The Schrödinger-Poisson system as a Newtonian N-body double

Fabio Briscese

Department of Mathematics, Università degli studi di Padova

ECT*
Nuclear and Atomic transitions as laboratories for high precision tests of Quantum Gravity inspired models
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Based on FB, Eur. Phys. J. C (2017) **77**:623

LSS

I deliberately use old data and simulations!

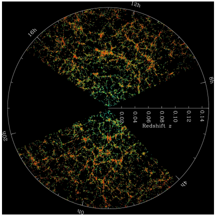


Image of 900.000 galaxies on 8400 deg^2 of the sky from the Sloan Digital Sky Survey up to $z \simeq 0.1 \sim 10^{22} km \sim 10^3 Mpc$

- DM is crucial for LSS formation
- at scales $\gtrsim$ 100 Mpc DM is successfully described as a cold pressureless fluid
- at smaller scales this model fails

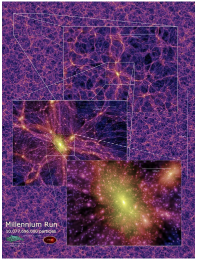
Newtonian N-body model for LSS formation

- Velocities are non-relativistic: Newtonian limit
- This justifies N-body numerical simulations
- Newton law in an expanding background with scale factor $a(t) \sim t^{2/3}$ during matter domination
- particles are DM halos of mass $10^6 - 10^{11}\, M_\odot$

Angulo, Hahn, Living Rev. Comput. Astrophys. (2022) 8:1

Millennium simulation

Millennium simulation kept busy the principal supercomputer at the Max Planck Society's Supercomputing Centre for more than a month. Springel et al. (2005)



- $N \sim 10^{10}$ identical bodies of mass $\sim 10^{9}\, M_{\odot}$
- Such bodies are huge agglomerates od DM particles (regardless of the specific model)
- Describes the formation of structures of different sizes (and quite good for our purposes)

Boltzmann-Vlasov equation

The phase-space probability density $f(x,p,t)$ of the N-body system satisfies the equation

$$\partial_t f(x,p,t) = -\frac{p}{a(t)^2 m}\nabla_x f(x,p,t) + m\nabla V \cdot \nabla_p f(x,p,t)$$

where the gravitational potential is given by the Poisson equation

$$\Delta V = \frac{4\pi G\rho_0}{a(t)}\left(\int f(x,p,t)d^3p - 1\right)$$

Angulo, Hahn, Living Rev. Comput. Astrophys. (2022) 8:1

Schrödinger-Poisson equation (SPE)

SPE is based on the hypothesis that it is possible to associate to the N-body system a wave function obeying the coupled Schrödinger-Poisson equations

$$\begin{aligned} i\hbar\partial_t\psi &= -\frac{\hbar^2}{2a(t)m}\Delta\psi + mV\,\psi \\ \Delta V &= \frac{4\pi G\rho_0}{a(t)}\,|\psi|^2 \end{aligned} \tag{1}$$

- $m \sim 10^9\, M_\odot$ and $\hbar$ effective free parameter chosen at will
- $\rho \simeq m \cdot |\psi|^2$
- $\Delta x \cdot \Delta p \sim \hbar$
- Effective Comton wavelength $\frac{\hbar}{m\, c} \sim 10^{-4} Mpc$

SPE can be recast as one nonlinear-nonlocal Schrödinger equation

A. Paredes, D. N. Olivieri, H. Michinel, Physica D: Nonlinear Phenomena **403** (2020) 132301, Physica D **403** (2020) 132301

SPE from ultralight DM particles

For elementary scalar DM, SPE is the low energy limit of the Klein-Gordon equation + gravitation

$$
\begin{aligned}
& i\hbar\partial_t\psi = -\tfrac{\hbar^2}{2a(t)m}\Delta\psi + mV\,\psi \\
& \Delta V = \tfrac{4\pi G\rho_0}{a(t)}\ |\psi|^2
\end{aligned}
\qquad (2)
$$

- In this case $\hbar$ is the Planck constant
- The scalar field must be ultralight, i. e. $m \sim 10^{-22}\,eV$

SPE as Newtonian N-body double

SPE $\longleftrightarrow$ Newtonian N-Body system

What are the theoretical bases of this correspondence?

We want to justify the use of SPE as N-body double and estimate the effective Planck constant in therms of the parameters of the N-body system using

- Nelson stochastic quantization
- Calogero conjecture

Nelson quantization

Assuming that the dynamics is Newtonian

$$m\,\ddot{\vec{x}} = -\vec{\nabla} V + \vec{B}(t)$$

where $\vec{B}(t)$ is a random variable with zero mean.

E. Nelson, Phys. Rev 150, 4 (1966). : the probability distribution $f(x) = |\psi|^2$ of the particle, is expressed in terms of a wave function

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\,\psi$$

A. M. Cetto, A. Valdes Hernandez, *The Emerging Quantum*, Springer (2015).

Calogero Conjecture

- Gravitation is universal
- Gravity has a long range
- The universe has a granular structure (is made of particles)
- Newtonian N-body system is chaotic

Every particle experiences a stochastic gravitational acceleration due to all the other particles in the universe. This is identified with the universal stochastic background force $\vec{B}(r)$ that is responsible of quantization a la Nelson.

By means of qualitative arguments, it is possible to estimate the effective $\hbar$ as

$$h \simeq \left(G\,R\,M^3\right)^{1/2} N^{-3/2}$$

Remarkably, if R, M, and N are respectively (order of magnitude) the radius, mass, and number of particles in the observable universe (within our Hubble horizon), $\hbar$ coincides (order of magnitude) with the Planck constant.

F. Calogero, Phys.Lett. A 228 (1997) 335-346, Int.J.Mod.Phys. B18 (2004) 519-526

Dimensional arguments

Relevant quantities

- Number of bodies N
- Gravitational constant G
- Two between M, D, and R

Characteristic units of time T, velocity V, acceleration a, energy E, and action A

- $T = \left(R^3/MG\right)^{1/2}$
- $V = R/T = (GM/R)^{1/2}$
- $a = \left(\int_0^R GDd^3r\right)/R^2 = 4\pi GDR$
- $E = GM^2/R = MV^2$
- $A = ET = \left(GM^3R\right)^{1/2}$

Number of bodies (particles) N

- We want to estimate the characteristic time-scale τ of the stochastic gravitational noise (i.e. of the stochastic motion).
- The stochastic character of this noise is due to the granularity of the universe. Thus it is plausible that the characteristic frequency be larger, the larges is N
- Since this is a collective effect, it is plausible to assume that such frequency is proportional to $N^{1/2}$

Indeed one assumes

$$\tau = T/N^{1/2}$$

Moreover, the gravitational energy ϵ of each particle is

$$\epsilon = E/N = GM^2/RN$$

Quantum of action h

The action associated with the stochastic motion is therefore

$$h = \epsilon\tau = A/N^{3/2} = \left(G\,R\,M^3\right)^{1/2} N^{-3/2}$$

The exponents of the dimensional quantities are fixed by dimensional considerations, while the exponent of N is based on a guess on the scaling $\tau \propto N^{-1/2}$

Assuming that

- The universe is made by nucleons, i. e. $N = M/m_p \sim 10^{78\pm8}$
- R is the Hubble radius $R \sim 10^{28\pm2}\,cm$
- $D \sim 10^{-30\pm2} g\,cm^{-3}$
- $M \sim 10^{54\pm8} g$

with very large uncertainties, one has

$$h \sim 10^{-25\pm1} g\,cm^2\,s^{-1}$$

to be compared with

$$h = 6 \times 10^{-27} g\,cm^2\,s^{-1}$$

Semi-quantitative argument

The acceleration a, timescale τ, velocity ν, and length l associated with the stochastic noisy force are

$$a \sim 10^{-8\pm4} cm/s^2\,, \quad \tau \sim 10^{-21\pm5} s\,, \quad \nu = a\,\tau \sim 10^{-29\pm1}\, cm/s, \quad l = \nu\tau \sim 10^{-50\pm6} cm$$

Conserved Hamiltonian

$$H = \sum_{j=1}^{N} \frac{m_j}{2} \left(\dot{r}_j\right)^2 - \frac{G}{2} \sum_{j=1}^{N} \sum_{k=1, k\neq j} m_j m_k |r_j - r_k|^{-1}$$

One separates the motion on the short time-scale $|t| \lesssim \tau$ and on the long time-scale

$$r_j(t) = R_j + V_j t + l_j(t)\,, \quad \dot{r}_j(t) = V_j + \nu_j(t)$$

where R_j and $V_j = O(V)$ are constant on the short time-scale τ, $|\nu_j| \sim O(\nu)$, $|l_j| \sim O(l)$. At zero-th order

$$H = \sum_{j=1}^{N} \frac{m_j}{2} \left(V_j\right)^2 - \frac{G}{2} \sum_{j=1}^{N} \sum_{k=1, k\neq j} m_j m_k |R_j - R_k|^{-1}$$

At $O(l) = O(\nu\tau)$ one has

$$W = \sum_{j=1}^{N} m_j \,(V_j \cdot \nu_j) = G_1 + G_2 + O(l^2)$$

with

$$G_1 = -\frac{G}{2} \sum_{j=1}^{N} \sum_{k=1, k\neq j} m_j m_k |R_j - R_k|^{-3} \,(R_j - R_k) \cdot (V_j - V_k) \; t$$

$$G_2 = -\frac{G}{2} \sum_{j=1}^{N} \sum_{k=1, k\neq j} m_j m_k |R_j - R_k|^{-3} \,(R_j - R_k) \cdot (l_j - l_k)$$

The sums in W and G_1 contains many terms with randomly alternating signs, so

$$W \sim N^{1/2}\, m\, V\, \nu\,, \quad G_1 \sim G\, N\, m^2\, R^{-2}\, V\, \tau$$

Assuming correlation between the stochastic and macroscopic displacements, i. e.

$$l_j - l_k = \lambda\,(R_j - R_k)$$

one has

$$G_2 \sim G\, m^2\, N^2\, R^{-2}\, l \sim G\, m^2\, N^2\, R^{-2}\, \nu\, \tau$$

Therefore, at $O(l)$ order one has

$$W \sim G_2 \Longrightarrow \tau \sim N^{-3/2} V R^2 G^{-1} m^{-1} \sim N^{-1/2} T$$

and thus

$$h \sim N^{-3/2} E T$$

The Calogero conjecture is based on the further hypothesis that the stochastic noise is coherent in space. One possible interpretation is that the stochastic noise is due to fluctuations of the scale factor, i. e.

$$l_j - l_k = \lambda (R_j - R_k) = \delta a(t) (R_j - R_k)$$

Limits of Nelson-Calogero quantization

At fundamental level

- Collapse of the wave function, entanglement, etc. can't be derived (at present stage of understanding)
- Doesn't include fields (radiation domination?)
- The treatment is non-relativistic (generalizations to relativity)

We are less ambitious

At cosmological level we can ignore these issues, as SPE are just an effective double of the Newtonian N-body system

SPE as stochastic quantization of a Newtonian N-body system

- $B(r)$ is the noise due to the gravitational field
- The chaotic nature of N-body dynamics implies that $B(t)$ is a random variable
- The universality of gravitation ensures that all the bodies feels the same $B(t)$
- $\hbar$ can be estimated as $\hbar \sim m^{5/3} G^{1/2} (N/D)^{1/6}$
- The scale factor $a(t)$ plays an important role

The dynamics of each particle (galaxy or DM Halo) is governed by the SPE equations

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2a(t)m}\Delta\psi + mV\,\psi \qquad \Delta V = \frac{4\pi G\, D}{a(t)}\,|\psi|^2 \tag{3}$$

We would like to verify this correspondence by numerical experiments

Numerical simulations

We use the Millennium simulation: the N-body problem is solved for $N \simeq 10^{10}$ particles of mass $M \simeq 10^9 M_\odot$, so that $D \simeq 3H_0^2/8\pi G \simeq 4 \times 10^{-26} Kg/m^3$, where $H_0 \simeq h^{-1} \times 100 \times km/sMpc$, with $h \simeq 0.73$, is the Hubble constant.

- One has $\hbar \simeq 2 \times 10^{66} Kg\, m^2/s$.
- This corresponds to a value $\hbar/M \simeq 10^{-4} Mpc \cdot c$ in the range of values used in numerical simulations:
  - $\hbar/M \sim 10^{-4} Mpc \cdot c$ in
    C. Uhlemann *et al.*, Phys. Rev. D 90, 023517 (2014) 50. A. Paredes, H. Michinel, Phys. Dark Univ. 12, 50-55 (2016).
  - $\hbar/M \sim 10^{-6} Mpc \cdot c$ in
    J. Zhang *et al.*, The Astrophysical Journal,**853**: 51 (2018); H.-Y. Schive *et al.*, Nat. Phys. 10, 496?499 (2014)

Conclusions

- New argument in support of the validity of SPE equations as numerical double of the N-body simulations of DM dynamics at large cosmological scales
- $\hbar$ is no more a free parameter, but it is estimated as

$$\hbar \sim m^{5/3} G^{1/2} (N/D)^{1/6}$$

  This justifies the huge value of $\hbar$ often used in numerical solutions of SPE
- These is remarkable, as this derivation is a practical application of the Calogero conjecture to a realistic physical problem.

Discussion and open questions

Is the correspondence between SPE and LSS N-body simulations robust?

- We need ad hoc numerical simulations to verify this coprrespondence
- Test the h scaling law with N, M, R
- Test the role of the scale factor $a(t)$

At a more fundamental level

- Is the Calogero conjecture on the cosmic origin of quantization correct?
- Does it play some role in the wave function collapse?
- How can it be generalized to fields?
- Can it be extended to a general relativistic framework?

INdAM-INFN joint research project

Thank you!