{(\mu}\Omega^{\nu)} + \xi_{TB} u^{(\mu} B^{\nu)} \,, \quad J^\mu_{cov} = J^\mu_s + \xi\,\Omega^\mu + \xi_B B^\mu \,.$$

Thermodynamic Kubo formulas

$$\langle J^x_{cov}(\mathrm{k}) T^{ty}(-\mathrm{k})\rangle = -i\xi\, k_z \,, \quad \langle J^x_{cov}(\mathrm{k}) J^y_{cons}(-\mathrm{k}))\rangle = -i\xi_B\, k_z \,,$$
$$\langle T^{tx}(\mathrm{k}) T^{ty}(-\mathrm{k})\rangle = -i\xi_T\, k_z \,, \quad \langle T^{tx}(\mathrm{k}) J^y_{cons}(-\mathrm{k})\rangle = -i\xi_{TB}\, k_z \,.$$

Chiral hydrodynamics

Eigenmodes, gapped

There are two gapped modes

$$\omega = \pm\frac{B_0^2}{w_0}\sigma_{12} - \frac{iB_0^2}{w_0}\sigma_{11} + v_{gap\pm} k \cos\theta - iD_c(\theta)k^2 \,,$$

where $w_0 = \epsilon_0 + p_0$ and $\sigma_{ab} = \delta_{ab}\sigma_\perp + \epsilon_{ab}\left(\tilde{\sigma}_\perp + \frac{n_0}{|B_0|}\right)$. The gapped modes have velocity

$$v_{gap\pm} = \frac{B_0^2 C\mu_0^3}{3w_0^2}(\sigma_{12} \pm i\sigma_{11}) \,,$$

and damping coefficient $D_c(\theta) = D_c(\theta)|_{C=0} \pm i\frac{C\mu_0^3}{3}\frac{v_{gap\pm}}{w_0}\cos^2\theta$.

Chiral hydrodynamics

Eigenmodes, gappless parallel

There are three gapless eigenmodes. For k ∥ B, they are

$$\omega = kv_\pm - i\frac{\Gamma_\parallel}{2}k^2\,, \quad \omega = kv_0 - iD_\parallel k^2\,,$$

where

$$v_0 = \frac{B_0 C}{\det(\chi)}\left(\frac{s_0 T_0}{v_s}\right)^2\,,$$

$$v_\pm = \pm v_s - \frac{v_0}{2} + \frac{B_0 C}{2}\gamma\,.$$

The speed of sound is given by

$$v_s^2 = \frac{n_0^2\chi_{11} + w_0^2\chi_{33} - 2n_0 w_0\chi_{13}}{\det(\chi)}\,,$$

and $\chi_{ab} = \frac{\delta\langle\varphi_a\rangle}{\delta\lambda^b}$ is the susceptibility matrix. Here, $\varphi_a = (T^{tt}, T^{ti}, J^t)$, and $\delta\lambda^a = (\delta T/T, \delta u^i, T\delta\frac{\mu}{T})$.

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$$\omega = kv_\pm - i\frac{\Gamma_\parallel}{2}k^2\,, \quad \omega = kv_0 - iD_\parallel k^2\,,$$

where

$$\Gamma_\parallel = \frac{3\zeta_1 + 10\eta_1 + 6\eta_2}{3w_0} + \frac{v_s^2\chi_{11} - w_0}{\det(\chi)}\frac{w_0}{v_s^2}\sigma_\parallel$$
$$+ CB_0\left(\Sigma_\eta(3\zeta_1 + 10\eta_1 + 6\eta_2) + \Sigma_\parallel\sigma_\parallel + \Sigma_\perp\sigma_\perp\right) + \mathcal{O}\left(B_0^2C^2\right)\,,$$
$$D_\parallel = \frac{w_0^2\sigma_\parallel}{v_s^2\det(\chi)} + \mathcal{O}\left(B_0^2C^2\right)\,.$$

Σ_η, $\Sigma_\parallel$ and $\Sigma_\perp$ are functions of the susceptibilities, other thermodynamic derivatives of the pressure, the chemical potential and the temperature.

Chiral hydrodynamics

Eigenmmode, gapless non-orthogonal

For modes propagating at an angle $\theta \neq \pi/2$ with respect to B_0

$$\omega = kv_\pm \cos\theta - \frac{i}{2}\Gamma(\theta)k^2\,, \quad \omega = kv_0\cos\theta - iD(\theta)k^2\,,$$

where

$$D(\theta) = D_\parallel \cos^2\theta + \left(\frac{n_0^2\, w_0^2\, \rho_\perp}{B_0^2 v_s^2 \det(\chi)^2} + \mathcal{O}(B_0 C)\right)\sin^2\theta\,,$$

$$\Gamma(\theta) = \Gamma_\parallel \cos^2\theta + \left(\frac{\eta_\parallel}{w_0} + \frac{(n_0\chi_{13} - w_0\chi_{33})^2 w_0^3}{B_0^2\, \det(\chi)^2}\rho_\perp + \mathcal{O}(B_0 C)\right)\sin^2\theta\,.$$

Chiral hydrodynamics

Eigenmodes, gappless perpendicular

For k ⊥ B, two diffusive modes

$$\omega = -i \left(\frac{w_0^3 \chi_{33} \rho_\perp}{\det(\chi) B_0^2} + \mathcal{O}\left(B_0^2 C^2\right) \right) k^2 ,$$

$$\omega = i \left(\frac{\eta_\parallel}{w_0} + \mathcal{O}\left(B_0^2 C^2\right) \right) k^2 ,$$

and one subdiffusive mode

$$\omega = -i \frac{\eta_\perp k^4}{B_0^2 \chi_{33}} .$$

Outline

Thermodynamics

$U(1)_V \times U(1)_A$ symmetry (w.i.p.)

Thermodynamic variables

$$T, \mu, \mu^5, u^\mu$$

Generating functional

$$W_s = \int d^4x \sqrt{-g}\, p(T, \mu, \mu^5) + \cdots$$

Constitutive relations

$$\delta W_s = \int d^4x \sqrt{-g} \left[\tfrac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu + J_5^\mu \delta A_\mu^5\right]$$

Thermodynamic correlation functions and Kubo formulas

$$\delta W_s = \int d^dx \sqrt{-g} \left[G_{J_a J_b} \delta A^a \delta A^b + \tfrac{1}{2} G_{T J_a} \delta g \delta A^a + \tfrac{1}{4} G_{TT} \delta g \delta g\right] .$$

Thermodynamics

$U(1)_V \times U(1)_A$ symmetry, strong magnetic field

Thermodynamic variables

$$T, \mu, \mu^5, B^2, u^\mu$$

Generating functional

$$W_s = \int d^4x \sqrt{-g}\, \tilde{p}(T, \mu, \mu^5, B^2) + \sum_1^7 \tilde{M}_n(T, \mu, \mu^5, B^2) s_n + \cdots$$

$$\delta W_s = \int d^4x \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta A_\mu + J_5^\mu \delta A_\mu^5 \right]$$

Chiral hydrodynamics

Consistent generating functional

Generating functional W not $U(1)_A$ invariant

$$\delta_{\alpha,\alpha_5} W = \frac{C}{24} \int d^4x \sqrt{-g} \epsilon^{\mu\nu\rho\sigma} \alpha_5 \left(F^5_{\mu\nu} F^5_{\rho\sigma} + 3 F_{\mu\nu} F_{\rho\sigma} \right) .$$

Chiral hydrodynamics

Consistent generating functional

Generating functional W not $U(1)_A$ invariant

$$\delta_{\alpha,\alpha_5} W = \frac{C}{24}\int d^4x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\alpha_5\left(F^5_{\mu\nu}F^5_{\rho\sigma} + 3F_{\mu\nu}F_{\rho\sigma}\right) .$$

Consistent generating functional

$$W_{cons} = W_s + \int d^4x\sqrt{-g}\Big[\tfrac{C}{3}\left(\mu B^\mu + \tfrac{1}{2}\mu^2\Omega^\mu\right)A^5_\mu$$

$$+C\left(\mu_5 B^\mu_5 + \tfrac{1}{2}\mu_5^2\Omega^\mu\right)A^5_\mu\Big] ,$$

$$\delta W_{cons} = \int d^4x\sqrt{-g}(J^\mu_{cons}\delta A_\mu + J^\mu_{5,cons}\delta A^5_\mu + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}) .$$

Chiral hydrodynamics

Consistent generating functional

Generating functional W not $U(1)_A$ invariant

$$\delta_{\alpha,\alpha_5} W = \frac{C}{24}\int d^4x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\alpha_5\left(F^5_{\mu\nu}F^5_{\rho\sigma} + 3F_{\mu\nu}F_{\rho\sigma}\right) .$$

Consistent generating functional

$$W_{cons} = W_s + \int d^4x\sqrt{-g}\Big[\tfrac{C}{3}\left(\mu B^\mu + \tfrac{1}{2}\mu^2\Omega^\mu\right)A^5_\mu + C\left(\mu_5 B^\mu_5 + \tfrac{1}{2}\mu_5^2\Omega^\mu\right)A^5_\mu\Big] ,$$

$$\delta W_{cons} = \int d^4x\sqrt{-g}(J^\mu_{cons}\delta A_\mu + J^\mu_{5,cons}\delta A^5_\mu + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}) .$$

Because $\delta_{\alpha_5} W_{cons} \neq 0$, but metric independent

$$\delta_{\alpha_5} J^\mu_{cons} , \delta_{\alpha_5} J^\mu_{5,cons} \neq 0 , \quad \delta_{\alpha_5} T^{\mu\nu} = 0 .$$

Chiral hydrodynamics

Anomaly inflow and covariant generating functional

Covariant generating functional

$$W_{cov} = W_{cons} - \frac{C}{24}\int_{\mathcal{M}} d^5x\sqrt{-G}\epsilon^{mnopq}A^5_m\left(F^5_{no}F^5_{pq} + 3F_{no}F_{pq}\right) ,$$

$$\delta W_{cov} = \int_{\partial\mathcal{M}} d^4x\sqrt{-g}(J^\mu_{cov}\delta A_\mu + J^\mu_{5,cov}\delta A^5_\mu + \tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu})$$
$$+ \int_{\mathcal{M}} d^5x\sqrt{-G}\left(J^m_H\delta A_m + J^m_{5,H}\delta A^5_m\right) .$$

Where

$$J^\mu_{cov} = J^\mu_{cons} - \tfrac{C}{2}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F_{\rho\sigma} , \quad J^\mu_{5,cov} = J^\mu_{5,cons} - \tfrac{C}{6}\epsilon^{\mu\nu\rho\sigma}A^5_\nu F^5_{\rho\sigma}$$

$$J^m_H = -\tfrac{C}{4}\epsilon^{mnopq}F_{no}F^5_{pq} , \quad J^m_{5,H} = -\tfrac{C}{8}\epsilon^{mnopq}\left(F_{no}F_{pq} + F^5_{no}F^5_{pq}\right)$$

Since $\delta_{\alpha_5}W_{cov} = 0$

$$\delta_{\alpha_5}J^\mu_{cov} = \delta_{\alpha_5}J^\mu_{5,cov} = 0 .$$

Chiral hydrodynamics

Hydrodynamic equations

From W_{cons} we find

Hydrodynamic equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu^{cons} + F_5^{\mu\nu} J_\nu^{5,cons} - A_5^\nu \nabla_\mu J^\mu_{cons}$$

$$\nabla_\mu J^\mu_{cons} = 0\,, \quad \nabla_\mu J^\mu_{5,cons} = -\frac{C}{24}\epsilon^{\mu\nu\rho\sigma}\left(F^5_{\mu\nu}F^5_{\rho\sigma} + 3F_{\mu\nu}F_{\rho\sigma}\right)\,.$$

From W_{cov} we find

Hydrodynamic equations

$$\nabla_\nu T_A^{\mu\nu} = F^{\mu\nu} J_\nu^{cov} + F_5^{\mu\nu} J_\nu^{5,cov}\,,$$

$$\nabla_\mu J^\mu_{cov} = -\frac{C}{4}\epsilon^{\mu\nu\rho\sigma} F^5_{\mu\nu} F_{\rho\sigma}\,,$$

$$\nabla_\mu J^\mu_{5,cov} = -\frac{C}{8}\epsilon^{\mu\nu\rho\sigma}\left(F^5_{\mu\nu}F^5_{\rho\sigma} + F_{\mu\nu}F_{\rho\sigma}\right)\,.$$

Chiral hydrodynamics

Constitutive relations and Kubo formulas

Equilibrium constitutive relations

$$T^{\mu\nu} = T_s^{\mu\nu} + \xi^T u^{(\mu}\Omega^{\nu)} + \xi_B^T u^{(\mu} B^{\nu)} + \xi_{B_5}^T u^{(\mu} B_5^{\nu)}$$

$$J^\mu_{cons} = J_s^\mu + C B\cdot A^5 u^\mu + \xi\,\Omega^\mu + (\xi_B - C\mu_5)\, B^\mu + \xi_{B_5} B_5^\mu + C\epsilon^{\mu\nu\rho\sigma} A_\nu u_\rho E_\sigma\,.$$

$$J^\mu_{5,cons} = J_s^\mu + \frac{C}{3} B_5\cdot A^5 u^\mu + \xi^5\,\Omega^\mu + \xi_B^5 B^\mu + \left(\xi^5_{B_5} - \frac{C}{3}\mu\right) B_5^\mu + \frac{C}{3}\epsilon^{\mu\nu\rho\sigma} A^5_\nu u_\rho E^5_\sigma\,.$$

$$J^\mu_{cov} = J_s^\mu + \xi\,\Omega^\mu + \xi_B B^\mu + \xi_{B_5} B_5^\mu\,.$$

$$J^\mu_{5,cov} = J_s^\mu + \xi^5\,\Omega^\mu + \xi_B^5 B^\mu + \xi^5_{B_5} B_5^\mu\,.$$

Chiral conductivities

$$\xi^T = C\left(\mu^2\mu_5 + \frac{1}{3}\mu_5^3\right)\,, \quad \xi^T_{B_5} = \xi^5 = \frac{C}{2}\left(\mu^2 + \mu_5^2\right)$$

$$\xi_B^T = \xi = C\mu\mu_5\,, \quad \xi_B = \xi^5_{B_5} = C\mu_5\,, \quad \xi_{B_5} = \xi^5_B = C\mu\,.$$

Chiral hydrodynamics

Non-equilibrium constitutive relations

Add the linearly independent terms that vanish in equilibrium $\mathcal{L}_V = 0$:

Constitutive relations

$$T^{\mu\nu} = \cdots\,, \quad J^\mu_V = \cdots\,, \quad J^\mu_A = \cdots\,.$$

Hydrodynamic transport coefficients

$$\eta_\perp\,,\ \eta_\parallel\,,\ \eta_1\,,\ \eta_2\,,\ \zeta_1\,,\ \zeta_2\,,\ \sigma^{ab}_\parallel\,,\ \sigma^{ab}_\perp\,,\ \tilde\eta_\perp\,,\ \tilde\eta_\parallel\,,\ \tilde\sigma^{ab}_\perp\,,$$
$$\sigma^a_1\,,\ \sigma^a_2\,,\ \sigma^a_3\,,\ \sigma^a_4\,,\ \zeta^a\,,\ \eta^a\,,\ \eta^a_\parallel\,,\ \tilde\eta^a_\parallel\,.$$

Entropy production constraints

$$S^\mu|_{\text{eq.}} = s u^\mu \text{ s.t. } \nabla_\mu S^\mu \geqslant 0 \Longrightarrow \text{linear} + \text{quadratic} + \text{qubic} + ?$$

Chiral hydrodynamics

To do

Thermodynamic Kubo formulas

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim M_n, \xi$$

Hydrodynamic Kubo formulas

$$\lim_{\omega\to 0} \mathrm{Im}\, G_{\mathcal{O}\mathcal{O}}(\omega, k=0) \sim \zeta, \eta, \sigma$$

Eigenmodes

$$\omega = \omega(k)$$

Recap

We overviewed

- Equilibrium generating functional at strong magnetic field
- Equilibrium and non-equilibrium constitutive relations
- Kubo formulas, eigenmodes
- Anomaly inflow and consistent vs covariant W and J

Work in progress and future work

- Hydro with $U(1)_V \times U(1)_A$ symmetry
- Weak gauging of $U(1)_V$: Chiral MHD

Thanks!