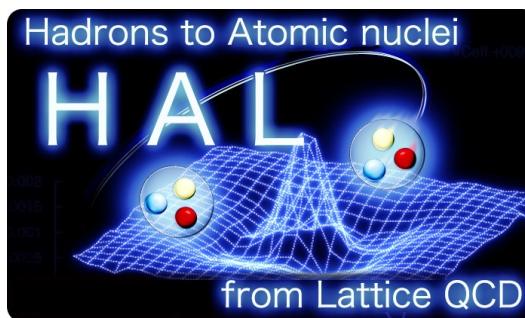


Baryon interactions from Lattice QCD

Kenji Sasaki (*YITP, Kyoto University*)

for HAL QCD Collaboration



HAL (Hadrons to Atomic nuclei from Lattice) QCD Collaboration

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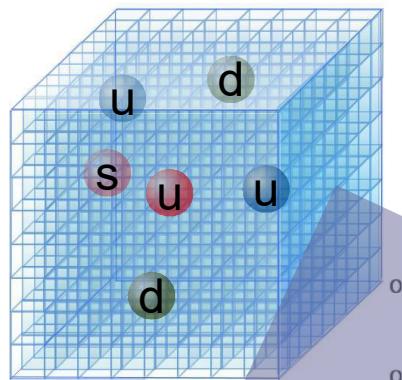
T.M. Doi
(*RIKEN*)

Contents

- **Introduction**
- **Baryon interactions from LQCD**
 - **Challenges in multi-baryon system**
- **Results**
- **Summary**

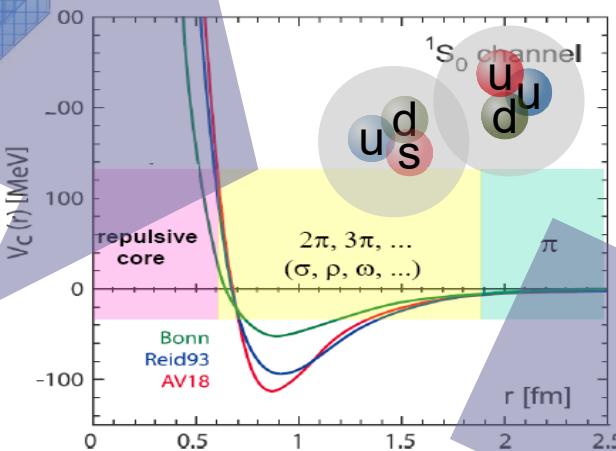
Introduction

From Quarks to Nuclei and Neutron stars



Derivation of hadron interaction

- Generalized baryon interaction
- Meson-meson,meson-baryon system
- Three-body interaction

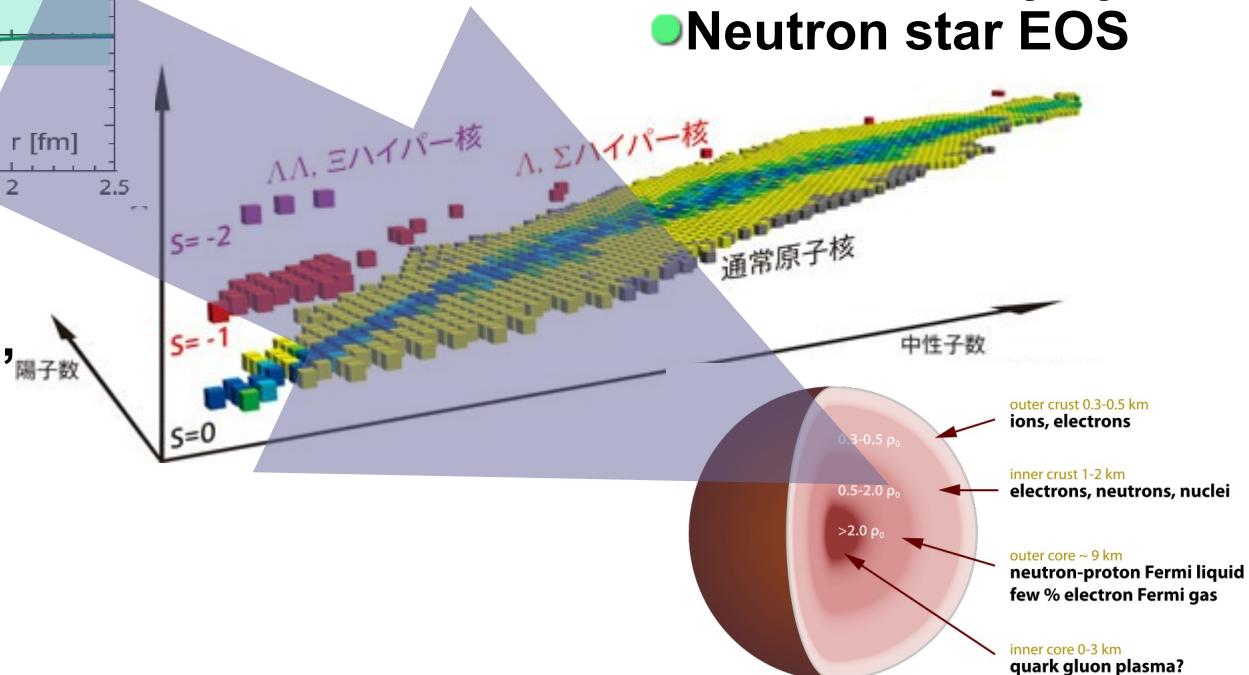


Applications

- Few-body system
- Medium heavy system
- Neutron star EOS

Technical improvements

- Unified Contraction Algorithm,
- Time dependent method,
- Higher partial waves



The process is independent of the experimental status

Baryon interaction from LQCD

BB interaction from QCD

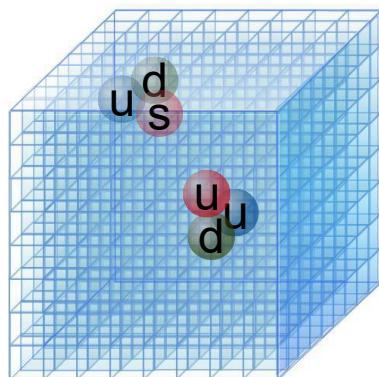
QCD is the fundamental theory of strong interactions

QCD Lagrangian

$$L_{QCD} = \bar{q}(i\gamma_\mu D^\mu - m)q + \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}$$



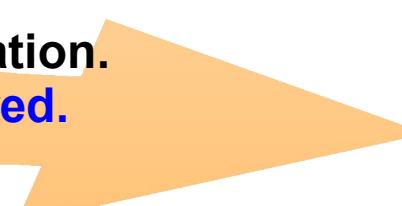
Lattice QCD simulation



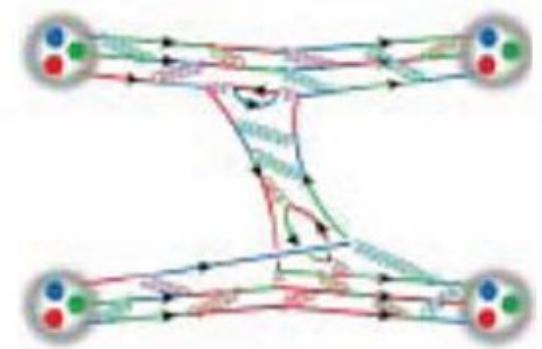
$$\left(\frac{1}{4m} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right) R(t, x) \\ = \int d^3y U(x, y) R(t, y)$$



- Non-perturbative calculation.
- Independent of experimental situation.
- Huge computer resource is required.



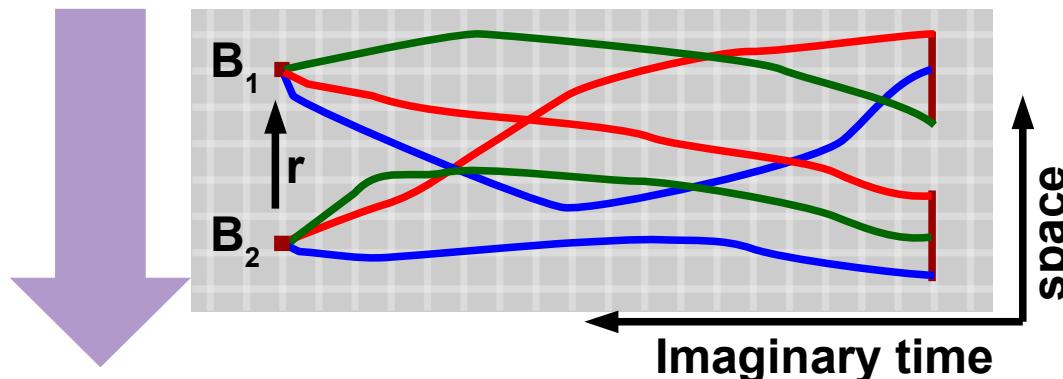
BB interaction



Baryon scattering on the lattice

In LQCD, the key quantity is baryon four-point correlator

$$\langle 0 | B_1 B_2(t, \vec{r}) \bar{I}(t_0) | 0 \rangle = A_0 \Psi(\vec{r}, E_0) e^{-E_0(t-t_0)} + \dots$$



Energy eigenvalues W_n and NBS (Nambu-Bethe-Salpeter) wave function

$$\Psi(E, \vec{r}) \simeq A \frac{\sin(pr + \delta(E))}{pr}$$

Asymptotic form of NBS wave function is characterized by phase shift

C.-J.D.Lin et al., NPB619 (2001) 467.

Lüscher's finite volume method

M. Lüscher, NPB354(1991)531

W_n from temporal correlation

HAL QCD method

Ishii, Aoki, Hatsuda, PRL99 (2007) 022001

Ψ from spacial correlation
 W_n from temporal correlation

S-matrix

Scattering observable from LQCD data

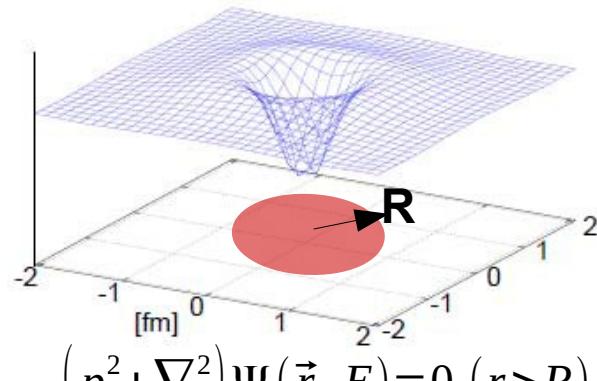
Lattice QCD simulation

$$\langle 0 | B_1 B_2(t, \vec{r}) \bar{I}(t_0) | 0 \rangle = A_0 \Psi(\vec{r}, E_0) e^{-E_0(t-t_0)} + \dots$$

HAL QCD method

Ishii, Aoki, Hatsuda, PRL99 (2007) 022001

NBS wave function



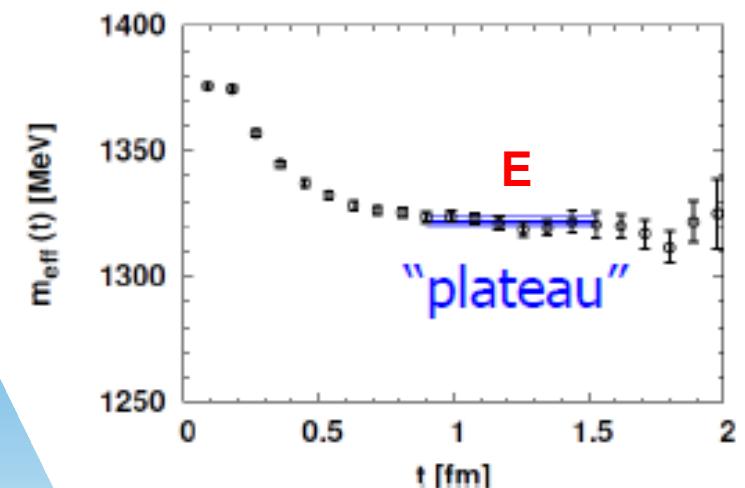
$$(p^2 + \nabla^2) \Psi^\alpha(E, \vec{x}) \equiv \int d^3y U_\alpha^\alpha(\vec{x}, \vec{y}) \Psi^\alpha(E, \vec{y})$$

- $U(x, y)$ is faithful to the S-matrix.
- $U(x, y)$ is energy independent but non-local.

Lüscher's finite volume method

M. Lüscher, NPB354(1991)531

Eigen energy is extracted
from temporal correlation



$$k_n \cot \delta(k_n) = \frac{4\pi}{L^3} \sum_{m \in \mathbb{Z}^3} \frac{1}{\bar{p}_m^2 - k_n^2}$$

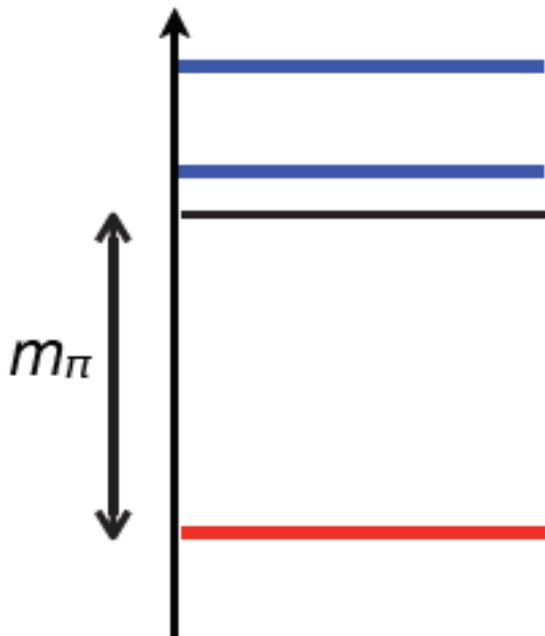
Scattering S-matrix

$$S \equiv e^{i\delta(k)}$$

Excited state contamination

Single baryon case

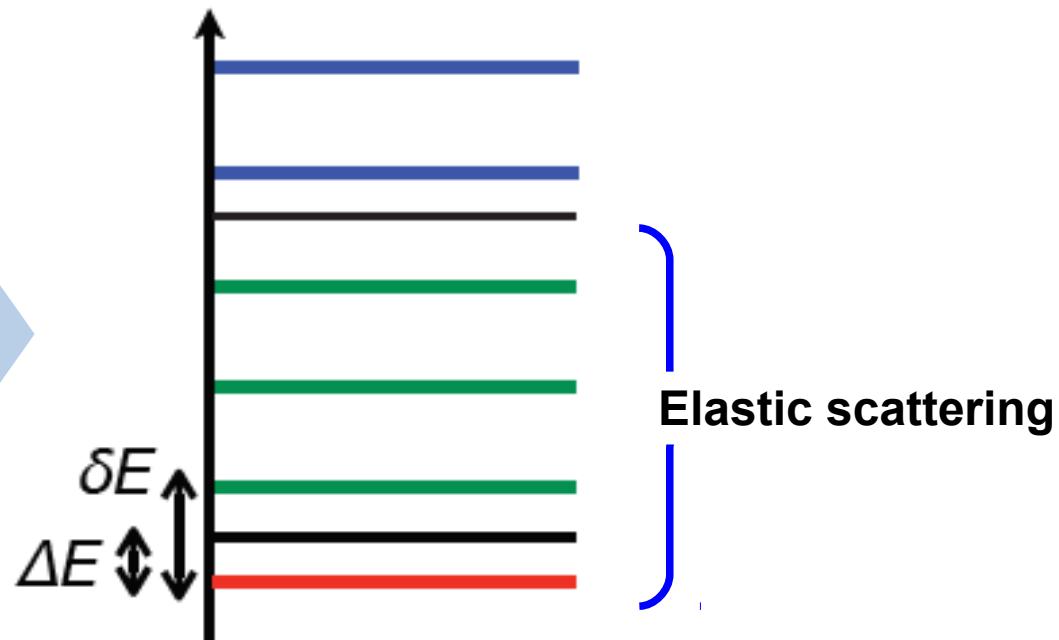
$$C_B(t) = a_0 e^{-m_B(t-t_0)} + a_1 e^{-(m_B+m_\pi)(t-t_0)} + \dots$$
$$\Rightarrow a_0 e^{-m_B t_{sat}}$$



$$t_{sat} \sim m_\pi^{-1} \sim 1 \text{ fm}$$

Two baryon case

$$C_{BB}(t) = b_0 e^{-W_0(t-t_0)} + b_1 e^{-W_1(t-t_0)} + \dots$$
$$\Rightarrow b_0 e^{-W_0 t_{sat}}$$



$$t_{sat} \sim \delta W^{-1} = m_B (L/2\pi)^2$$

**In contrast to the determination of the single baryon mass,
The measurement of ground state energy is hindered by elastic excited states.**

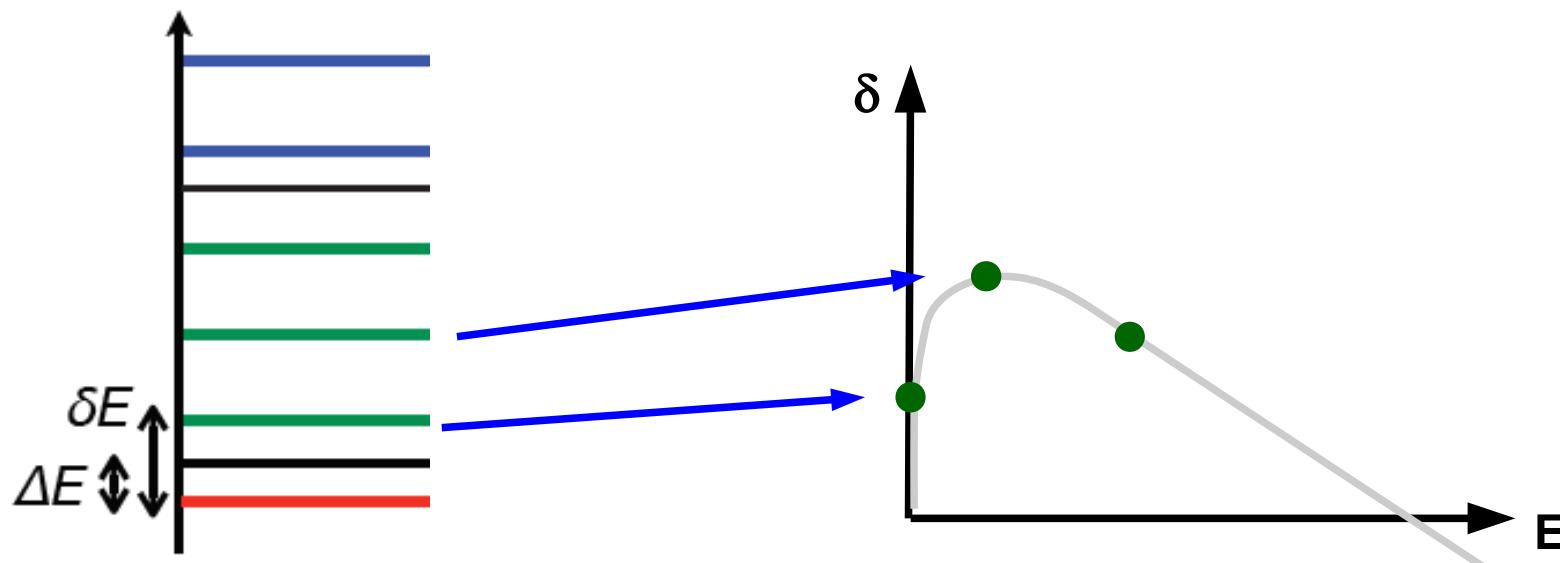
Management of excited states

Lüscher's finite volume method

M. Lüscher, NPB354(1991)531

It is mandatory to find the well-optimized source operator which strictly couples to the specific energy.

$$\langle 0 | B_1 B_2(t, \vec{r}) \bar{I}_{E_i}^{opt}(t_0) | 0 \rangle = A_i \Psi(\vec{r}, E_i) e^{-E_i(t-t_0)}$$



By way of caution

If the optimization of source operator is not enough, one encounters the “mirage plateau” and irrelevant observable through the Lüscher’s formula.

T. Iritani et al. (HAL) JHEP1610(2016)101
T. Iritani et al. (HAL) PRD96(2017)034521

Time-dependent HAL QCD method

Considering the normalized four-point correlator,

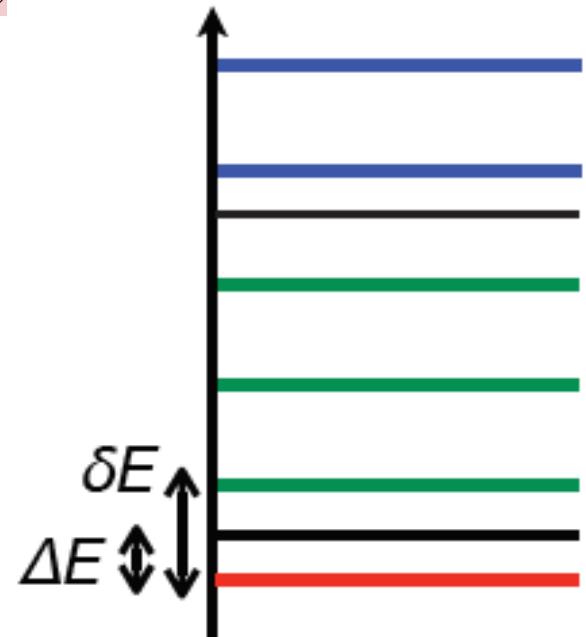
$$R_I^{B_1 B_2}(t, \vec{r}) = F_I^{B_1 B_2}(t, \vec{r}) e^{(m_1 + m_2)t} \\ = A_0 \Psi(\vec{r}, E_0) e^{-(E_0 - m_1 - m_2)t} + A_1 \Psi(\vec{r}, E_1) e^{-(E_1 - m_1 - m_2)t} + \dots$$

$$\left(\frac{p_0^2}{2\mu} + \frac{\nabla^2}{2\mu} \right) \Psi(\vec{r}, E_0) = \int U(\vec{r}, \vec{r}') \Psi(\vec{r}', E_0) d^3 r'$$

$$E_n - m_1 - m_2 \approx \frac{p_n^2}{2\mu} \quad \left(\frac{p_1^2}{2\mu} + \frac{\nabla^2}{2\mu} \right) \Psi(\vec{r}, E_1) = \int U(\vec{r}, \vec{r}') \Psi(\vec{r}', E_1) d^3 r'$$

A single state saturation is not required!!

$$\left(-\frac{\partial}{\partial t} + \frac{\nabla^2}{2\mu} \right) R_I^{B_1 B_2}(t, \vec{r}) = \int U(\vec{r}, \vec{r}') R_I^{B_1 B_2}(t, \vec{r}') d^3 r'$$



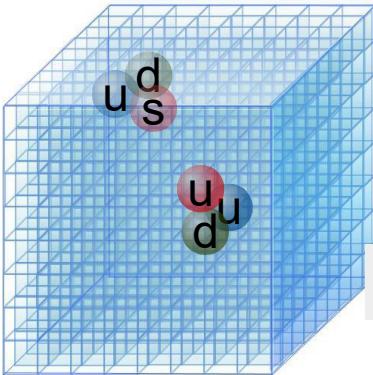
All elastic energies contribute as a signal of energy indep. pot.

Derivative (velocity) expansion of U

$$U(\vec{r}, \vec{r}') = [V_C(r) + S_{12} V_T(r)] + [\vec{L} \cdot \vec{S}_s V_{LS}(r) + \vec{L} \cdot \vec{S}_a V_{ALS}(r)] + O(\nabla^2)$$

Lüscher's method (coupled-channel)

Lattice QCD simulation



Scattering S-matrix

$$S(E) = \begin{pmatrix} \eta e^{2i\delta_1} & i\sqrt{1-\eta^2}e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2} \end{pmatrix}$$

$$\langle 0 | B_1 B_2(t, \vec{r}) \bar{I}(t_0) | 0 \rangle = A_0 \Psi(\vec{r}, E_0) e^{-E_0(t-t_0)} + \dots$$

Lüscher's finite volume method

M. Lüscher, NPB354(1991)531

Two-channel S-matrix has 3-parameters

$$\delta_1(E), \quad \delta_2(E), \quad \eta(E)$$

These are related to the energy E by an eigenvalue equation (s-wave)

$$\cos(\Delta_1 + \Delta_2 - \delta_1^0 - \delta_2^0) = \eta \cos(\Delta_1 - \Delta_2 - \delta_1^0 + \delta_2^0)$$

$$\frac{1}{\tan \Delta_i} = \frac{4\pi}{k_i} \cdot \frac{1}{L^3} \sum_p \frac{1}{p^2 - k_i^2}$$

Unlike the single channel case,

the number of equations is less than the number of parameters in S-matrix.

Extra-information (relation) is necessary to solve coupled channel scattering

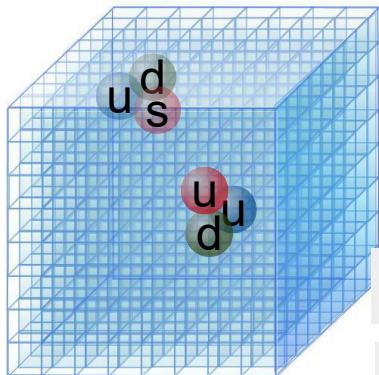
Relations of parameters

Assumption of Interaction

Fixed form of K-matrix

Coupled-channel HAL QCD method

Lattice QCD simulation



HAL QCD method

S.Aoki et al [HAL] Proc. Jpn. Acad., Ser.B, 87 509

$$\langle 0 | (B_1 B_2)^\alpha(t, \vec{r}) \bar{I}(t_0) | 0 \rangle = A_0 \Psi^\alpha(\vec{r}, E_0) e^{-E_0(t-t_0)} + \dots$$

$$\langle 0 | (B_1 B_2)^\beta(t, \vec{r}) \bar{I}(t_0) | 0 \rangle = C_0 \Psi^\beta(\vec{r}, E_0) e^{-E_0(t-t_0)} + \dots$$

Scattering S-matrix

$$S(E) = \begin{pmatrix} \eta e^{2i\delta_1} & i\sqrt{1-\eta^2}e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2} \end{pmatrix}$$

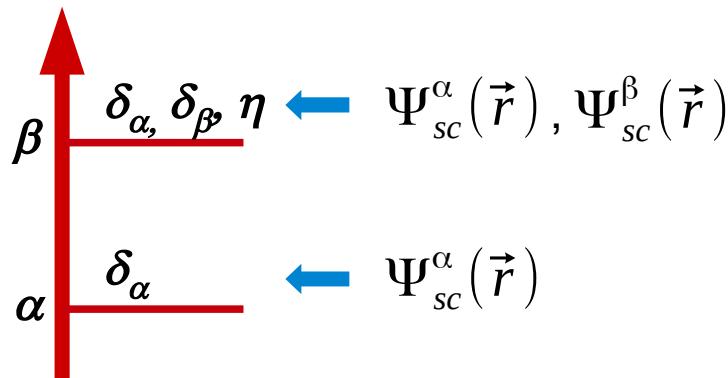
NBS wave function for each channel

$$\Psi^\alpha(\vec{r}, E_i) e^{-E_i t} = \langle 0 | (B_1 B_2)^\alpha(\vec{r}) | E_i \rangle$$

$$\Psi^\beta(\vec{r}, E_i) e^{-E_i t} = \langle 0 | (B_1 B_2)^\beta(\vec{r}) | E_i \rangle$$

Coupled-channel Schrödinger equation

$$(p_\alpha^2 + \nabla^2) \Psi^\alpha(E, \vec{x}) \equiv \int d^3 y U_\beta^\alpha(\vec{x}, \vec{y}) \Psi^\beta(E, \vec{y})$$



- **$U(x,y)$ is faithful to the S-matrix beyond the threshold of channel β .**
- **$U(x,y)$ is energy independent until the higher energy threshold opens.**
- **Derivative (velocity) expansion is used.**

Results

Numerical setup

► 2+1 flavor gauge configurations.

- Iwasaki gauge action & O(a) improved Wilson quark action
- $a = 0.084 [fm]$, $a^{-1} = 2.333 \text{ GeV}$.
- $96^3 \times 96$ lattice, $L = 8.12 [fm]$.
- 414 confs x 96 sources x 4 rotations.

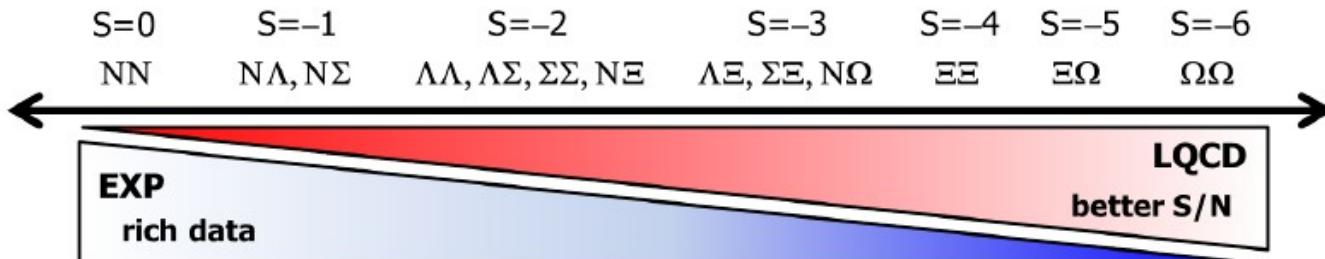


PACS Coll., PoS LAT2015, 075

► Wall source is considered to produce S-wave B-B state.

• Measurement

- All of NN/YN/YY for central/tensor forces in $P=(+)$ (S, D-waves)



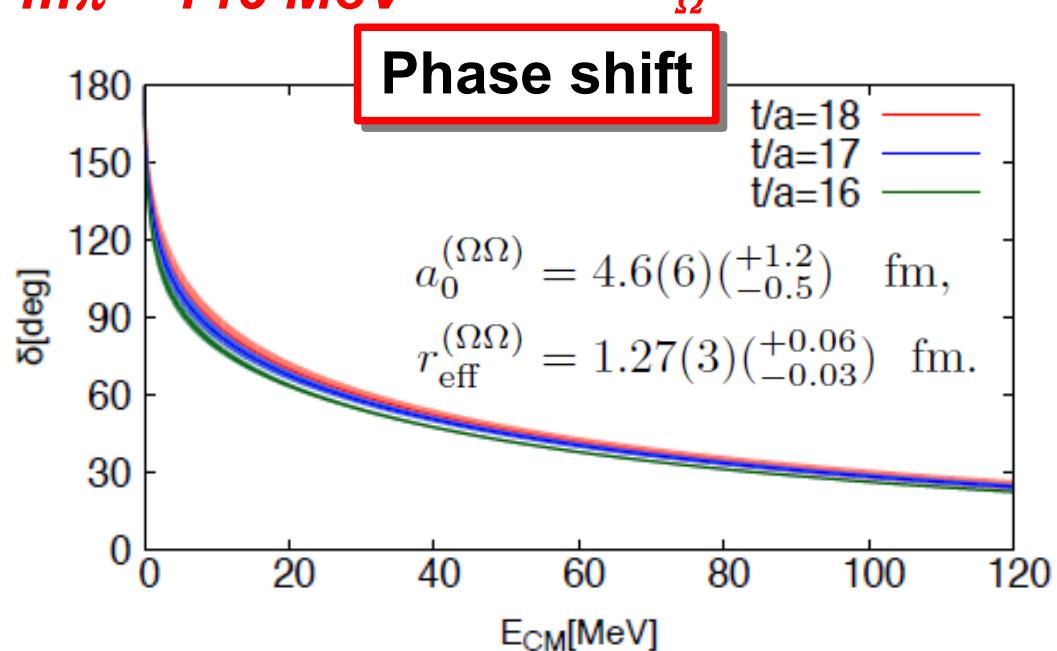
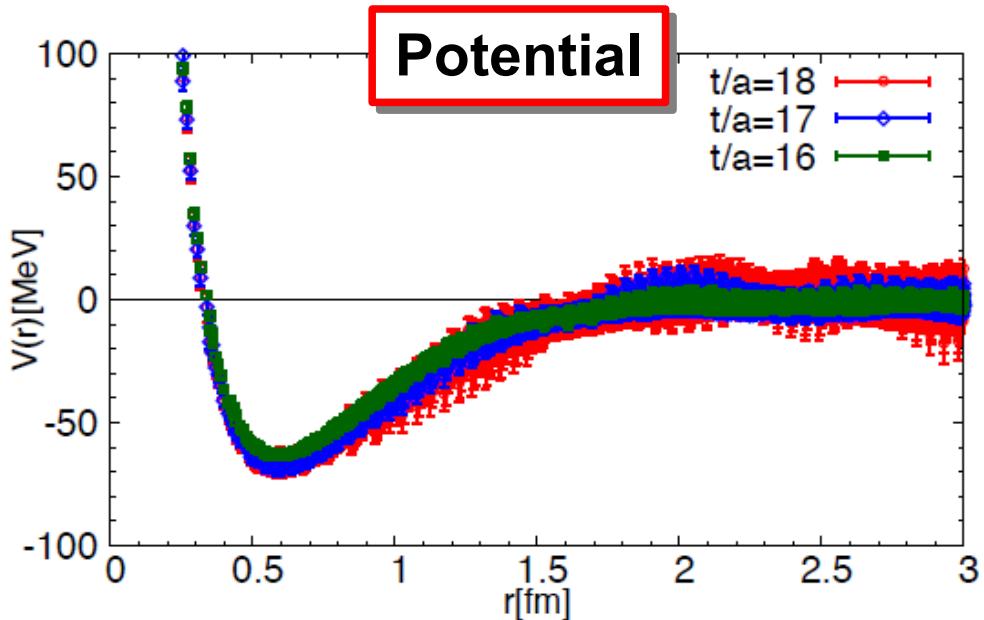
K-computer + FX100@ RIKEN + HA-PACS@ Tsukuba

	Mass [MeV]
π	146
K	525
m_p/m_K	0.28
N	958 ± 3
Λ	1140 ± 2
Σ	1223 ± 2
Ξ	1354 ± 1

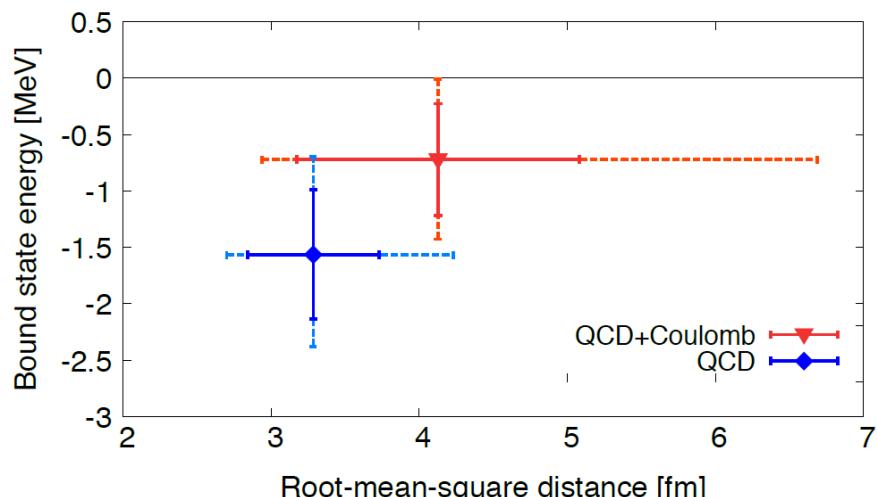
$\Omega\Omega J^p(I) = 0^+(0)$ state near the physical point

► $N_f = 2+1$ full QCD with $L = 8\text{fm}$, $m_\pi = 146 \text{ MeV}$

$m_\Omega = 1712 \text{ MeV}$

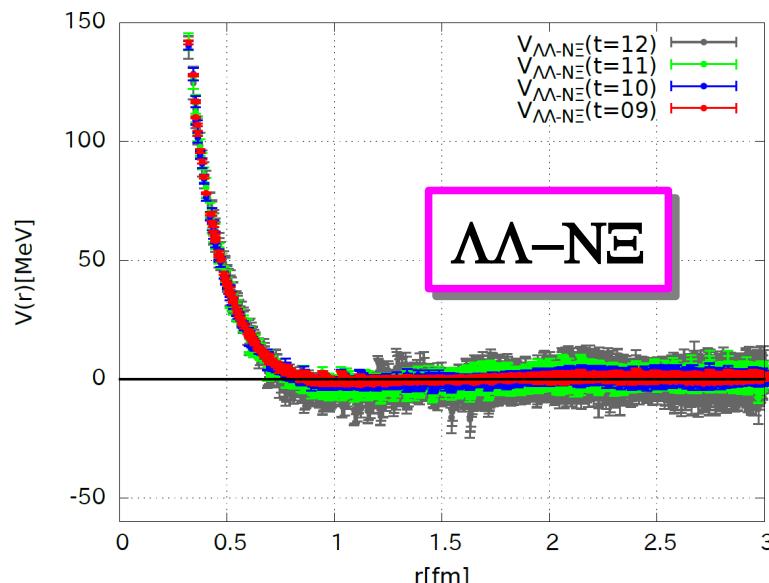
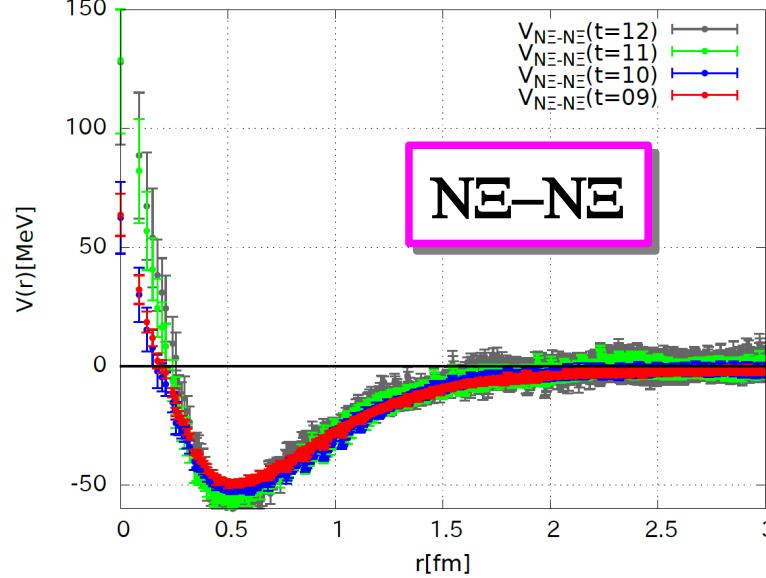
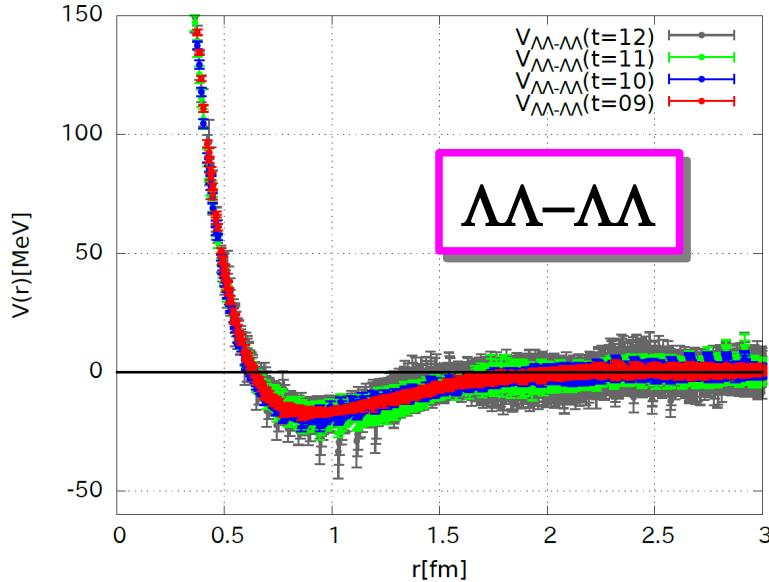


- Short range repulsion and attractive pocket is found.
- Calculated phase shift indicates a bound $\Omega\Omega$ state [Most strange dibaryon].
- Physical $\Omega\Omega$ state in $J^p(I) = 0^+(0)$ is very close to unitary region.



$\Lambda\Lambda$, $N\Xi$ ($I=0$) 1S_0 potential

► $N_f = 2+1$ full QCD with $L = 8.1\text{fm}$, $m\pi = 146\text{ MeV}$



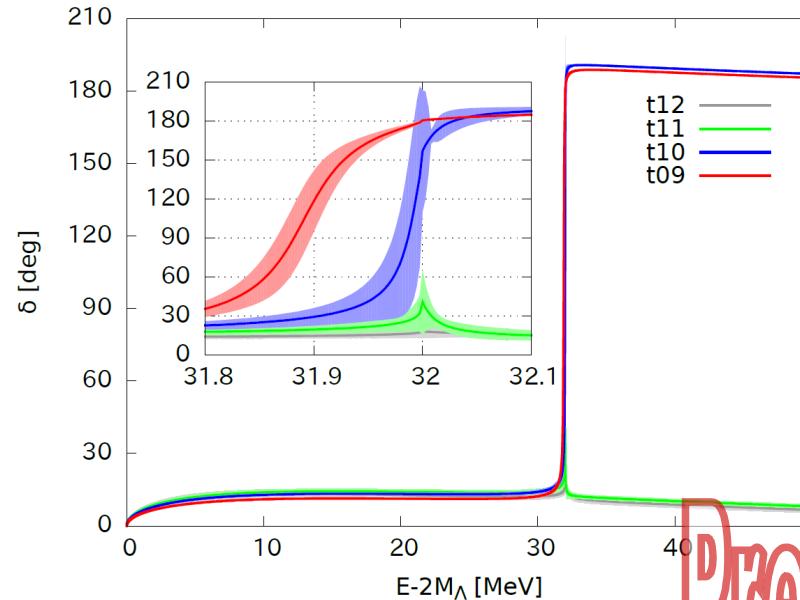
Preliminary!

- Coupled-channel $\Lambda\Lambda$ and $N\Xi$ potentials are plotted.
- Long range part of potential is almost stable against the time slice.

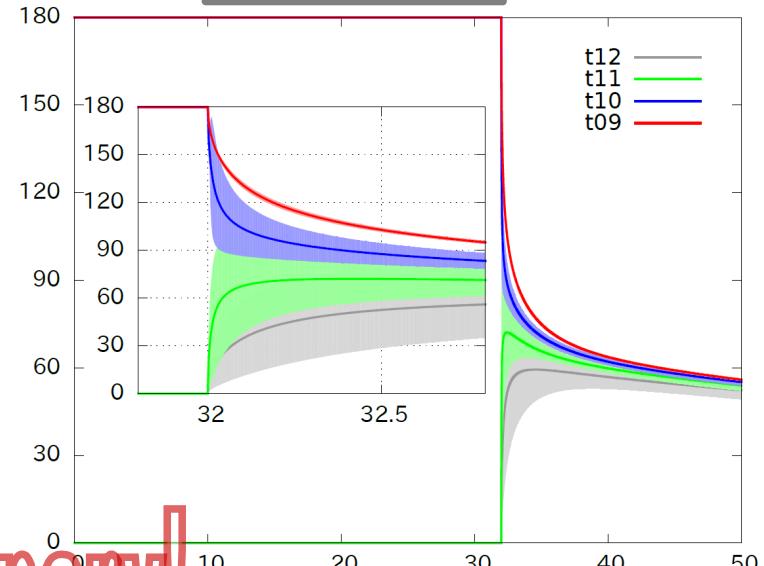
$t=09$
 $t=10$
 $t=11$
 $t=12$

$\Lambda\Lambda$ and $N\Xi$ phase shift and inelasticity

$\Lambda\Lambda$ phase shift

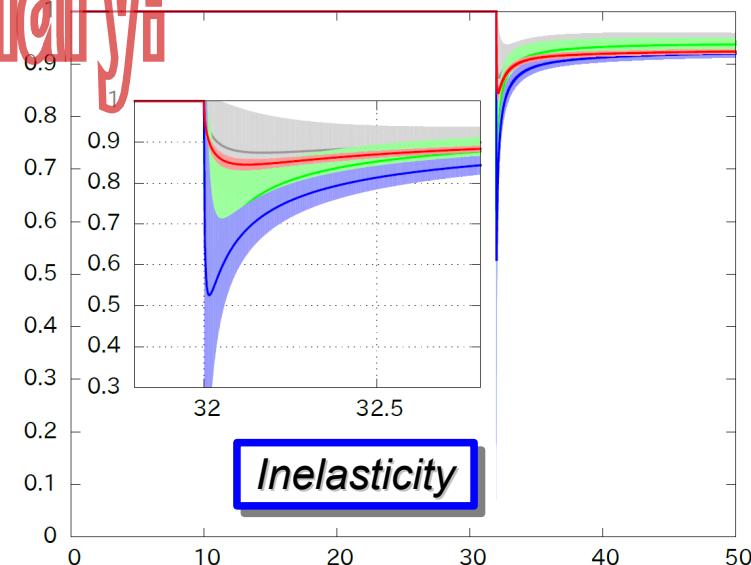


$N\Xi$ phase shift



Preliminary

- $\Lambda\Lambda$ and $N\Xi$ phase shift is calculated.
- A sharp resonance is found below the $N\Xi$ threshold for $t=9 - 10$.

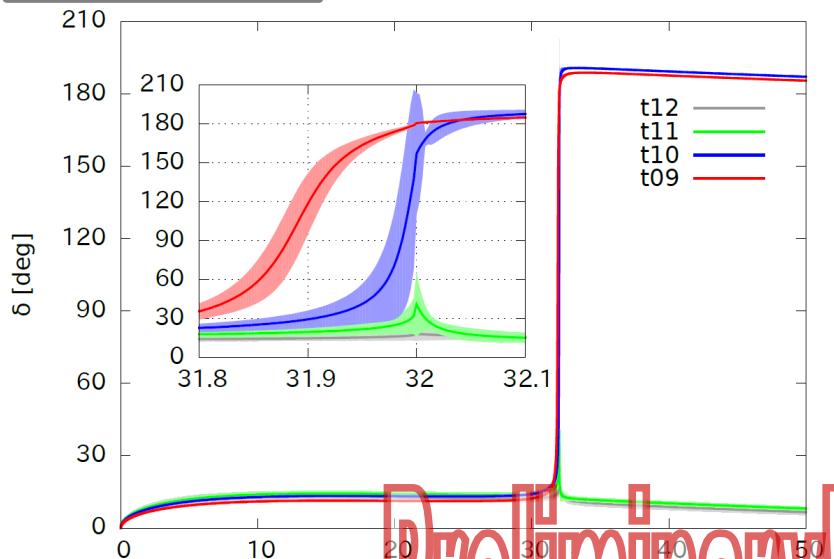


Inelasticity

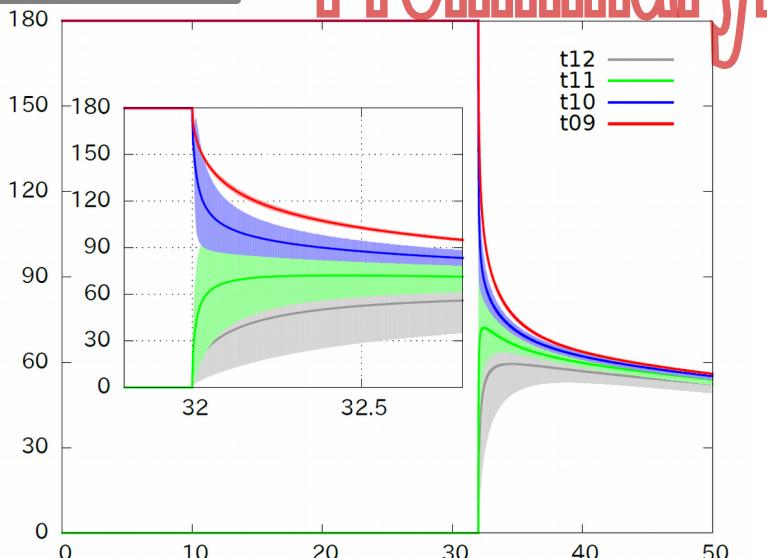
$t=09$
 $t=10$
 $t=11$
 $t=12$

$\Lambda\Lambda$ and $N\Xi$ phase shift –comparison--

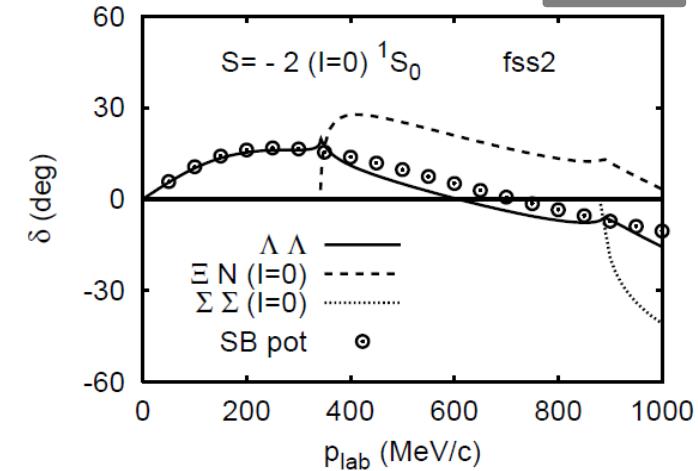
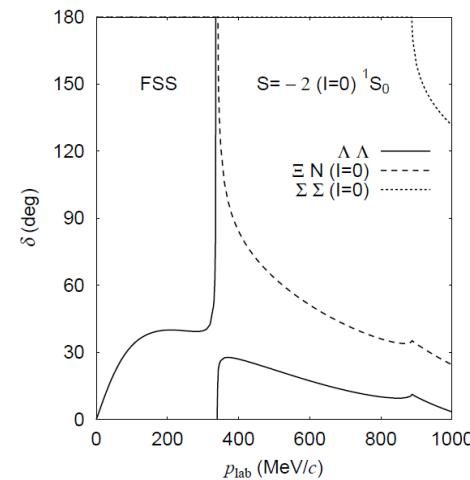
$\Lambda\Lambda$ phase shift



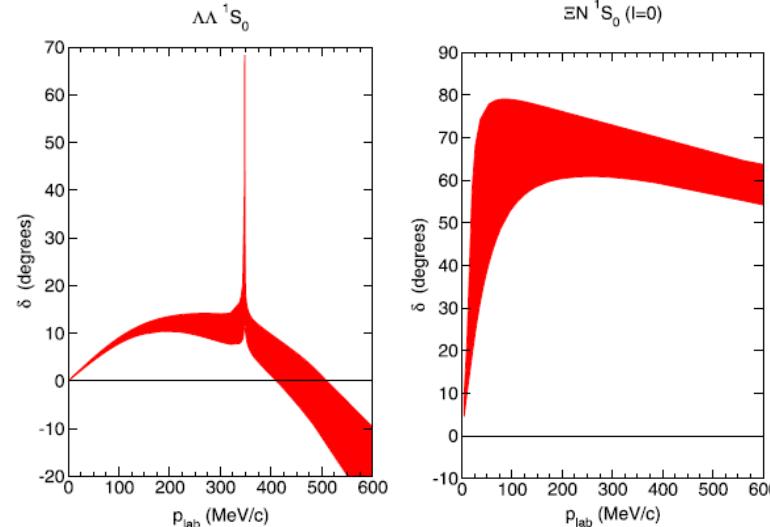
$N\Xi$ phase shift



Preliminary!



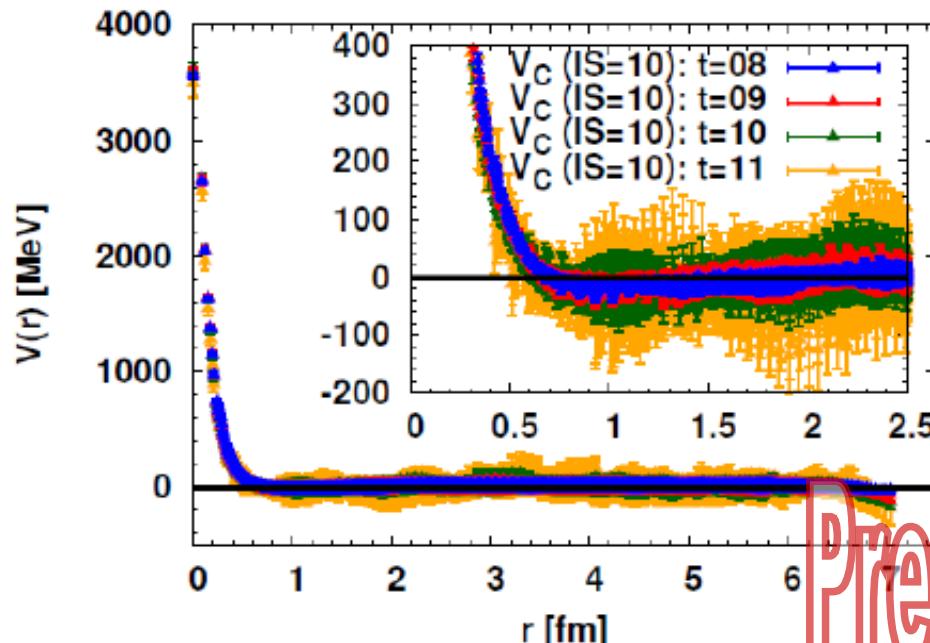
Y.Fujiwara et al, PPNP58(2007)439



J. Haidenbauer et al, NPA954(2016)273

Our results are compatible
with the phenomenological ones.

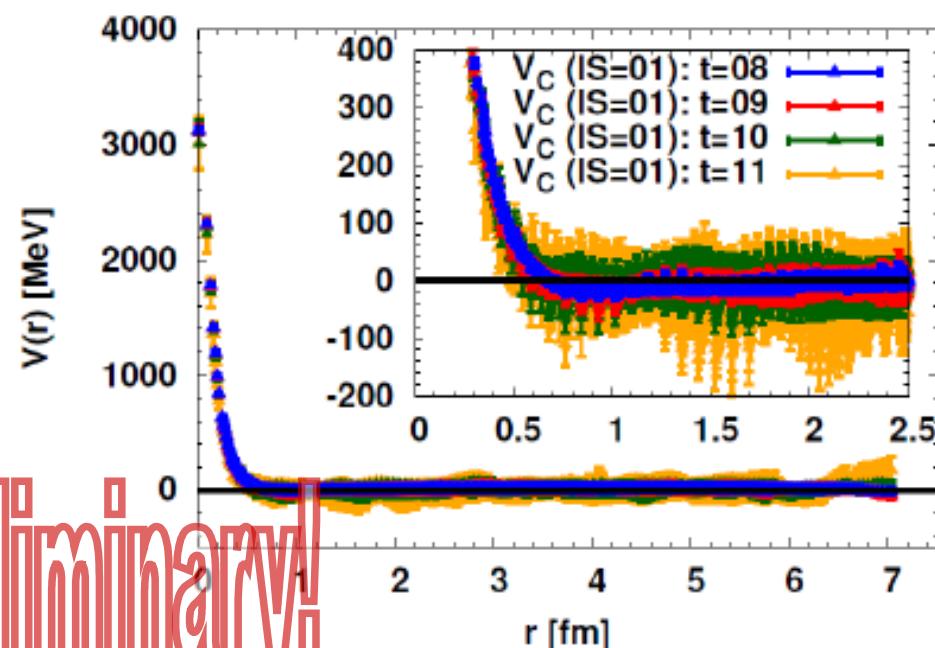
$NN J^p(l) = 0^+(1)$ and $1^+(0)$ state



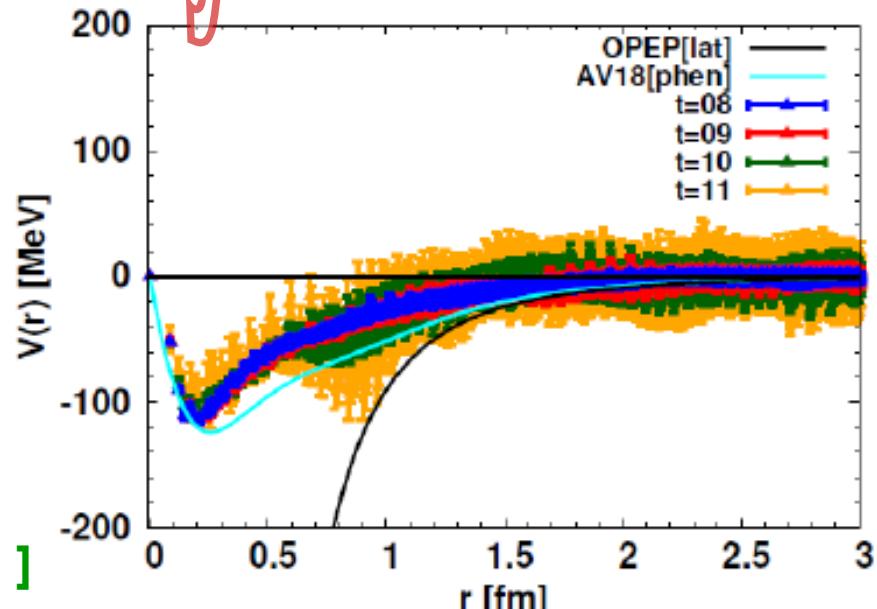
Repulsive core
observed

Attraction at
mid-long range

Strong Tensor Force is
clearly visible !



Preliminary!



[T. Doi]

Impact on dense matter

LQCD YN/YY-forces + Phen NN-forces (AV18)
used in Brueckner-Hartree-Fock (BHF)

→ Single-particle energy of Hyperon in nuclear matter

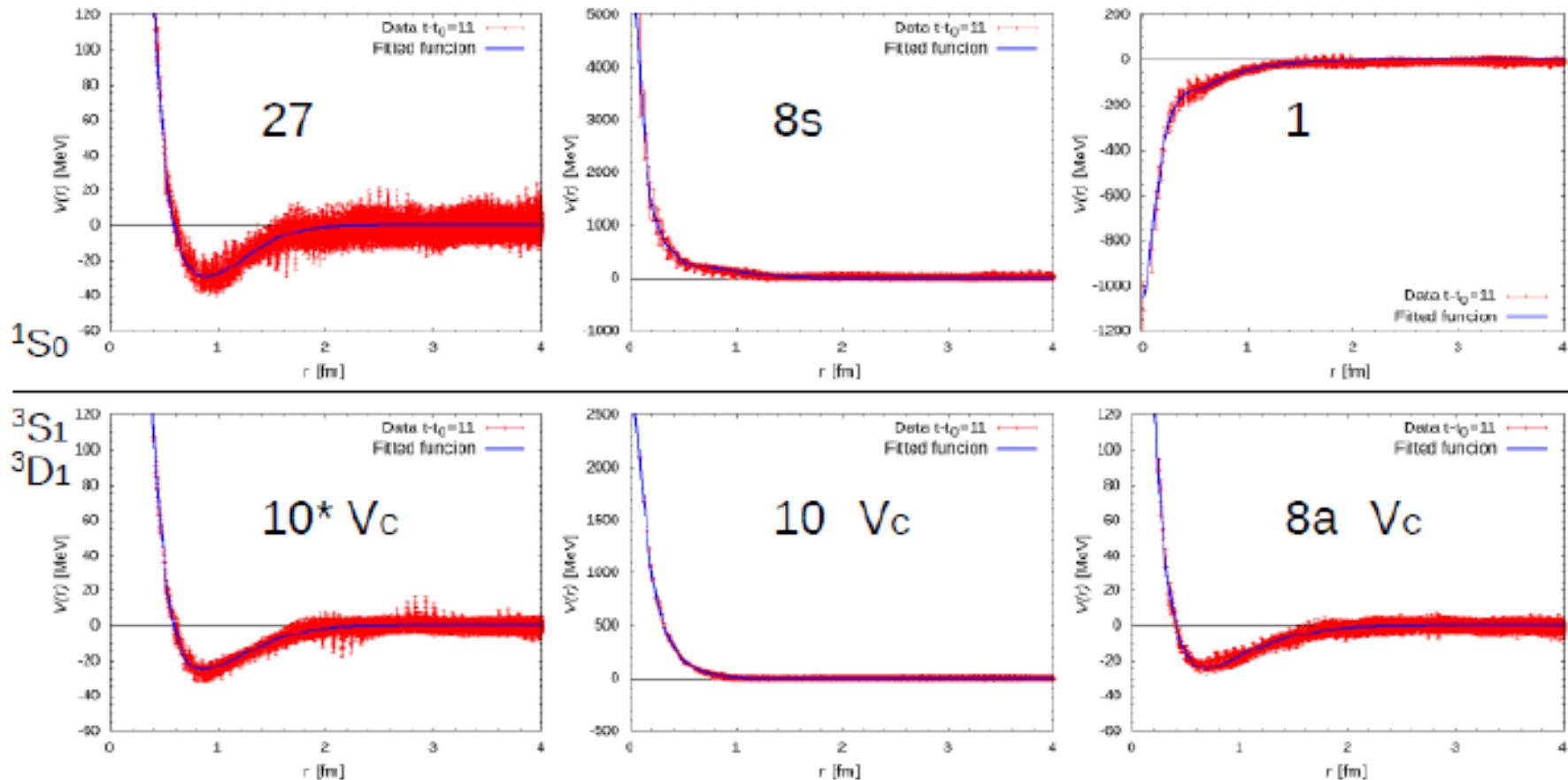
(Only diagonal YN/YY forces in SU(3) irrep used)

51

$S=-2$ interactions suitable to grasp whole NN/YN/YY interactions

Central Force in Irrep-base (diagonal)

$$8 \times 8 = \frac{27 + 8s + 1 + 10^* + 10 + 8a}{^1S_0} + \frac{3S_1, ^3D_1}{}$$

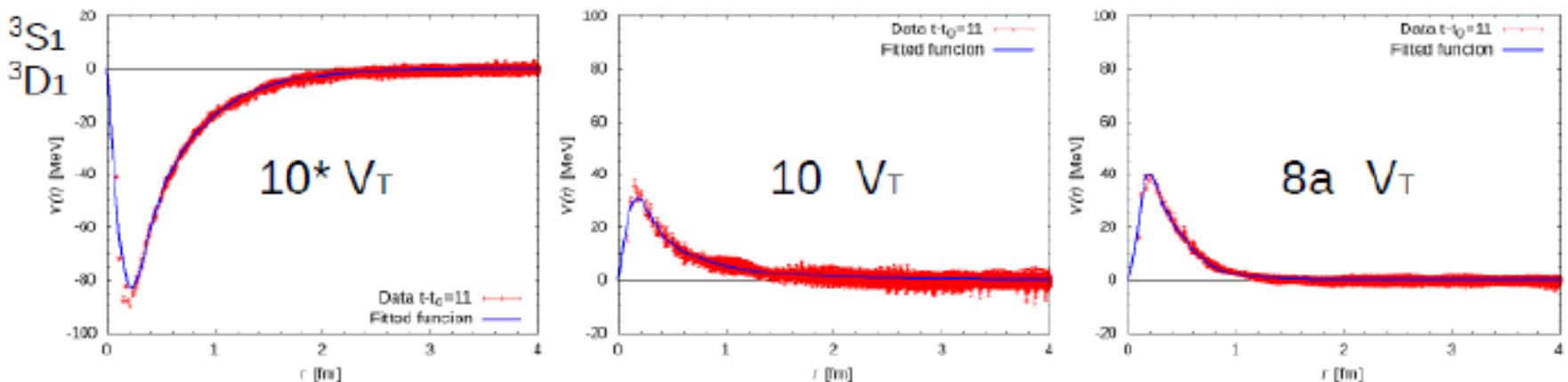


(off-diagonal component is small)

S=-2 interactions suitable to grasp whole NN/YN/YY interactions

Tensor Force in Irrep-base (diagonal)

$$8 \times 8 = \frac{27 + 8s + 1}{^1S_0} + \frac{10^* + 10 + 8a}{^3S_1, ^3D_1}$$



→ We calculate single-particle energy of hyperon
in nuclear matter w/ LQCD baryon forces

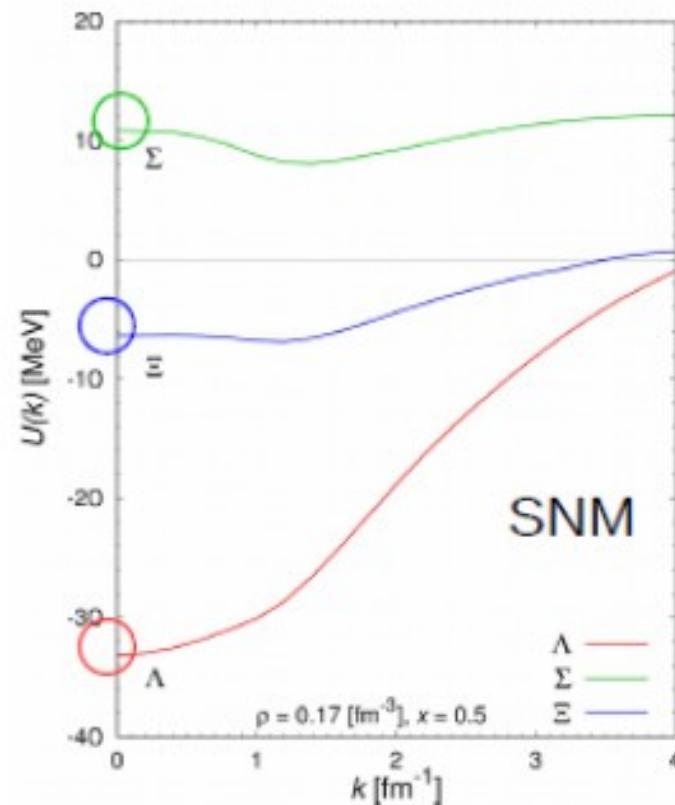
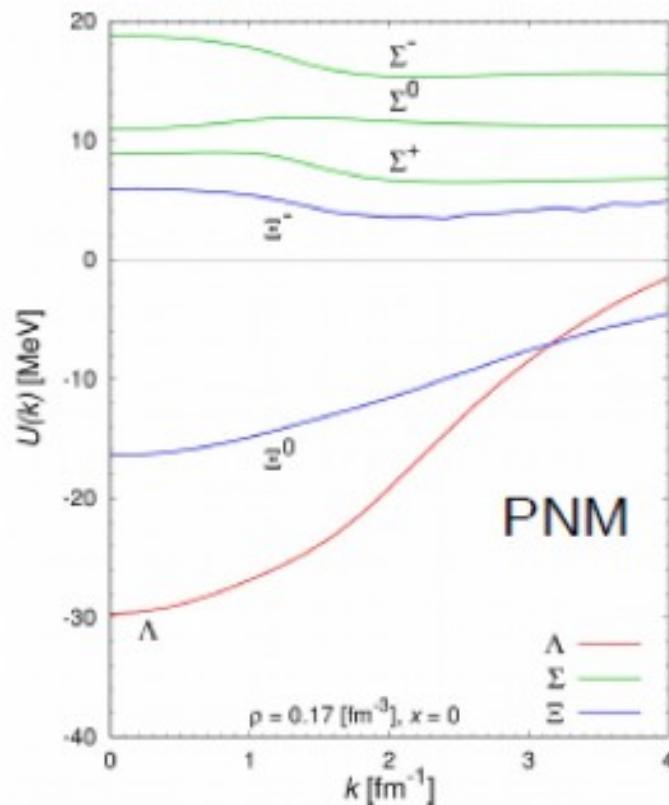
(off-diagonal component neglected)

We fit by

$$V(r) = a_1 e^{-a_2 r^2} + a_3 e^{-a_4 r^2} + a_5 \left[\left(1 - e^{-a_6 r^2}\right) \frac{e^{-a_7 r}}{r} \right]^2 \quad (\text{central})$$

$$V(r) = a_1 \left(1 - e^{-a_1 r^2}\right)^2 \left(1 + \frac{3}{a_3 r} + \frac{3}{(a_3 r)^2}\right) \frac{e^{-a_5 r}}{r} + a_4 \left(1 - e^{-a_4 r^2}\right)^2 \left(1 + \frac{3}{a_6 r} + \frac{3}{(a_6 r)^2}\right) \frac{e^{-a_6 r}}{r} \quad (\text{tensor})$$

Hyperon single-particle potentials



Preliminary

- obtained by using YN, YY forces from QCD.
- Results are compatible with experimental suggestion.

$$U_{\Lambda}^{\text{Exp}}(0) \simeq -30, \quad U_{\Xi}(0)^{\text{Exp}} \simeq -10, \quad U_{\Sigma}^{\text{Exp}}(0) \geq +20 \quad [\text{MeV}]$$

attraction attraction small repulsion

1

[T. Inoue]

Summary

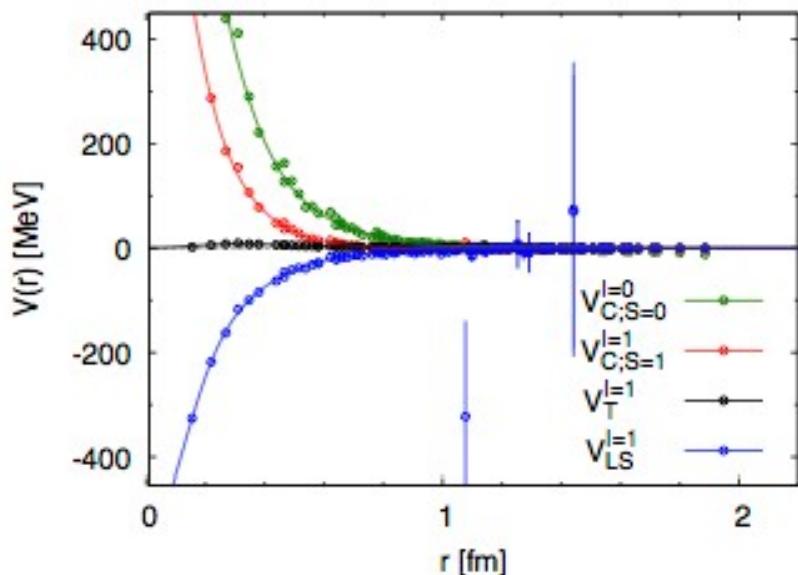
- **Baryon interactions:**
Bridge between particle/nuclear/astro-physics
- **HAL QCD method crucial for a reliable calculation**
 - Direct method suffers from excited state contaminations
- **The 1st LQCD calculation for Baryon interactions at almost physical point is reported**
 - $m(\pi) \sim= 146$ MeV, $L \sim= 8$ fm, $1/a \sim= 2.3$ GeV
 - Central/Tensor forces for NN/YN/YY in $P=(+)$ channel
- **Prospects:**
 - Exascale computing Era ~ 2020
 - LS-forces, $P=(-)$ channel, 3-baryon forces, etc., & EoS

NN-forces in P=(-) channel

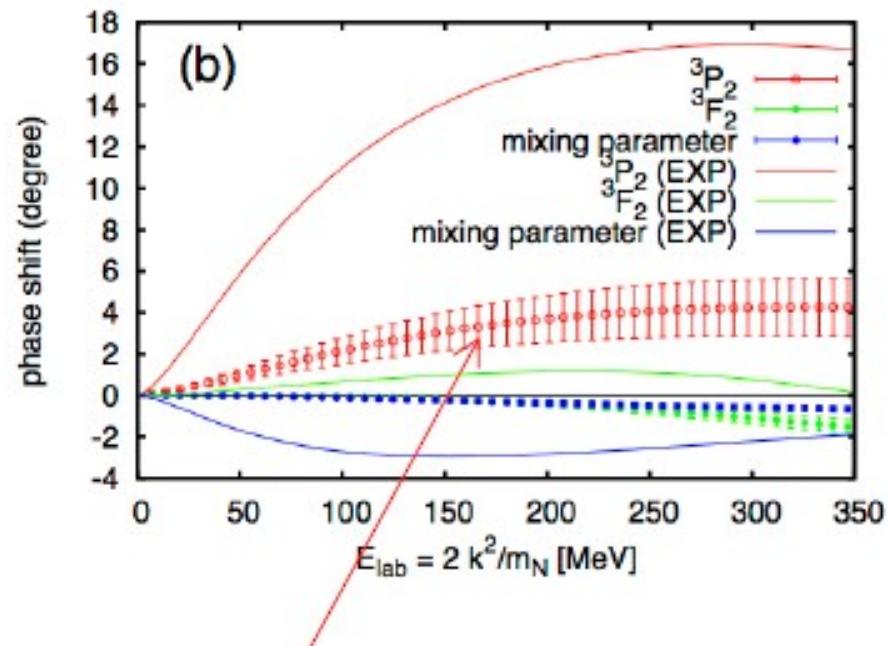
($m_\pi = 1.1 \text{ GeV}$)

- Central, tensor & LS forces

$^1P_1, ^3P_0, ^3P_1, ^3P_2 - ^3F_2$



Phase shifts



Superfluidity 3P_2 in neutron star
↔ neutrino cooling

↔ observation of Cas A NS

K.Murano et al., PLB735(2014)19

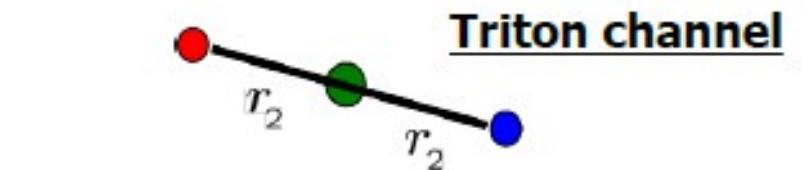
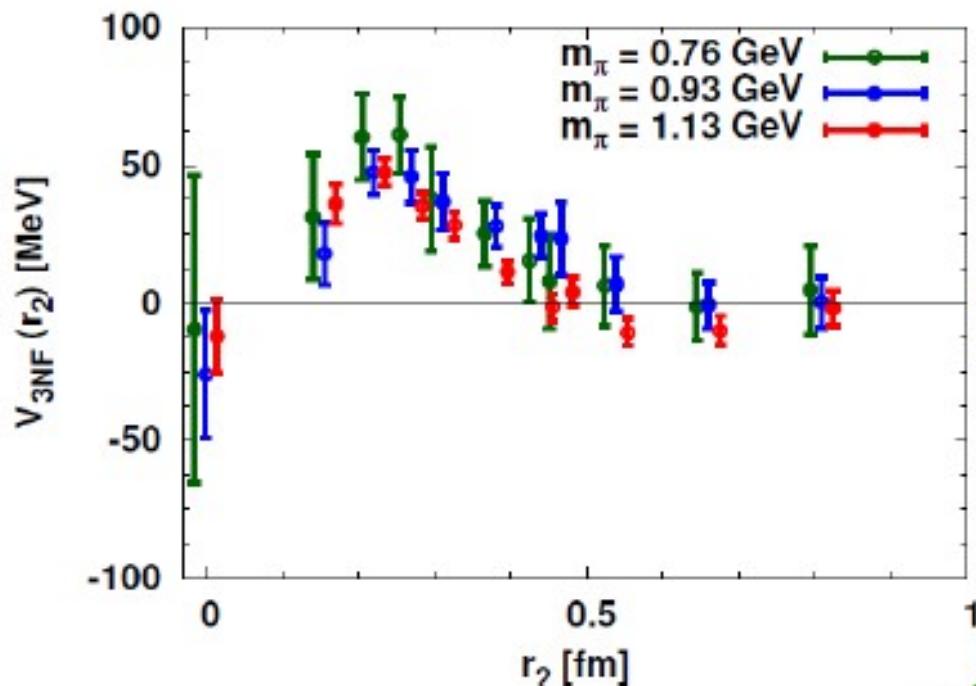
c.f. CalLat Coll. PLB765(2017)285

Attractive in 3P_2

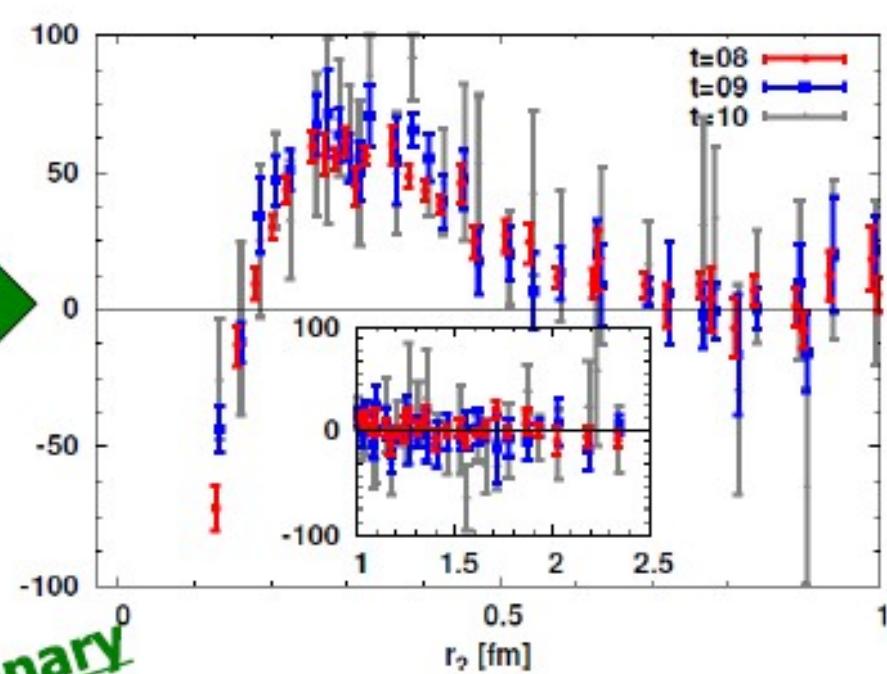
Qualitatively good, but strength is weak
(We also observe potentials glow by lighter mass)

3N-forces (3NF)

Nf=2, m π =0.76-1.1 GeV



Nf=2+1, m π =0.51 GeV



Preliminary

Magnitude of 3NF is similar for all masses
Range of 3NF tend to get longer (?) for $m(\pi)=0.5\text{GeV}$

Kernel: ~50% efficiency achieved !

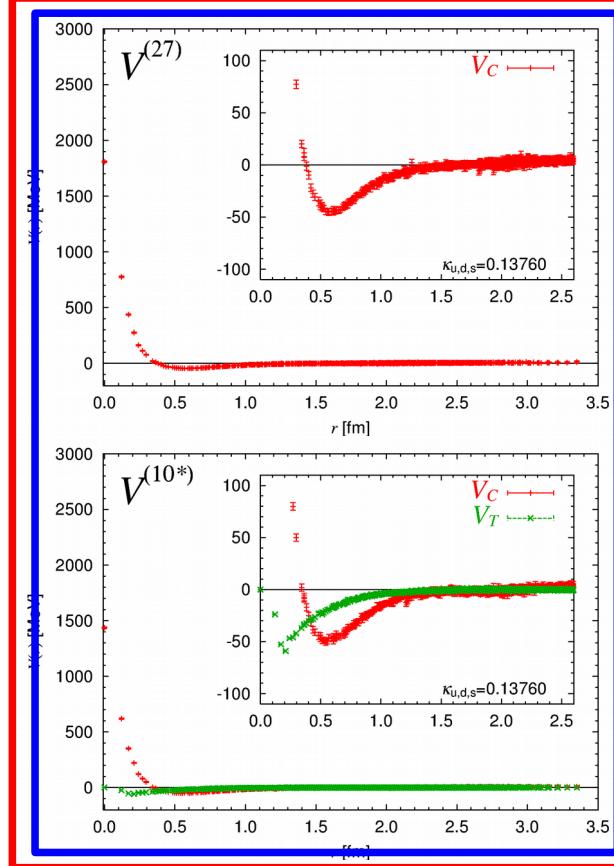
57

Backup

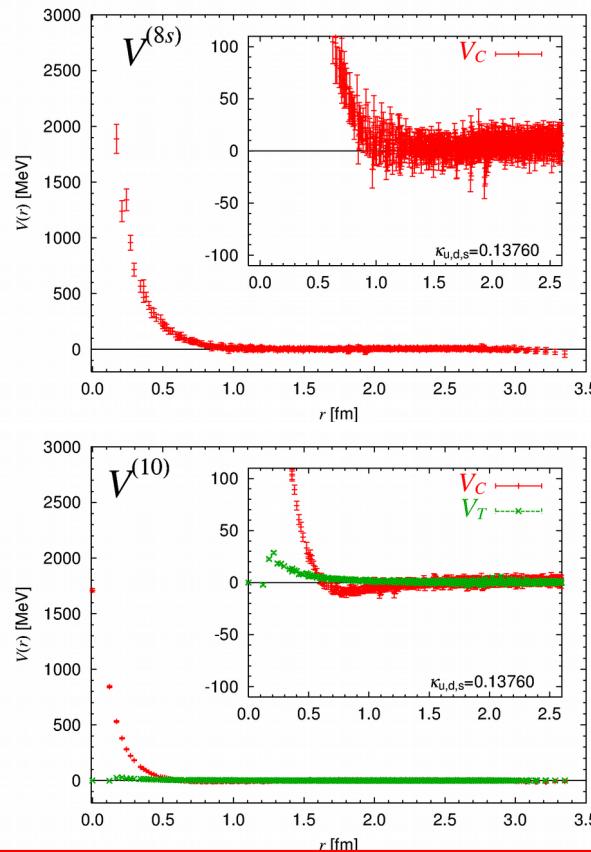
B-B potentials in SU(3) limit

$m_\pi = 469 \text{ MeV}$

J=0



Two-flavors



Three-flavors

- Quark Pauli principle can be seen at around short distances
 - ✓ No repulsive core in flavor singlet state
 - ✓ Strongest repulsion in flavor 8s state
- Possibility of bound H-dibaryon in flavor singlet channel.

Nambu-Bethe-Salpeter wave function

Definition : equal time NBS w.f.

$$\Psi^a(E, \vec{r}) e^{-Et} = \sum_{\vec{x}} \langle 0 | H_1^a(t, \vec{x} + \vec{r}) H_2^a(t, \vec{x}) | E \rangle$$

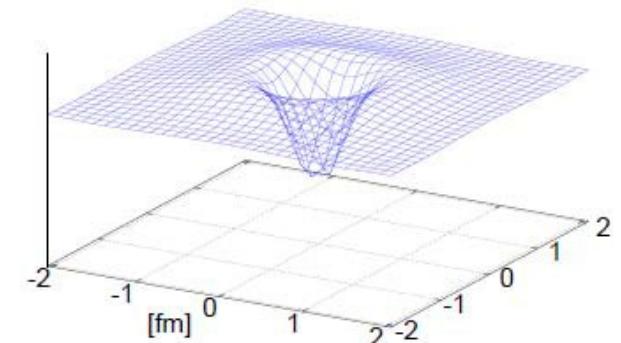
E : Total energy of the system

Local composite interpolating operators

$$B_\alpha = \epsilon^{abc} (q_a^T C \gamma_5 q_b) q_{c\alpha} \quad D_{\mu\alpha} = \epsilon^{abc} (q_a^T C \gamma_\mu q_b) q_{c\alpha}$$

$$M = (\bar{q}_a \gamma_5 q_a)$$

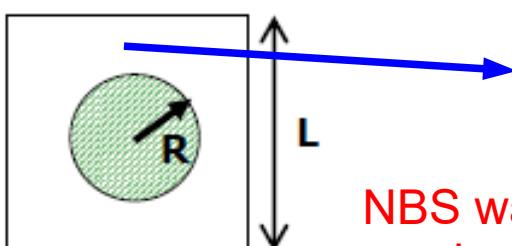
Etc.....



- It satisfies the Helmholtz eq. in asymptotic region : $(p^2 + \nabla^2) \Psi(E, \vec{r}) = 0$
- Using the reduction formula,

C.-J.D.Lin et al., NPB619 (2001) 467.

$$\Psi^a(E, \vec{r}) = \sqrt{Z_{H_1}} \sqrt{Z_{H_2}} \left(e^{i \vec{p} \cdot \vec{r}} + \int \frac{d^3 q}{2 E_q} \frac{T(q, p)}{4 E_p (E_q - E_p - i\epsilon)} e^{i \vec{q} \cdot \vec{r}} \right)$$



$$\Psi(E, \vec{r}) \simeq A \frac{\sin(pr + \delta(E))}{pr}$$

NBS wave function has a same asymptotic form with quantum mechanics.

Phase shift is defined as
 $S \equiv e^{i\delta}$

(NBS wave function is characterized from phase shift)

Potential in HAL QCD method

We define potentials which satisfy Schrödinger equation

$$(p^2 + \nabla^2) \Psi^\alpha(E, \vec{x}) \equiv \int d^3y U_\alpha^\alpha(\vec{x}, \vec{y}) \Psi^\alpha(E, \vec{y})$$

Energy independent potential

$$(p^2 + \nabla^2) \Psi^\alpha(E, \vec{x}) = K^\alpha(E, \vec{x})$$

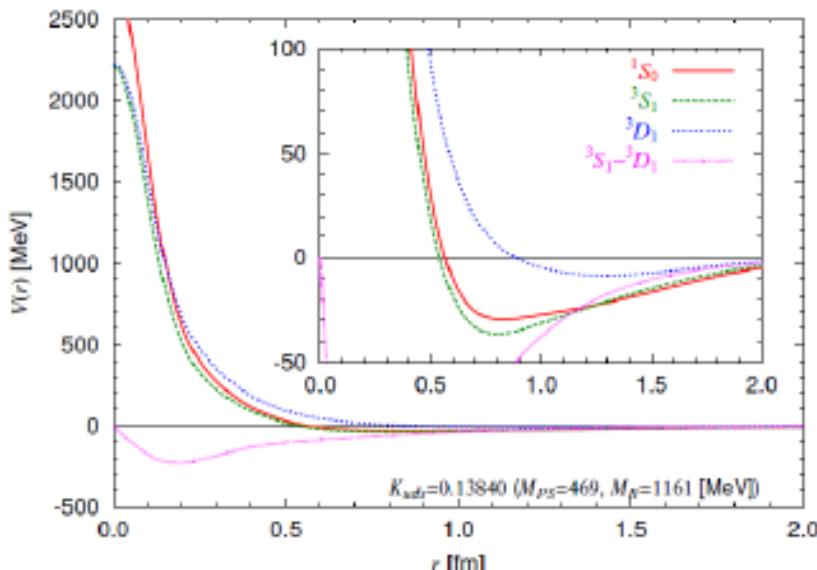
$$\begin{aligned} K^\alpha(E, \vec{x}) &\equiv \int dE' K^\alpha(E', \vec{x}) \int d^3y \tilde{\Psi}^\alpha(E', \vec{y}) \Psi^\alpha(E, \vec{y}) \\ &= \int d^3y \left[\int dE' K^\alpha(E', \vec{x}) \tilde{\Psi}^\alpha(E', \vec{y}) \right] \Psi^\alpha(E, \vec{y}) \\ &= \int d^3y U_\alpha^\alpha(\vec{x}, \vec{y}) \Psi^\alpha(E, \vec{y}) \end{aligned}$$

We can define **an energy independent potential** but it is fully non-local.

This potential automatically reproduce the scattering phase shift

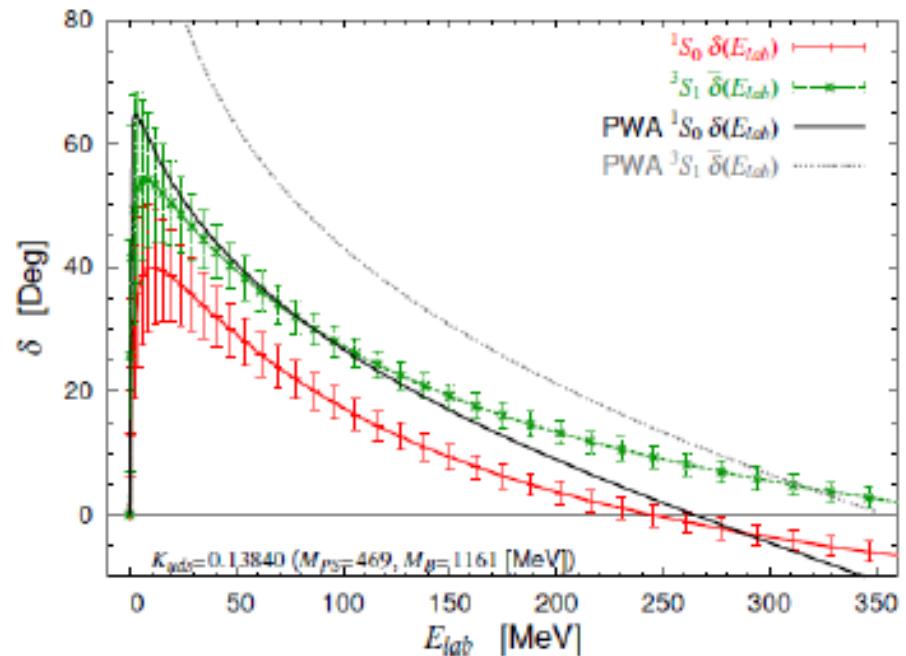
From LQCD to NN phase shifts

Lat NN forces



(SU(3), $m(\text{PS})=0.47\text{GeV}$)

Phase shifts



NN : unbound (1S_0 , 3S_1 - 3D_1)

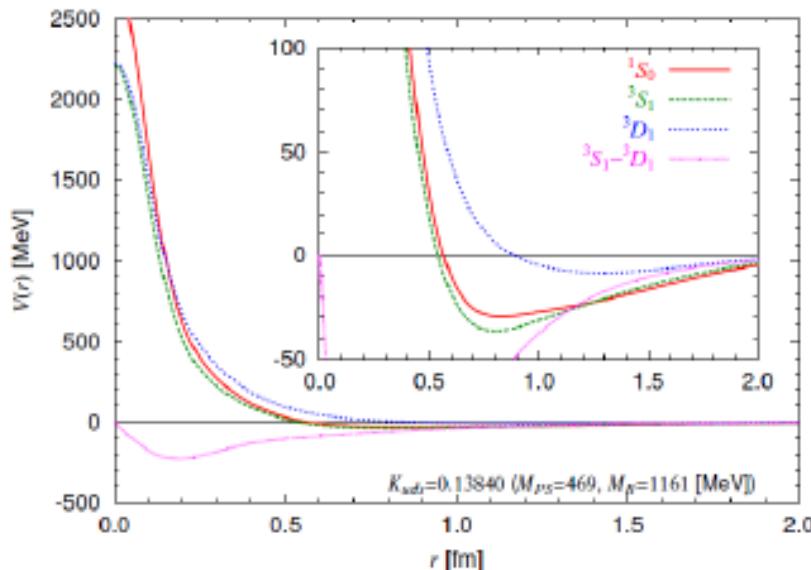
T.Inoue et al. (HAL Coll.) PRL111(2013)112503

T.Inoue et al. (HAL Coll.), NPA881(2012)28

28

From LQCD to EoS / Neutron Star

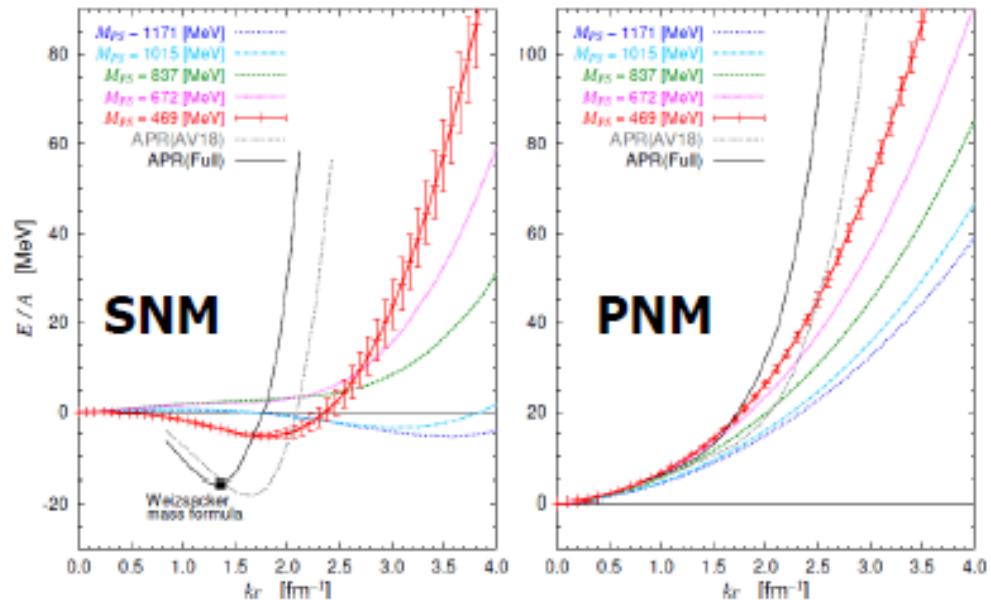
Lat NN forces



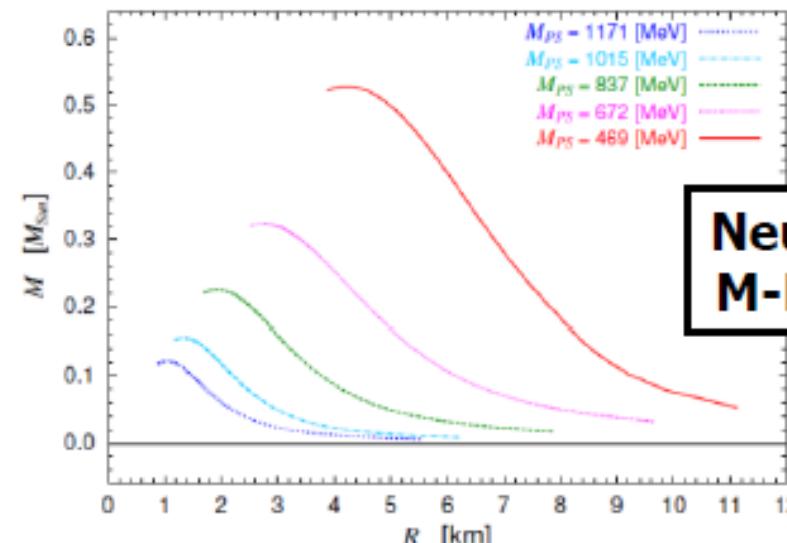
(SU(3), $m(\text{PS}) = 0.47\text{GeV}$)

BHF

EoS of nuclear matter



TOV



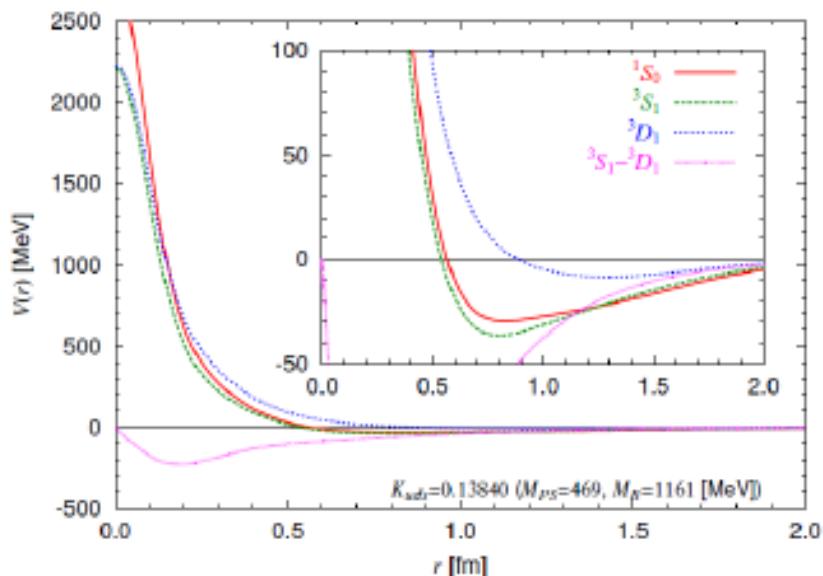
Neutron Star M-R relation

T.Inoue et al. (HAL Coll.) PRL111(2013)112503

T.Inoue et al. (HAL Coll.), PRC91(2015)011001

From LQCD to Nuclei (^{16}O , ^{40}Ca)

Lat NN forces

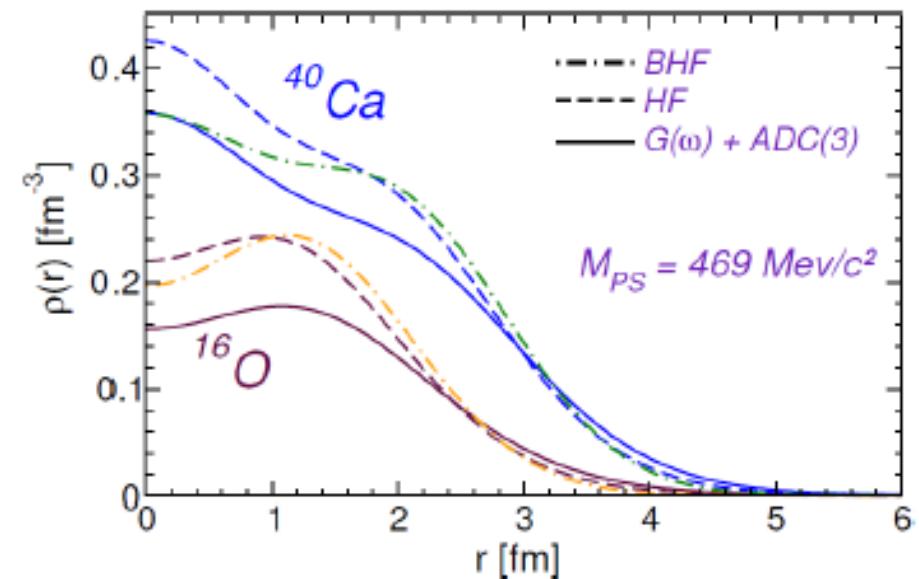


(SU(3), $m(\text{PS})=0.47\text{GeV}$)

Ab initio
SCGF



Density Distribution



C. McIlroy et al., 1701.02607, PRC

E_0^A [MeV]	^4He	^{16}O	^{40}Ca
BHF [22]	-8.1	-34.7	-112.7
G(ω) + ADC(3)	-4.80(0.03)	-17.9 (0.3) (1.8)	-75.4 (6.7) (7.5)
Exact Result [51]	-5.09	-	-
Separation into ^4He clusters:	-2.46 (0.3) (1.8)	24.5 (6.7) (7.5)	

Particle Physics
First-principles LQCD calc
HAL Coll. @ Japan



Nuclear Physics
Ab initio many-body calc
Univ. of Surrey @ UK

30

Interactions on the Lattice

- Direct method (Luscher's method)
 - Phase shift & B.E. from temporal correlation in finite V
- HAL QCD method
 - “Potential” from spacial (& temporal) correlation
 - Phase shift & B.E. by solving Schrodinger eq in infinite V

M.Luscher, CMP104(1986)177
CMP105(1986)153
NPB354(1991)531

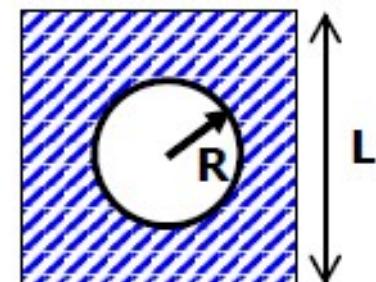
Ishii-Aoki-Hatsuda, PRL99(2007)022001, PTP123(2010)89
HAL QCD Coll., PTEP2012(2012)01A105

Luscher's formula: Scatterings on the lattice

- Consider Schrodinger eq at asymptotic region

$$(\nabla^2 + k^2)\psi_k(r) = mV_k(r)\psi_k(r)$$

$$V_k(r) = 0 \text{ for } r > R$$



- (periodic) Boundary Condition in finite V
→ constraint on energies of the system
- Energy E and phase shift (at E) are related

$$E = 2\sqrt{m^2 + k^2} \quad (\text{QFT: } \psi_k(r) \rightarrow \text{NBS w.f.})$$

$$k \cot \delta_E = \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2), \quad q = \frac{kL}{2\pi}$$

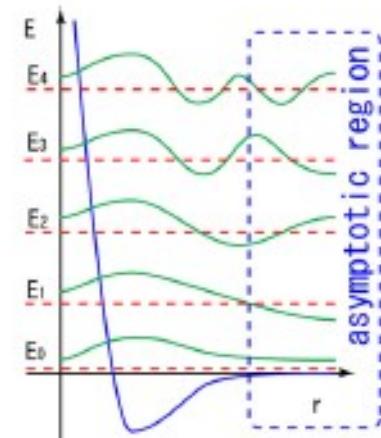
$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{n \in \mathbf{Z}^3} \frac{1}{(n^2 - q^2)^s}$$

Large V expansion

$$\Delta E = E - 2m = -\frac{4\pi \mathbf{a}}{mL^3} \left[1 + c_1 \frac{a}{L} + c_2 \left(\frac{a}{L} \right)^2 + \mathcal{O}\left(\frac{1}{L^3}\right) \right]$$

\mathbf{a} : scattering length

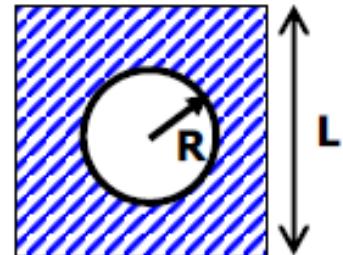
c_1, c_2 : geometric constants



Luscher's formula: Scatterings on the lattice

NBS equation in (3+1)-dimension:

$$(\nabla^2 + k^2)\psi_k(r) = mV_k(r)\psi_k(r)$$



Wave function at $|x| > R$
for infinite L :

$$\psi_\infty^k(r) = A_k \sin(kr + \underline{\delta(k)})/(kr)$$

Wave function at $|x| > R$
for finite L with PBC:

$$\begin{aligned} \psi_L^k(r) &= \frac{1}{L^3} \sum_{\vec{n} \in \mathbb{Z}^3} \frac{e^{i\vec{p}_n \cdot \vec{x}}}{\vec{p}_n^2 - k^2}, \quad \vec{p}_n = 2\pi/L \cdot \vec{n} \\ &= g_{00}(k) \frac{1}{\sqrt{4\pi}} j_0(kr) + \frac{k}{4\pi} n_0(kr) + \dots (j_{l \geq 1}(kr)) \end{aligned}$$

Quantization condition
for finite L with PBC:

$$k \cot \underline{\delta(\mathbf{k})} = \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2), \quad q = \frac{kL}{2\pi}$$

$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{(\mathbf{n}^2 - q^2)^s}$$

Lucsher's formula in (3+1)-D

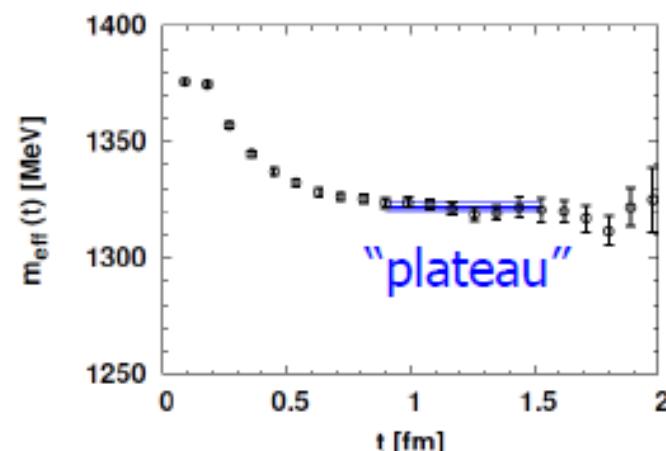
Practical procedure w/ Luscher's formula

- Calculate the energy spectrum of 2-hadron on finite V lattice
 - Temporal correlation in Euclidean time \rightarrow energy

$$G(t) = \langle 0 | \mathcal{O}(t) \overline{\mathcal{O}}(0) | 0 \rangle = \sum_n A_n e^{-E_n t} \rightarrow A_0 e^{-E_0 t} \quad (t \rightarrow \infty)$$

- Convert the **energy shift** to **phase shift** by Luscher's formula
 $E \rightarrow \Delta E = E - 2m$ (effect of int.) $\rightarrow k$ (asymp. mom.) $\rightarrow \delta_E$
 - Determination of energies
 - Take $t \gg 1/(E_1 - E_0)$ and find a “plateau” (G.S. saturation)

$$E_{\text{eff}}(t) = \ln \left[\frac{G(t)}{G(t+1)} \right] \xrightarrow[t \rightarrow \infty]{} E_0$$



[HAL QCD method]

- “Potential” defined through phase shifts (S-matrix)
- Nambu-Bethe-Salpeter (NBS) wave function

$$\psi(\vec{r}) = \langle 0 | N(\vec{x} + \vec{r}) N(\vec{x}) | N(k) N(-k); W \rangle$$

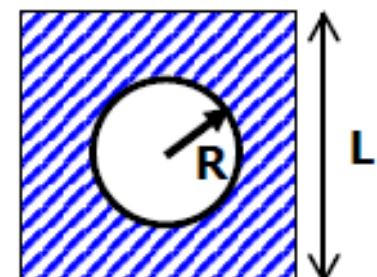
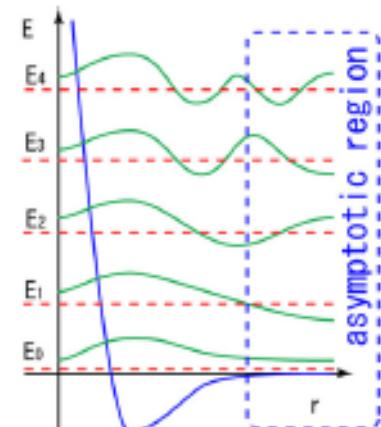
$$(\nabla^2 + k^2)\psi(\vec{r}) = 0, \quad r > R \quad W = 2\sqrt{m^2 + k^2}$$

– Wave function \leftrightarrow phase shifts

$$\psi(r) \simeq A \frac{\sin(kr - l\pi/2 + \delta(k))}{kr}$$

(below inelastic threshold)

Extended to multi-particle systems



M.Luscher, NPB354(1991)531

Ishizuka, PoS LAT2009 (2009) 119

C.-J.Lin et al., NPB619(2001)467

Aoki-Hatsuda-Ishii PTP123(2010)89

CP-PACS Coll., PRD71(2005)094504

S.Aoki et al., PRD88(2013)014036

“Potential” as a representation of S-matrix

- Consider the wave function at “interacting region”

$$(\nabla^2 + k^2)\psi(r) = m \int dr' U(\mathbf{r}, \mathbf{r}') \psi(r'), \quad r < R$$

- $U(\mathbf{r}, \mathbf{r}')$: faithful to the phase shift by construction

- $U(\mathbf{r}, \mathbf{r}')$: NOT an observable, but well defined

- $U(\mathbf{r}, \mathbf{r}')$: E-independent, while non-local in general

- “Proof of Existence”: Explicit form can be given as

$$U(\mathbf{r}, \mathbf{r}') = \frac{1}{m} \sum_{n,n'}^{n_{\text{th}}} (\nabla_{\mathbf{r}}^2 + k_n^2) \psi_n(\mathbf{r}) \mathcal{N}_{nn'}^{-1} \psi_{n'}^*(\mathbf{r}') \quad \mathcal{N}_{nn'} = \int d\mathbf{r} \psi_n^*(\mathbf{r}) \psi_{n'}(\mathbf{r})$$

- Non-locality → derivative expansion

Okubo-Marshak(1958)

$$U(\vec{r}, \vec{r}') = \left[V_c(r) + S_{12} V_T(r) + \vec{L} \cdot \vec{S} V_{LS}(r) + \mathcal{O}(\nabla^2) \right] \delta(\vec{r} - \vec{r}')$$

LO **LO** **NLO** **NNLO**

Aoki-Hatsuda-Ishii PTP123(2010)89

11

Check on convergence: K.Murano et al., PTP125(2011)1225

New method to diagnose LQCD data in plateau method

-- “Sanity check” with Lüscher’s finite volume formula --

Our proposal

Iritani et al. [HAL QCD], arXiv:1703.07210.

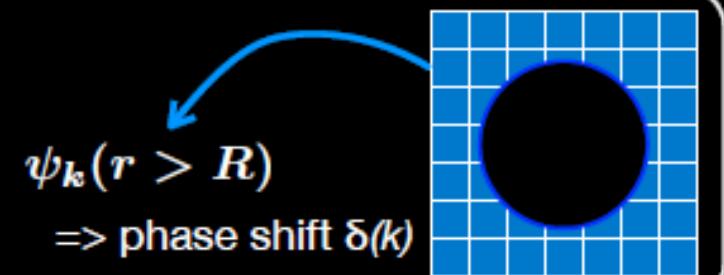
see also talk by Aoki (Thur.)

1. extract $k \cot \delta(k)$ through Lüscher’s formula

- Lüscher’s finite V formula

► energy $\Delta E \rightarrow$ momentum $k \rightarrow k \cot \delta(k)$

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{\vec{n}^2 - (kL/2\pi)^2}$$

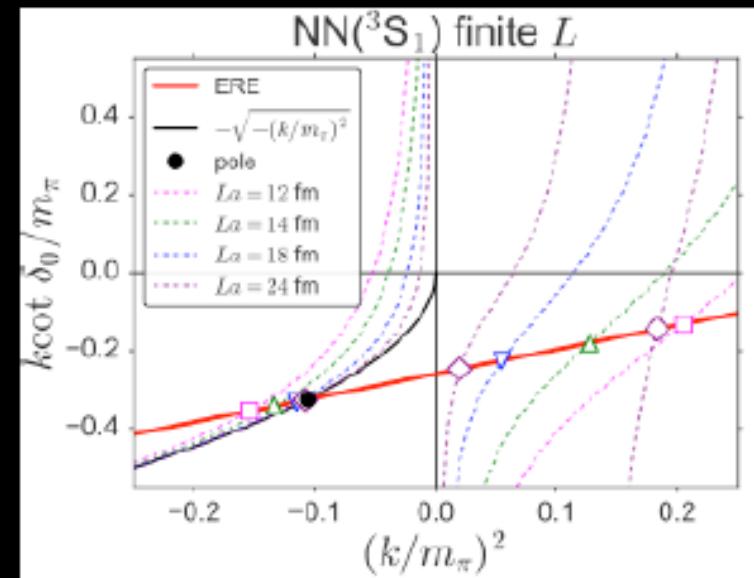


2. combine the results with effective range expansion (ERE)

- Effective range expansion (ERE)

$$k \cot \delta(k) = \frac{1}{a} + \frac{1}{2} r_e k^2 + \dots$$

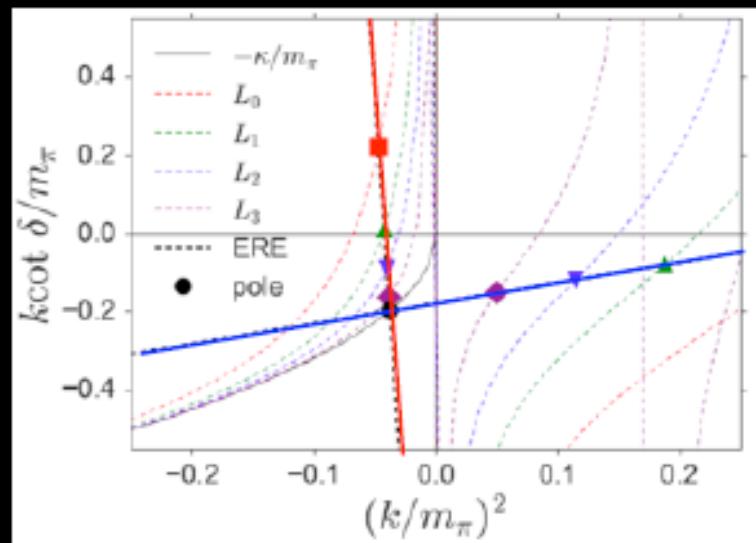
- An example of ERE w/ Lüscher’s formula (dotted lines) for bound state problem
- If the energy is extracted correctly, all data should be aligned on ERE line at low-energy



“Sanity check” for all existing data

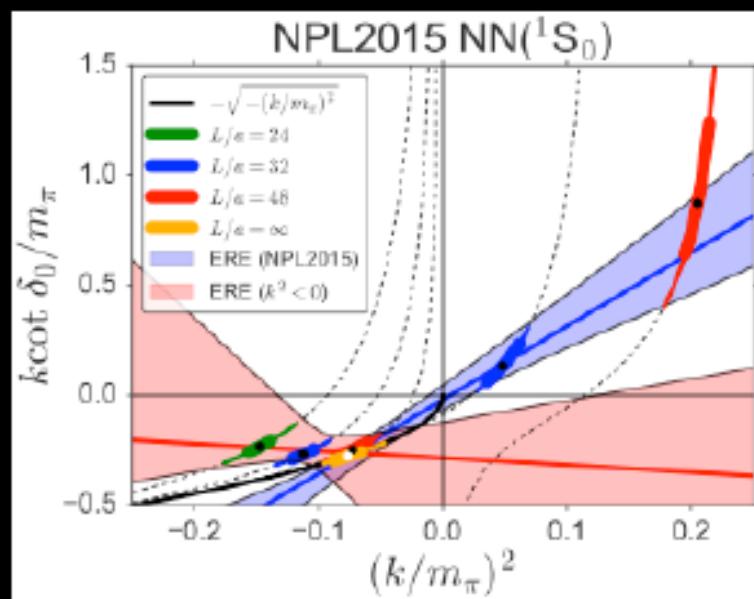
An exotic example

Iritani et al. [HAL QCD], arXiv:1703.07210.

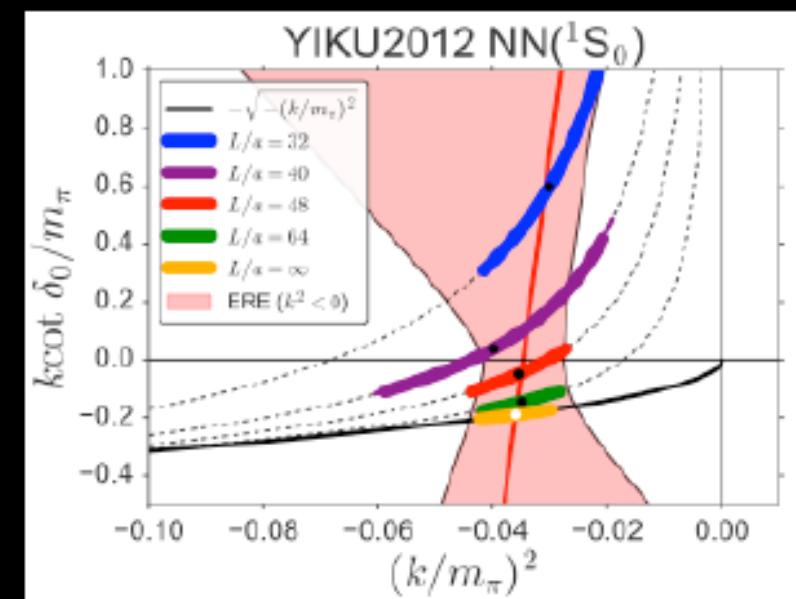


$$k \cot \delta(k) = \frac{1}{a} + \frac{1}{2} r_e k^2 + \dots$$

- (i) inconsistent ERE for $k^2 < 0$ and for $k^2 > 0$
 - (ii) singular parameterizations $\rightarrow r_e \simeq \pm\infty$
 - (iii) unphysical residue at pole position
- ➡ indicate the problem in LQCD data

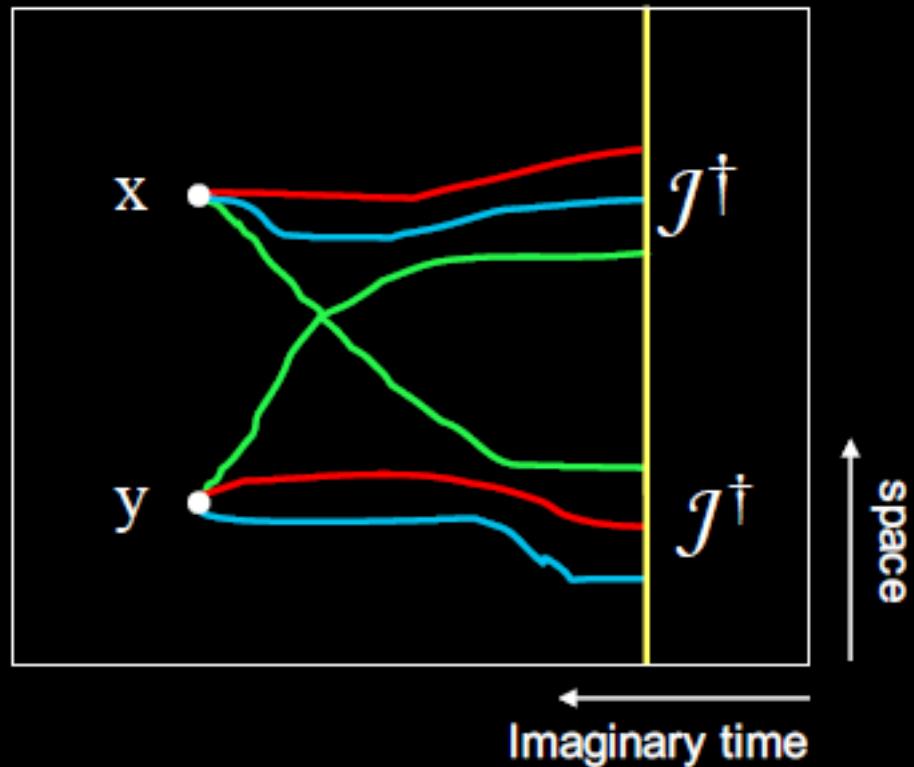


(i) inconsistent ERE, (iii) unphysical residue



(ii) singular behavior

Scattering observables from LQCD



$$\langle N_1(\mathbf{x}, t) N_2(\mathbf{y}, t) \mathcal{J}_1^\dagger(0) \mathcal{J}_2^\dagger(0) \rangle = \sum_n \langle 0 | N_1(\mathbf{x}) N_2(\mathbf{y}) | n \rangle a_n e^{-E_n t} \xrightarrow{t > t^*} \phi(\mathbf{r}, t) = \sum_{n < n^*} b_n \phi_n(\mathbf{r}) e^{-E_n t}$$

Finite Volume Method

$$E_n(L)$$

→ phase shift, binding energy

Luescher, Nucl. Phys. B354 (1991) 531

HAL QCD Method

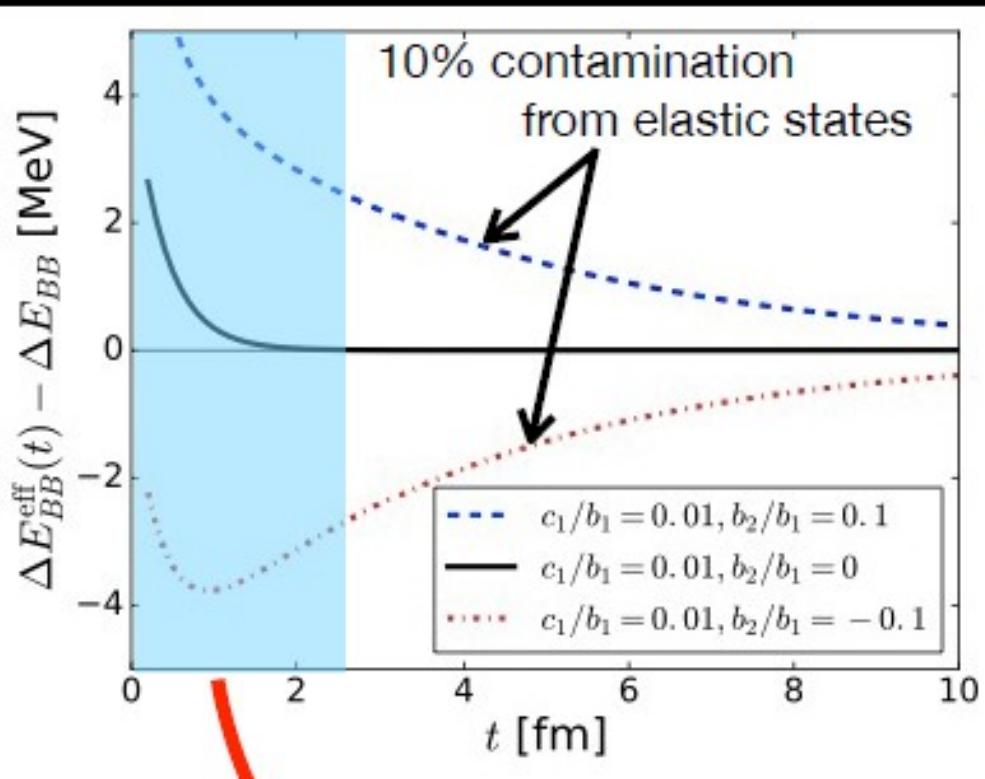
$\phi(\mathbf{r}, t) \rightarrow$ 2PI kernel ($T = U + GUT$)
→ phase shift, binding energy

Ishii, Aoki & Hatsuda, PRL 99 (2007) 022001
Ishii et al. [HAL QCD Coll.], PLB 712 (2012) 437

Demonstration of plateau method by mock-up data

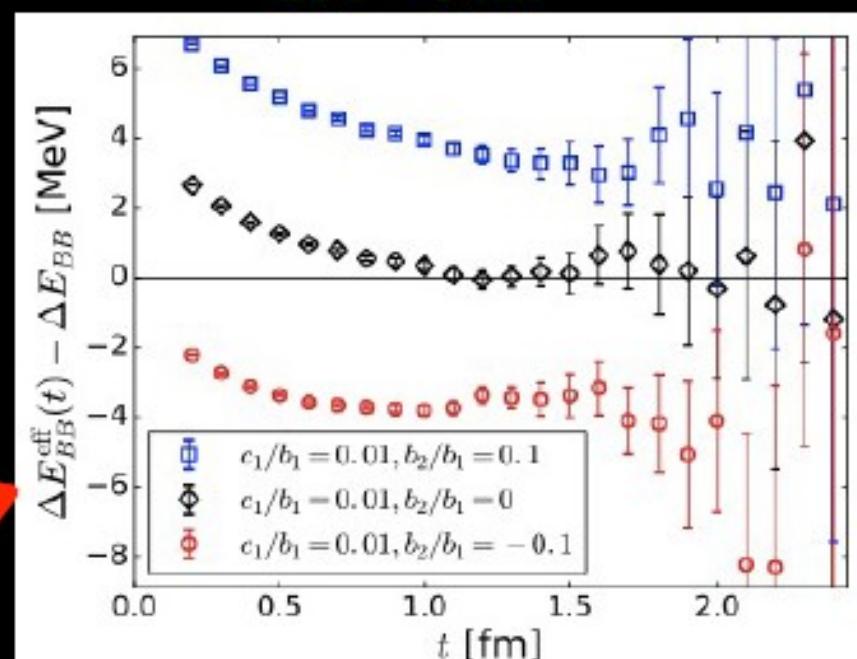
“Mirage in temporal correlation functions for baryon-baryon interactions in lattice QCD”

Iritani, Doi et al. [HAL QCD], JHEP10 (2016) 101.



→ True ground state for $t > 8$ fm
with 10% contamination

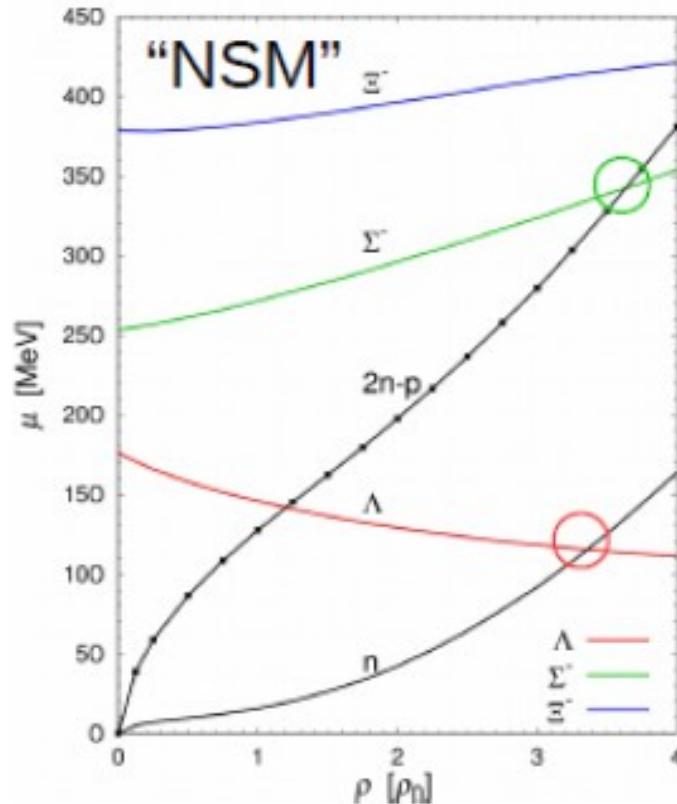
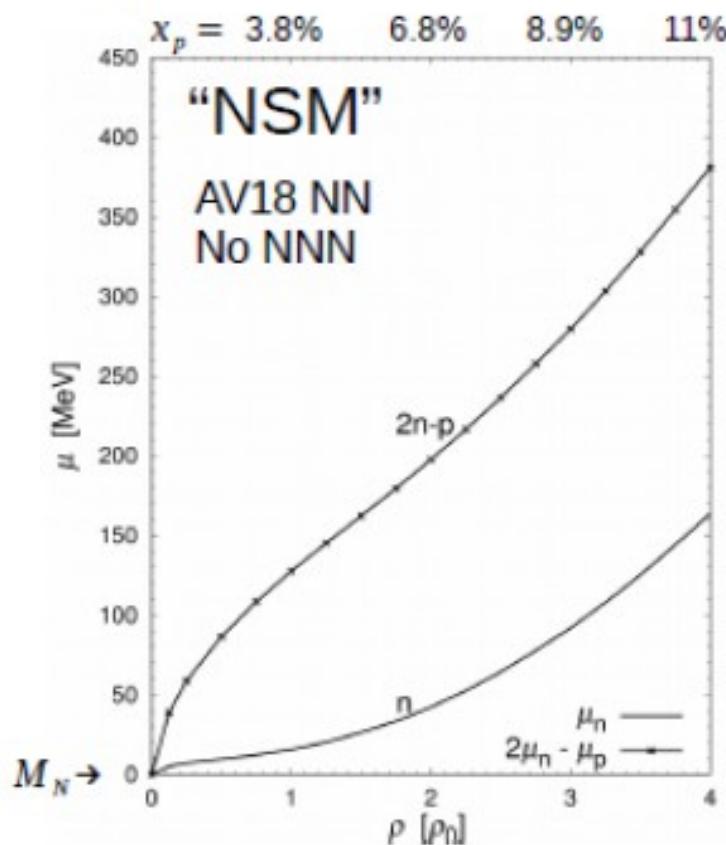
Fake plateaux or “Mirage” appear
at $t \sim 1$ fm



Zoom + typical stat. error

Hyperon onset

(just for a demonstration)



S-wave YN only

Preliminary

- “NSM” is matter w/ n, p, e, μ under β -eq and $Q=0$.

[T. Inoue]

[Missing]
P-wave/LS forces
3-baryon forces

4

55