



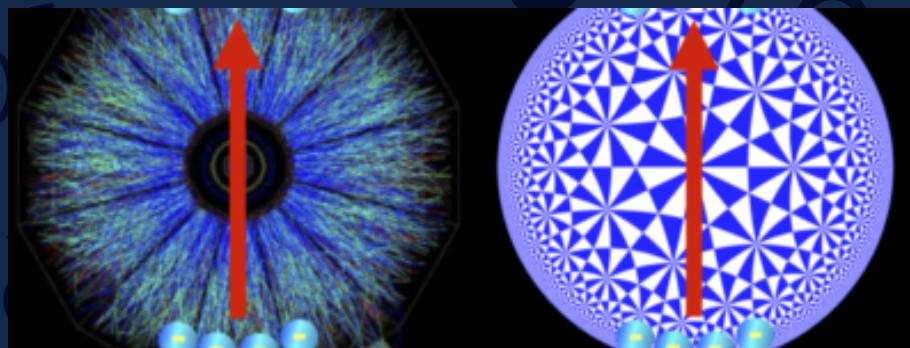
Illinois Center for Advanced Studies of the Universe



A causal and stable theory of chiral hydrodynamics

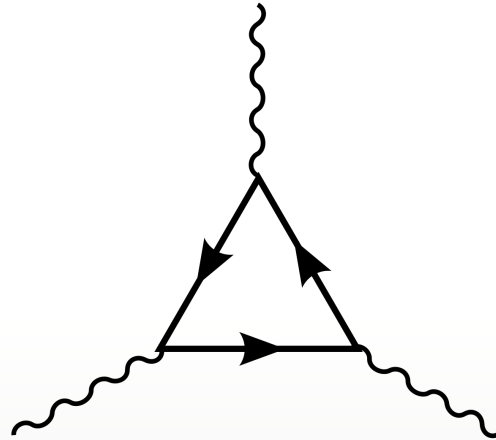
Jorge Noronha

Holographic Perspectives on Chiral Transport, ECT*
Trento, March 2023





Fluid



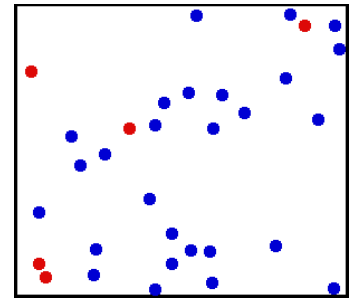
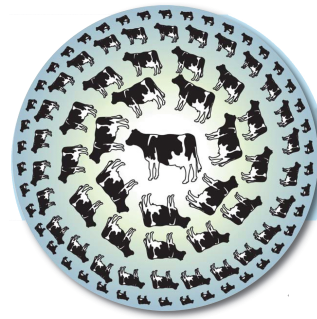
Anomaly



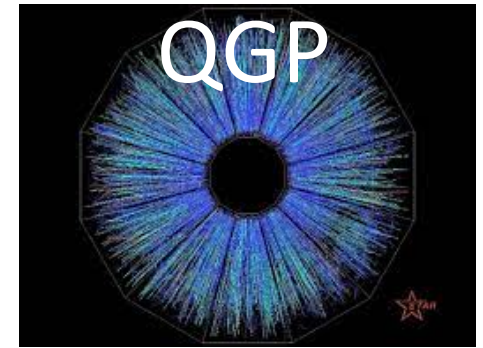
vorticity

*Fig. from Rezzolla and
Zanotti's book*

- Chiral hydro derived from:



- How does the anomaly/vorticity affect the hydrodynamic evolution?
Observable consequences?



- How does one include such novel effects in nonlinear hydrodynamic simulations?
- Are there new constraints that must be imposed to ensure proper evolution?

Ideal chiral hydrodynamics

*Banerjee, Bhattacharyya, Dutta, Loganayagam, Surowka, JHEP (2011); Erdmenger, Haack, Kaminski, Yarom, JHEP (2009)
Son, Surowka, PRL (2009), Neiman, Oz, JHEP (2011)*

- Most general ideal chiral hydro at 1st order ^{for simplicity} (no E&M)

Energy-momentum tensor: general hydrodynamic frame

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P \Delta^{\mu\nu} + \xi_T (\omega^\mu u^\nu + \omega^\nu u^\mu)$$

$$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad u_\mu u^\mu = -1$$

Vector (“baryon”) current:

$$J_V^\mu = n_V u^\mu + \xi_V \omega^\mu$$

Axial current:

$$J_A^\mu = n_A u^\mu + \xi_A \omega^\mu$$

Entropy current:

$$S^\mu = s u^\mu + \xi_S \omega^\mu$$

- Presence of vorticity tensor

$$\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$$

- New coefficients

$\xi_S, \xi_T, \xi_V, \xi_A$ depend on T, μ_V, μ_A

$$\nabla_\mu S^\mu = 0$$

Ideal chiral hydrodynamics

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Axial current:

$$J_A^\mu = n_A u^\mu + \xi_A \omega^\mu$$

Entropy current:

$$S^\mu = s u^\mu + \xi_S \omega^\mu$$

Equations of motion (EOM):

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

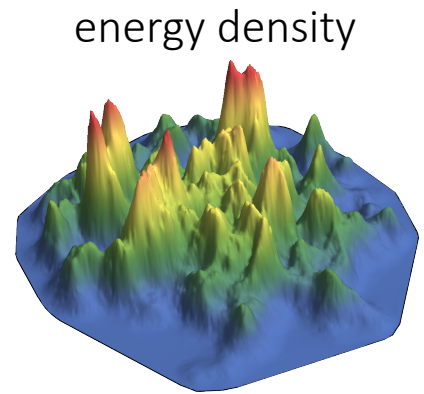
Conservation of currents

$$\partial_\mu J_V^\mu = 0, \quad \partial_\mu J_A^\mu = 0.$$

- Differently than ideal hydro, most properties of ideal chiral hydro are not known.

- New nonlinear PDEs, existence/uniqueness of solutions is not known.

- Considering heavy-ion simulations we need to know if EOM can be solved under general conditions.



- Is ideal chiral hydro ready to be simulated?

- Do solutions exist? Is the evolution causal?

Causality (in the sense of PDEs)

See, e.g., Choquet-Bruhat, Wald

Consider a system of (linear or nonlinear) PDE's

N unknowns

$$P_K^I \Psi^K = 0$$

$$\Psi^K$$

System is causal if for any point x in the future of initial Cauchy surface Σ

$$\Psi^K(x) \text{ depends only on } J^-(x) \cap \Sigma$$

causal past of x

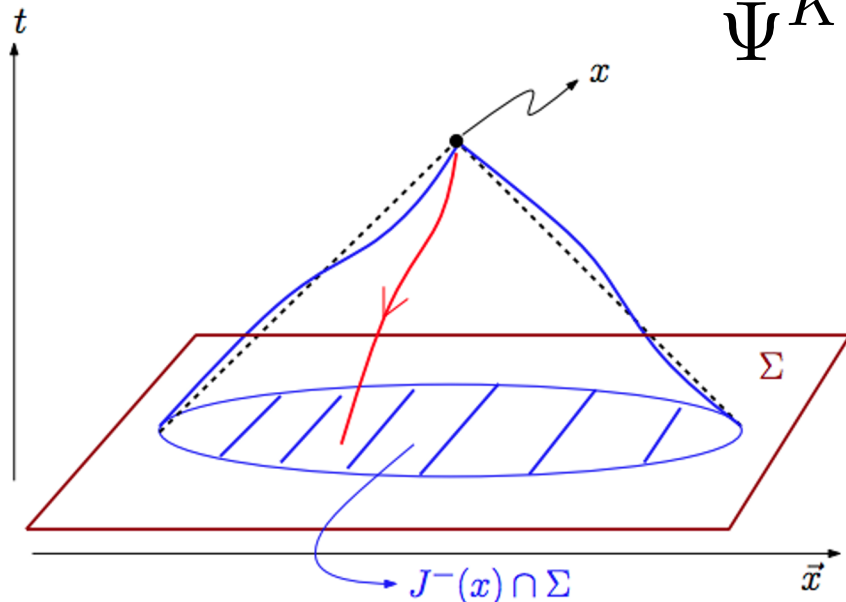


FIG. 1: (color online) Illustration of causality. In curved spacetime $J^-(x)$ looks like a distorted light-cone opening to the past (blue region); in flat spacetime the cone would be straight (dotted line). Points inside $J^-(x)$ can be joined to a point x in spacetime by a causal past directed curve (e.g. the red line). The Cauchy surface Σ supports the initial data and the value of the field $\Psi(x)$ depends only on the initial data on $J^-(x) \cap \Sigma$.

Quasilinear PDEs: $\mathcal{A}(\Psi, \partial)\Psi = \mathcal{B}$

Characteristic surface $\phi(x^\mu) = 0$, with normal $\varphi_\alpha = \partial_\alpha \phi$ (real)

- **Causality** holds if solution of $\det(A^\alpha \varphi_\alpha) = 0$ gives $\varphi_\alpha \varphi^\alpha \geq 0$

- Example: “sound wave” equation $u_\mu u_\nu \partial^\mu \partial^\nu \Psi - \beta \Delta_{\mu\nu} \partial^\mu \partial^\nu \Psi = 0$,

principal part

$$\mathcal{A}(\Psi, \partial) = (u_\mu u_\nu - \beta \Delta_{\mu\nu}) \partial^\mu \partial^\nu$$

$$\text{Imposing } \varphi_\alpha \varphi^\alpha \geq 0$$

characteristic determinant

$$\det[\mathcal{A}(\Psi, \varphi)] = (u_\mu \varphi^\mu)^2 - \beta \Delta_{\mu\nu} \varphi^\mu \varphi^\nu = 0$$

$$\text{gives } 0 \leq \beta \leq 1$$

- This is why for sound, causality gives $0 \leq c_s^2 \leq 1$.

General quasilinear system of PDEs

unknowns

$$\Psi(x) \in \mathbb{R}^N$$

$$\mathcal{A}(\Psi, \partial)\Psi = \mathcal{B}$$

principal part

$$\mathcal{A}(\Psi, \partial)$$

Initial value problem (IVP):

- Find a solution with initial values of Ψ and its lower-order derivatives along the initial hypersurface Σ .
- IVP is locally well-posed if, for arbitrary initial data on Σ , there exists a unique solution in the neighborhood of Σ .
- For relativistic systems, causality must also hold.

See, e.g., Hawking, Ellis; Choquet-Bruhat; Witten, RMP (2020).

Why should one care about LWP?

Imagine that Einstein's eqs were not LWP. Then, at least one of the three happens: (i) solutions don't exist, (ii) are not unique, or (iii) don't depend continuously on initial data.

- If (i), then what you are generating in the computer has nothing to do with “GR” because you are producing solutions to “something” but this theory has no solution.
- If (ii), how do you know which solution you get with the simulation?
- If (iii), a numerical solution probably has nothing to do with the actual solution because small numerical errors should give you something close to the actual solution, but if there is no continuous dependence, then the two things are completely different.

What happens in ideal chiral hydro?

Challenges in solving ideal chiral hydro

Speranza, Bemfica, Disconzi, JN, PRD (2023)

$$\text{EOM: } \mathcal{A}(\Psi, \partial)\Psi = \mathcal{B} \qquad \Psi = (\varepsilon, n_V, n_A, u^\nu)$$

variables

- Higher-order derivatives should be expressible in terms of lower-order derivatives.
- This cannot happen if for any covector $\varphi_\mu = \partial_\mu \phi$.

$$\Sigma = \{\varphi(x) = 0\}$$

Arbitrary initial data

$$\Psi_0$$

Ill-posed initial value problem

$$\det[\mathcal{A}(\Psi_0, \varphi)] = 0.$$

Challenges in solving ideal chiral hydro

Speranza, Bemfica, Disconzi, JN, PRD (2023)

For the most general ideal chiral hydro (fully nonlinear EOM)

$$\begin{aligned} \det[\mathcal{A}(\Psi, \varphi)] &= \det \begin{bmatrix} b + \xi_{T,\varepsilon} c & \xi_{T,n_V} c & \xi_{T,n_A} c & 0_{1 \times 4} \\ \frac{1}{3} v^\mu + \xi_{T,\varepsilon} b \omega^\mu & (\xi_T)'_{n_V} b \omega^\mu & \xi_{T,n_A} b \omega^\mu & \frac{1}{2} \xi_T b u_\lambda v_\alpha \epsilon^{\lambda\alpha\mu}{}_\nu \\ \xi_{V,\varepsilon} c & b + \xi_{V,n_V} c & \xi_{V,n_A} c & 0_{1 \times 4} \\ \xi_{A,\varepsilon} c & \xi_{A,n_V} c & b + \xi_{A,n_A} c & 0_{1 \times 4} \end{bmatrix} \\ &= \left(\frac{\xi_T b}{2} \right)^4 \det \begin{bmatrix} b + \xi_{T,\varepsilon} c & \xi_{T,n_V} c & \xi_{T,n_A} c \\ \xi_{V,\varepsilon} c & b + \xi_{V,n_V} c & \xi_{V,n_A} c \\ \xi_{A,\varepsilon} c & \xi_{A,n_V} c & b + \xi_{A,n_A} c \end{bmatrix} \left(\det [u_\lambda v_\alpha \epsilon^{\lambda\alpha\mu}{}_\nu] = 0 \right) \end{aligned}$$

- One finds a vanishing characteristic determinant for any initial hypersurface: **Ill-posed initial-value problem.**
- This occurs for any hydro frame where $\xi_T \neq 0$.
- True for no drag, entropy, and thermodynamic frames.

- A natural modification: Landau frame

Speranza, Bemfica, Disconzi, JN, PRD (2023)

$$u_\mu T^{\mu\nu} = -\varepsilon u^\nu \qquad \xi_T = 0$$

$$T^{\mu\nu} = (\varepsilon + P) u_L^\mu u_L^\nu + g^{\mu\nu} P, \qquad \begin{aligned} J_V^\mu &= n_V u_L^\mu + \xi_{VL} \omega_L^\mu \\ J_A^\mu &= n_A u_L^\mu + \xi_{AL} \omega_L^\mu \end{aligned}$$

- Leads to a causal and well-posed theory if

$$\frac{\partial \xi_{AL}}{\partial n_A} \sqrt{\omega_\mu \omega^\mu} \leq 1 \qquad \text{within the regime of validity of derivative expansion}$$

- Here, the chiral anomaly constraints the magnitude of the vorticity (at 1st order in gradients).

Towards a causal and stable 1st order theory of viscous chiral hydrodynamics

Abboud, Speranza, JN, to appear soon

Causal and stable 1st order hydro

Bemfica, Disconzi, JN, PRD (2018), PRD (2019), PRX (2022)
Kovtun, JHEP (2019)

Most general tensor decomposition (e.g., without anomaly)

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu + \mathcal{T}^{\mu\nu}$$

NO ANOMALY

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu$$

Systematic expansion in Knudsen number/derivatives

$$T_{\mu\nu} = T_{\mu\nu}^{ideal} + K_N T_{\mu\nu}^{(1)} + \mathcal{O}(K_N^2)$$

$$J_\mu = J_\mu^{ideal} + K_N J_\mu^{(1)} + \mathcal{O}(K_N^2)$$

Causal and stable 1st order hydro

- Write the most general expansion at 1st order in derivatives compatible with symmetries.
- Most general hydrodynamic frame.

$$\mathcal{E} = \varepsilon + \varepsilon_1 \frac{u^\alpha \nabla_\alpha T}{T} + \varepsilon_2 \nabla_\alpha u^\alpha + \varepsilon_3 u^\alpha \nabla_\alpha (\mu/T),$$

$$\mathcal{P} = P + \pi_1 \frac{u^\alpha \nabla_\alpha T}{T} + \pi_2 \nabla_\alpha u^\alpha + \pi_3 u^\alpha \nabla_\alpha (\mu/T),$$

$$\mathcal{N} = n + \nu_1 \frac{u^\alpha \nabla_\alpha T}{T} + \nu_2 \nabla_\alpha u^\alpha + \nu_3 u^\alpha \nabla_\alpha (\mu/T),$$

$$\mathcal{Q}^\mu = \theta_1 \frac{\Delta^{\mu\nu} \nabla_\nu T}{T} + \theta_2 u^\alpha \nabla_\alpha u^\mu + \theta_3 \Delta^{\mu\nu} \nabla_\nu (\mu/T),$$

$$\mathcal{J}^\mu = \gamma_1 \frac{\Delta^{\mu\nu} \nabla_\nu T}{T} + \gamma_2 u^\alpha \nabla_\alpha u^\mu + \gamma_3 \Delta^{\mu\nu} \nabla_\nu (\mu/T)$$

$$\pi_{\mu\nu} = -2\eta\sigma_{\mu\nu}$$

NO ANOMALY

- Causality
- Stability
- Strong hyperbolicity
- LWP IVP
- 2nd law of thermo
- Fully coupled to GR
- Solution is globally hyperbolic spacetime.



Bemfica, Disconzi, JN, PRX (2022)

Viscous chiral hydro at 1st order

Abboud, Speranza, JN, to appear soon

- Write the most general expansion at 1st order in derivatives compatible with symmetries.

Weyl derivative

- Most general hydrodynamic frame, with anomaly.

\mathcal{D}

conformal invariant

$$\mathcal{A} = a_1 \mathcal{D}\varepsilon + a_2 \mathcal{D}n_V + a_3 \mathcal{D}n_A$$

$$\mathcal{Q}^\mu = b_1 \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + b_2 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + b_3 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_T \omega^\mu$$

$$\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$

$$\mathcal{N}_V = c_{V1} \mathcal{D}\varepsilon + c_{V2} \mathcal{D}n_V + c_{V3} \mathcal{D}n_A$$

$$\mathcal{J}_V^\mu = e_{V1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{V2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{V3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_V \omega^\mu$$

$$\mathcal{N}_A = c_{A1} \mathcal{D}\varepsilon + c_{A2} \mathcal{D}n_V + c_{A3} \mathcal{D}n_A$$

$$\mathcal{J}_A^\mu = e_{A1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{A2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{A3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_A \omega^\mu$$

Viscous chiral hydro at 1st order

Abboud, Speranza, JN, to appear soon

- For simplicity, we choose the class of hydrodynamic frames where:

$$\left. \begin{aligned} \mathcal{A} &= a_1 \mathcal{D}\varepsilon \\ \mathcal{Q}^\mu &= b_1 \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon \\ \pi^{\mu\nu} &= -2\eta \sigma^{\mu\nu} \end{aligned} \right\} \text{and } \xi_T = 0 \quad \left. \begin{array}{l} \text{Well-understood conformal limit} \\ \text{Bemfica, Disconzi, JN, PRD (2018)} \end{array} \right\}$$

$$\mathcal{N}_V = c_{V1} \mathcal{D}\varepsilon + c_{V2} \mathcal{D}n_V + c_{V3} \mathcal{D}n_A$$

$$\mathcal{J}_V^\mu = e_{V1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{V2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{V3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_V \omega^\mu$$

$$\mathcal{N}_A = c_{A1} \mathcal{D}\varepsilon + c_{A2} \mathcal{D}n_V + c_{A3} \mathcal{D}n_A$$

$$\mathcal{J}_A^\mu = e_{A1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{A2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{A3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_A \omega^\mu.$$

- We consider the most general form for the conserved currents.

Viscous chiral hydro at 1st order

Abboud, Speranza, JN, to appear soon

- Causality ✓
- Stability of homogeneous equilibrium ✓
- This theory can be used in simulations ✓
- Stability of rotating equilibrium state?
- Strong hyperbolicity/LWP?



Conclusions

- Chiral anomaly/vorticity has a drastic impact on hydro even in the ideal regime.
- Energy diffusion $\xi_T(u^\mu\omega^\nu + u^\nu\omega^\mu)$ in chiral hydro leads to ill-posedness, acausality.
- No drag frame, entropy, thermodynamic frame display this issue.
- A causal and stable 1st order theory with entropy production can be formulated with chiral anomaly.
- Generalize analysis to consider spin hydrodynamics.

ADDITIONAL SLIDES

- If $\xi_T \neq 0$ in $\xi_T(u^\mu \omega^\nu + u^\nu \omega^\mu)$ the theory has an ill-posed initial-value problem (and, hence, acausal).
- No solution to arbitrary initial data, or the solution is not unique.
- When $\xi_T \neq 0$, theory cannot be used to systematically determine how the anomaly or vorticity affect the hydrodynamic evolution.

Hydro frame transformations

The most general set of constitutive relations at $\mathcal{O}(\partial)$ are

$$T^{\mu\nu} = (\varepsilon + \mathcal{A}) \left(u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3} \right) + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}$$

$$J_V^\mu = (n_V + \mathcal{N}_V) u^\mu + \mathcal{J}_V^\mu$$

$$J_A^\mu = (n_A + \mathcal{N}_A) u^\mu + \mathcal{J}_A^\mu$$

where

$$\mathcal{A} = a_1 \mathcal{D}\varepsilon + a_2 \mathcal{D}n_V + a_3 \mathcal{D}n_A$$

$$\mathcal{Q}^\mu = b_1 \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + b_2 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + b_3 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_T \omega^\mu$$

$$\pi^{\mu\nu} = -2\eta \sigma^{\mu\nu}$$

$$\mathcal{N}_V = c_{V1} \mathcal{D}\varepsilon + c_{V2} \mathcal{D}n_V + c_{V3} \mathcal{D}n_A$$

$$\mathcal{J}_V^\mu = e_{V1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{V2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{V3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_V \omega^\mu$$

$$\mathcal{N}_A = c_{A1} \mathcal{D}\varepsilon + c_{A2} \mathcal{D}n_V + c_{A3} \mathcal{D}n_A$$

$$\mathcal{J}_A^\mu = e_{A1} \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + e_{A2} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + e_{A3} \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \xi_A \omega^\mu.$$

Hydro frame transformations

The most general frame transformation at $\mathcal{O}(\partial)$ is

$$\varepsilon \rightarrow \varepsilon + \delta\varepsilon \quad n_V \rightarrow n_V + \delta n_V \quad n_A \rightarrow n_A + \delta n_A \quad u^\mu \rightarrow u^\mu + \delta u^\mu.$$

where $\delta\varepsilon = \alpha_1 \mathcal{D}\varepsilon + \alpha_2 \mathcal{D}n_V + \alpha_3 \mathcal{D}n_A$

$$\delta u^\mu = \beta_1 \Delta^{\mu\lambda} \mathcal{D}_\lambda \varepsilon + \beta_2 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_V + \beta_3 \Delta^{\mu\lambda} \mathcal{D}_\lambda n_A + \beta_4 \omega^\mu$$

$$\delta n_V = \gamma_{V1} \mathcal{D}\varepsilon + \gamma_{V2} \mathcal{D}n_V + \gamma_{V3} \mathcal{D}n_A$$

$$\delta n_A = \gamma_{A1} \mathcal{D}\varepsilon + \gamma_{A2} \mathcal{D}n_A + \gamma_{A3} \mathcal{D}n_A.$$

coefficients get shifted

$$a_i \rightarrow a_i - \alpha_i$$

$$b_i \rightarrow b_i - \frac{4}{3} \varepsilon \beta_i$$

$$\xi_T \rightarrow \xi_T - \frac{4}{3} \varepsilon \beta_4$$

$$\eta \rightarrow \eta$$

$$c_{Vi} \rightarrow c_{Vi} - \gamma_{Vi}$$

$$e_{Vi} \rightarrow e_{Vi} - n \beta_i$$

$$\xi_V \rightarrow \xi_V - n \beta_4$$

$$c_{Ai} \rightarrow c_{Ai} - \gamma_{Ai}$$

$$e_{Ai} \rightarrow e_{Ai} - n \beta_i$$

$$\xi_A \rightarrow \xi_A - n \beta_4$$

$$\xi_S \rightarrow \xi_S - n \beta_4$$

$$K \equiv 2T\xi_S + 2\mu_V\xi_V + 2\mu_A\xi_A - 3\xi_T$$

$$\begin{aligned} K &\rightarrow K - [2Ts + 2\mu_V n_V + 2\mu_A n_A + 4\varepsilon] \beta_4 \\ &= \frac{20}{3} \varepsilon \beta_4, \end{aligned}$$

So no drag, thermodynamic, and entropy frame have $\xi_T \neq 0$

Define **equilibrium** with 5 variables: T, μ, u_μ constraint
 $u_\mu u^\mu = -1$

$$T_{\mu\nu} = \varepsilon u_\mu u_\nu + P \Delta_{\mu\nu}$$

$$\varepsilon = \varepsilon(T, \mu)$$

$$n = n(T, \mu)$$

$$P = P(\varepsilon, n)$$

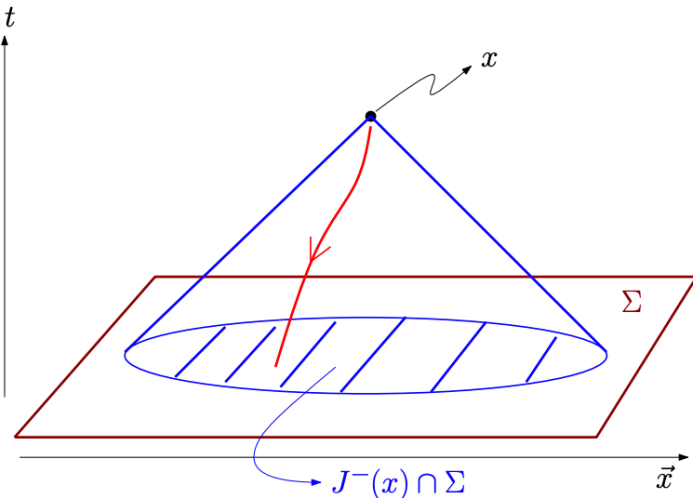
Equation of State

$$J_\mu = n u_\mu$$

5 EOM: Nonlinear PDEs

Conservation laws

- Compatible with relativity (causal) if $c_s^2 = \left(\frac{\partial P}{\partial \varepsilon} \right)_s \leq 1$
nonnegative



- Given initial T, μ, u_μ
- Locally well-posed initial-value problem.
 $\varepsilon > 0 \quad \varepsilon + P > 0$
- Well tested numerical algorithms.

Quasilinear PDEs: $\mathcal{A}(\Psi, \partial)\Psi = \mathcal{B}$

Characteristic surface $\phi(x^\mu) = 0$, normal $\phi_\alpha = \partial_\alpha \phi$

$$\phi_\alpha \in \mathbb{R}$$

- **Causality** holds if solution of $\det(A^\alpha \phi_\alpha) = 0$ gives $\phi_\alpha \phi^\alpha \geq 0$

- **Strong hyperbolicity**: Let $\xi_\alpha \in \mathbb{R}$ be any timelike vector. Then if

(i) $\det(A^\alpha \xi_\alpha) \neq 0$

LWP can be established

Bemfica, Disconzi, Graber, Jameson, Commun. Pure Appl. Anal. (2021)

(ii) The eigenvalue problem $(\zeta_\alpha + \Lambda \xi_\alpha) A^\alpha R = 0$ has only real eigenvalues Λ and a complete set of right-eigenvectors R , for any spacelike vector ζ_α .

Ideal chiral hydrodynamics

- Derivable from chiral kinetic theory

Chen, Son, Stephanov, PRL (2015)

- Local equilibrium distrib. for charged massless fermions

$$f_{\lambda}^{eq}(x, p) = \frac{2}{(2\pi\hbar)^3} \sum_{q=\pm 1} \theta[-q(u \cdot p)] f_D(g_{\lambda,q}) \delta(p^2)$$

where

$$g_{\lambda,q}(x, p) = -q\beta \cdot p - q \frac{\mu_{\lambda}}{T} - \frac{\hbar}{2} S^{\mu\nu} \varpi_{\mu\nu}$$

thermal vorticity

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_{\mu}\beta_{\nu} - \partial_{\nu}\beta_{\mu})$$

spin-dipole moment tensor

$$S^{\mu\nu} = \frac{\lambda}{2} \frac{\epsilon^{\mu\nu\rho\sigma} p_{\rho} n_{\sigma}}{p \cdot n}$$

frame
vector

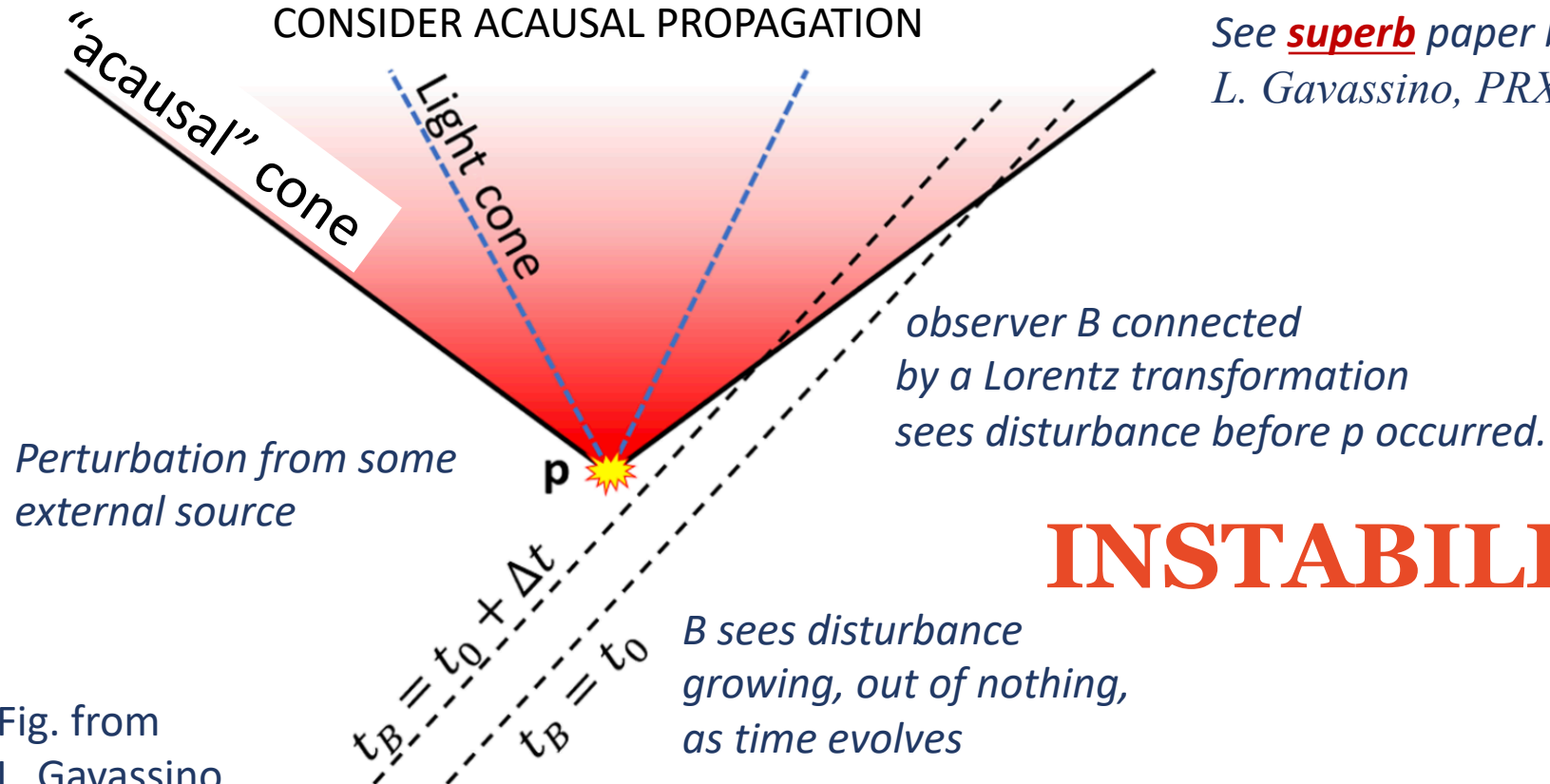
Causality is necessary for stability

Bemfica, Disconzi, JN, PRX (2022)

Dissipation is only a Lorentz invariant concept in causal theories

CONSIDER ACAUSAL PROPAGATION

See superb paper by
L. Gavassino, PRX (2022)



INSTABILITY!

Fig. from
L. Gavassino

Causality is not sufficient for stability

Consider the following nonlinear PDE

$$\partial_t^2 \varphi + \alpha \partial_t \varphi - \nabla^2 \varphi + \varphi^2 = S$$

Causality is not sufficient for stability

Consider the following nonlinear PDE

$$\partial_t^2 \varphi + \alpha \partial_t \varphi - \nabla^2 \varphi + \varphi^2 = S$$

For any α , this nonlinear equation is:

- Hyperbolic
- Causal

Causality is not sufficient for stability

Consider the following nonlinear PDE

$$\partial_t^2 \varphi + \alpha \partial_t \varphi - \nabla^2 \varphi + \varphi^2 = S$$

For any α , this nonlinear equation is:

- Hyperbolic

Linearization around
equilibrium state: $\varphi = 0$

Stable if $\alpha \geq 0$

- Causal

Unstable if $\alpha < 0$

Thermodynamic stability implies causality (linear)

L. Gavassino, M. Antonelli, B. Haskell, PRL (2022)

Linearization around the equilibrium state: assume thermodynamic stability

$$E[\Sigma] := -\delta\Phi[\Sigma] = \int_{\Sigma} E^a n_a d\Sigma \geq 0$$

with $E^a = -\delta(s^a + \alpha_I^* J^{Ia}) = -\delta s^a - \alpha_I^* \delta J^{Ia}$

$$E = \left(\begin{array}{c} \text{Ignorance at} \\ \text{equilibrium} \end{array} \right) - \left(\begin{array}{c} \text{Ignorance in the} \\ \text{perturbed state} \end{array} \right)$$

$$E = T\Omega$$

Thermodynamic equilibrium in relativity

Minimized in equilibrium

- (i) - For any unit vector n^a , time-like and past-directed ($n^a n_a = -1$, $n^0 < 0$), we have

$$E^a n_a \geq 0. \quad (4)$$

- (ii) - For the same n^a as in (i), $E^a n_a = 0$ on any point where the perturbation to every observable is zero, and only on these points.

- (iii) - The four-divergence of E^a is non-positive:

$$\nabla_a E^a \leq 0. \quad (5)$$



There is a proof that disturbances remain inside the light cone (causality)

Causality violation and instabilities

- **W. Israel:** *“If the source of an effect can be delayed, it should be possible for a system to borrow energy from its ground state, and this implies instability”*
- **Hawking-Ellis vacuum conservation theorem:** *if energy can enter an empty region faster than the speed of light, then the dominant energy condition is violated, and the energy density may become negative in some reference frame.*
- **Thermal version, L. Gavassino PRX (2022):** *“If the source of an effect can be delayed, it should be possible for a system to borrow entropy from its equilibrium state, and this implies instability”.*

Example: Einstein-Israel-Stewart theory

Bemfica, Disconzi, JN, PRL (2019)

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi) \Delta^{\mu\nu}$$

ONLY

Bulk viscosity: Π

+ coupling to Einstein's equations

Conservation laws

$$\nabla_\mu T^{\mu\nu} = 0 \quad \nabla_\mu (n u^\mu) = 0$$

Israel-Stewart-like equation

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi + \zeta \nabla_\mu u^\mu = 0$$

Israel, Stewart, Ann. Phys. (1979)

- Causality, strong hyperbolicity, stability constraint:

Connects EOS, transport, and dissipative fields

$$\left[\frac{\zeta}{\tau_\Pi} + n \left(\frac{\partial P}{\partial n} \right)_\varepsilon \right] \frac{1}{\varepsilon + P + \Pi} \leq 1 - \left(\frac{\partial P}{\partial \varepsilon} \right)_n$$

Nonlinear constraints for DNMR/Israel-Stewart eqs

Bemfica, Disconzi, Hoang, JN, Radosz, [PRL \(2021\)](#)

(First general result of the kind)

- Computation of characteristic velocities (nonlinear problem)
- Causality in the **nonlinear regime**: shear + bulk effects

Novel nonlinear constraints for dissipative quantities

$$(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0$$

(4a)

$$\varepsilon + P + \Pi - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}\Lambda_3 \geq 0,$$

(4b)

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) + \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_a + \Lambda_d) \geq 0, \quad a \neq d,$$

(4c)

$$\varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_d + \Lambda_a) \geq 0, \quad a \neq d$$

(4d)

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) + \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d + \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi\Pi} + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d]$$

(4e)

$$+ \frac{\zeta + \delta_{\Pi\Pi\Pi} + \lambda_{\Pi\Pi}\Lambda_d}{\tau_{\Pi}} + (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0,$$

$$\varepsilon + P + \Pi + \Lambda_d - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d - \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi\Pi} + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d]$$

(4f)

$$- \frac{\zeta + \delta_{\Pi\Pi\Pi} + \lambda_{\Pi\Pi}\Lambda_d}{\tau_{\Pi}} - (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0,$$

$$(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_3 \geq 0, \quad (5a)$$

$$(2\eta + \lambda_{\pi\Pi\Pi}) - \tau_{\pi\pi}|\Lambda_1| > 0, \quad (5b)$$

$$\tau_{\pi\pi} \leq 6\delta_{\pi\pi}, \quad (5c)$$

$$\frac{\lambda_{\Pi\Pi}}{\tau_{\Pi}} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_{\pi}} \geq 0, \quad (5d)$$

$$\frac{1}{3\tau_{\pi}}[4\eta + 2\lambda_{\pi\Pi\Pi} + (3\delta_{\pi\pi} + \tau_{\pi\pi})\Lambda_3] + \frac{\zeta + \delta_{\Pi\Pi\Pi} + \lambda_{\Pi\Pi}\Lambda_3}{\tau_{\Pi}} + |\Lambda_1| + \Lambda_3 c_s^2$$

$$+ \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_{\pi}} \left(\frac{\lambda_{\Pi\Pi}}{\tau_{\Pi}} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_{\pi}} \right) (\Lambda_3 + |\Lambda_1|)^2}{\varepsilon + P + \Pi - |\Lambda_1| - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_3} \leq (\varepsilon + P + \Pi)(1 - c_s^2), \quad (5e)$$

$$\frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi\Pi} + (\tau_{\pi\pi} - 6\delta_{\pi\pi})|\Lambda_1|] + \frac{\zeta + \delta_{\Pi\Pi\Pi} - \lambda_{\Pi\Pi}|\Lambda_1|}{\tau_{\Pi}} + (\varepsilon + P + \Pi - |\Lambda_1|)c_s^2 \geq 0, \quad (5f)$$

$$1 \geq \frac{\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_{\pi}} \left(\frac{\lambda_{\Pi\Pi}}{\tau_{\Pi}} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_{\pi}} \right) (\Lambda_3 + |\Lambda_1|)^2}{\left[\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}|\Lambda_1| \right]^2} \quad (5g)$$

$$\frac{1}{3\tau_{\pi}}[4\eta + 2\lambda_{\pi\Pi\Pi} - (3\delta_{\pi\pi} + \tau_{\pi\pi})|\Lambda_1|] + \frac{\zeta + \delta_{\Pi\Pi\Pi} - \lambda_{\Pi\Pi}|\Lambda_1|}{\tau_{\Pi}} + (\varepsilon + P + \Pi - |\Lambda_1|)c_s^2$$

$$\geq \frac{(\varepsilon + P + \Pi + \Lambda_2)(\varepsilon + P + \Pi + \Lambda_3)}{3(\varepsilon + P + \Pi - |\Lambda_1|)} \left\{ 1 + \frac{2 \left[\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi\Pi}) + \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_3 \right]}{\varepsilon + P + \Pi - |\Lambda_1|} \right\}, \quad (5h)$$

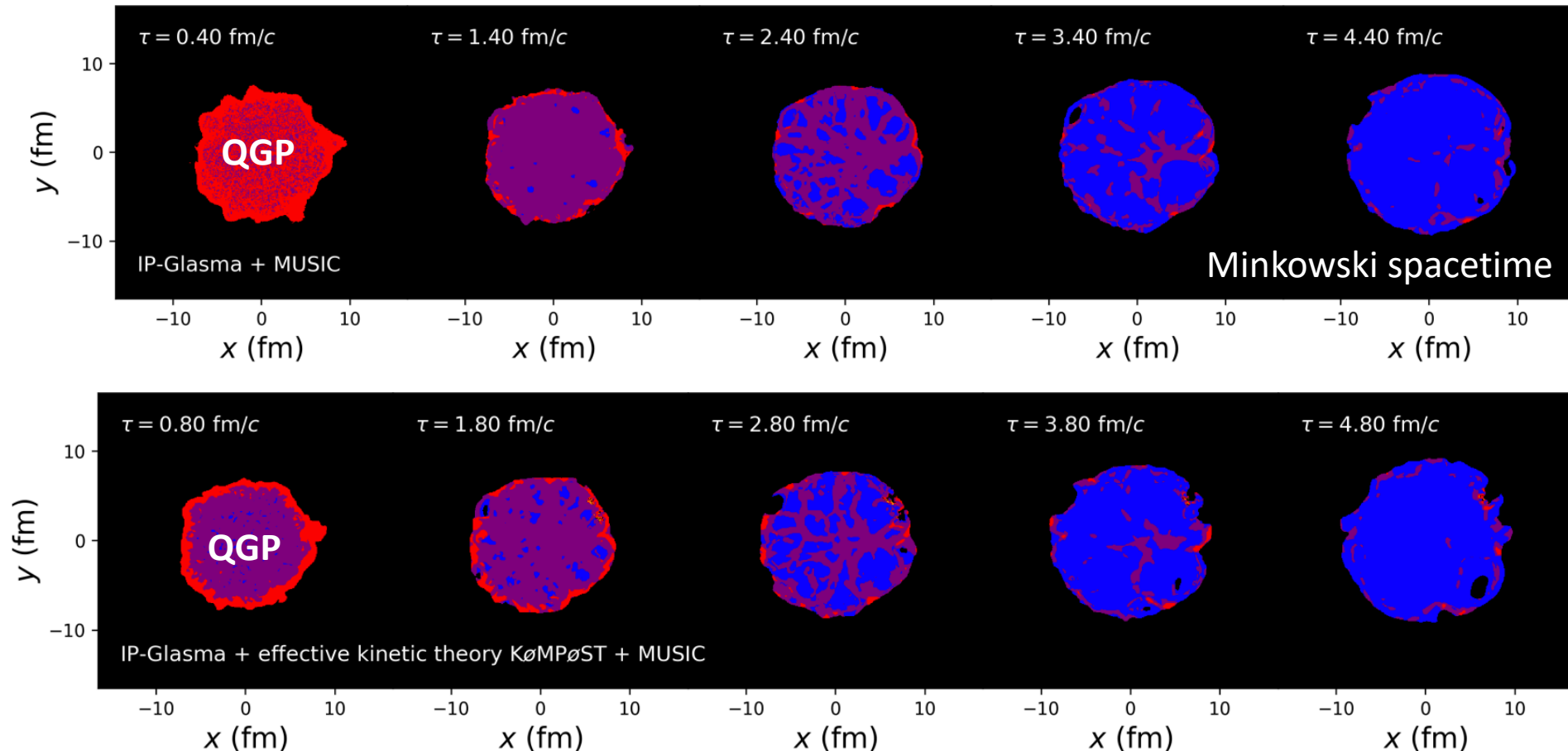
Causality violation in state-of-the-art simulations

Bemfica, Disconzi, Hoang, JN, Radosz, PRL (2021)

Plumberg, Almaalol, Dore, JN, Noronha-Hostler, PRC Letters (2022)

Very large deviations from equilibrium in the initial state

Diagnostics: **Red: Acausal** **Purple: Unknown** **Blue: Causal**



In GR, you need to be able to prescribe data on any spacelike surface. If you have a system that is superluminal, it will have characteristics that are spacelike. From the GR point of view you should be able to prescribe data on that spacelike surface, but for the acausal system you cannot because you can't prescribe arbitrary data on a characteristics surface.

Definition 8.6 (Strongly causal spacetime)

A spacetime \mathcal{M} is strongly causal if given an arbitrarily chosen event $p \in \mathcal{M}$ for each $U \subset \mathcal{M}$ open neighborhood of p there exist another open neighborhood of p , $V \subset U$, such that no causal curve intersects it more than once.

Definition 8.7 (Inextendible causal curve)

A causal curve γ_C is called future (resp. past) inextendible if it is impossible to find an event $p \in \mathcal{M}$ such that for all $U \subset \mathcal{M}$, U neighborhood of p , there exist a t' such that $\gamma_C(t) \in U$ for all $t > t'$ (resp $t < t'$)

In more concrete words, this means that γ_C has no future (resp. past) endpoint.

Definition 8.10 (Domains of dependence)

Let \mathcal{A} be a closed achronal set. The set $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$) of all spacetime events p such that every past (resp. future) inextendible causal curve passing through p intersects \mathcal{A} is called the future (resp. past) domain of dependence of \mathcal{A} . The set $D(\mathcal{A}) = D^+(\mathcal{A}) \cup D^-(\mathcal{A})$, union of the past and of the future domains of dependence is the domain of dependence of \mathcal{A} .

Definition 8.11 (Cauchy surface and global hyperbolicity)

Let $\mathcal{A} \subset \mathcal{M}$ be an achronal set such that $D(\mathcal{A}) = \mathcal{M}$. Then \mathcal{A} is called a Cauchy surface (we instead use the denomination partial Cauchy surface for a closed achronal set without edge). A spacetime \mathcal{M} which admits a Cauchy surface is called globally hyperbolic.

Analytic functions obey ($\alpha = \text{multi-index}$):

$$|\partial^\alpha f| \leq C^{|\alpha|+1} \alpha!$$

The **Gevrey** class $\gamma^{(\sigma)}$, $\sigma > 1$, consists of C^∞ functions that obey the **weaker** inequality:

$$|\partial^\alpha f| \leq C^{|\alpha|+1} (\alpha!)^\sigma.$$

Advantage: large class of functions, including compactly supported (not determined by values on an open set).

The larger the σ , the larger the space. Larger σ : more general results.

$\gamma^{(\infty)}$ = Sobolev space.

$\gamma^{(\sigma)}$: used in the study of non-relativistic viscous fluids; also have had applications in General Relativity (magneto-hydrodynamics).

Sobolev H_s

$$||f||_s^2 = \sum_{|j| \leq s} ||\partial^j u||_{L^2}^2$$

Characteristics.

Consider the linear differential operator:

$$Lu = a^{\mu\nu}(x) \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b(x, \partial u)$$

or, more generally (α =multi-index),

$$Lu = \sum_{|\alpha|=m} a^\alpha(x) \partial_\alpha u + b(x, \partial^{m-1} u, \dots, \partial u, u).$$

We define the **characteristic cone** V_x of L at T_x^*M by

$$h(x, \xi) \equiv \sum_{|\alpha|=m} a^\alpha(x) \xi_\alpha = 0.$$

$h(x, \xi)$ (= **characteristic polynomial**) is a homogeneous polynomial of degree m .

Hyperbolic polynomials (Leray).

$h(x, \xi)$ is called a **hyperbolic polynomial** (at x) if there exists $\zeta \in T_x^*M$ such that every line through ζ that does not pass through the origin intersects V_x at m real distinct points ($m = \text{degree of } h = \text{order of } L$).

In this case, the set of $\zeta \in T_x^*M$ with this property forms the interior of two opposite **convex half-cones** Γ_x^\pm .

The differential operator L is called **hyperbolic** (at x) if $h(x, \xi)$ is hyperbolic.

Dualizing, one obtains $C_x^\pm \subset T_x M$. For example

$$C_x^+ = \{v \in T_x M \mid \zeta(v) \geq 0 \text{ for all } \zeta \in \Gamma_x^+\}.$$

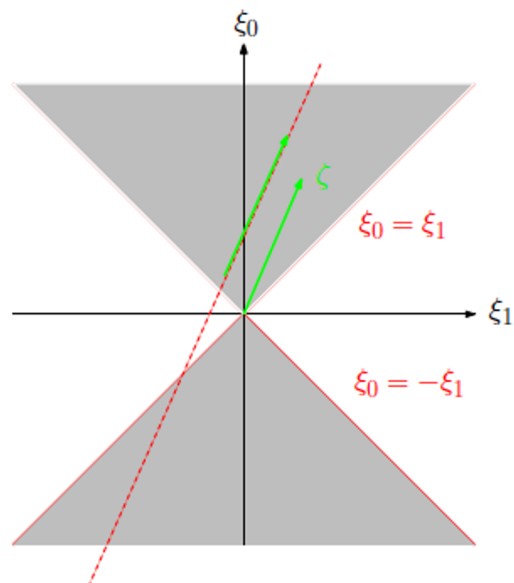
$\Sigma^n = \{\varphi(x) = 0\} \subset M^{n+1}$ is **characteristic for** L if

$$\sum_{|\alpha|=m} a^\alpha(x) \partial_\alpha \varphi = 0.$$

Wave equation: characteristics.

Consider $Lu = u_{tt} - u_{xx}$, $\xi = (\xi_0, \xi_1)$. Then:

$$\xi_0^2 - \xi_1^2 = 0 \Rightarrow \xi_0 = \pm \xi_1.$$



Γ_x and C_x are both given by the “light-cone”.

Hyperbolic and weakly hyperbolic operators.

Hyperbolic operators (sometimes called strictly hyperbolic) have a **Cauchy problem** that is well-posed in Sobolev spaces.

When the definition of a hyperbolic polynomial is weakened to:

there exists $\zeta \in T_x^*M$ such that every line through ζ that does not pass through the origin intersects V_x at m , **not necessarily distinct**, real points, we obtain **weakly** hyperbolic polynomials and operators ($m = \text{degree of } h = \text{order of } L$).

Weakly hyperbolic operators are well-posed in Gevrey spaces, but there are counter-examples to well-posed in Sobolev spaces.