Particle-vibration coupling effects on dipole strength function at finite temperature in the covariant density functional theory framework

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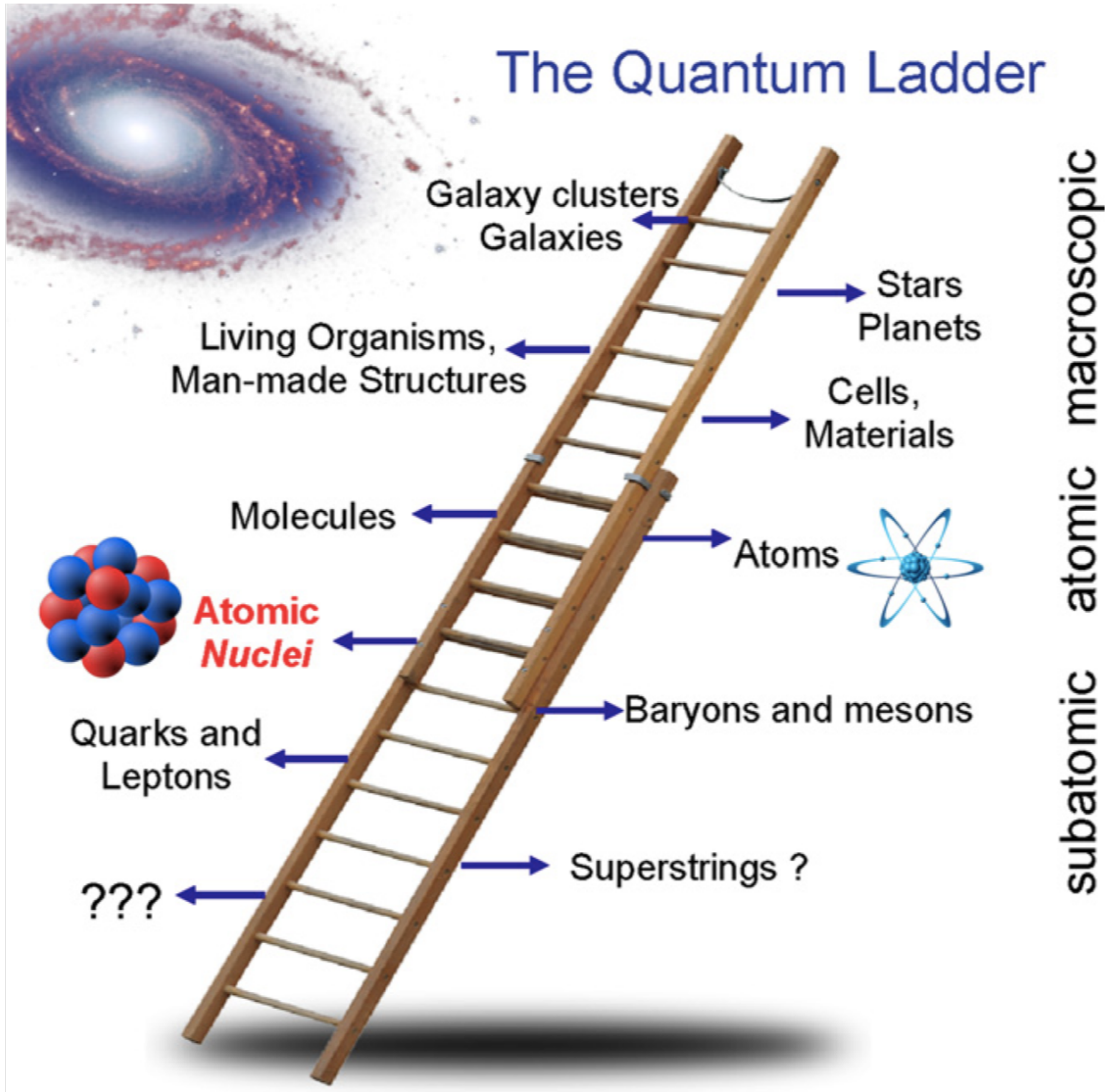
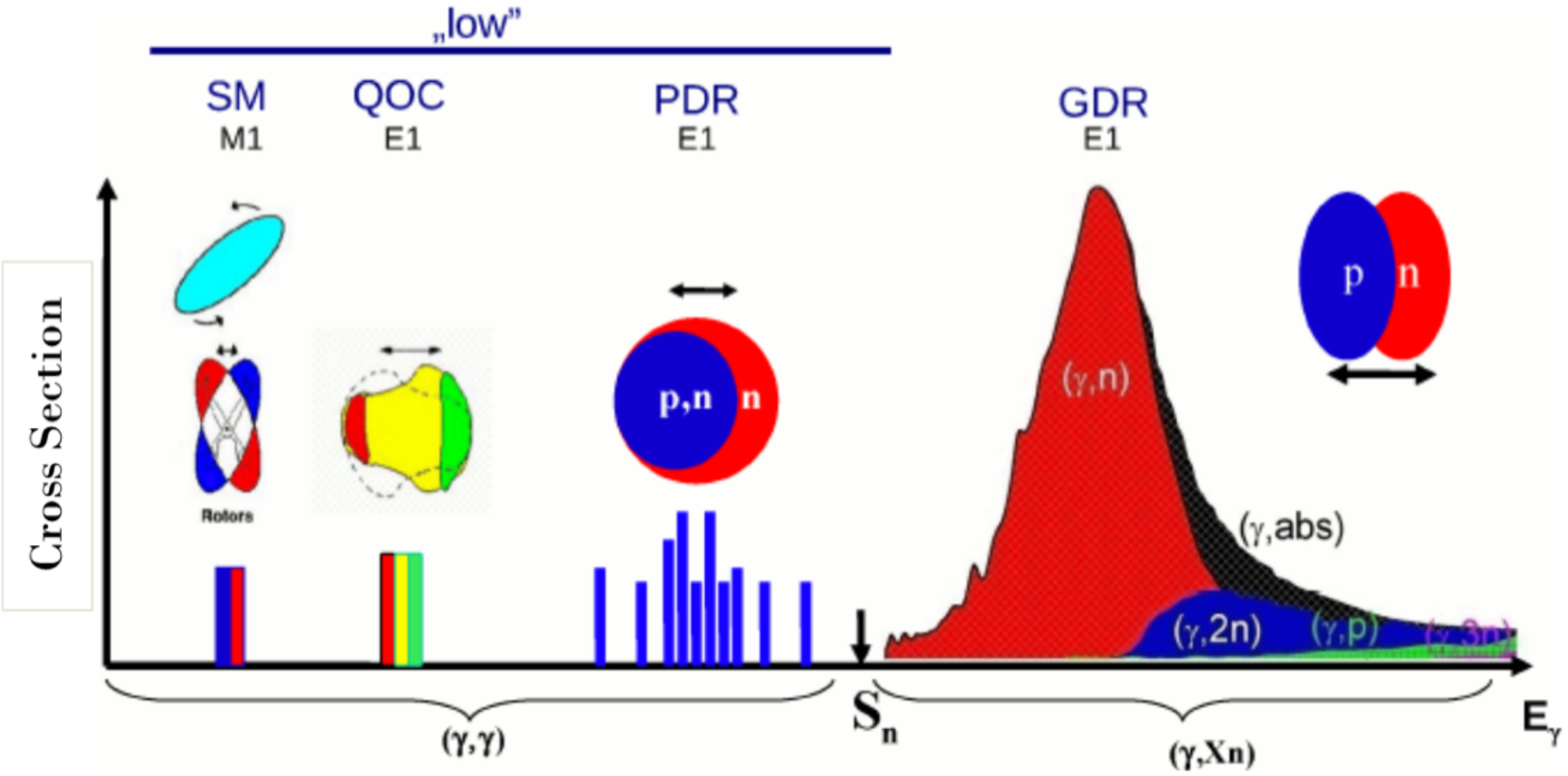
Outline

- Motivation
- Nuclear mean field at finite temperature
- Correlations beyond mean field
- Numerical results
- Summary and outlooks

Motivation

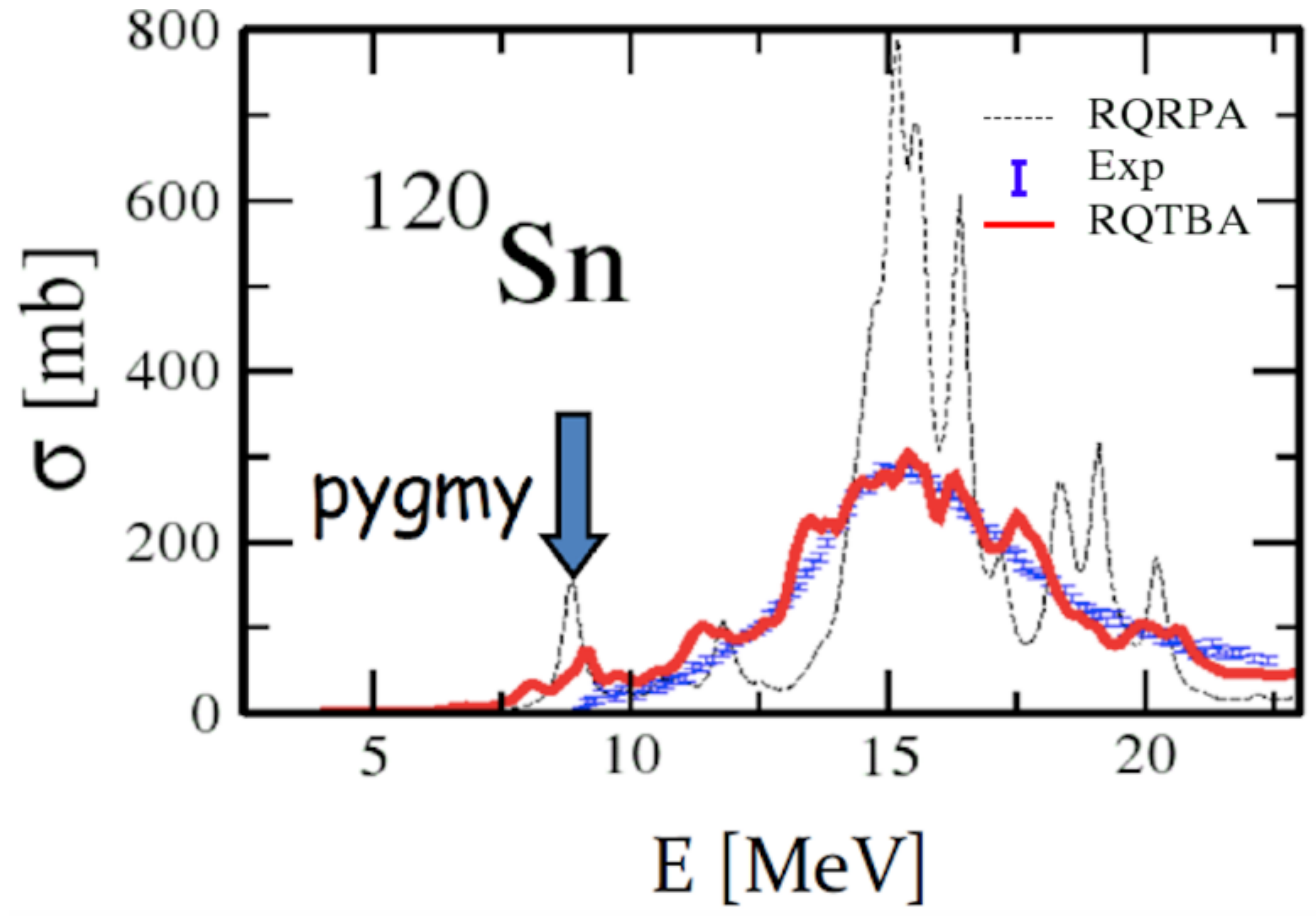


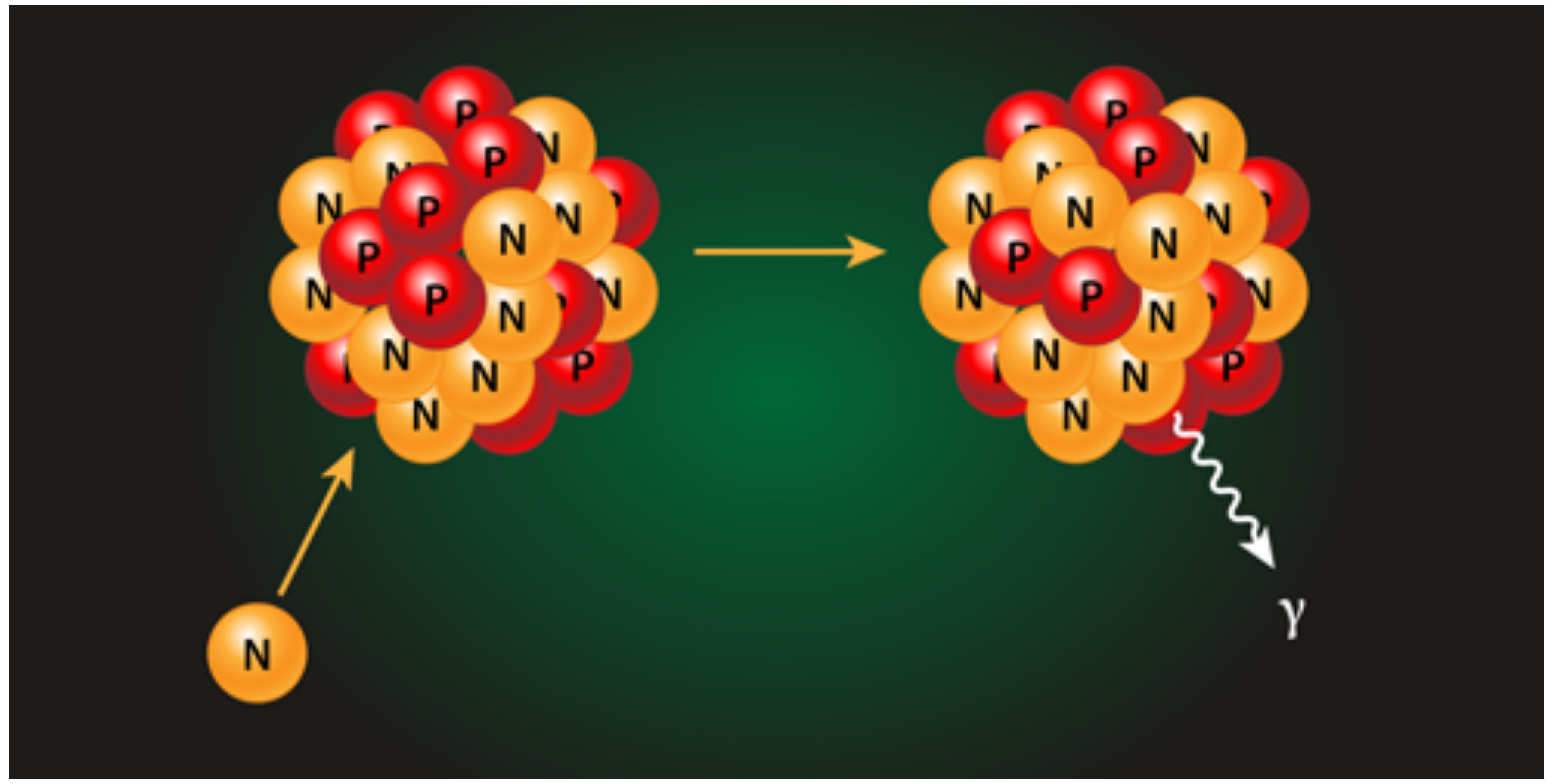
W. Nazarewicz, Nucl. Part. Phys. **43**, 044002 (2016)



D. Habs et.al. AIP Conf. Proc. **1462**, 177 (2012)

- The atomic nucleus occupies the threshold between the fundamental and the emergence.
- Nuclear excitation spectra can be studied by subjecting the atomic nucleus to an external perturbation and observing its response.





Credit: Alan Stonebraker/APS

- Models based on the one-loop approximation (such as (quasiparticle) random phase approximation or (Q)RPA) cannot describe the spreading width of giant resonances and low-lying strength distributions.
- The most important correlations beyond RPA are induced by the particle-vibration coupling (PVC). The leading PVC contribution can be provided by the relativistic (quasiparticle) time blocking approximation (R(Q)TBA) [E. Litvinova, P. Ring, and V. Tselyaev, PRC **78** 014312 (2008) and PRC **75**, 064302 (2007)].
- The radiative neutron capture reaction rates of the r-process nucleosynthesis are immensely affected by the microscopic structure of the low-energy spectra of compound nuclei [S. Goriely and E. Khan, Nucl. Phys. A **706**, 217 (2002)].
- Motivation: to understand the temperature evolution of the excitation spectra of spherical medium-mass and heavy nuclei, in particular, the low-energy sector.

Hot Nucleus and Nuclear Temperature

- A hot nucleus is a highly excited compound nucleus formed as an intermediate state during heavy-ion fusion reaction or radiative neutron capture process.

$$\rho(A, E^*) \simeq \frac{e^{2\sqrt{aE^*}}}{\sqrt{48}E^*}; \quad a \approx A/(8-12)\ \mathrm{MeV}^{-1} \qquad \text{(Bethe's Fermi gas formula)}$$

- For example, for $A = 100$ and $E^* = 10$ MeV, the Bethe's Fermi gas formula estimates the density of levels $\rho(A, E^*)$ of 10^7 states per MeV. Here, we have used $a \approx A/10\ \mathrm{MeV}^{-1}$.

- Definition of temperature:

$$T = \left(\frac{1}{\rho}\frac{\partial \rho}{\partial E^*}\right)^{-1} \quad \rightarrow$$

Relation between E^* and T

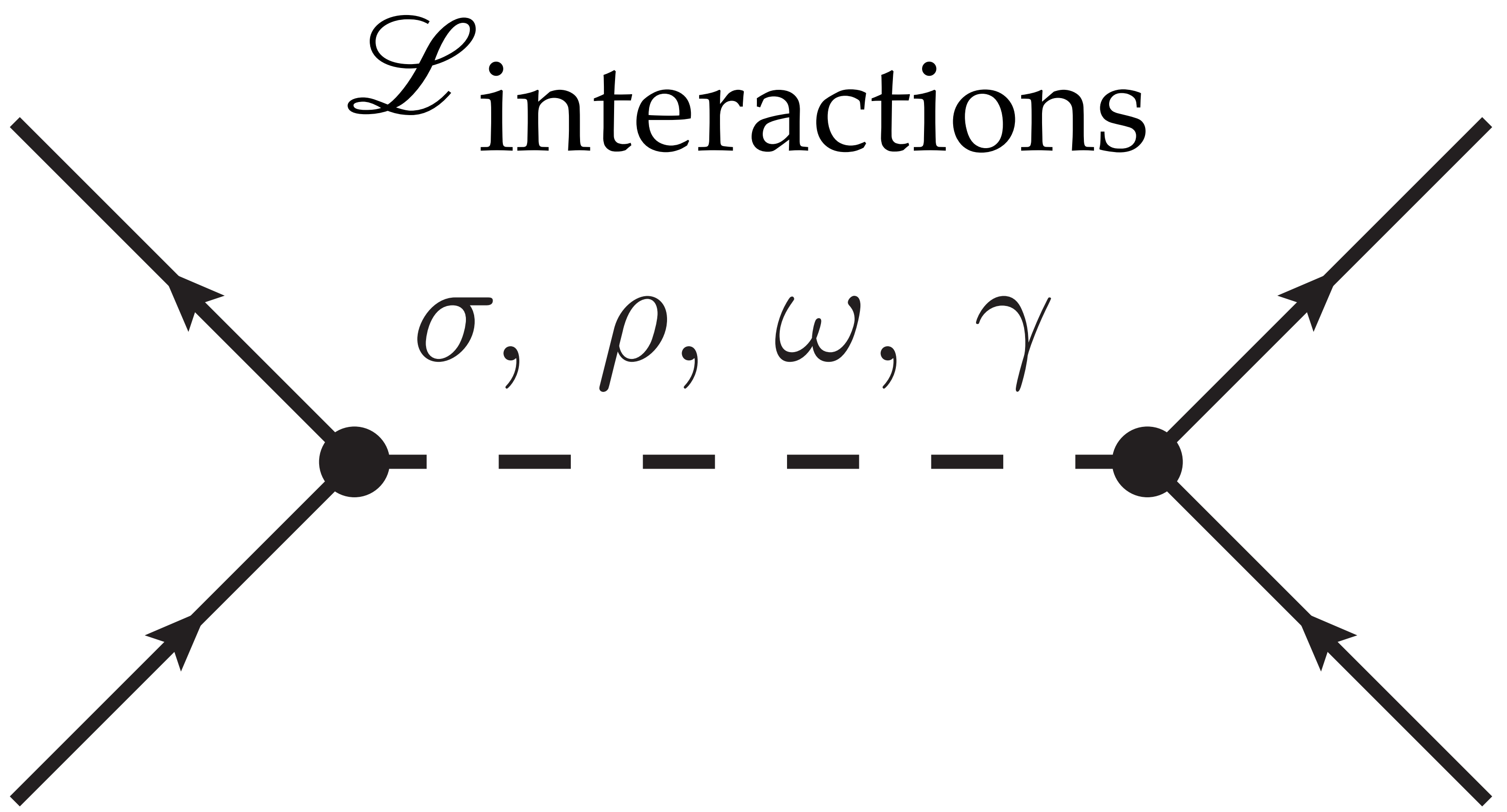
$$E^* \approx aT^2$$

P. F. Bortignon, A. Bracco, and R. A. Broglia, *Giant Resonances: Nuclear Structure at Finite Temperature*

Nuclear mean field at finite temperature

- The hot nucleus is considered as a system of Dirac nucleons interacting via effective meson-exchange and photon exchange at finite temperature.
- Effective Lagrangian density $\mathscr{L}_{\text{effective}}$:

$$\mathscr{L}_{\text{effective}} = \mathscr{L}_{\text{nucleons}} + \mathscr{L}_{\text{mesons}} + \mathscr{L}_{\text{interactions}}$$



Serot and Walecka, Advance in Nuclear Physics, Vol. 16 (1986); Ring, Prog. Part. Nucl. Phys. **37**, 193 (1996)

Quantum numbers $(J,\ \pi,\ T)$

Meson	J	π	T
Sigma (σ)	0	+1	0
Omega (ω)	1	−1	0
Rho (ρ)	1	−1	1

Mean-field approximation

- Mean-field approximation:
  $\phi_{\text{mesons}} \rightarrow \langle \phi_{\text{mesons}} \rangle$
- Equation of motion for nucleons:

Dirac Equation

$$\hat{h}^D \varphi_k(\mathbf{r}) = \varepsilon_k \varphi_k(\mathbf{r})$$

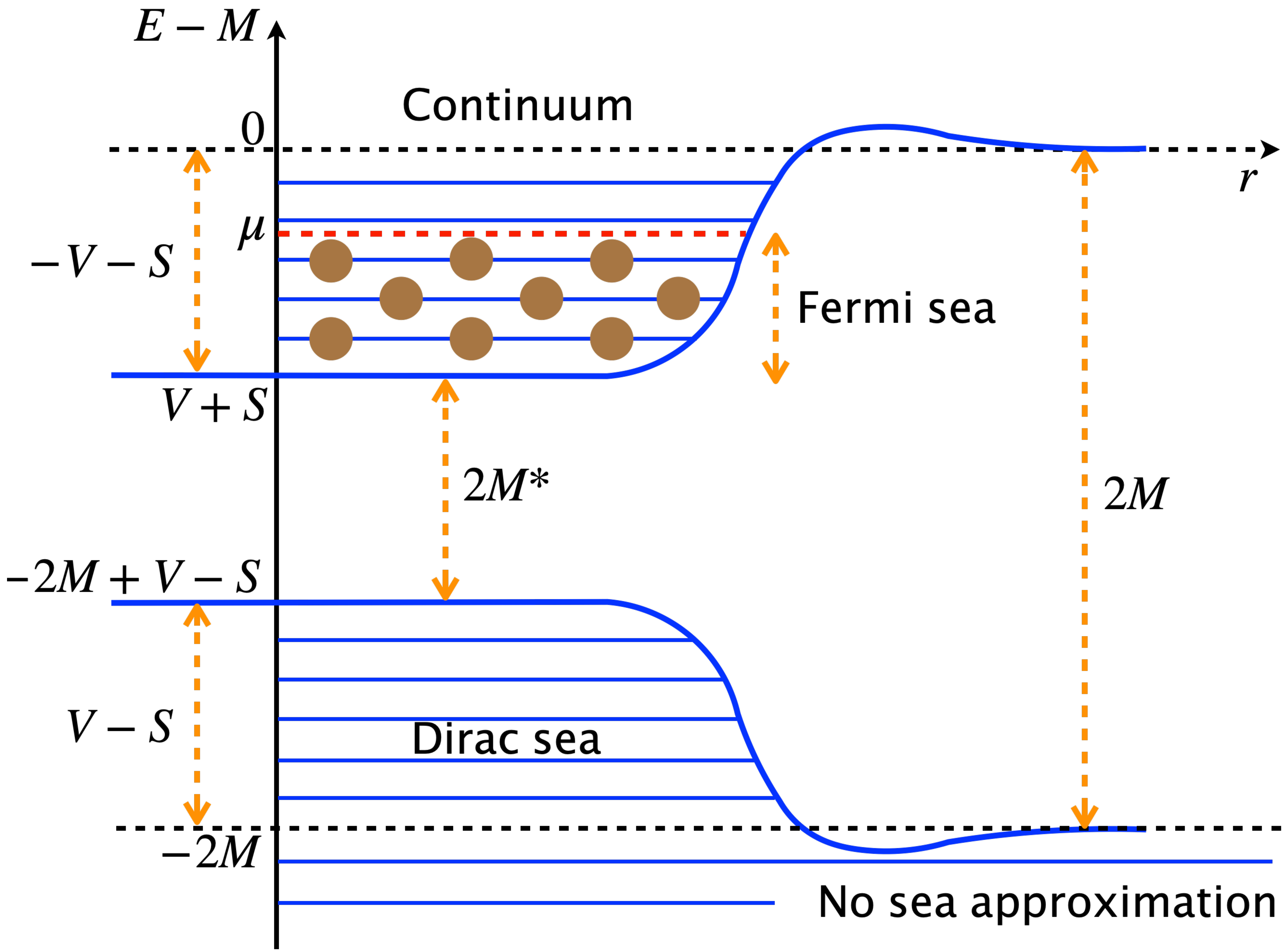
$$\hat{h}^D = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta(M + \widetilde{\Sigma}(\mathbf{r}))$$

Relativistic Mass Operator

$$\widetilde{\Sigma}(\mathbf{r}) = S(\mathbf{r}) + \beta V(\mathbf{r})$$

$$S(\mathbf{r}) = g_\sigma \sigma(\mathbf{r})$$

$$V(\mathbf{r}) = g_\omega \omega^0(\mathbf{r}) + g_\rho \tau_3 \rho_3^0(\mathbf{r}) + \frac{1}{2}(1 + \tau_3) e A^0(\mathbf{r})$$



Klein-Gordon Equations

$$\left(-\nabla^2 + m_\sigma^2\right)\sigma(\mathbf{r}) = -g_\sigma \rho_s(\mathbf{r}) - dU(\sigma)/d\sigma$$

$$\left(-\nabla^2 + m_\omega^2\right)\omega^0(\mathbf{r}) = g_\omega \rho_v(\mathbf{r})$$

$$\left(-\nabla^2 + m_\rho^2\right)\rho_3^0(\mathbf{r}) = g_\rho \rho_3(\mathbf{r})$$

$$-\nabla^2 A^0(\mathbf{r}) = e\rho_c(\mathbf{r})$$

$$U(\sigma) = \frac{1}{3}g_2\sigma^3 + \frac{1}{4}g_3\sigma^4$$

Boguta and Bodmer, Nucl. Phys. A **292**, 413 (1977)

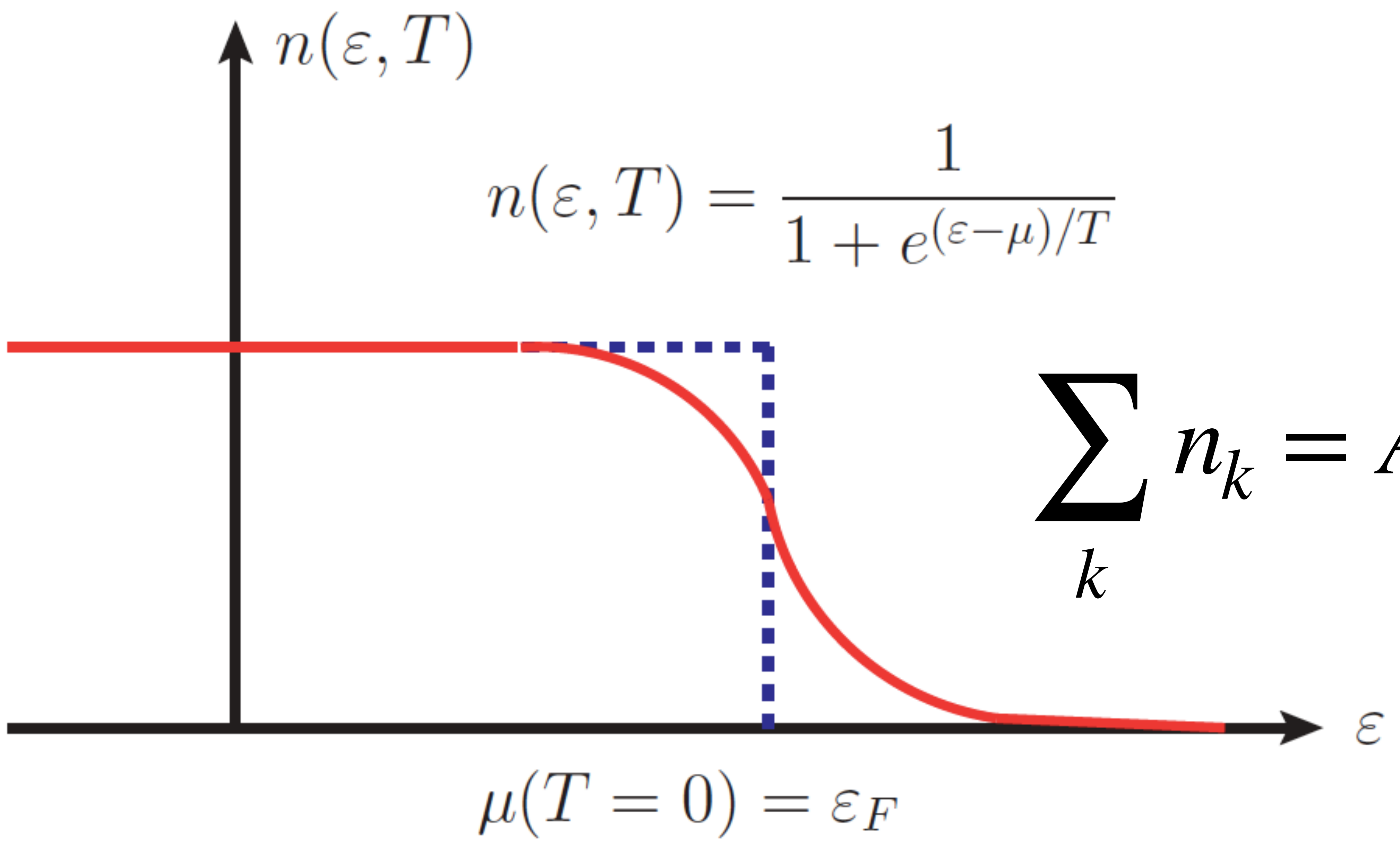
Four Baryonic Densities

$$\rho_s(\mathbf{r}) = \sum_k n_k \overline{\varphi}_k(\mathbf{r})\varphi_k(\mathbf{r})$$

$$\rho_v(\mathbf{r}) = \sum_k n_k \varphi_k^\dagger(\mathbf{r})\varphi_k(\mathbf{r})$$

$$\rho_3(\mathbf{r}) = \sum_k n_k \varphi_k^\dagger(\mathbf{r})\tau_3\varphi_k(\mathbf{r})$$

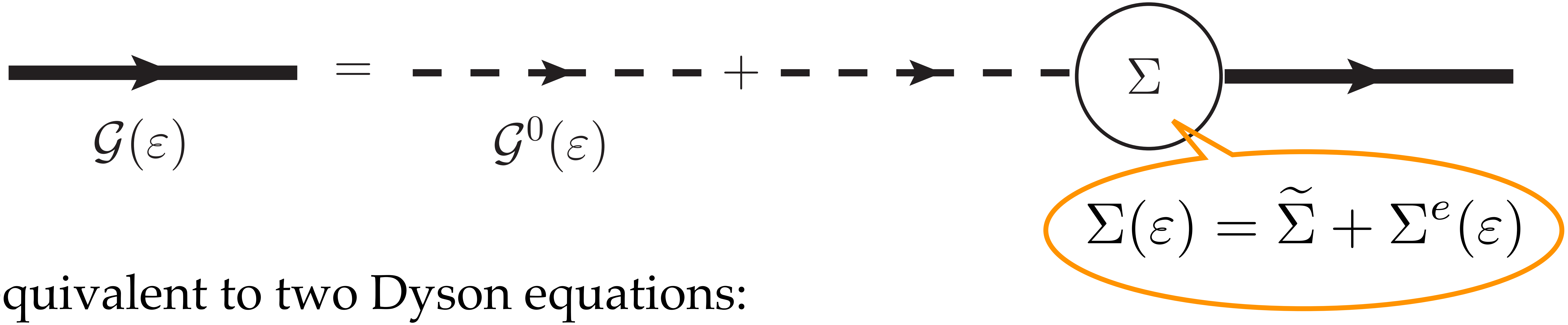
$$\rho_c(\mathbf{r}) = \sum_k n_k \varphi_k^\dagger(\mathbf{r})\frac{1}{2}(1+\tau_3)\varphi_k(\mathbf{r})$$



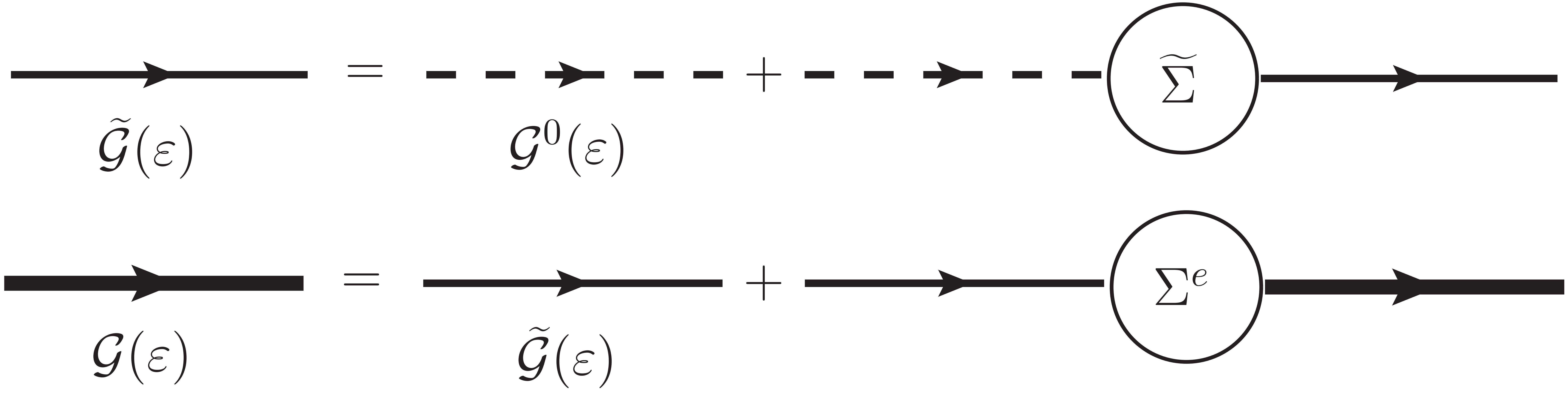
$$\sum_k n_k = A$$

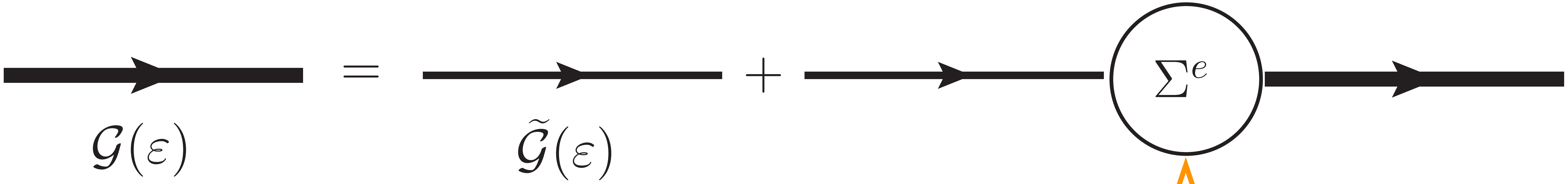
Correlations beyond mean field

Full Dyson equation:



is equivalent to two Dyson equations:

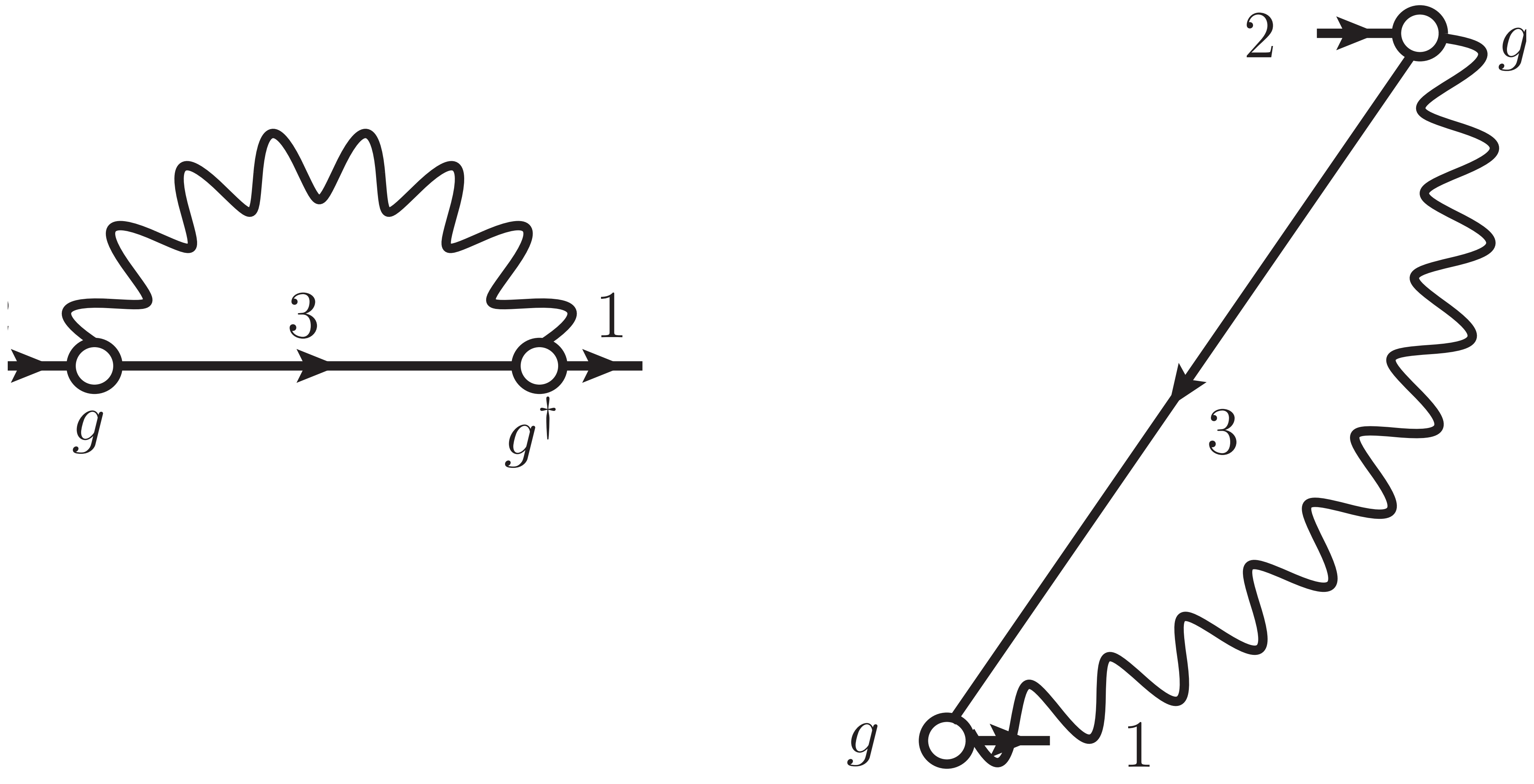




$$\tilde{\mathcal{G}}_{k_2 k_1}(\varepsilon_\ell) = \frac{\delta_{k_2 k_1}}{i\varepsilon_\ell - \varepsilon_{k_1} + \mu}$$

$$\varepsilon_\ell = (2\ell + 1)\pi T$$

T. Matsubara, Prog. Theor. Phys. **14**, 351 (1995)



Particle-vibration coupling (PVC) self-energy

Phonon Vertices

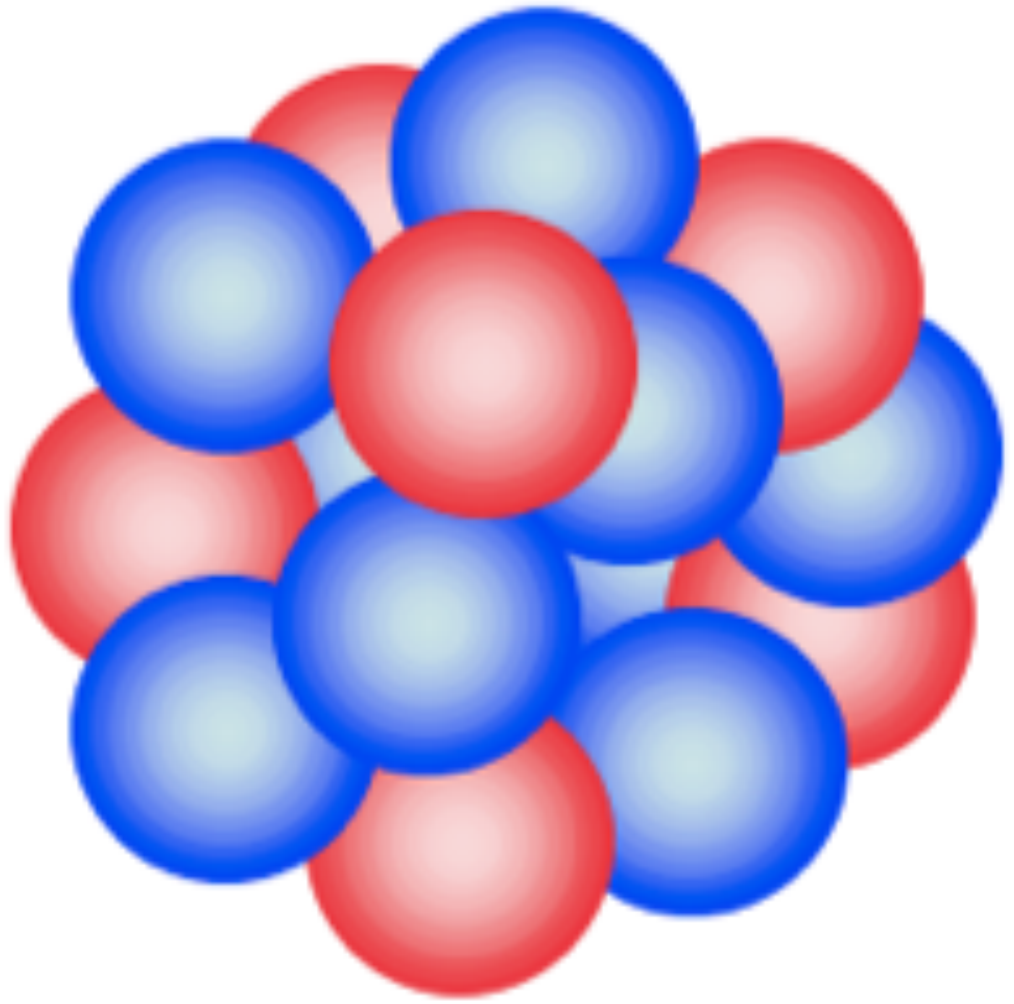
$$g^m_{12} = \sum_{34} \widetilde{\mathscr{U}}_{12,34}\rho^m_{34}$$

$$\widetilde{\mathscr{U}}_{12,34} = \frac{\delta\widetilde{\Sigma}_{34}}{\delta\rho_{12}}$$

The phonon energies ω_m and the phonon transition densities ρ^m can be determined from Finite Temperature Relativistic RPA (FT-RRPA).

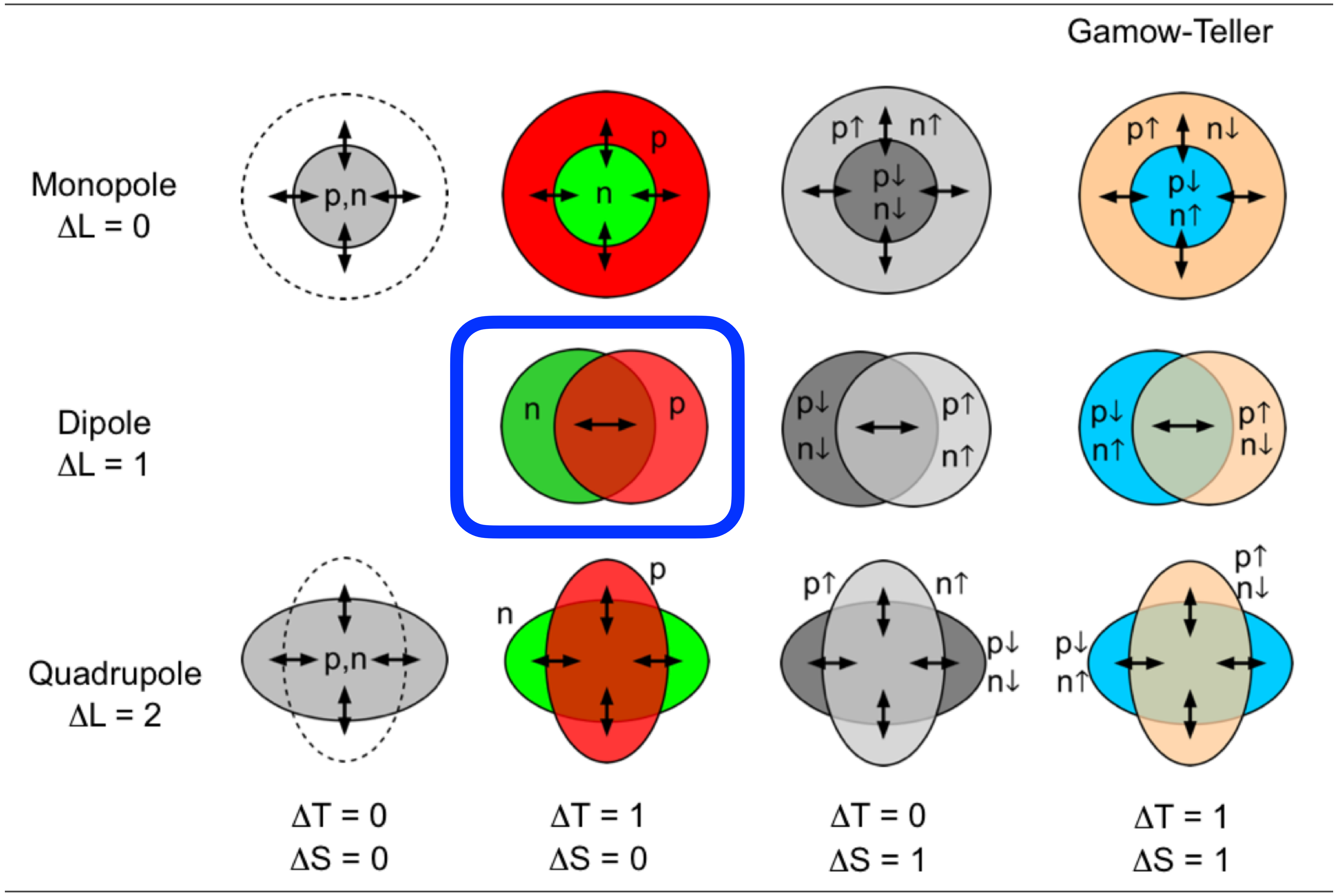
Nuclear Response at Finite Temperature

$$\hat{V}^{0}e^{-i\omega t} + \hat{V}^{0\dagger}e^{i\omega t}$$



Hot Nucleus

Nuclear Excitation Modes



* M. N. Harakeh and A. van der Woude: *Giant Resonances*

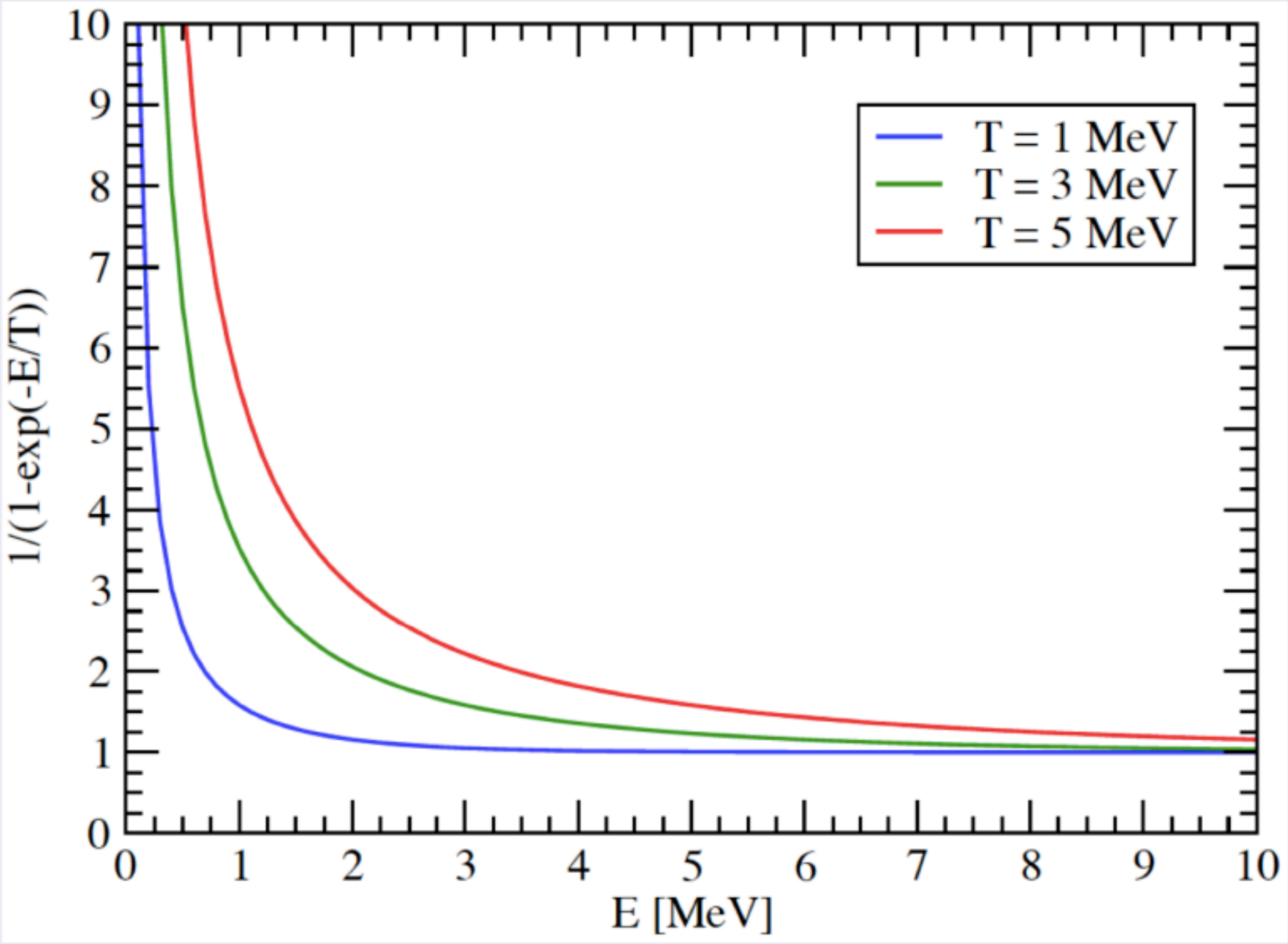
- E. Litvinova, C. Robin, and H. Wibowo, Phys. Lett. B **800**, 135134 (2020).
- H. Wibowo and E. Litvinova, Phys. Rev. C **100**, 024307 (2019).
- E. Litvinova, P. Schuck, and H. Wibowo, EPJ Web of Conferences **223**, 01033 (2019).
- E. Litvinova and H. Wibowo, Eur. Phys. J. A **55**, 223 (2019).
- E. Litvinova and H. Wibowo, Phys. Rev. Lett. **121**, 082501 (2018).

Transition strength distribution at $T > 0$

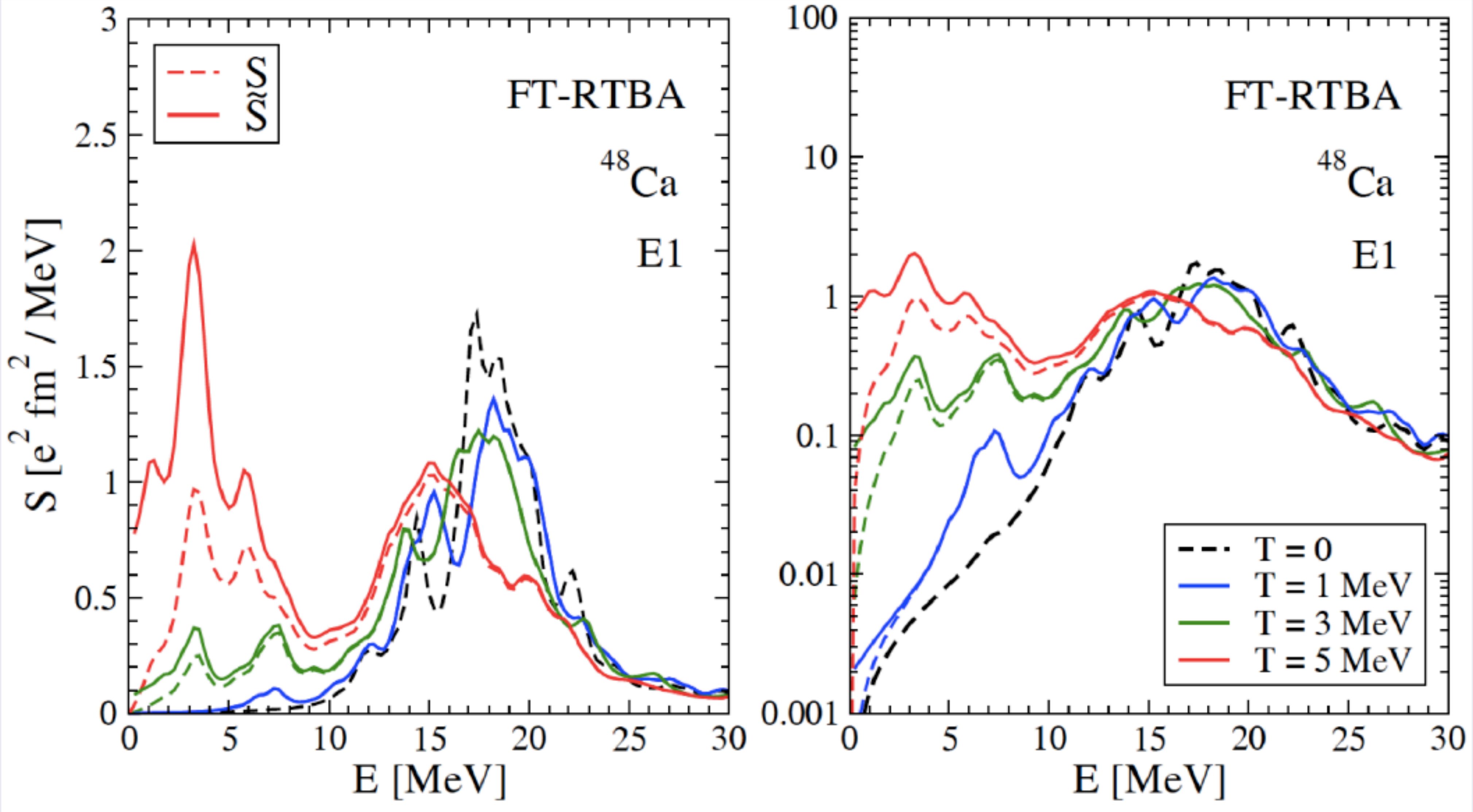
Strength Function at $T > 0$

$$\widetilde{S}(E) = \frac{1}{1 - e^{-E/T}} \lim_{\Delta \to +0} \frac{1}{\pi} \mathrm{Im} \sum_{1234} V^{0*}_{21} \mathscr{R}_{12,34}(E + i\Delta) V^{0}_{43}$$

Exponential Factor



The Role of Exponential Factor



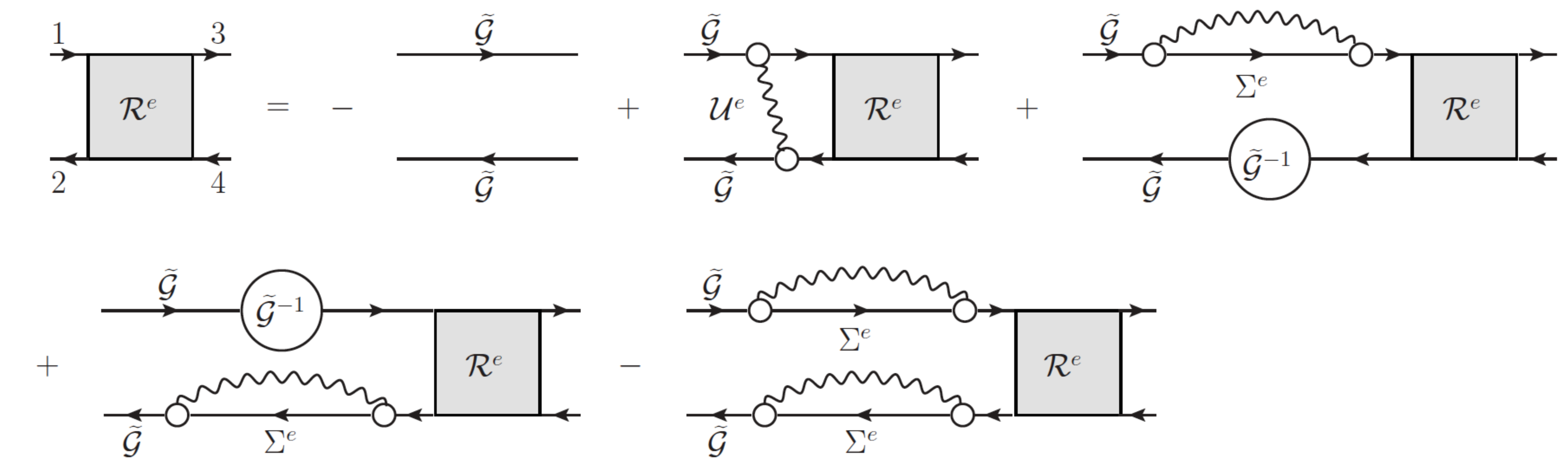
E. Litvinova and **H. Wibowo**, EPJA **55**, 223 (2019)

Bethe-Salpeter Equation (BSE) for particle-hole $\mathscr{R}$

BSE for Full Response $\mathscr{R}$

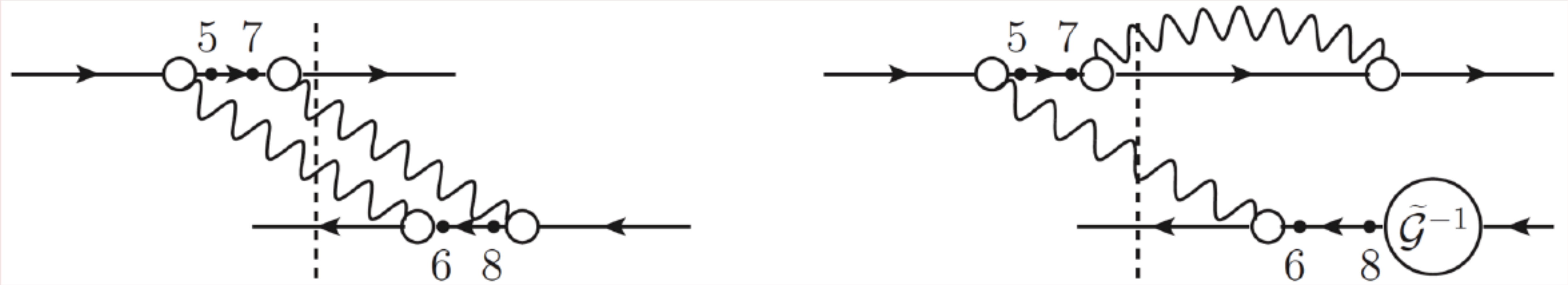
$$\mathscr{R} = \mathscr{R}^e - \mathscr{R}^e \widetilde{\mathscr{U}} \mathscr{R}$$

BSE for Correlated Propagator $\mathscr{R}^e$



Imaginary-time blocking approximation

Problem: Diagrams with no imaginary-time ordering

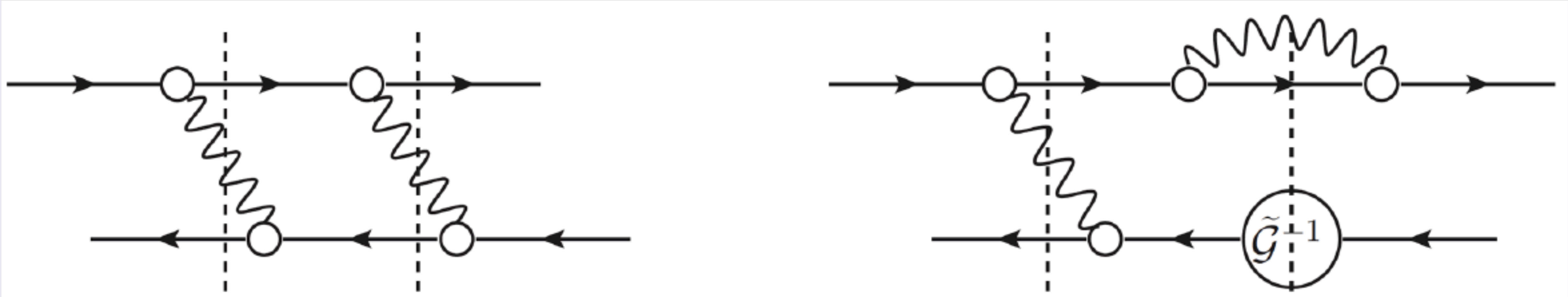


Imaginary-time Projection Operator

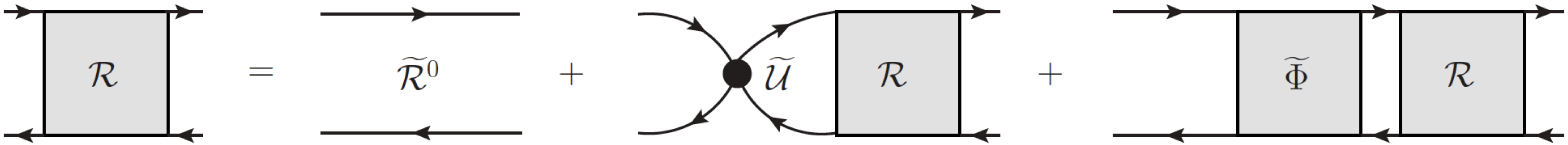
$$\widetilde{\mathscr{D}}(12,34) = \Theta(12,34;\,T)\widetilde{\mathscr{R}}^{\,0\sigma_1\sigma_2}(12,34)$$

$$\Theta(12,34;\,T) = \delta_{\sigma_1,-\sigma_2}\theta(\sigma_1\tau_{41})\theta(\sigma_1\tau_{32})[n(\sigma_1\varepsilon_2,\,T)\theta(\sigma_1\tau_{12}) + n(\sigma_2\varepsilon_1,\,T)\theta(\sigma_2\tau_{12})]$$

Allowed Diagrams



Final equation for the response function



$$\mathscr{R}(\omega) = \widetilde{\mathscr{R}}^0(\omega) + \widetilde{\mathscr{R}}^0(\omega)\Big[\widetilde{\mathscr{U}} + \widetilde{\Phi}(\omega) - \widetilde{\Phi}(0)\Big]\mathscr{R}(\omega)$$

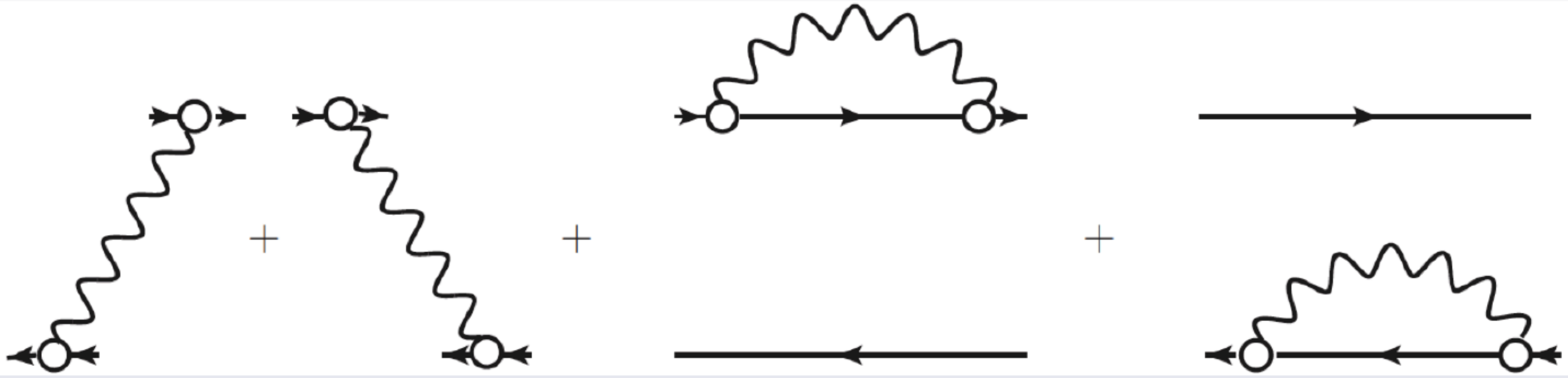
Subtraction technique

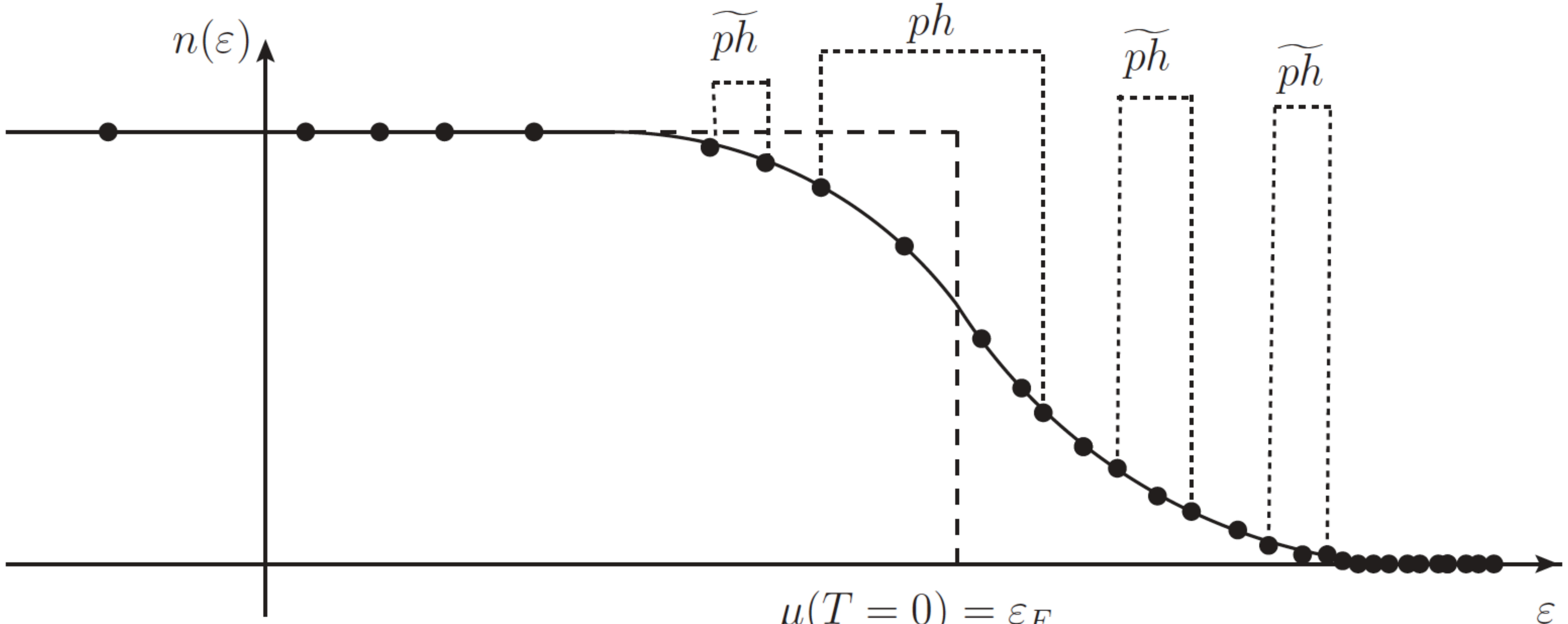
V. I. Tselyaev, PRC **88**, 1 (2013)

Free Response Function and PVC Amplitude

$$\widetilde{\mathscr{R}}^0_{12,34}(\omega) = -\delta_{13}\delta_{24}\frac{n_2(T) - n_1(T)}{\omega - \varepsilon_1 + \varepsilon_2}$$

$\widetilde{\Phi}(\omega) =$





$$\widetilde{\mathscr{R}}^{0}_{12,34}(\omega)=\delta_{13}\delta_{24}\frac{n_2(T)-n_1(T)}{\omega-\varepsilon_1+\varepsilon_2}$$

Strength Function and Transition Density

Density matrix variation: $$\delta\rho_{k_1k_2}(\omega) = -\sum_{k_3k_4}\mathcal{R}_{k_1k_2,k_3k_4}(\omega)V^0_{k_4k_3}$$

Uncorrelated density matrix variation: $$\delta\rho^0_{k_1k_2}(\omega) = -\sum_{k_3k_4}\widetilde{\mathcal{R}}^0_{k_1k_2,k_3k_4}(\omega)V^0_{k_4k_3}$$

BSE equation in terms of $\delta\rho(\omega)$ and $\delta\rho^0(\omega)$:

$$\delta\rho(\omega) = \delta\rho^0(\omega) + \widetilde{\mathcal{R}}^0(\omega)\left[\widetilde{\mathcal{U}} + \widetilde{\Phi}(\omega) - \widetilde{\Phi}(0)\right]\delta\rho(\omega)$$

Spectral density $S(E)$: $$S(E) = -\frac{1}{\pi}\lim_{\Delta\to+0}\mathrm{Im}\sum_{k_1k_2}V^{0*}_{k_2k_1}\delta\rho_{k_1k_2}(E+i\Delta)$$

Transition density $\rho^{fi}_{k_1k_2}$: $$\rho^{fi}_{k_1k_2} = \lim_{\Delta\to+0}\sqrt{\frac{\Delta}{\pi\cdot S(\omega_{fi})}}\mathrm{Im}\delta\rho_{k_1k_2}(\omega_{fi}+i\Delta)$$

FT-RTBA generalized normalization condition:

$$1 = \sum_{k_1k_2k_3k_4} \rho^{fi*}_{k_1k_2} \left[\mathcal{N}_{k_1k_2,k_3k_4} - \left.\frac{d\widetilde{\Phi}_{k_1k_2,k_3k_4}(\omega)}{d\omega}\right|_{\omega=\omega_{fi}} \right] \rho^{fi}_{k_3k_4}$$

FT-RPA norm:

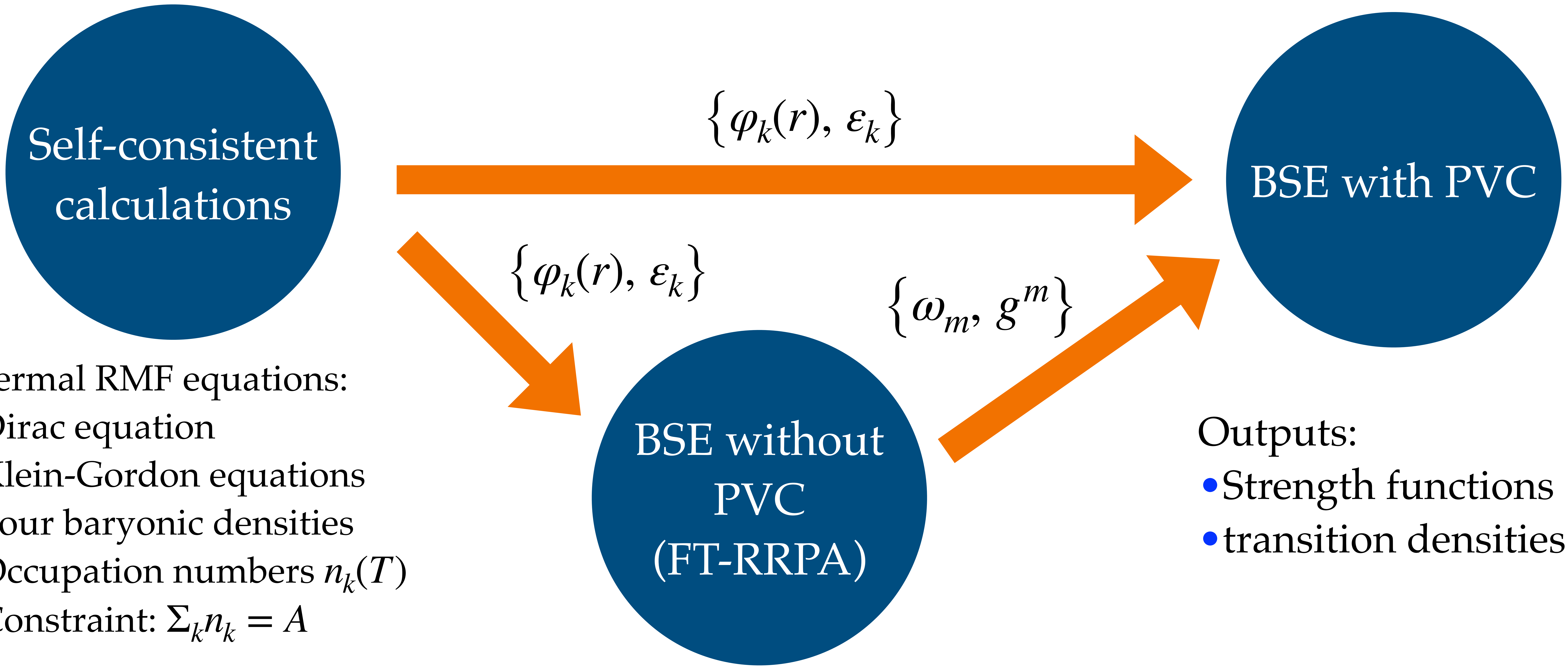
$$\mathcal{N}_{k_1k_2,k_3k_4} = \frac{\delta_{k_1k_3}\delta_{k_2k_4}}{n(\varepsilon_{k_2}, T) - n(\varepsilon_{k_1}, T)}$$

FT-RPA normalization condition:

$$\sum_{ph} \frac{|\rho^{fi}_{ph}|^2 - |\rho^{fi}_{hp}|^2}{n_h(\varepsilon_h, T) - n_p(\varepsilon_p, T)} = 1$$

z^{fi}_{ph}

Numerical Scheme (Finite-temperature Relativistic Time Blocking Approximation → FT-RTBA)



Thermal RMF equations:

- ✓Dirac equation
- ✓Klein-Gordon equations
- ✓Four baryonic densities
- ✓Occupation numbers $n_k(T)$
- ✓Constraint: $\Sigma_k n_k = A$

Outputs:

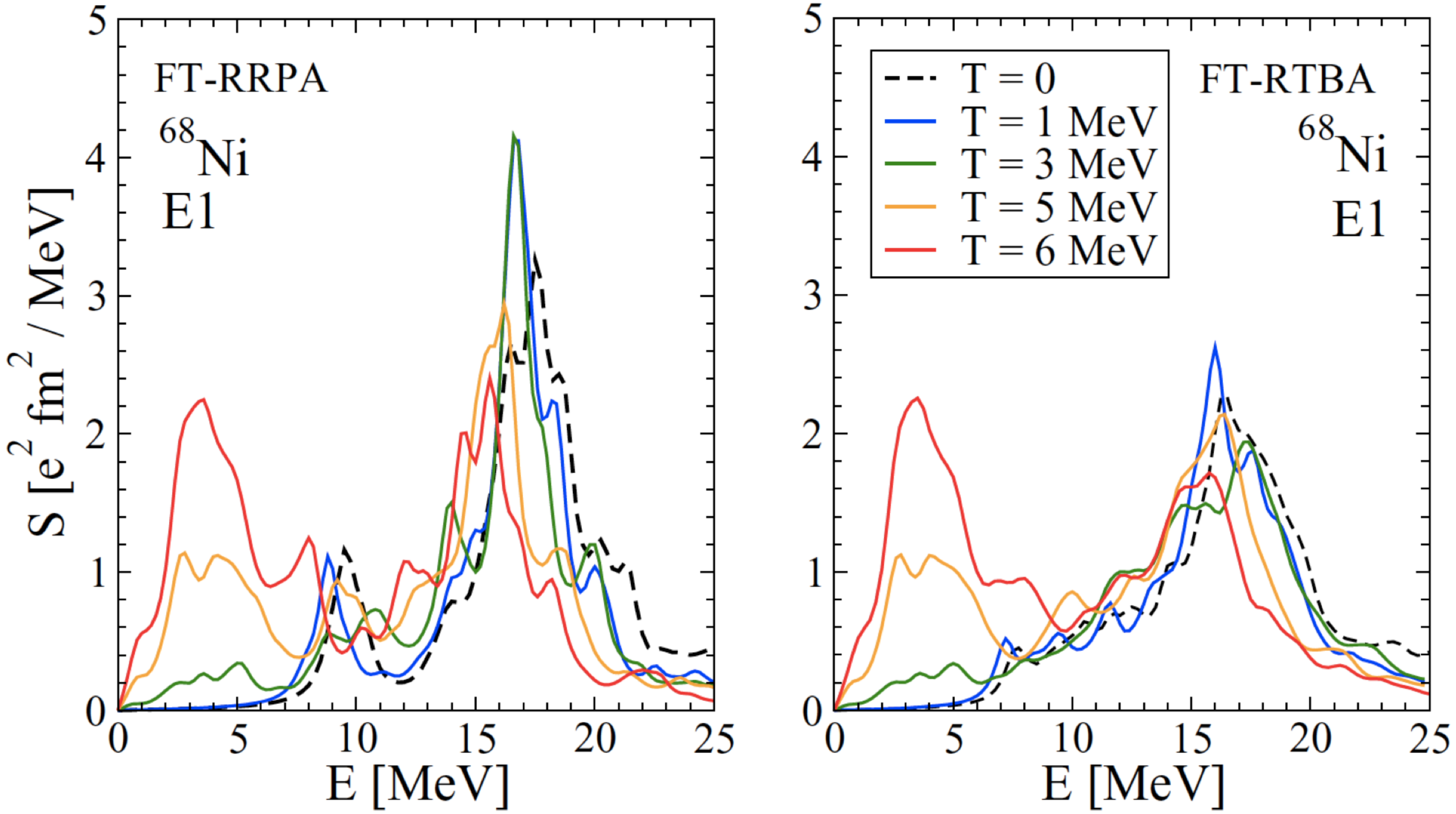
- Strength functions
- transition densities

NL3 parametrization
(G. A. Lalazissis, J. König, P. Ring, PRC 55, 540, 1997)

$m = 2^+, 3^-, 4^+, 5^-, 6^+$

Electromagnetic dipole response of ^{68}Ni

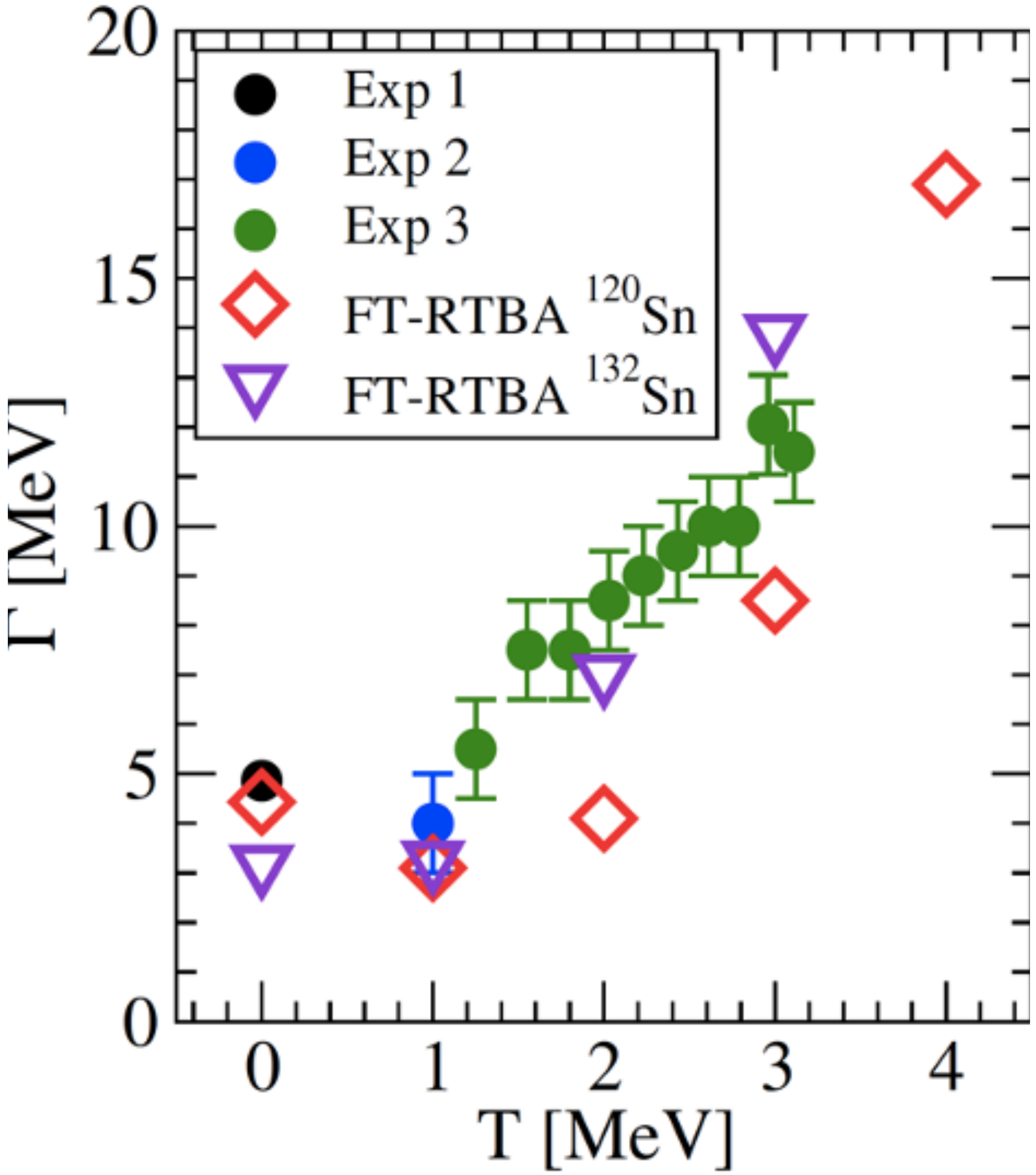
$$V^0_{1M} = \frac{eN}{A}\sum_{i=1}^{Z} r_i Y_{1M}(\hat{\mathbf{n}}_i) - \frac{eZ}{A}\sum_{i=1}^{N} r_i Y_{1M}(\hat{\mathbf{n}}_i)$$



H. Wibowo and E. Litvinova, PRC **100**, 024307 (2019)

- The enhancement of the low-energy spectral density becomes stronger as temperature increases.
- The high-frequency peak remains fragmented because of the PVC at all temperatures.
- The giant dipole resonance starts to “disappear” at temperature 6 MeV.

Width of giant dipole resonance in ^{120,132}Sn



E. Litvinova and H. Wibowo, PRL **121**, 082501 (2018)

$\Gamma(T)$ [MeV] of the GDR in ^{120}Sn nucleus

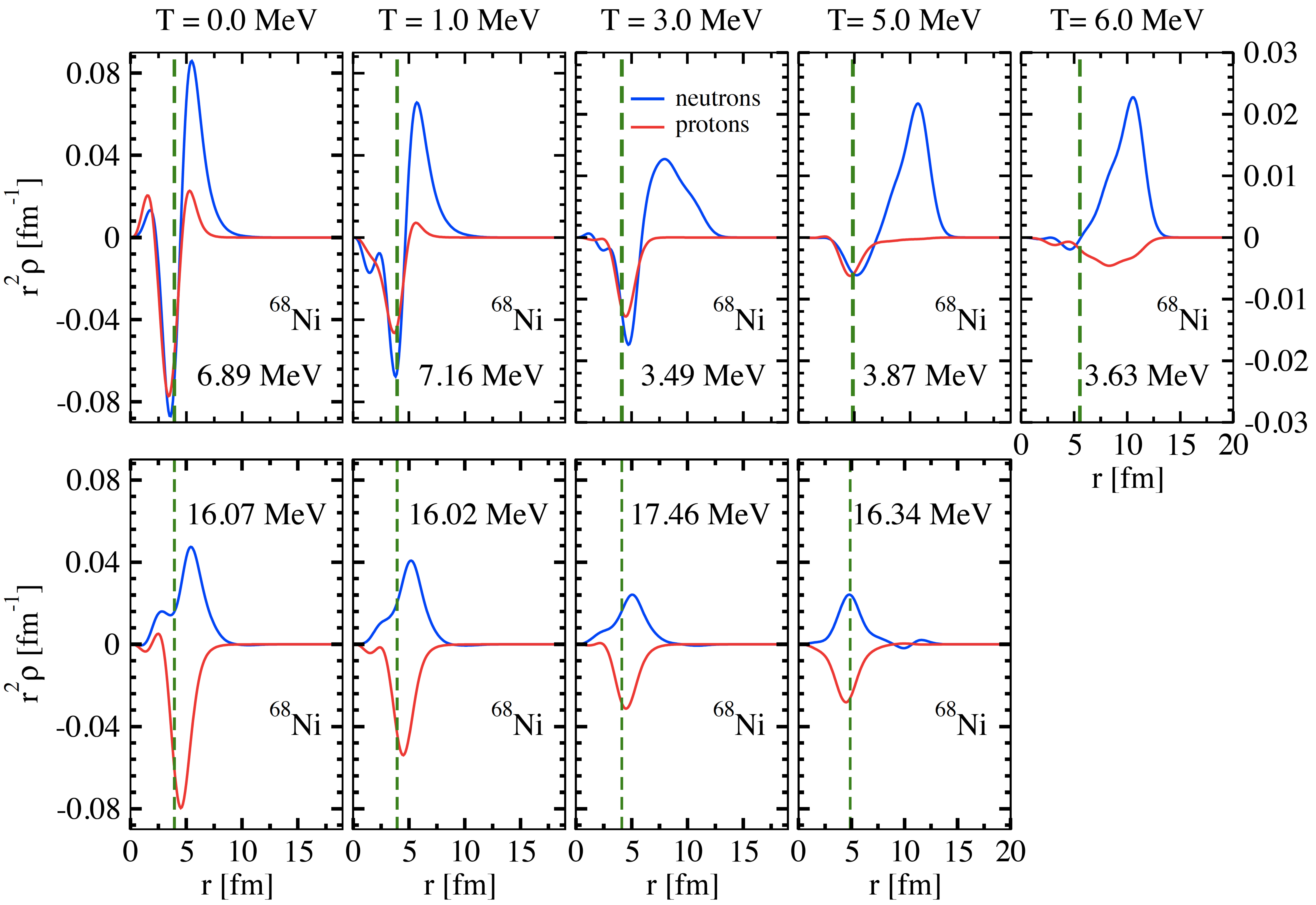
T [MeV]	FT-RRPA	FT-RTBA
0	2.70	4.43
1.0	2.26	3.08
2.0	3.09	4.07
3.0	6.94	8.46
4.0	14.46	16.92

- The width $\Gamma(T)$ at $T = 1$ MeV in ^{120}Sn is smaller than at $T = 0$
- After $T = 1$ MeV in ^{132}Sn and $T = 2$ MeV in ^{120}Sn, there is a fast increase of $\Gamma(T)$.

Proton and neutron transition densities in ^{68}Ni

Most prominent peaks at $E < 10$ MeV

GDR peaks



Proton (p) and neutron (n) ph and $\widetilde{ph}$ configurations for low-energy peaks

$T=0$; $E=6.89$ MeV		$T=1$ MeV; $E=7.16$ MeV		$T=2$ MeV; $E=7.70$ MeV		$T=3$ MeV; $E=3.49$ MeV	
10.3%	$(2p_{3/2} \to 2d_{5/2})\ n$	56.8%	$(2p_{1/2} \to 3s_{1/2})\ n$	4.9%	$(1f_{5/2} \to 2d_{5/2})\ n$	31.1%	$(3s_{1/2} \to 3p_{3/2})\ n$
9.8%	$(2s_{1/2} \to 2p_{3/2})\ p$	4.4%	$(1f_{7/2} \to 1g_{9/2})\ n$	3.2%	$(1f_{7/2} \to 1g_{9/2})\ n$	15.7%	$(2d_{5/2} \to 3p_{3/2})\ n$
7.1%	$(1f_{7/2} \to 1g_{9/2})\ p$	2.2%	$(1f_{5/2} \to 2d_{5/2})\ n$	2.9%	$(2p_{3/2} \to 2d_{5/2})\ n$	0.1%	$(3s_{1/2} \to 3p_{1/2})\ n$
6.2%	$(1f_{5/2} \to 2d_{5/2})\ n$	1.4%	$(1f_{7/2} \to 1g_{9/2})\ p$	2.1%	$(1f_{5/2} \to 2d_{3/2})\ n$	0.01%	$(1f_{7/2} \to 1g_{9/2})\ n$
6.1%	$(1f_{7/2} \to 1g_{9/2})\ n$	1.0%	$(1f_{5/2} \to 2d_{3/2})\ n$	1.7%	$(1f_{7/2} \to 1g_{9/2})\ p$	0.01%	$(1g_{9/2} \to 1h_{11/2})\ n$
4.6%	$(1f_{5/2} \to 2d_{3/2})\ n$	0.9%	$(2p_{3/2} \to 3s_{1/2})\ n$	1.3%	$(2p_{1/2} \to 2d_{3/2})\ n$		
1.0%	$(2p_{1/2} \to 2d_{3/2})\ n$	0.9%	$(2p_{1/2} \to 2d_{3/2})\ n$	1.1%	$(2s_{1/2} \to 2p_{3/2})\ p$		
0.9%	$(1d_{3/2} \to 2p_{1/2})\ p$	0.7%	$(2p_{1/2} \to 4s_{1/2})\ n$	0.9%	$(2p_{3/2} \to 3s_{1/2})\ n$		
0.9%	$(1d_{3/2} \to 2p_{3/2})\ p$	0.5%	$(2p_{3/2} \to 2d_{5/2})\ n$	0.2%	$(1d_{3/2} \to 2p_{1/2})\ p$		
0.7%	$(2p_{3/2} \to 3s_{1/2})\ n$	0.3%	$(1d_{3/2} \to 2p_{3/2})\ p$	0.2%	$(1d_{3/2} \to 2p_{3/2})\ p$		
0.4%	$(1f_{5/2} \to 3d_{3/2})\ n$	0.2%	$(1d_{3/2} \to 2p_{1/2})\ p$	0.1%	$(1f_{5/2} \to 3d_{3/2})\ n$		
0.3%	$(2s_{1/2} \to 2p_{1/2})\ p$	0.1%	$(2p_{1/2} \to 5s_{1/2})\ n$				
0.2%	$(2p_{3/2} \to 3d_{5/2})\ n$						
0.2%	$(1f_{5/2} \to 3d_{5/2})\ n$						
0.2%	$(2p_{3/2} \to 2d_{3/2})\ n$						
0.2%	$(1f_{7/2} \to 2d_{5/2})\ p$						
0.1%	$(1f_{7/2} \to 2d_{5/2})\ n$						
49.2%		69.4%		18.6%		46.92%	

$T = 4$ MeV; $E = 2.55$ MeV		$T = 5$ MeV; $E = 3.87$ MeV		$T = 6$ MeV; $E = 3.63$ MeV	
66.1%	$(2f_{7/2} \rightarrow 2g_{9/2})\ n$	61.9%	$(2g_{9/2} \rightarrow 2h_{11/2})\ n$	21.2%	$(1i_{11/2} \rightarrow 1j_{13/2})\ n$
5.1%	$(3p_{1/2} \rightarrow 3d_{3/2})\ n$	3.0%	$(3f_{7/2} \rightarrow 4d_{5/2})\ n$	9.5%	$(2d_{5/2} \rightarrow 2f_{7/2})\ p$
0.7%	$(2f_{5/2} \rightarrow 2g_{7/2})\ n$	0.4%	$(2g_{7/2} \rightarrow 3f_{5/2})\ n$	8.8%	$(1i_{13/2} \rightarrow 1j_{15/2})\ n$
0.4%	$(2d_{3/2} \rightarrow 3p_{1/2})\ n$	0.3%	$(2d_{3/2} \rightarrow 2f_{5/2})\ p$	3.2%	$(2d_{3/2} \rightarrow 2f_{5/2})\ n$
0.1%	$(1g_{7/2} \rightarrow 2f_{5/2})\ n$	0.2%	$(1h_{11/2} \rightarrow 1i_{13/2})\ n$	0.1%	$(2g_{9/2} \rightarrow 3f_{7/2})\ n$
		0.1%	$(3d_{3/2} \rightarrow 2f_{5/2})\ n$		
72.4%		65.9%		42.8%	

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Summary

- We model a compound nucleus in the framework of the thermal relativistic mean-field theory. The single-particle occupation probabilities are given by the Fermi-Dirac occupation number.
- We have generalized the time blocking method for finite-temperature case and derived the single-frequency Bethe-Salpeter equation to calculate the response function of a thermally excited compound nucleus.
- Our method called the finite-temperature relativistic time-blocking approximation (FT-RTBA) has been implemented numerically for calculating dipole response of medium-mass and heavy nuclei.

Outlooks

- To include the PVC effects into Relativistic Continuum Quasiparticle RPA (RCQRPA).
- To include the finite-temperature effects into the RCQRPA+PVC framework using Matsubara Green's function formalism.
- To perform accurate calculations of radiative strength function (RSFs) for the modeling of the neutron-capture cross-sections in the r-process nucleosynthesis.

$$\Sigma^{e}_{k_1 k_2}(\varepsilon_\ell) \;=\; \sum_{k_3,m}\left\{ g^{m*}_{k_1 k_3} g^{m}_{k_2 k_3} \frac{N(\omega_m,T)+1-n(\varepsilon_{k_3},T)}{i\varepsilon_\ell-\varepsilon_{k_3}+\mu-\omega_m} + g^{m}_{k_3 k_1} g^{m*}_{k_3 k_2} \frac{n(\varepsilon_{k_3},T)+N(\omega_m,T)}{i\varepsilon_\ell-\varepsilon_{k_3}+\mu+\omega_m} \right\}$$

$$N(\omega_m,T) = \frac{1}{e^{\omega_m/T}-1}$$

$$\mathscr{U}^{e}_{k_1 k_2, k_3 k_4}(\omega_n,\varepsilon_\ell,\varepsilon_{\ell'}) = \sum_m \frac{g^{m}_{k_4 k_2} g^{m*}_{k_3 k_1}}{i\varepsilon_\ell - i\varepsilon_{\ell'} + \omega_m} - \sum_m \frac{g^{m*}_{k_2 k_4} g^{m}_{k_1 k_3}}{i\varepsilon_\ell - i\varepsilon_{\ell'} - \omega_m}$$

$$\widetilde{\Phi}_{k_1k_2,k_3k_4}(\omega_n) = \widetilde{\Phi}^{(1)}_{k_1k_2,k_3k_4}(\omega_n) + \widetilde{\Phi}^{(2)}_{k_1k_2,k_3k_4}(\omega_n) + \widetilde{\Phi}^{(3)}_{k_1k_2,k_3k_4}(\omega_n)$$

$$\begin{aligned}
\widetilde{\Phi}^{(1)}_{k_1k_2,k_3k_4}(\omega_n) &= \frac{\delta_{\sigma_{k_1},-\sigma_{k_2}}\sigma_{k_1}}{n(\varepsilon_{k_4},T) - n(\varepsilon_{k_3},T)} \Bigg\{ \sum_m g^m_{k_4k_2} g^{m*}_{k_3k_1} \left[N(\omega_m,T) + n(\varepsilon_{k_4},T)\right] \\
&\times \frac{n(\varepsilon_{k_1},T) - n(\varepsilon_{k_4} - \omega_m,T)}{i\omega_n - \omega_m - \varepsilon_{k_1k_4}} + \sum_m g^m_{k_4k_2} g^{m*}_{k_3k_1} \left[N(\omega_m,T) + n(\varepsilon_{k_3},T)\right] \\
&\times \frac{n(\varepsilon_{k_3} - \omega_m,T) - n(\varepsilon_2,T)}{i\omega_n + \omega_m - \varepsilon_{k_3k_2}} + \sum_m g^m_{k_1k_3} g^{m*}_{k_2k_4} \left[N(\omega_m,T) + 1 - n(\varepsilon_{k_3},T)\right] \\
&\times \frac{n(\omega_m + \varepsilon_{k_3},T) - n(\varepsilon_{k_2},T)}{i\omega_n - \omega_m - \varepsilon_{k_3k_2}} + \sum_m g^m_{k_1k_3} g^{m*}_{k_2k_4} \left[N(\omega_m,T) + 1 - n(\varepsilon_4,T)\right] \\
&\times \frac{n(\varepsilon_{k_1},T) - n(\omega_m + \varepsilon_{k_4},T)}{i\omega_n + \omega_m - \varepsilon_{k_1k_4}} \Bigg\},
\end{aligned} \tag{3.112}$$

$$
\begin{aligned}
\widetilde{\Phi}^{(2)}_{k_1k_2,k_3k_4}(\omega_n) &= \frac{\delta_{\sigma_{k_1},-\sigma_{k_2}}\sigma_{k_1}\delta_{k_2k_4}}{n(\varepsilon_{k_4},T)-n(\varepsilon_{k_3},T)}\Bigg\{\sum_{k_5,m} g^{m*}_{k_3k_5}g^{m}_{k_1k_5}[N(\omega_m,T)+1-n(\varepsilon_{k_5},T)] \\
&\times \frac{n(\varepsilon_{k_4},T)-n(\varepsilon_{k_5}+\omega_m,T)}{i\omega_n-\omega_m+\varepsilon_{k_4k_5}} + \sum_{k_5,m} g^{m}_{k_5k_3}g^{m*}_{k_5k_1}[N(\omega_m,T)+n(\varepsilon_{k_5},T)] \\
&\times \frac{n(\varepsilon_{k_4},T)-n(\varepsilon_{k_5}-\omega_m,T)}{i\omega_n+\omega_m+\varepsilon_{k_4k_5}}\Bigg\}, \qquad (3.113)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Phi}^{(3)}_{k_1k_2,k_3k_4}(\omega_n) &= \frac{\delta_{\sigma_{k_1},-\sigma_{k_2}}\sigma_{k_1}\delta_{k_3k_1}}{n(\varepsilon_{k_4},T)-n(\varepsilon_{k_3},T)}\sum_{k_6,m}\Bigg\{g^{m*}_{k_2k_6}g^{m}_{k_4k_6}[N(\omega_m,T)+1-n(\varepsilon_{k_6},T)] \\
&\times \frac{n(\omega_m+\varepsilon_{k_6},T)-n(\varepsilon_{k_3},T)}{i\omega_n+\omega_m-\varepsilon_{k_3k_6}} + g^{m}_{k_6k_2}g^{m*}_{k_6k_4}[N(\omega_m,T)+n(\varepsilon_{k_6},T)] \\
&\times \frac{n(\varepsilon_{k_6}-\omega_m,T)-n(\varepsilon_{k_3},T)}{i\omega_n-\omega_m-\varepsilon_{k_3k_6}}\Bigg\}, \qquad (3.114)
\end{aligned}
$$

$$\xi^{m\eta_m}_{k_1k_2,k_3k_4} = \delta_{k_1k_3} g^{m(\eta_m)}_{k_4k_2} - g^{m(\eta_m)}_{k_1k_3} \delta_{k_4k_2}$$

$$g^{m(\sigma_k)}_{k_1k_2} = \delta_{\sigma_k,+1} g^{m}_{k_1k_2} + \delta_{\sigma_k,-1} g^{m*}_{k_2k_1}$$

$$\begin{aligned} \widetilde{\Phi}_{k_1k_2,k_3k_4}(\omega) &= \frac{\delta_{\sigma_{k_1},-\sigma_{k_2}} \sigma_{k_1}}{n(\varepsilon_{k_4},T) - n(\varepsilon_{k_3},T)} \sum_{k_5,k_6,m} \sum_{\eta_m=\pm 1} \eta_m \xi^{m\eta_m}_{k_1k_2,k_5k_6} \xi^{m\eta_m *}_{k_3k_4,k_5k_6} \\ &\times \frac{[N(\eta_m\omega_m,T) + n(\varepsilon_{k_6},T)][n(\varepsilon_{k_6} - \eta_m\omega_m,T) - n(\varepsilon_{k_5},T)]}{\omega - \varepsilon_{k_5} + \varepsilon_{k_6} - \eta_m\omega_m} \end{aligned}$$