Critical Phenomena and Renormalization Group

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General Discussion

Topic may be important to highlight and strengthen connections between different concepts & methods in other topics (Essentials, Algorithms, Thermodynamics)

Also useful to illustrate general topics in Statistical Mechanics (spin models, conformal transition, KT transitions and quantum transitions in general) and Complex Systems. E.g., spin glasses, which are also relevant to quantum annealing and machine learning: phase diagram of QCD in the Nf plane shares similarities with that of a quantum glass in the temperature vs. magnetic field plane

Motivations

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B) Continuum limit may be viewed as a critical point, with scaling and universality properties (e.g. invariance of the limit for different actions/discretizations, as explored in the topic of improvement) given by the RG as a consequence. Material already available in A. Hasenfratz's lectures

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3) Phenomenologically interesting phase transitions and near-critical systems, what is known, experimental/observational implications (QCD, electroweak, condensed matter)
   $\Rightarrow$ Overlap with Thermodynamics

Outline

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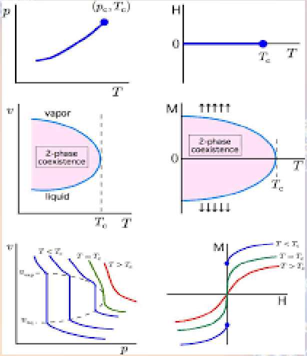
1. Critical Behavior & Continuum Limit
   - Phase Transitions: Ising Model
   - Criticality, Scaling, Universality
   - Wilsonian RG
   - Applications
2. Numerical Methods
   - Finite Size Scaling
   - Reweighting
   - Critical Slowing-Down in Simulations
   - Worm Algorithm
3. Interesting Phase Transitions

Ising Model

Simplest model to exhibit nontrivial behavior, SSB

$$H = -J \sum_{\langle i,j \rangle} S_i S_j$$

Phase Diagram



Fluctuations

Very useful to illustrate important concepts

$$f(T) = -\frac{1}{\beta} \ln \left(\mathrm{tr}\ \mathrm{e}^{-\beta \mathrm{H}} \right)$$

Hamiltonian $\Rightarrow$ partition function $\Rightarrow$ free energy

Order parameter

$$M = \sum_i S_i$$

obtained from first derivative of f with respect to magnetic field, second derivative gives susceptibility, related to fluctuation $\langle M^2 \rangle - \langle M \rangle^2$

Also: correlation functions related to length scale ξ

Kramers-Wannier duality in 2d, Onsager solution, high-T expansion

Spontaneous Symmetry Breaking

Symmetric physical laws give rise to asymmetric state (ferromagnetism, Higgs mechanism via "sombrero" potential, chiral symmetry)

Is this really so natural!?

Historical-epistemological analysis of the notion of "spontaneous symmetry breaking"

> *Between symmetry and asymmetry: spontaneous symmetry breaking as narrative knowing*
>
> Arianna Borrelli, Synthese 198, 3919–3948 (2021)

Monte Carlo Simulation

Time to compute observables by deterministic method: Number of terms in the partition function is $2^{\text{volume}} \sim 2^{1000} \sim 10^{300}$ operations, hence roughly 10^{266} times the AGE OF THE UNIVERSE!!

Monte Carlo simulation ensures computation of observable (center value of ensemble average) correct to within $O(1/\sqrt{N})$ using N independent samples of the system's configuration

Actually: impossible to sample Boltzmann distribution directly, so generate samples from Markov chain evolution (i.e. SIMULATION) in configuration space, taking "time" correlation into account

Exercise: simulate 2d Ising model and compare to exact solution

Correlations, Fluctuations, Scaling

Loss of scale (diverging correlation length) leads to singularity in response functions at criticality, e.g. diverging susceptibility, and SCALING LAWS

Critical Exponents: for Magnetization M (order parameter) and Susceptibility χ

$$M_{h=0} \sim |t|^{\beta}, \qquad \chi_{h=0} \sim |t|^{-\gamma}, \qquad M_{t=0} \sim h^{1/\delta}$$

where $t = (T - T_c)/T_0, \; h = H/H_0$.

Scaling Function: given T_0 and H_0, M is described by the universal function f_M for all systems in the universality class

$$M/h^{1/\delta} = f_M(t/h^{1/\beta\delta})$$

Also $\xi \sim |t|^{-\nu}$, so correlations do not decay exponentially anymore!

Universality

Same critical exponents for class of systems

Also implies that different lattice discretizations have same continuum limit

Dictionary: statistical mechanics $\Leftrightarrow$ euclidean QFT

Renormalization

Lattice "distances" $\xi_{\rm latt}$ given by

$$\xi_{\rm phys} = \frac{1}{m_{\rm phys}} = \xi_{\rm latt}\, a$$

Thus must have $\xi_{\rm latt} \to \infty$ as $a \to 0 \Rightarrow$ continuum limit is critical point!

How is this picture manifest in QCD? Other possibilities? conformal fixed points? UV-stable fixed points?

Expect behavior given by RG expression, asymptotic scaling
See e.g. Lectures by G. Münster at PSI Zuoz Summer School, 2000

So we trust that we know what we are doing, and renormalization is achieved by apropriately "removing" the cutoff a

Interesting reading on history of renormalization in QFT, by W. Bietenholz

Wilsonian RG & Applications

Material in Anna Hasenfratz's lectures

Video from ICTS Bangalore, 2022
Continuum limit, critical phenomena, blocking transformations, RG flow

Applications to Ising model, QCD-like systems, conformal systems, exercises

Outline

Numerical Methods

Finite-size Scaling, Reweighting, Multicanonical Simulations

Critical Slowing-Down

Autocorrelation time grows with system size as L^z

See e.g. Lectures from Introduction to Lattice Simulations course at Swieca School in Brazil, 2009

Important in order to develop more efficient algorithms, since the dynamics is chosen, rather than imposed by the physical system!

References (MC Errors):

- Book Chapter by A. Sokal "Monte Carlo Methods in Statistical Mechanics: Foundations and New Algorithms", 1997
- Article by U. Wolff "Monte Carlo errors with less errors", 2004

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Interesting Phase Transitions

Scalar field theory, Higgs theory, electroweak symmetry breaking

Triviality(?) of scalar field theory, Triviality bounds on Higgs mass

QCD phase phase transition (what symmetries are in play?)

Semimetal-insulator transition in graphene (invite Buividovich)