lattice
virtual
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Lattice field theory with worldlines and worldsheets: Part I - scalar field

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Section: *XXXXXXXX*
Part: 1/??

Overview

- Many lattice field theories can be rewritten exactly in terms of worldlines and worldsheets.

- This new representation highlights different physical aspects and in some cases solves complex action problems / sign problems.

- As a first example we derive and discuss the worldline representation of the charged scalar field with a chemical potential.

Charged scalar ϕ^4 field

Continuum action ($\phi(x) \in \mathbb{C}$):

$$S = \int d^4x \left[-\phi(x)^* \triangle \phi(x) + [m^2 - \mu^2] \, |\phi(x)|^2 + \lambda \, |\phi(x)|^4 \right] + i\mu N$$

Action on the lattice ($\kappa = 8 + m^2$):

$$S = \sum_x \left[\kappa \, |\phi_x|^2 + \lambda \, |\phi_x|^4 - \sum_{j=1}^{3} \left(\phi_x^\star \, \phi_{x+\hat{j}} + \phi_x^\star \, \phi_{x-\hat{j}} \right) - \phi_x^\star \, e^{-\mu} \, \phi_{x+\hat{4}} - \phi_x^\star \, e^{\mu} \, \phi_{x-\hat{4}} \right]$$

$$= \sum_x \left[\kappa |\phi_x|^2 + \lambda \, |\phi_x|^4 \right] - \sum_{x,\nu} \left[e^{-\mu \, \delta_{\nu 4}} \, \phi_x^\star \, \phi_{x+\hat{\nu}} + e^{\mu \, \delta_{\nu 4}} \, \phi_x \, \phi_{x+\hat{\nu}}^\star \right]$$

Complex action problem:

- For $\mu \neq 0$ the action S has an imaginary part.
- e^{-S} is complex and cannot be used as a probability in a stochastic process.
- No direct application of Monte Carlo techniques.

First steps towards the worldline representation

Lattice action:

$$S \;=\; \sum_x \left[\kappa\,|\phi_x|^2 \;+\; \lambda\,|\phi_x|^4 \right] - \sum_{x,\nu} \left[e^{-\mu\,\delta_{\nu 4}}\,\phi_x^\star\,\phi_{x+\widehat{\nu}} \;+\; e^{\mu\,\delta_{\nu 4}}\,\phi_x\,\phi_{x+\widehat{\nu}}^\star \right]$$

$$e^{-S} \;=\; \prod_x e^{-\kappa|\phi_x|^2 \,-\, \lambda|\phi_x|^4} \prod_{x,\nu} \exp\left(e^{-\mu\,\delta_{\nu 4}}\,\phi_x^\star\,\phi_{x+\widehat{\nu}} \right) \times \exp\left(e^{\mu\,\delta_{\nu 4}}\,\phi_x\,\phi_{x+\widehat{\nu}}^\star \right)$$

Expand the nearest neighbor terms of e^{-S}:

$$\prod_{x,\nu} \exp\left(e^{-\mu\,\delta_{\nu 4}}\,\phi_x^\star\,\phi_{x+\widehat{\nu}} \right) \times \exp\left(e^{\mu\,\delta_{\nu 4}}\,\phi_x\,\phi_{x+\widehat{\nu}}^\star \right)$$

$$= \prod_{x,\nu} \sum_{j_{x,\nu}=0}^{\infty} \frac{\left(e^{-\mu\,\delta_{\nu 4}}\right)^{j_{x,\nu}}}{j_{x,\nu}\,!} \left(\phi_x^\star\,\phi_{x+\widehat{\nu}}\right)^{j_{x,\nu}} \times \sum_{\overline{j}_{x,\nu}=0}^{\infty} \frac{\left(e^{\mu\,\delta_{\nu 4}}\right)^{\overline{j}_{x,\nu}}}{\overline{j}_{x,\nu}\,!} \left(\phi_x\,\phi_{x+\widehat{\nu}}^\star\right)^{\overline{j}_{x,\nu}}$$

$$= \sum_{\{j,\overline{j}\}} e^{-\mu \sum_x (j_{x,4}-\overline{j}_{x,4})} \prod_{x,\nu} \frac{1}{j_{x,\nu}!\,\overline{j}_{x,\nu}!} \prod_x \phi_x^{\sum_\nu (\overline{j}_{x,\nu}+j_{x-\widehat{\nu},\nu})}\,\phi_x^{\star\,\sum_\nu (j_{x,\nu}+\overline{j}_{x-\widehat{\nu},\nu})}$$

The $j_{x,\nu}$ and $\overline{j}_{x,\nu}$ will turn into the new worldline degrees of freedom.

Worldline representation - integrating out the fields

Integral over ϕ_x at site x: ($\Sigma_j, \overline{\Sigma}_j$ are the sums of $j_{y,\nu}, \overline{j}_{y,\nu}$ connected to x)

$$\int_{\mathbb{C}} d\,\phi_x \;\; e^{-\kappa\,|\phi_x|^2 - \lambda\,|\phi_x|^4} \; (\phi_x)^{\Sigma_j} \; (\phi_x^\star)^{\overline{\Sigma}_j}$$

Polar coordinates $\phi_x = r e^{i\theta}$ to separate radial and U(1) parts (symmetry):

$$\int_0^\infty dr \; r^{\,\Sigma_j + \overline{\Sigma}_j + 1} \; e^{-\kappa\, r^2 - \lambda\, r^4} \int_{-\pi}^{\pi} d\theta \; e^{\,i\theta\,(\Sigma_j - \overline{\Sigma}_j)} \; = \; \mathcal{I}(\Sigma_j + \overline{\Sigma}_j) \; \delta(\Sigma_j - \overline{\Sigma}_j)$$

At every site there is a weight factor $\mathcal{I}(\Sigma_j + \overline{\Sigma}_j)$ and a constraint.

The constraint $\delta(\Sigma_j - \overline{\Sigma}_j)$ enforces vanishing flux of the new link-based worldline variables

$$k_{x,\nu} \; = \; j_{x,\nu} - \overline{j}_{x,\nu} \; \in \; \mathbb{Z}$$

"forward hoppings minus backward hoppings"

Worldline representation - final form

The original partition function is mapped exactly to a sum over configurations of the variables $j_{x,\nu}, \overline{j}_{x,\nu} \in \mathbb{N}_0$ with $k_{x,\nu} = j_{x,\nu} - \overline{j}_{x,\nu} \in \mathbb{Z}$.

$$Z \;=\; \sum_{\{j,\overline{j}\}} \mathcal{W}[\,j,\overline{j}\,]\,\mathcal{C}[\,k\,]$$

The weight factor from radial d.o.f. and combinatorics is real !!! :

$$\mathcal{W}[\,j,\overline{j}\,] = e^{-\mu \sum_x (j_{x,4} - \overline{j}_{x,4})} \prod_{x,\nu} \frac{1}{j_{x,\nu}!\,\overline{j}_{x,\nu}!} \prod_x \mathcal{I}\Big(\sum_\nu \Big[j_{x,\nu} + j_{x-\widehat{\nu},\nu} + \overline{j}_{x,\nu} + \overline{j}_{x-\widehat{\nu},\nu} \Big] \Big)$$

$$\mathcal{I}(n) \;=\; \int_0^\infty dr\; r^{\,n+1}\, e^{-\kappa\, r^2 - \lambda\, r^4}$$

(pre-computed numerically)

Constraint from integrating over the symmetry group ($k_{x,\nu} = j_{x,\nu} - \overline{j}_{x,\nu}$):

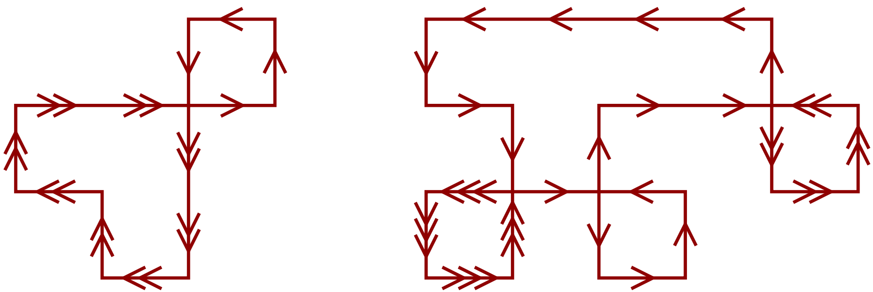
$$\mathcal{C}[\,k\,] \;=\; \prod_x \delta\Big(\sum_\nu \,[\,k_{x,\nu} - k_{x-\widehat{\nu},\nu}\,] \Big)$$

Admissible configurations are loops of worldlines:

Constraint from integrating over the symmetry group:

$$\forall\, x \;:\qquad \vec{\nabla}\vec{k}_x \;=\; \sum_{\nu} \left[\, k_{x,\nu} - k_{x-\widehat{\nu},\nu} \,\right] \;=\; 0$$

Admissible configurations of dual variables are oriented loops of flux:



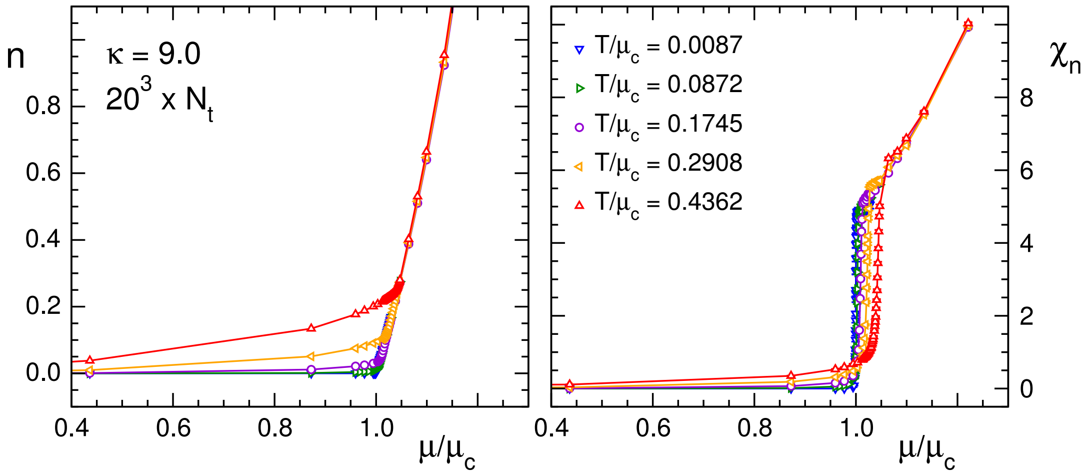
Chemical potential μ couples to the temporal winding number ω of $k_{x,\nu}$-flux.

$\Rightarrow$ Net particle number n is the topological invariant ω.

MC simulations directly in terms of world lines overcome the complex action problem.

Example result:

Onset of condensation:



C. Gattringer, T. Kloiber, Nuclear Physics B 869 (2013)

Update of the constrained worldline variables with worm algorithms $\Rightarrow$ separate lecture.