Topology in SU(N) gauge theories

Michael Teper (Oxford) - ECT 2022 - NEDM Workshop

A.Athenodorou,MT 2106.00364v2,v3, 2007.06422; MT 2202.02528

- continuum topology : a brief sketch
- topology : continuum $\longrightarrow$ lattice
- calculating topology on the lattice
- 'freezing' of topology as $a \to 0$ and/or $N \to \infty$
- some further methods for lattice topology

continuum topology : a brief sketch

In a finite volume with periodic boundary conditions, the gauge field possesses a topological charge

$$Q = \int d^4x Q(x) = \text{integer}$$

where

$$Q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}\{F_{\mu\nu}(x)F_{\rho\sigma}(x)\}.$$

The minimum action $Q = 1$ field in $SU(2)$ is the 'instanton':

$$A^I_\mu(x) = \frac{x^2}{x^2+\rho^2}g^{-1}(x)\partial_\mu g(x) \quad ; \quad g(x) = \frac{x_0 I + i x_j \sigma_j}{(x_\mu x_\mu)^{1/2}}$$

where $g(x)$ covers the $SU(2)$ group once as x_μ covers the surface at $x_\mu x_\mu = \infty$. To obtain an instanton in a periodic finite but large volume make the singular gauge transformation $g^\dagger(x)$ (or equivalent).

The instanton action is $S_I = 8\pi^2/g^2$ and $S_I(x), Q(x) \neq 0$ for $x^2 \leq O(\rho)$

One obtains an $SU(N)$ instanton by embedding $SU(2)$ in $SU(N)$

density of instantons - classical:

$$D(\rho)\frac{d\rho}{\rho} \propto \frac{d\rho}{\rho}\frac{1}{\rho^4}\frac{v(N)}{g^{4N}}\exp\left\{-\frac{8\pi^2}{g^2}\right\}$$

$\Longrightarrow$

density of instantons - quantum (1 loop):

$$D(\rho) \propto \frac{1}{\rho^4}\frac{v(N)}{g^{4N}}\exp\left\{-\frac{8\pi^2}{g^2(\rho)}\right\}$$

$\stackrel{N\to\infty}{\Longrightarrow}$

$$D(\rho) \propto \frac{1}{\rho^4}\left\{\frac{const}{\lambda^2}\exp\left\{-8\pi^2/\lambda(\rho)\right\}\right\}^N \quad ; \quad \lambda = g^2 N$$

$\Longrightarrow$

small instantons disappear exponentially with N: larger instantons?

plausibly not ...

Interlaced θ-vacua in SU(N) gauge theories

Consider the gauge action with a θ term

$$S[g^2,\theta] = \frac{1}{4g^2}\int d^4x \mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr} F_{\mu\nu}F_{\rho\sigma}$$

Now

$$\frac{1}{16\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr} F_{\mu\nu}F_{\rho\sigma} = Q = \text{integer} \quad \Longrightarrow \quad E(\theta) = E(\theta+2\pi)\ \forall N$$

But for a smooth $N \to \infty$ limit, we need to factor N from S so that the couplings to keep fixed are $1/g^2N$, θ/N, ... i.e.

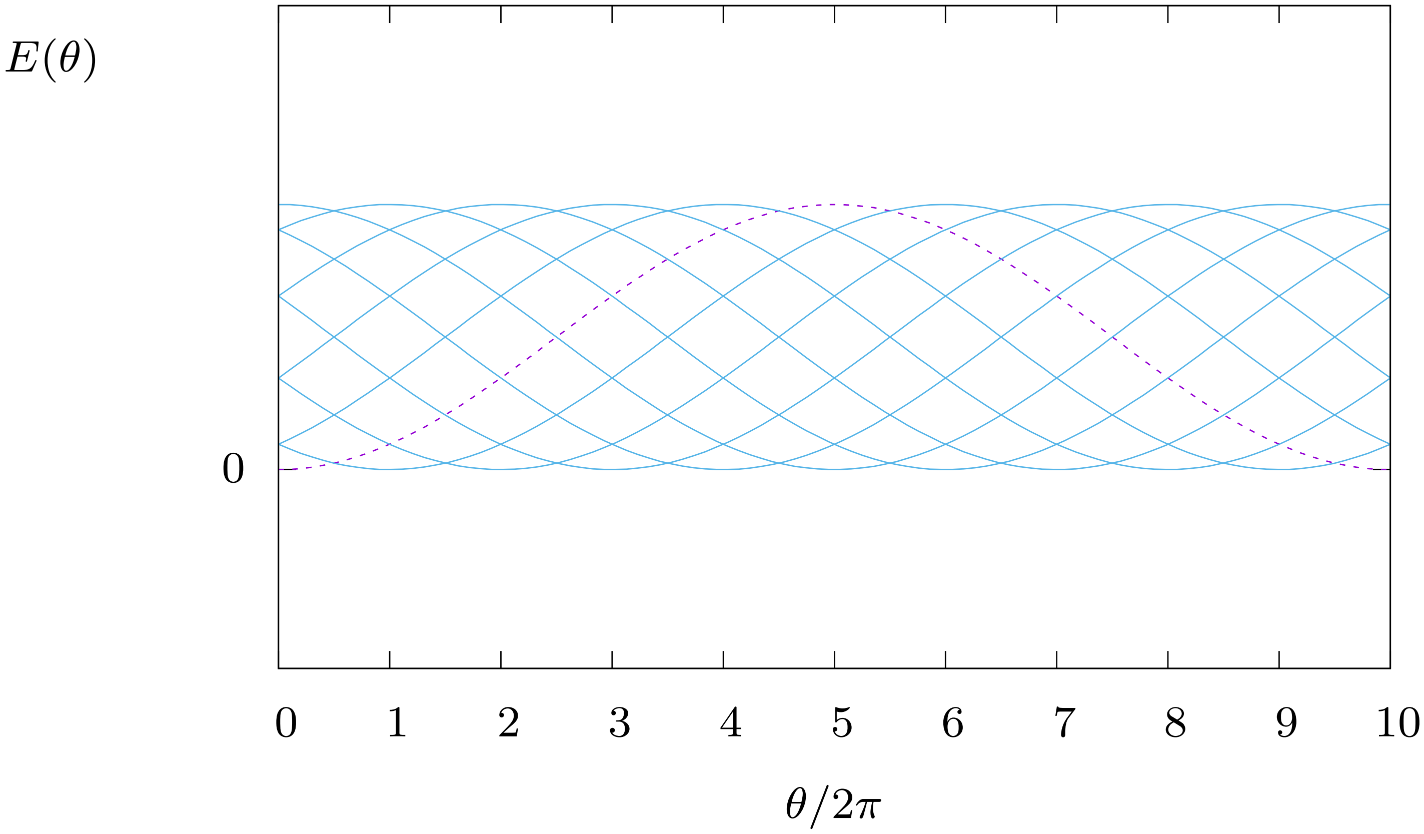
$$E(\theta) = N^2 h(\theta/N)$$

$\Longrightarrow$ E(θ) is a multi-branched function E.Witten hep-th/9807109

$$E_k(\theta) = N^2 h\left(\frac{\theta+2\pi k}{N}\right) \quad ; \quad E(\theta) = \min_k E_k(\theta)$$

so that: $E(\theta) = E(\theta+2\pi)$ while each $E_k(\theta)$ is periodic in $2\pi N$

e.g. $SU(10)$:



domain wall tension between different 'k-vacua' is $O(N)$ so as $N \to \infty$ these will all become stable ... Witten: AdS/CFT ; Shifman: $\mathcal{N} = 1$ SUSY

topology : continuum $\longrightarrow$ lattice

Let $U_{\mu\nu}(x)$ be the plaquette at x in the $\mu\nu$ plane. On smooth fields

$$U_{\mu\nu}(x) = 1 + a^2 F_{\mu\nu}(x) + ...$$

so on smooth fields:

$$Q_L(x) \equiv \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\{U_{\mu\nu}(x)U_{\rho\sigma}(x)\} = a^4 Q(x) + O(a^6)$$

However $Q_L(x)$ is not a topological quantity and is not protected from local UV fluctuations that are $O(1/\beta^3)$ and these will swamp the $O(a^4)$ physical piece on rough Monte Carlo fields, particularly since

$$\langle Q_L(x)\rangle_Q = Z(\beta)Q(x) \ll Q(x) \quad ; \quad Z_{1-loop} \stackrel{SU3}{\simeq} 1 - 5.45/\beta + O(1/\beta^2)$$

Practical strategy: smoothen the lattice gauge field so that $Q_L \simeq Q$ e.g. cooling, gradient flow, ... – here I shall use ‘cooling’ i.e. a few sweeps where heat bath is replaced by action minimisation/reduction ...

an instanton on the lattice

- continuum $SU(2)$ instanton of size ρ:

$$A^I_\mu(x) = \frac{x^2}{x^2+\rho^2} g^{-1}(x)\partial_\mu g(x) \quad ; \quad g(x) = \frac{x_0 I + i x_j \sigma_j}{(x_\mu x_\mu)^{1/2}}$$

- corresponding lattice field:

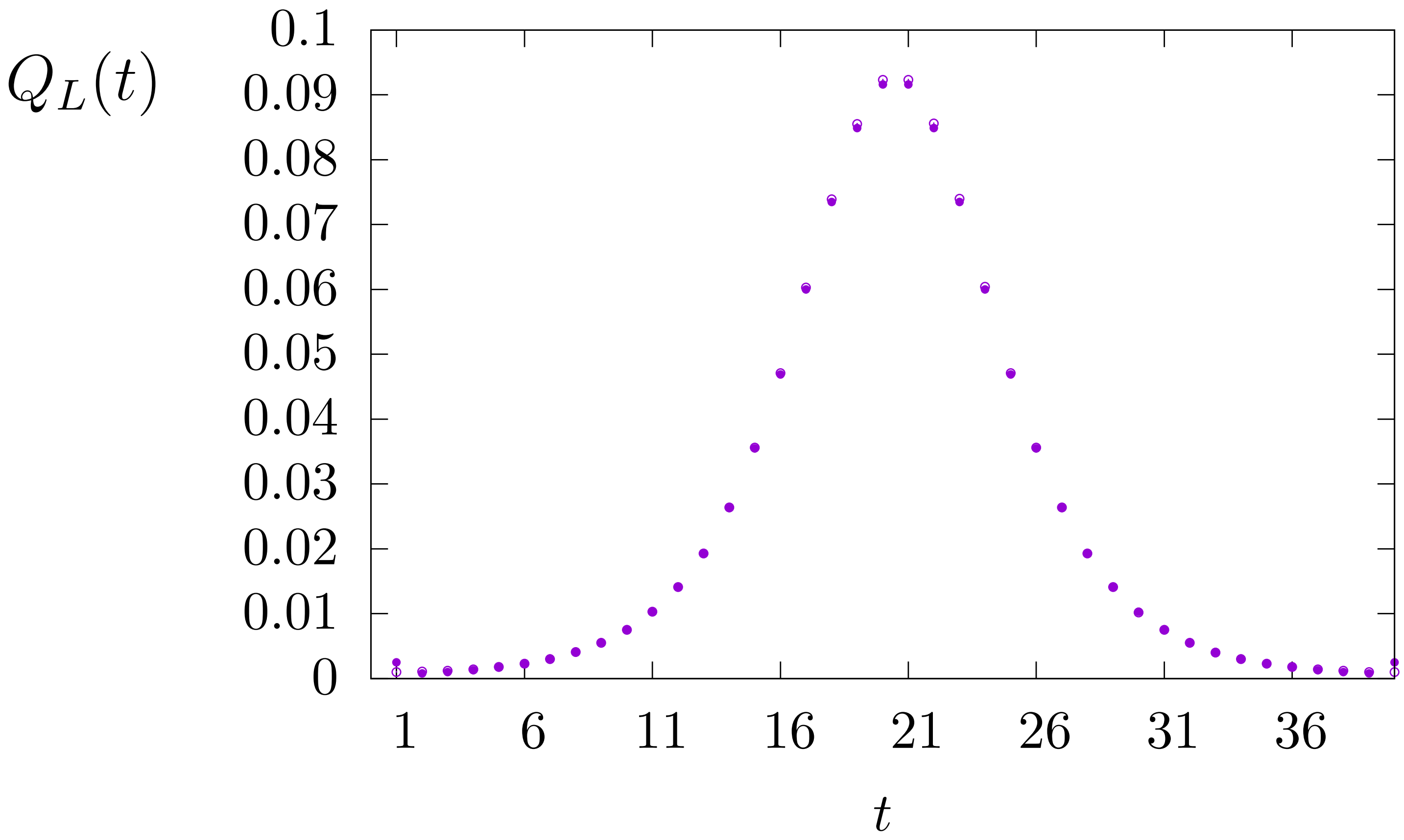
$$U^I_\mu(x) = \mathcal{P}\left\{\exp \int_x^{x+a\hat{\mu}} A^I_\mu(x) dx\right\} \quad ;$$

divide link in sections, exponentiate at centre of each section, then multiply matrices in order recalling $\exp\{i\theta n_k \sigma_k\} = I\cos(\theta) + i n_k \sigma_k \sin(\theta)$;
perform the gauge transformation $g^\dagger(x)$ so that $U^I_\mu(x) \simeq I$ at boundary;
perform a few cooling/smoothening sweeps to iron out any remaining 'bumps' at boundary

- for an $SU(N)$ instanton:

e.g. take the $N \times N$ unit matrix and replace the top left hand 2×2 submatrix by the $SU(2)$ instanton $U^I_\mu(x)$

profile $Q_L(t) = \sum_{\bar{x}} Q_L(t, \bar{x})$ of a $\rho = 8a$ instanton on a 40^4 lattice



For the original lattice field, •, and after 20 cooling sweeps, ∘.

calculating topology of a 'rough' lattice gauge field

e.g.

• cooling – smoothening by a few sweeps where heat bath is replaced by action minimisation/reduction ...

• gradient flow – an 'RG invariant' smoothening

• zero modes of (Neuberger) overlap Dirac operator – or 'near' zero modes with other lattice Dirac operators

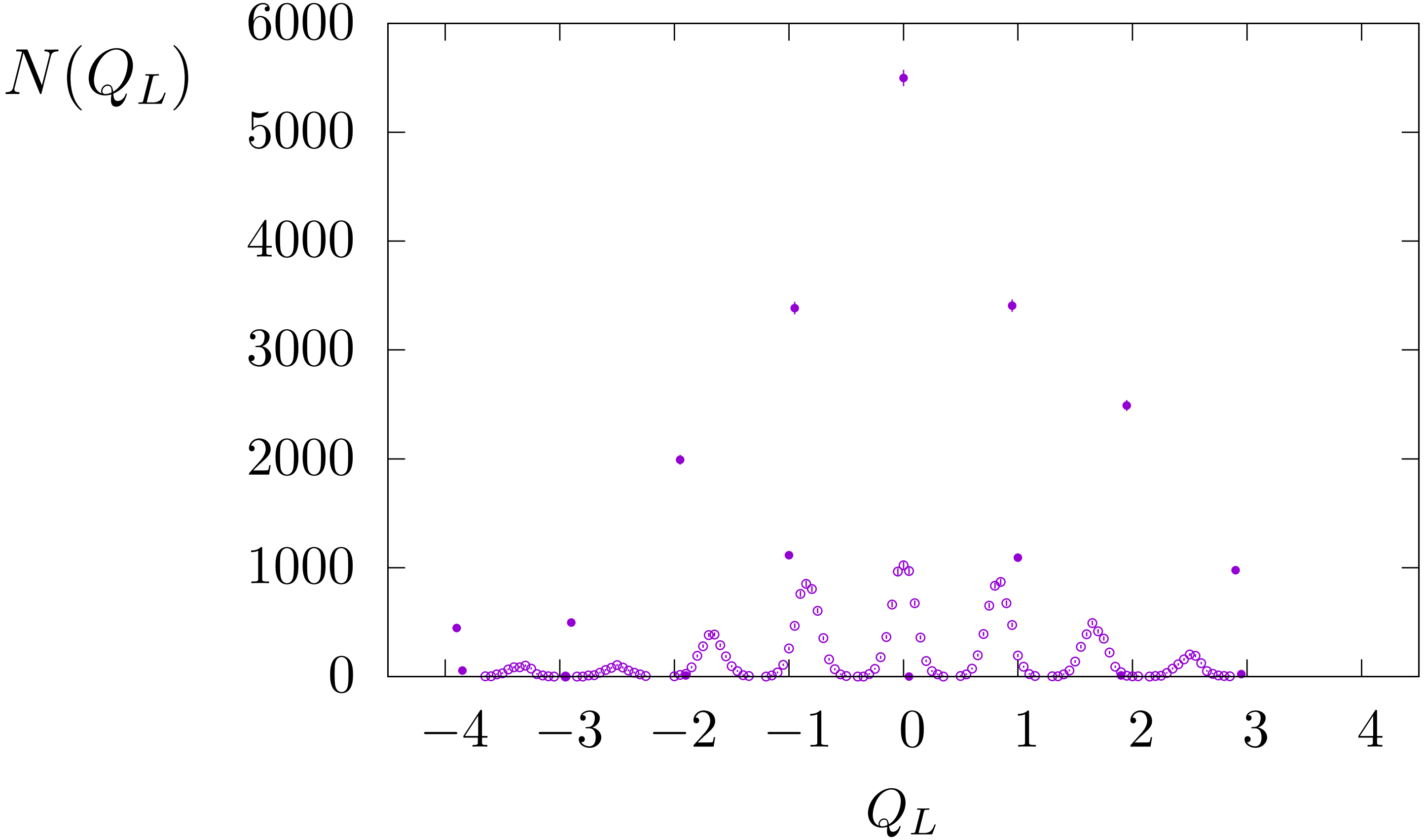
$\cdots \quad \Longrightarrow$

cooling and gradient flow lead to $\sim$same results : Bonati,D'Elia 1401.2441, Alexandrou, Athenodorou,Jansen et al 1509.04259

cooling and Dirac spectral methods lead to $\sim$same results : Alexandrou et al, 1708.00696, Cundy et al, hep-lat/0203030

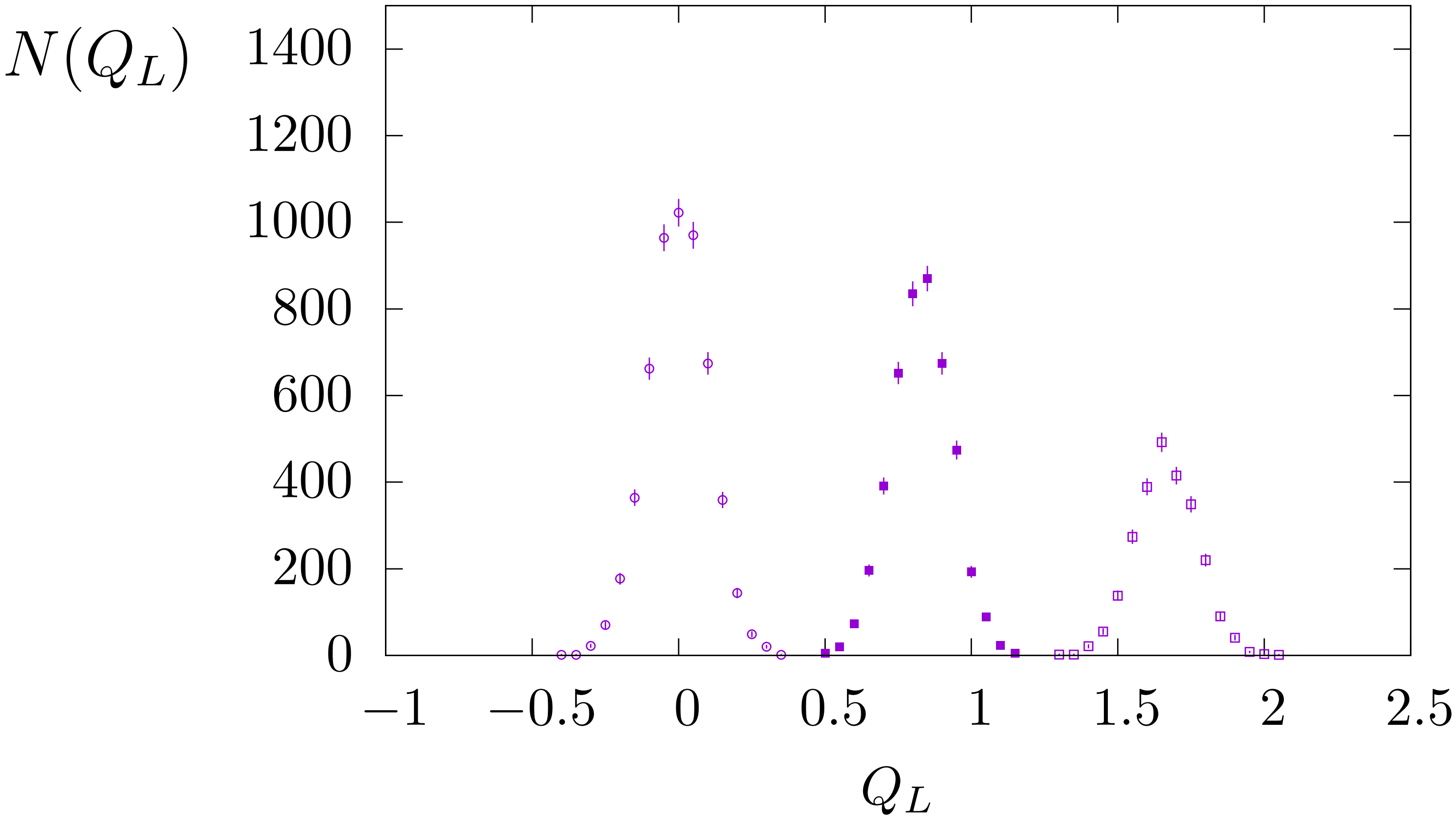
Cooling: $SU(8)$ lattice fields on a $20^3 30$ lattice with $a\sqrt{\sigma} \simeq 0.133$:

Q_L after 2 ($\circ$) and 20 ($\bullet$) cooling sweeps.



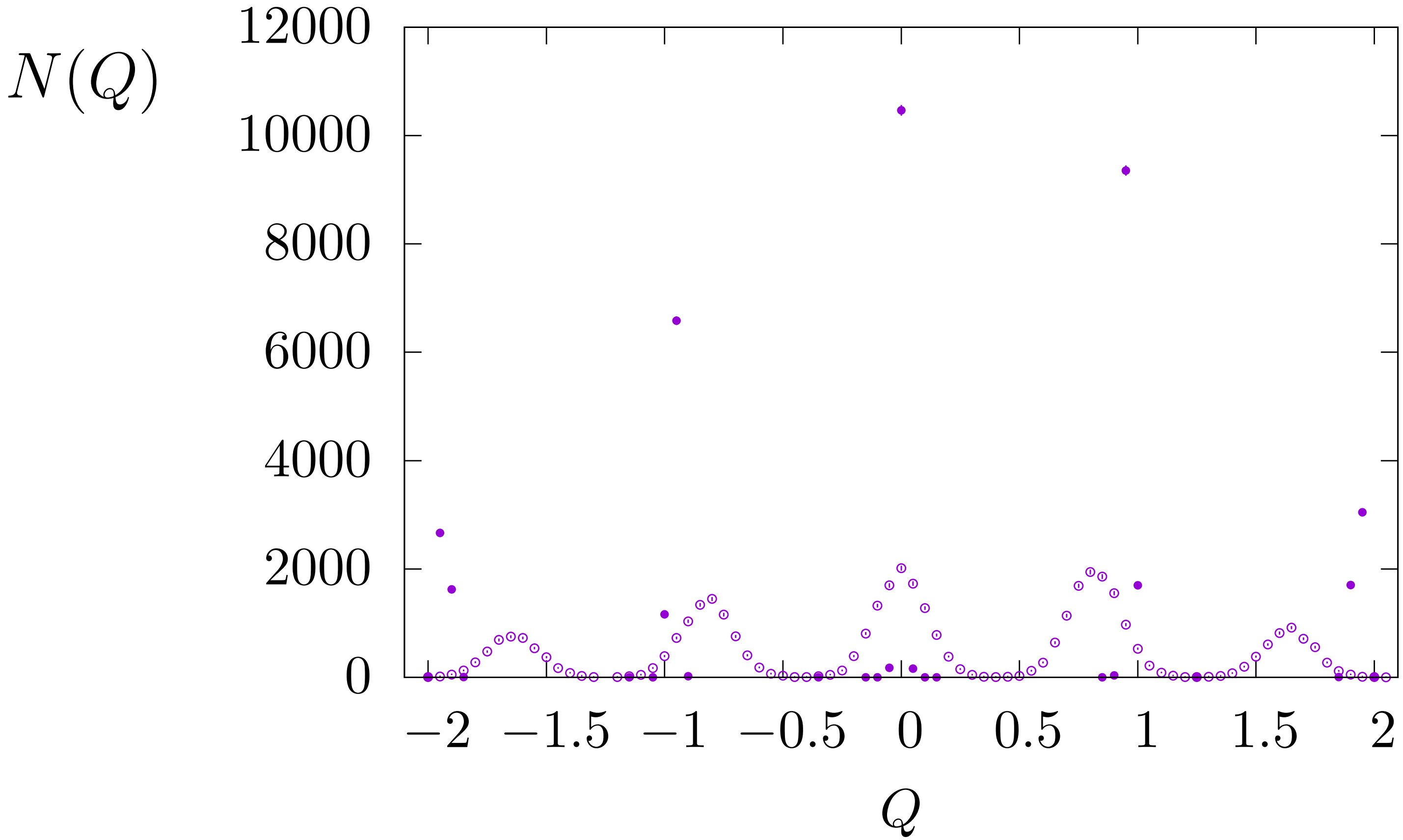
$SU(8)$ lattice fields on a $20^3 30$ lattice with $a\sqrt{\sigma} \simeq 0.133$:

Q_L after 2 cooling sweeps for fields with $Q_L = 0, 1, 2$ ($\circ, \blacksquare, \square$) after 20 cooling sweeps.

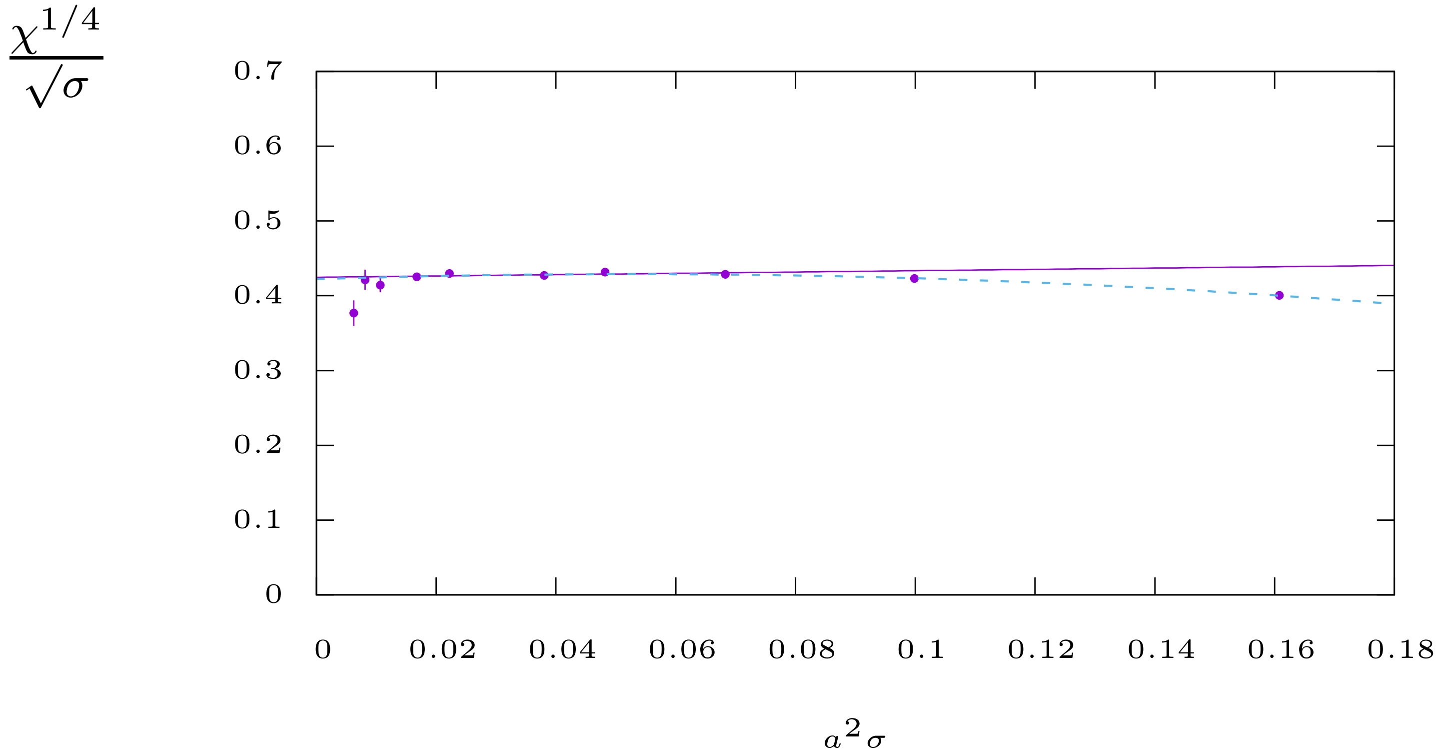


Cooling: $SU(5)$ $20^3 24$ lattice at $\beta = 18.04$ with $a\sqrt{\sigma} \simeq 0.156$:

Q_L after 2 ($\circ$) and 20 ($\bullet$) cooling sweeps.

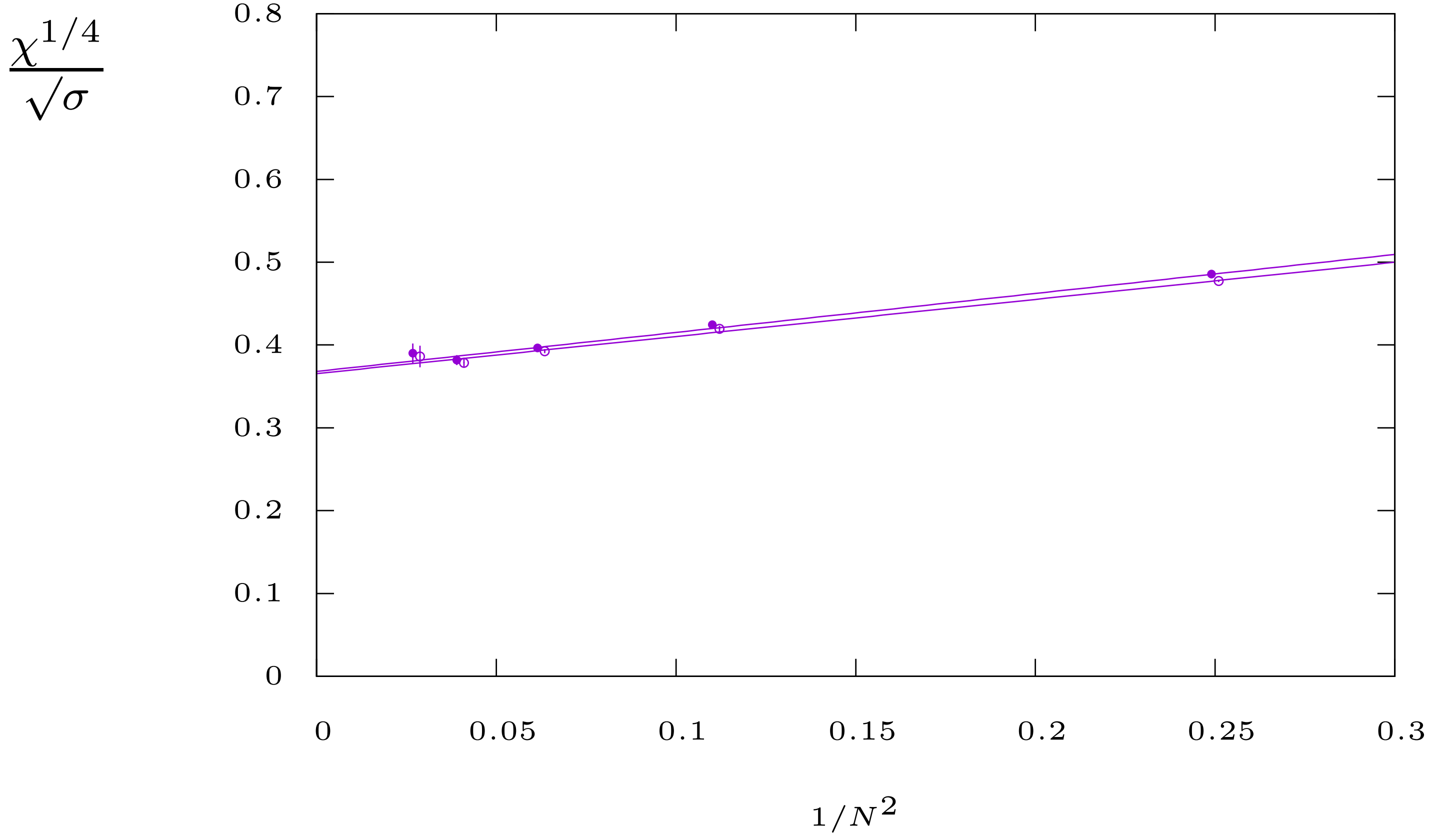


topological susceptibility: $SU(3)$ $\frac{\chi^{1/4}}{\sqrt{\sigma}} = 0.4246(36) + 0.09(8)a^2\sigma$
(AA,MT: 2106.00364v2,2007.06422)



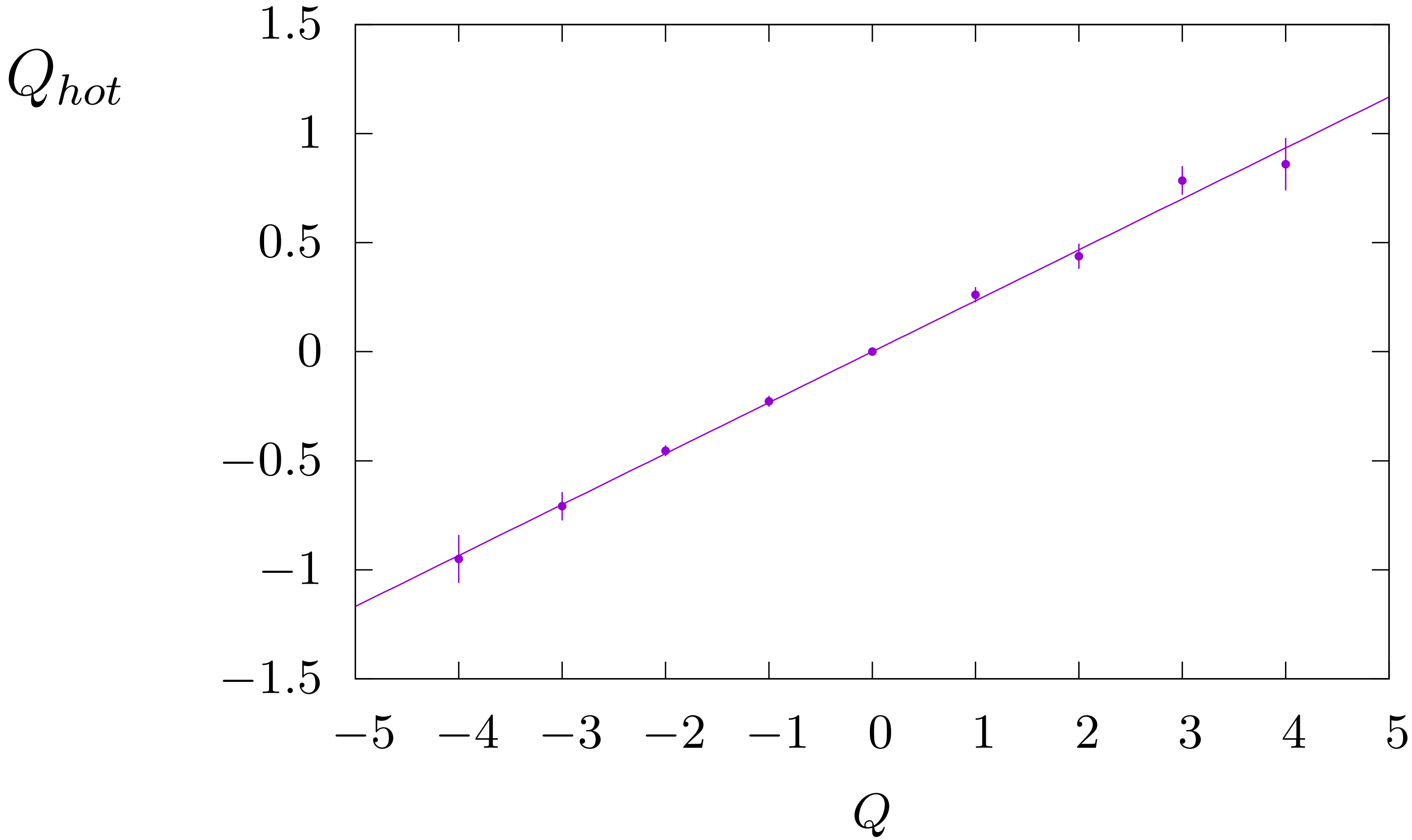
$\frac{\chi^{1/4}}{\sqrt{\sigma}} = 0.4246(36)|_{su3} \longrightarrow \chi^{1/4} = 206(4)\text{MeV}$
using $r_0\sqrt{\sigma} = 1.160(6)$ and $r_0 = 0.472(5)\text{fm}$ (Sommer: 1401.3270)

topological susceptibility: $SU(N)$ $\frac{\chi^{1/4}}{\sqrt{\sigma}} = 0.368(3) + \frac{0.47(2)}{N^2}$ (AA,MT: 2106.00364)



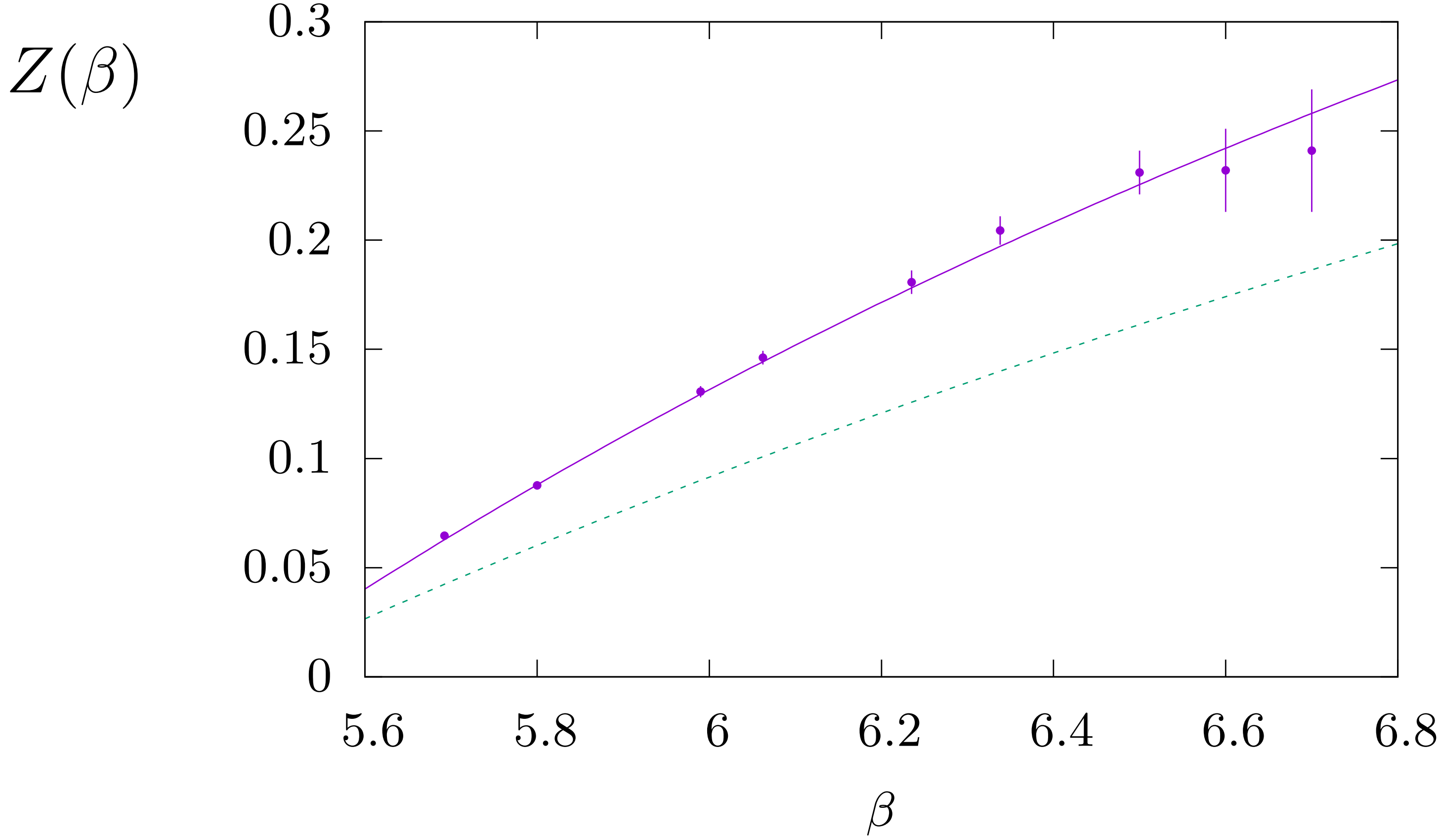
Q integer (•), Q raw (∘) (AA,MT: 2106.00364)

$Q_{hot} = Z(\beta)Q$ on $26^3 38$ lattices at $\beta = 6.5$ in $SU(3)$



Average topological charge on lattice fields which have a charge Q after 20 cools

$Q_{hot} = Z(\beta)Q$ versus β in $SU(3)$



$Z(\beta)$ interpolating fit (solid line) and one-loop perturbative result (dashed line) interpolating fits for $Z(\beta)$ in $SU(N)$ for **ranges** of $\beta = 2N/g^2$ shown:

$Z_Q^{int} = 1 - z_0 g^2 N - z_1 (g^2 N)^2$				
N	z_0	z_1	$\beta \in$	χ^2/n_{df}
2	0.190(30)	0.023(9)	[2.45,2.80]	1.17
3	0.162(10)	0.0425(31)	[5.69,6.70]	0.62
4	0.156(20)	0.047(7)	[10.70,11.60]	1.32
5	0.203(21)	0.035(7)	[16.98,18.37]	2.76
6	0.205(30)	0.036(11)	[24.67,26.71]	1.37
8	0.187(24)	0.043(9)	[44.10,47.75]	1.71
10	0.141(44)	0.060(16)	[69.20,73.35]	1.05
12	0.182(24)	0.071(22)	[99.86,105.95]	2.22

can be useful for $\theta \neq 0$ calculations where $\exp\{i\theta Q\} \simeq \exp\{i\theta Z(\beta)^{-1} Q_{hot}\}$

‘freezing’ of topology as $a \to 0$ and/or $N \to \infty$

basic idea: $Q \to Q-1$ involves an instanton shrinking from $\rho \sim O(1)$fm to $\rho \sim a$ and then disappearing within a hypercube, so upper bound is probability of finding very small I with $\rho \sim a \times few$:

$$D(\rho) \propto \frac{1}{\rho^5}\frac{1}{g^{4N}} \exp\left\{-\frac{8\pi^2}{g^2(\rho)}\right\} \overset{N\to\infty}{\propto} \frac{1}{\rho^5}\left\{\exp\left\{-\frac{8\pi^2}{g^2(\rho)N}\right\}\right\}^N \overset{\rho\sim a}{\propto} (a\Lambda)^{\frac{11N}{3}-5}.$$

so let: τ_Q = average number sweeps for $Q \to Q \pm 1$

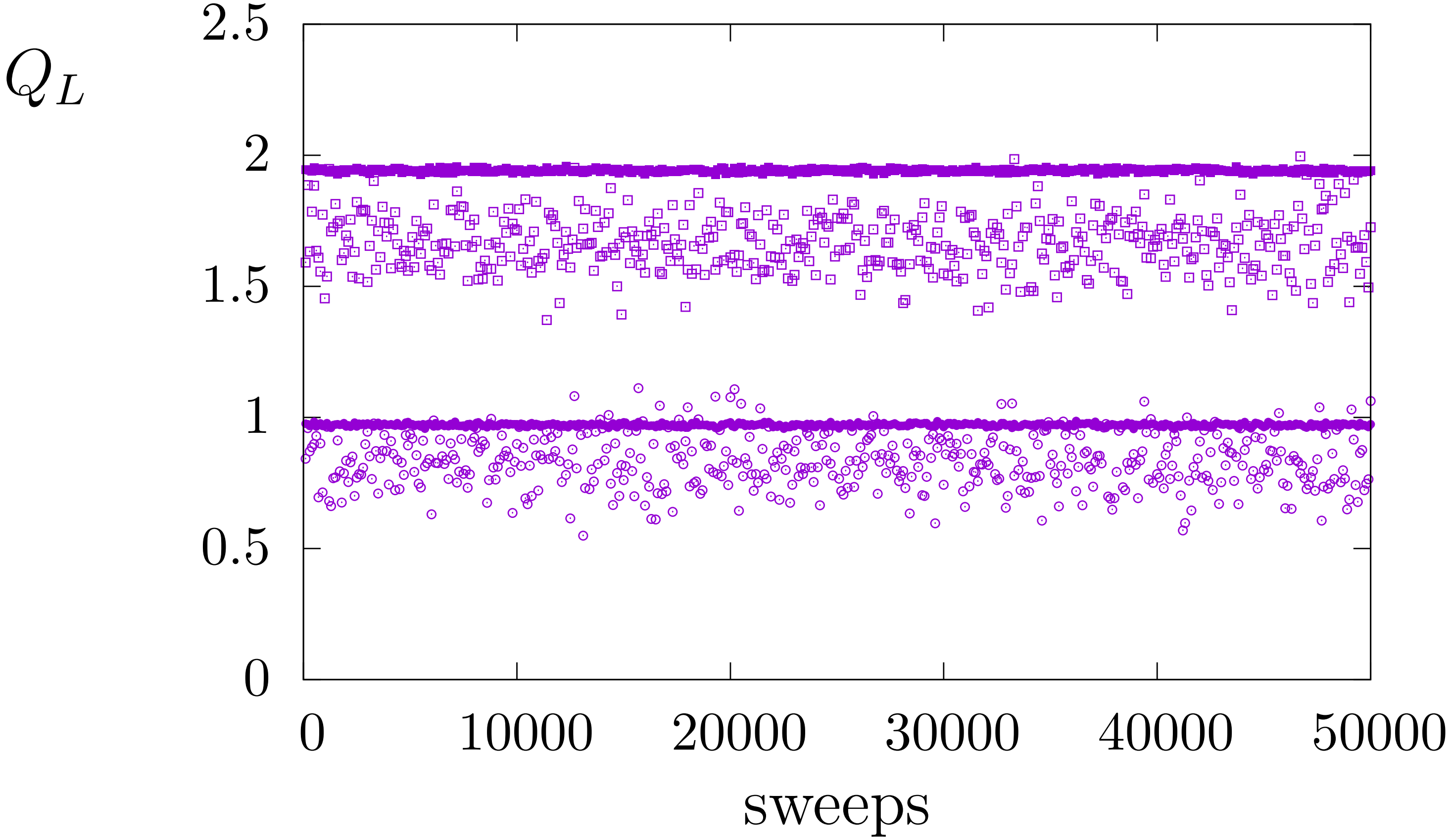
$\Longrightarrow$

$\tau_Q \twoheadrightarrow \infty$ for $a \to 0$ at fixed N *or* for $N \to \infty$ at fixed a

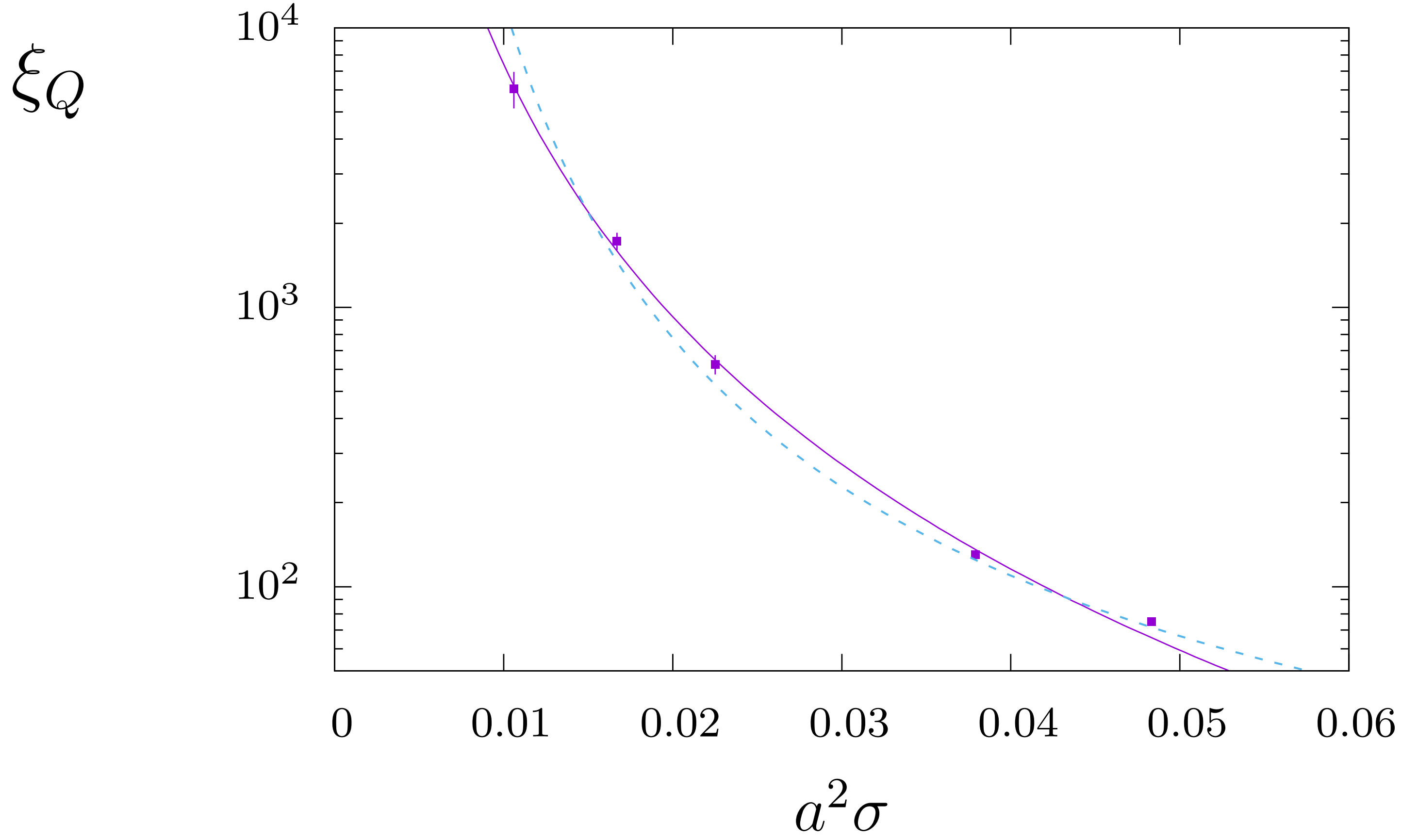
topological freezing at fixed a and large N:

two sequences of $SU(8)$ lattice fields on a $20^3 30$ lattice with $a\sqrt{\sigma} \simeq 0.133$:

Q_L after 2 ($\circ, \square$) and 20 ($\bullet, \blacksquare$) cooling sweeps.

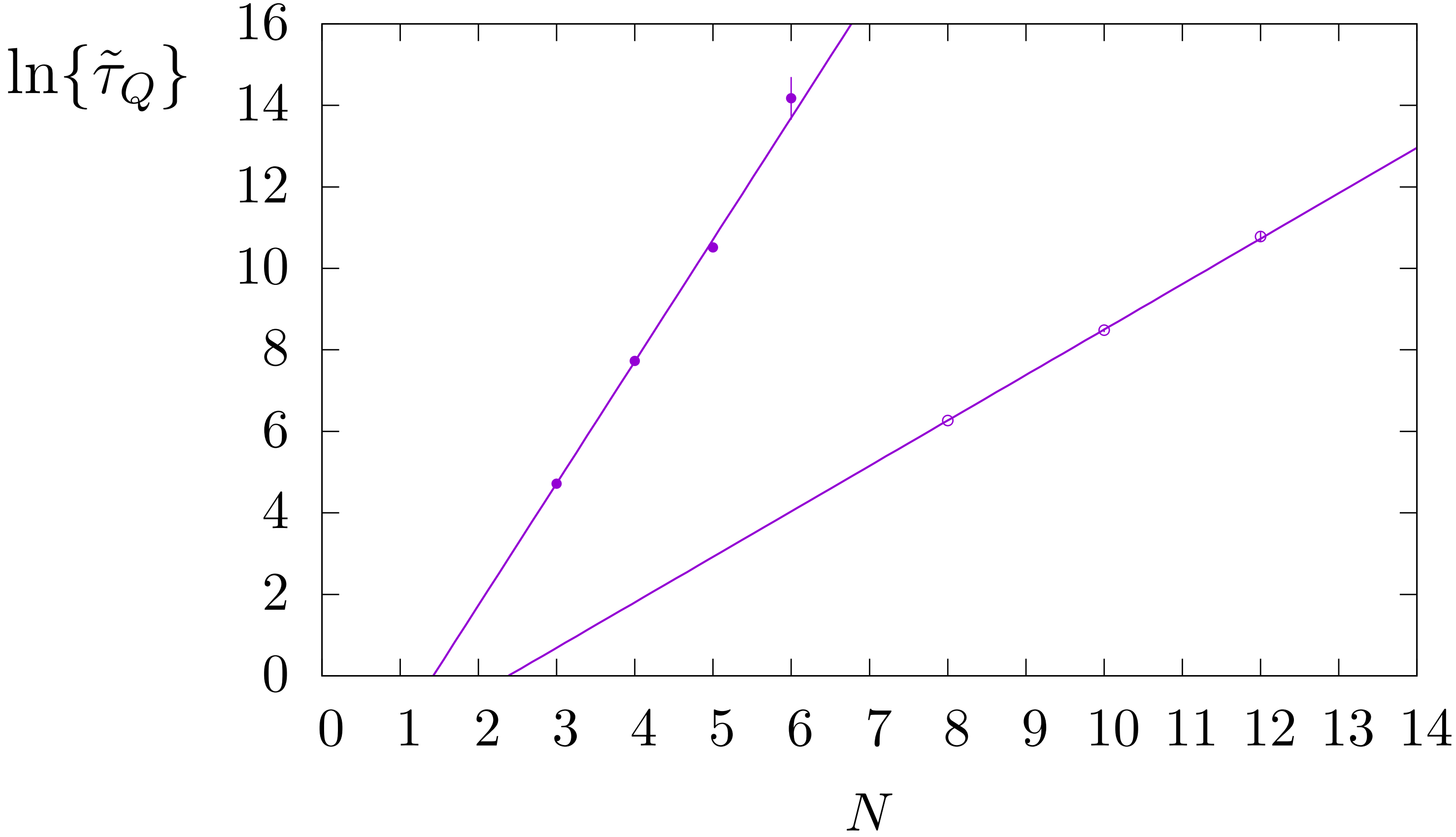


topological freezing at fixed $N = 3$ and decreasing a, $\beta \in [5.99, 6.50]$:
correlation length ξ_Q : $\langle Q(is)Q(is + \xi_Q)\rangle / \langle Q^2 \rangle = e^{-1}$



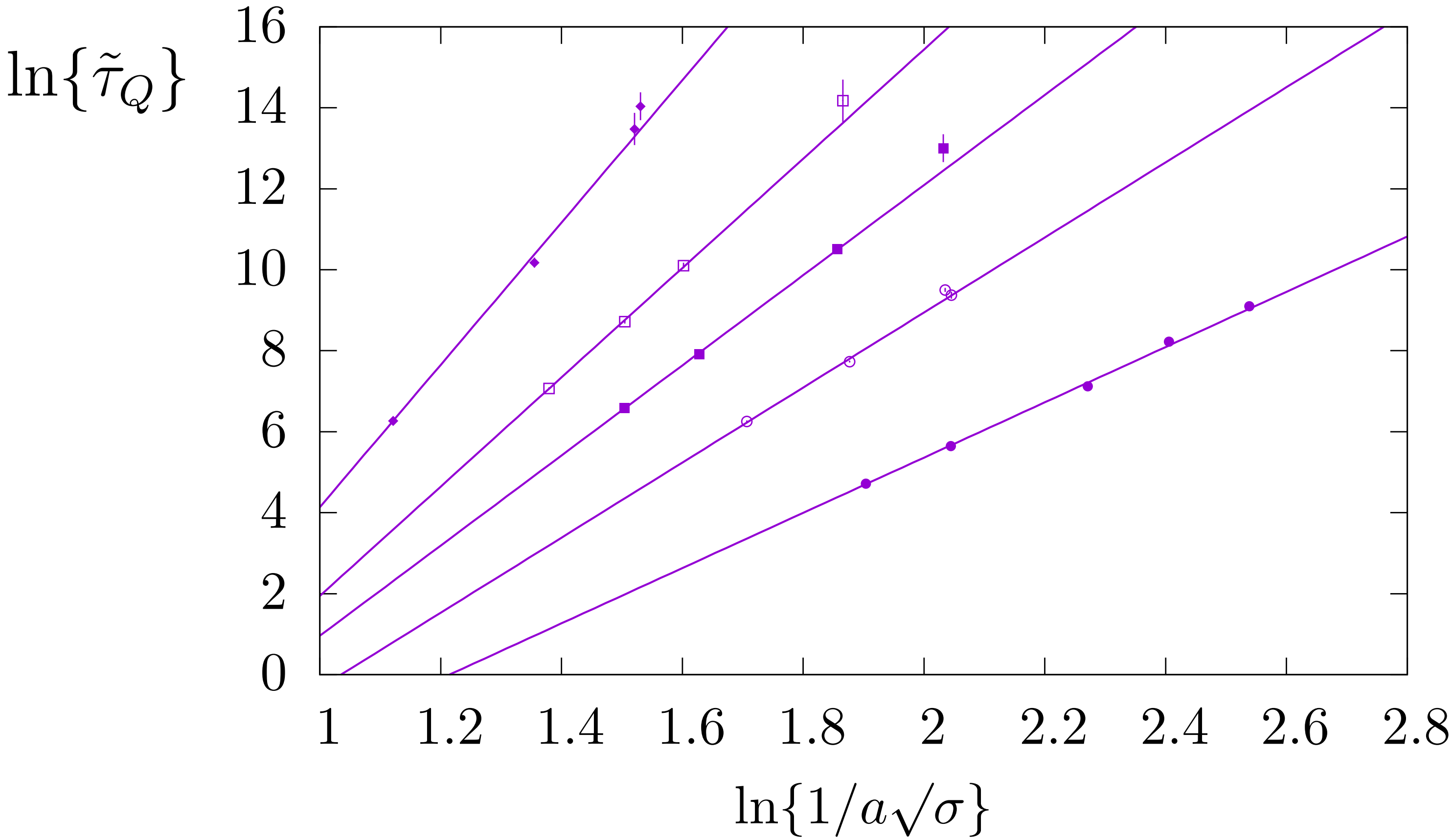
Solid line is $\xi_Q \propto 1/(a\sqrt{\sigma})^6$; dashed line is $\xi_Q \propto \exp\{c/a\sqrt{\sigma}\}$.

τ_Q vs N with fits $\tau_Q = b\exp\{cN\}$:



$a\sqrt{\sigma} \sim 0.15$ (●) and $a\sqrt{\sigma} \sim 0.33$ (○).

τ_Q vs a with fits $\tau_Q = b\{1/a\sqrt{\sigma}\}^c$:



$SU(3)$ (●), $SU(4)$ (○), $SU(5)$ (■), $SU(6)$ (□), $SU(8)$ (♦) on volume $= (3/\sqrt{\sigma})^4$.

does the freezing matter here?

• not for large N: $\frac{\langle C(t)Q^2\rangle}{\langle C(t)\rangle\langle Q^2\rangle} \sim 1 + O(1/N^2)$ (Witten's interlaced θ-vacua)

• for $N \leq 5$ and most $N = 6$ no freezing issue in our calculations

• for $N \geq 8$ freezing, but explicit check $\Rightarrow$ no visible effect

• improvement: multiple parallel sequences starting with different Q with a 'reasonable' distribution

BUT: cannot calculate Q-dependent properties, e.g. susceptibility, for $N \geq 8$ (or even 6) dealing better with freezing:

- very large (physical) volumes : computationally expensive!

- open (non-periodic) boundary : only partial success

- introduce a suitable defect (M. Hasenbusch 1706.04443, C. Bonanno et al 2205.06190) : computationally expensive

Of course changes in Q are a lattice artifact, albeit a useful one!

Some further methods for lattice topology MT 2202.02528

•We have many mutually consistent methods for calculating the total topological charge Q of a lattice field

•But calculating the charge density $Q(x)$ is more tricky: alterred by any smoothing

•Problem: given a lattice field $\{U_l\}_0$, how to calculate its physical density $Q(x)$?

$\Longrightarrow$Some extra methods: 'repetition', blocking, smearing
MT Phys.Lett. B232 (1989) 227 – see also DeGrand,Hasenfratz,Kovacs hep-lat/9711032

‘Repetition’ - relatively simple and unambiguous if (computationally) expensive

$\{U_l\}_0 \rightarrow \{U_l\}_{i_h}$ with i_h heat bath sweeps at same β

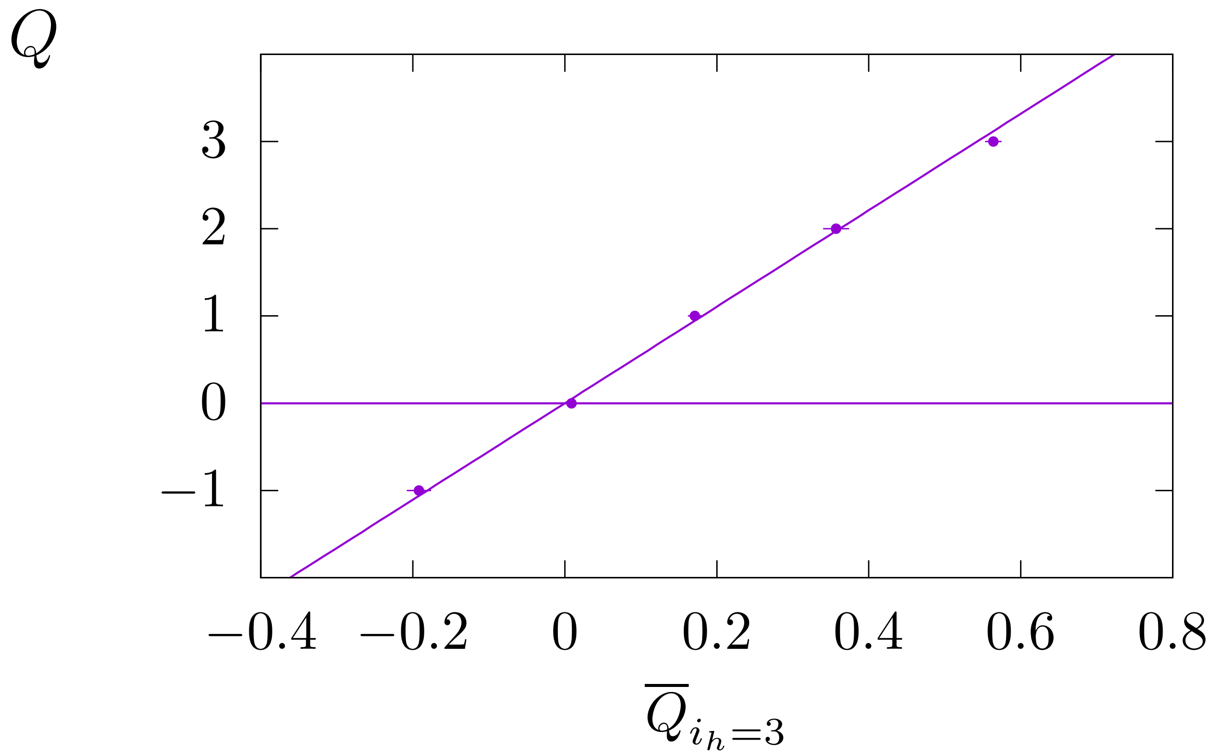
repeat with different random numbers $\rightarrow$ generate an ensemble of n_r such fields $\{U_l\}^j_{i_h}; j = 1, ..., n_r$ each just i_h heat bath sweeps from $\{U_l\}_0$

calculate the average density:
$\overline{Q}_{i_h}(x) = \frac{1}{n_r} \sum_{j=1}^{n_r} Q^j_{i_h}(x)$

for i_h very small, e.g. $i_h = 3$, this will average the most UV fluctuations but not those on physical length scales

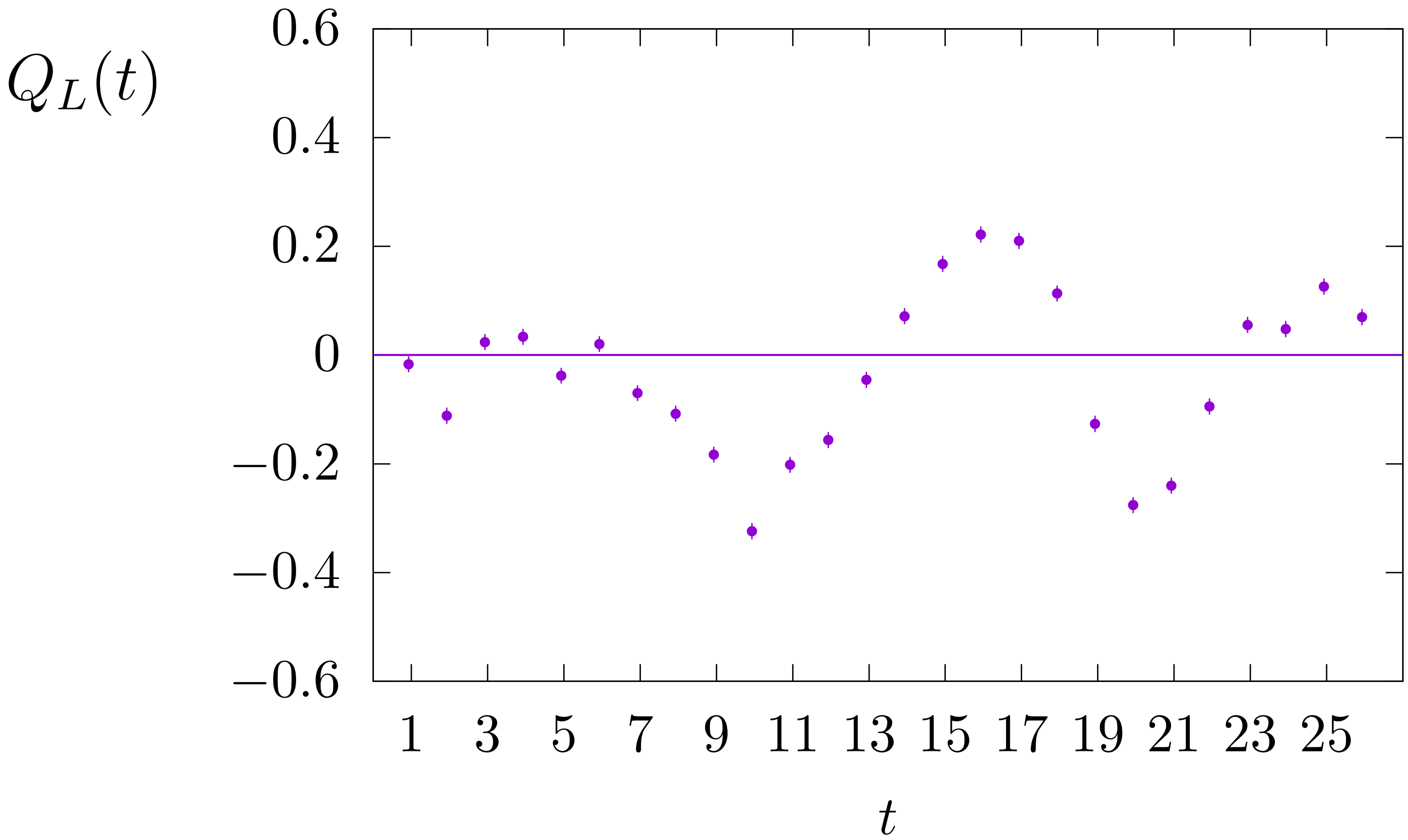
expensive : good for calibrating other faster methods

Q averaged over 10^4 repetitions of 3 heat bath sweeps starting from five separate starting fields with $Q = -1, 0, 1, 2, 3$, generated at $\beta = 6.235$ on a $18^3 26$ lattice:

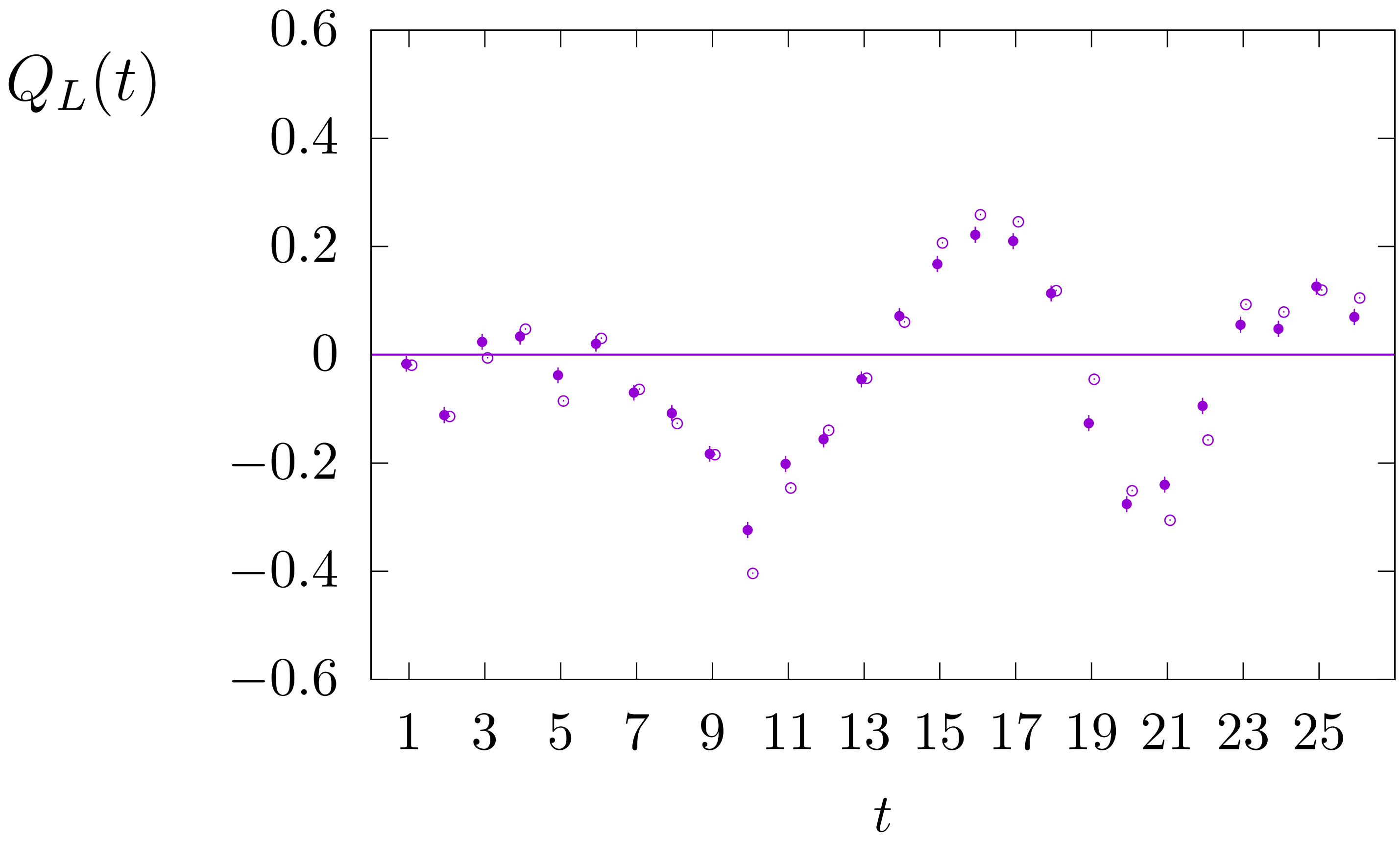


Diagonal line is $\overline{Q}_L = Z(\beta)Q$ with correct $Z(\beta = 6.235) = 0.1808$

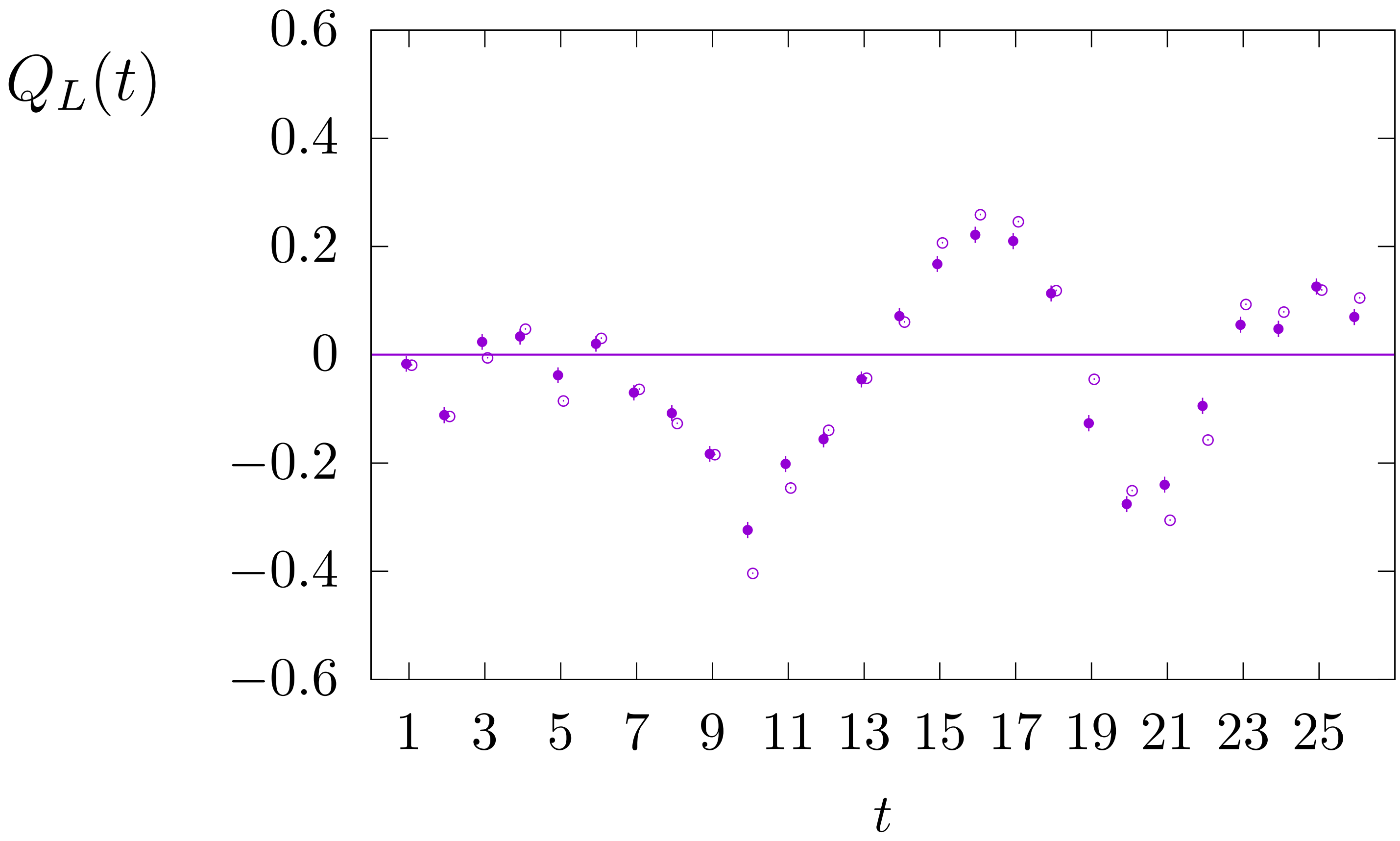
Profile of $\overline{Q}_L$ ($\bullet$) from 10^4 fields each 3 heat bath sweeps from a single $Q = -1$ $SU(3)$ lattice field generated at $\beta = 6.235$.(Normalised)



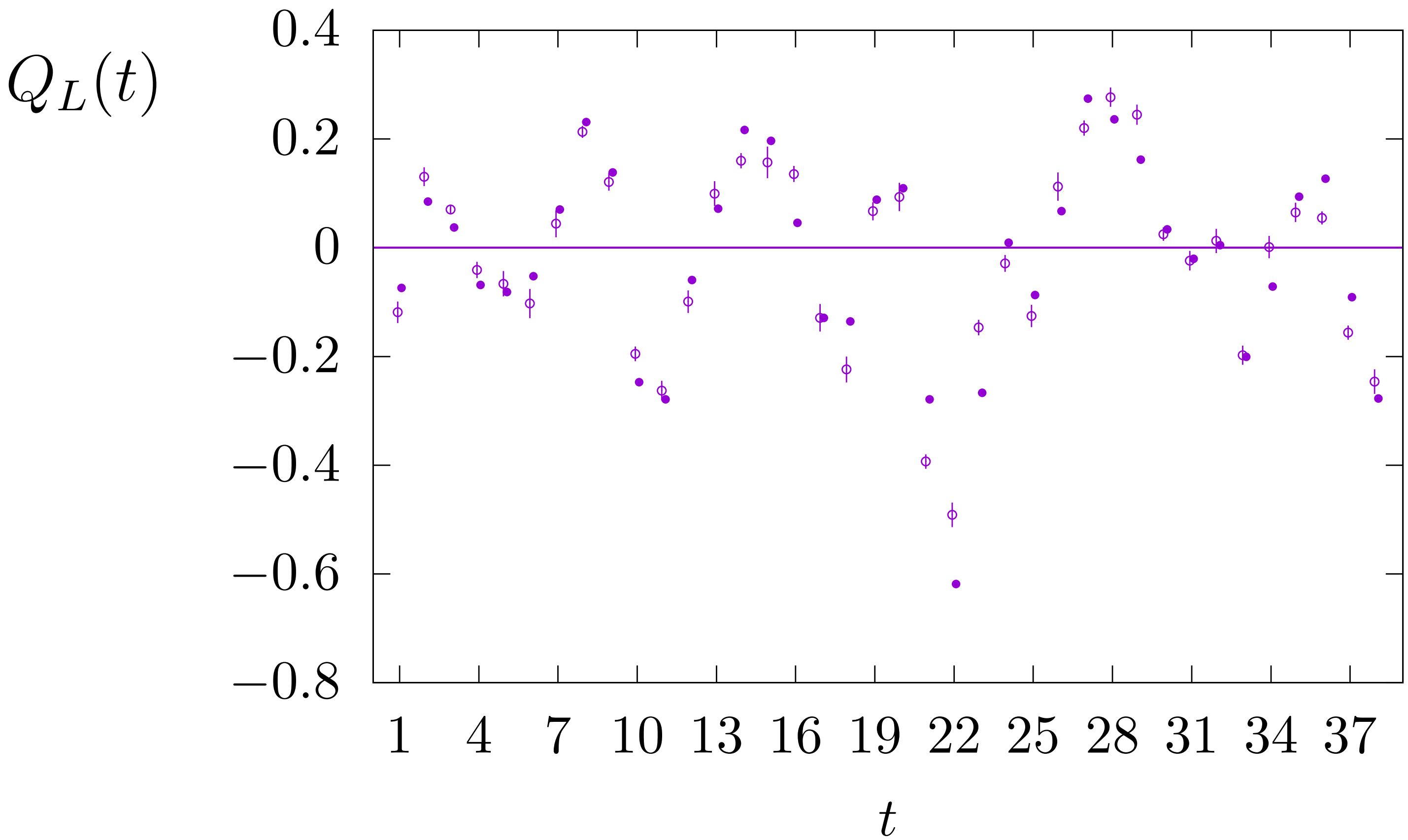
Profile in t of $\overline{Q}_L$ ($\bullet$) from 10000 fields each 3 heat bath sweeps from a single $Q = -1$ $SU(3)$ lattice field generated at $\beta = 6.235$, compared to profile of original $Q = -1$ field after 2 cooling sweeps ($\circ$).(Normalised to same Q.)



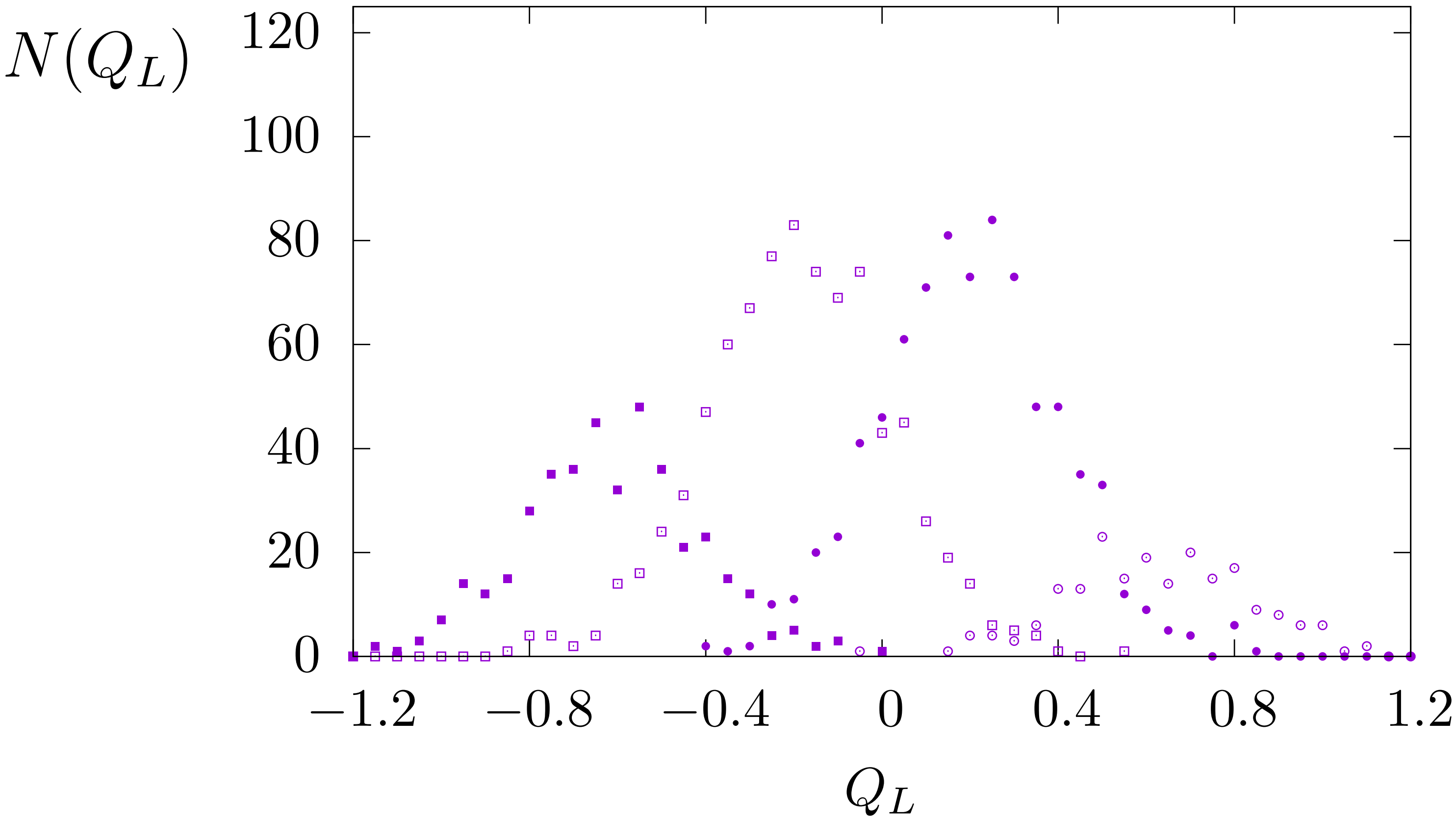
Profile in t of $\overline{Q}_L$ ($\bullet$) from 10000 fields each 3 heat bath sweeps from a single $Q = -1$ $SU(3)$ lattice field generated at $\beta = 6.235$, compared to profile of original $Q = -1$ field after 2 cooling sweeps ($\circ$).(Normalised to same Q.)



Smearing: profile in t of $\overline{Q}_L$ ($\bullet$) from 10000 fields each 3 heat bath sweeps from a single $Q = -1$ $SU(3)$ lattice field generated at $\beta = 6.50$, compared to profile of original $Q = -1$ field after 6 smearing sweeps ($\circ$).(Normalised to same Q.)



Blocking: Q_L on a sequence of doubly blocked $SU(3)$ gauge fields generated at $\beta = 6.70$. Histograms of fields with $Q = 3$ ($\circ$), $Q = 1$ ($\bullet$), $Q = -1$ ($\square$),$Q = -3$ ($\blacksquare$), where the value of Q is obtained after 20 cooling sweeps of the unblocked fields



Final remarks