Dispersion corrections to elastic electron-nucleus scattering cross sections

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July, 2022

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Contents:

1. Potential scattering from spin-zero nuclei
2. Nuclear excitation of ^{12}C
3. Influence of dispersion on elastic $e+^{12}$C scattering

Spin asymmetry at small momentum transfer

1a. Potential scattering and diffraction structures

Scattering operator

$$\hat{f}_{\rm coul}(k_i,\theta) \;= A \;+\; B\;\boldsymbol{n}\cdot\boldsymbol{\sigma} \qquad\qquad \boldsymbol{n} \upuparrows \boldsymbol{k}_i \times \boldsymbol{k}_f$$

Scattering amplitudes from phase-shift theory

$$A = \frac{1}{2ik_i}\sum_{l=0}^{\infty} f_1(\delta_l)\,P_l(\cos\theta), \quad B = \frac{1}{2k_i}\sum_{l=1}^{\infty} f_2(\delta_l)\,P_l^1(\cos\theta)$$

Differential cross section for unpolarized electrons

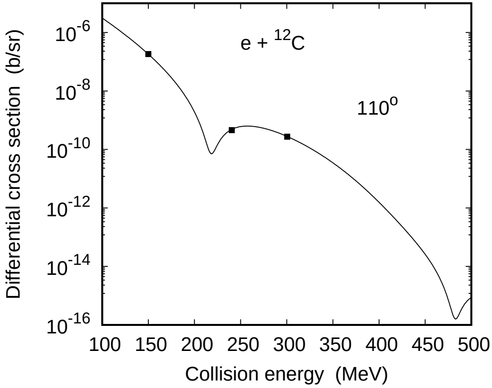
$$\frac{d\sigma_{\rm coul}}{d\Omega} = |A|^2 \,+\, |B|^2$$

Born approximation:

$$\frac{d\sigma_{\rm coul}^{B1}}{d\Omega} \;=\; \sigma_{\rm Mott}\;|F_0^C(q)|^2, \qquad \sigma_{\rm Mott} \;=\; \frac{Z^2\cos^2(\theta/2)}{(2ck_i\sin^2(\theta/2))^2}$$

Charge form factor $F_0^C(q) = \frac{4\pi}{Z} \int_0^\infty r_N^2 dr_N \, \varrho_0(r_N) \, \frac{\sin(qr_N)}{qr_N}$

Diffraction structures:



momentum transfer:

$$q \approx 2k_i \sin(\theta/2)$$

1^{st} diffraction minimum:
$q \approx 1.82 \text{ fm}^{-1}$

Exp: Reuter et al (1982)

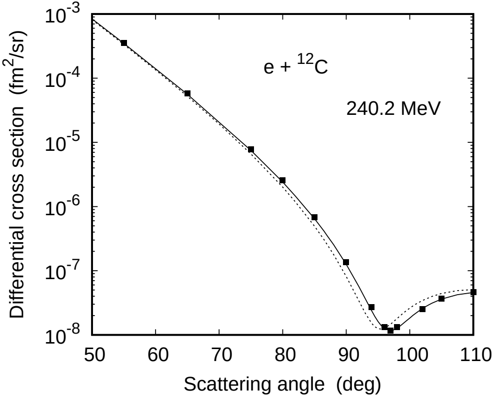
Change q by variation of energy $E_i \approx k_i c$ or scattering angle θ

1b. Recoil effects

Recoil energy of nucleus: $E_R \approx \frac{q^2}{2M}$

Modified cross section: $\frac{d\sigma_{\rm coul}}{d\Omega} = \frac{k_f}{k_i}\,\frac{1}{f_{\rm rec}}\left(|A|^2 + |B|^2\right)$

Reduced collision energy $\overline{E}_i = \sqrt{E_i E_f}, \quad E_f = E_i - E_R$



$f_{\rm rec} \approx 1 + \frac{E_R}{E_i}$

Exp: Reuter et al (1982)

2a. DWBA formalism for nuclear excitation

Excitation cross section of spin-zero nucleus to $L^\pi(\omega_L)$:

$$\frac{d\sigma}{d\Omega} = \frac{k_f}{k_i} \frac{4\pi^3 E_i E_f}{f_{\rm rec} \;\; c^2} \frac{1}{2} \sum_{\sigma_i \sigma_f} \sum_{M_L} \left| A_{fi}^{\rm coul} + A_{fi}^{\rm mag} \right|^2$$

$$\begin{pmatrix} A_{fi}^{\rm coul} \\ A_{fi}^{\rm mag} \end{pmatrix} = \frac{1}{c} \int d\boldsymbol{r}_N d\boldsymbol{r}_e \, (\psi_{k_f}^{(\sigma_f)+}(\boldsymbol{r}_e) \begin{pmatrix} -1 \\ \boldsymbol{\alpha} \end{pmatrix} \psi_{k_i}^{(\sigma_i)}(\boldsymbol{r}_e))$$

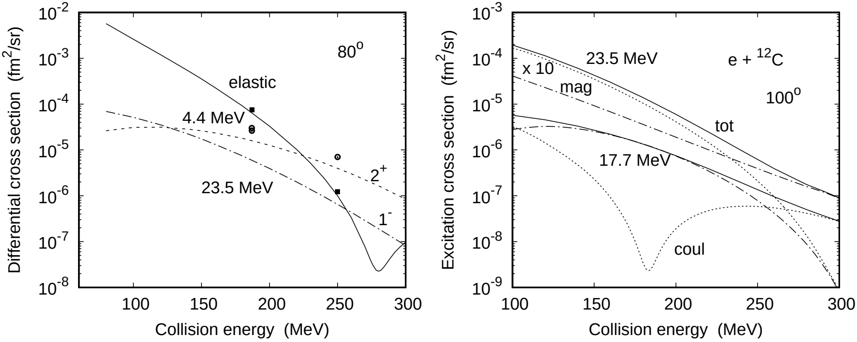
$$\times \frac{e^{i\omega_L/c|\boldsymbol{r}_e - \boldsymbol{r}_N|}}{|\boldsymbol{r}_e - \boldsymbol{r}_N|} \begin{pmatrix} \varrho_L(r_N) \; Y_{LM_L}^*(\Omega_N) \\ -i \sum_\lambda J_{L\lambda}(r_N) \, \boldsymbol{Y}_{L\lambda}^{M_L *}(\Omega_N) \end{pmatrix}$$

For parity $\pi = (-1)^L$:
Nuclear transition densities $\varrho_L, \; J_{L,L+1}, \; J_{L,L-1}$
calculated from QRPA, QPM models

(Ponomarev)

2b. Importance of magnetic scattering

Dipole (1^-) and quadrupole (2^+) excitation of ^{12}C



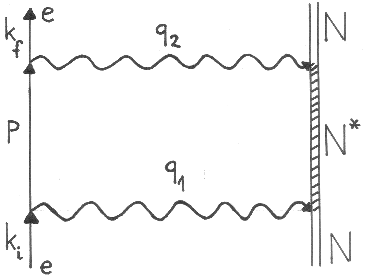
Elastic scattering dominant at small E_i, excitation dominant at large E_i

Magnetic scattering dominant at high E_i or large angles (strongly dependent on L, ω_L)

Experiment: Fregeau (1956); Crannell, Griffy (1964)

3. Dispersion in elastic scattering

Box diagram : second Born (Friar, Rosen 1972)



Virtual nuclear excitation to state L, M

Nuclear transition matrix element $T_{\mu\nu}$

$$T_{00} = \langle 0|\varrho(\boldsymbol{q}_2)|LM\omega_L\rangle\langle LM\omega_L|\varrho(\boldsymbol{q}_1)|0\rangle \sim (F_L^c\, Y_{LM})(\boldsymbol{q}_2)\cdot(F_L^c\, Y_{LM}^*)(\boldsymbol{q}_1)$$

$$T_{0m} \sim (F_L^c\, Y_{LM})(\boldsymbol{q}_2) \cdot \sum_{\lambda=L\pm1} (F_{L\lambda}^{te}\, \boldsymbol{Y}_{L\lambda}^{M*})_m(\boldsymbol{q}_1)$$

Charge form factor

$$F_L^c(q) = \int r_N^2 dr_N\, \varrho_L(r_N)\, j_L(qr_N)$$

Transverse form factor

$$F_{L\lambda}^{te}(q) = \int r_N^2 dr_N\, J_{L\lambda}(r_N)\, j_\lambda(qr_N)$$

$$T_{mn} \sim \sum_{\lambda=L\pm1} (F^{te}_{L\lambda}\, \boldsymbol{Y}^{M}_{L\lambda})_m(\boldsymbol{q}_2) \cdot \sum_{\lambda'=L\pm1} (F^{te}_{L\lambda'}\, \boldsymbol{Y}^{M*}_{L\lambda'})_n(\boldsymbol{q}_1)$$

Dispersion amplitude

$$A^{\mathrm{box}}_{fi} = \frac{\sqrt{E_i E_f}}{\pi^2\, c^3} \sum_{LM,\omega_L} \int d\boldsymbol{p} \frac{1}{(q_2^2 + i\epsilon)(q_1^2 + i\epsilon)} \sum_{\mu,\nu=0}^{3} t_{\mu\nu}\; T^{\mu\nu}$$

Electron transition matrix element

$$t_{\mu\nu} = c\, u^{(\sigma_f)+}_{k_f} \gamma_0 \gamma_\mu\; \frac{E_p + c\boldsymbol{\alpha}\boldsymbol{p} + \beta m c^2}{E_p^2 - \boldsymbol{p}^2 c^2 - m^2 c^4 + i\epsilon}\; \gamma_0 \gamma_\nu\; u^{(\sigma_i)}_{k_i}$$

intermediate electron energy $E_p \approx E_i - \omega_L$

Photon momenta $\boldsymbol{q}_1 = \boldsymbol{k}_i - \boldsymbol{p},\ \boldsymbol{q}_2 = \boldsymbol{p} - \boldsymbol{k}_f$

Gauge-invariant decomposition

$$\frac{g^{\mu\nu}}{q^2 + i\epsilon} \; = \; - \; \frac{1}{\boldsymbol{q}^2} \; \delta_{\mu 0}\delta_{\nu 0} \; - \; \frac{\delta_{mn} - \hat{q}^m \hat{q}^n}{q^2 + i\epsilon} \; \delta_{\mu m}\delta_{\nu n}$$

leads to

$$\sum_{\mu,\nu=0}^{3} t_{\mu\nu} \, \frac{1}{(q_2^2 + i\epsilon)(q_1^2 + i\epsilon)} \, T^{\mu\nu} \; = \; \frac{1}{\boldsymbol{q}_2^2 \boldsymbol{q}_1^2} \; t_{00} \; T_{00}$$

$$+ \; \frac{1}{\boldsymbol{q}_2^2 (q_1^2 + i\epsilon)} \sum_{m=1}^{3} t_{0m} \; f(\boldsymbol{q}_1, \, T_{0k}) + \frac{1}{(q_2^2 + i\epsilon)\boldsymbol{q}_1^2} \sum_{m=1}^{3} t_{m0} \; f(\boldsymbol{q}_2, \, T_{n0})$$

$$+ \; \frac{1}{(q_2^2 + i\epsilon)(q_1^2 + i\epsilon)} \sum_{m,k=1}^{3} f(\boldsymbol{q}_2, t_{lk}) \; f(\boldsymbol{q}_1, \, T_{mj})$$

$=$ one Coulombic $+$ three magnetic contributions

$$A_{fi}^{\rm box} = \sum_{LM\omega_L} \left[M_{fi}^{C} + M_{fi}^{te1} + M_{fi}^{te2} + M_{fi}^{te3} \right]$$

Total transition amplitude: $A_{fi} = f_{\rm coul} + A_{fi}^{\rm box}$

Differential cross section including dispersion

$$\frac{d\sigma^{\rm box}}{d\Omega} = \frac{d\sigma_{\rm coul}}{d\Omega} + \frac{k_f}{k_i} \frac{1}{f_{\rm rec}} \frac{1}{2} \sum_{\sigma_i \sigma_f} 2 \text{ Re } \{ f_{\rm coul}^{*} \cdot A_{fi}^{\rm box} \}$$

Friar-Rosen theory:

1) No magnetic terms ($M_{fi}^{te1} = M_{fi}^{te2} = M_{fi}^{te3} = 0$)

2) Mean excitation energy ($\omega_L = \bar{\omega} = 15$ MeV)

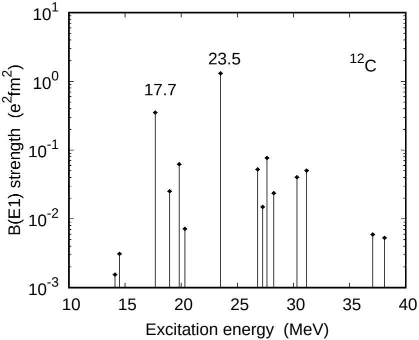
$\Longrightarrow$ Closure approximation

$$\sum_{LM\omega_L} M_{fi}^{C} \sim \sum_{LM\omega_L} T_{00} = \langle 0 | \varrho(\boldsymbol{q}_2) \underbrace{\sum_{LM\omega_L} | LM\omega_L \rangle \langle LM\omega_L |}_{1 - |0\rangle\langle 0|} \varrho(\boldsymbol{q}_1) | 0 \rangle$$

Improvement:

(1) $M_{fi}^{te(1-3)} \neq 0$,
(2) Explicit consideration of excited states

Dipole strength distribution of ^{12}C (Ponomarev):



Prominent states of the giant dipole (1^-) resonance:

$\omega_L = 23.5$ MeV
$\omega_L = 17.7$ MeV

In addition: quadrupole (2_1^+) at $\omega_L = 4.439$ MeV

Additivity of contributions from different L, ω_L:

$$\frac{d\sigma^{\rm box}}{d\omega} = \frac{d\sigma_{\rm coul}}{d\Omega} + \frac{k_f}{k_i}\frac{1}{f_{\rm rec}}\frac{1}{2}\sum_{\sigma_i,\sigma_f}\sum_{L,\omega_L} 2\ \mathrm{Re}\ \{f^*_{\rm coul}\ A^{\rm box}_{fi}(L,\omega_L)\}$$

$$A^{\rm box}_{fi}(L,\omega_L) = \sum_{M=-L}^{L}\left\{M^C_{fi}(LM\omega_L) + (M^{te1}_{fi} + M^{te2}_{fi} + M^{te3}_{fi})(LM\omega_L)\right\}$$

Challenging term:

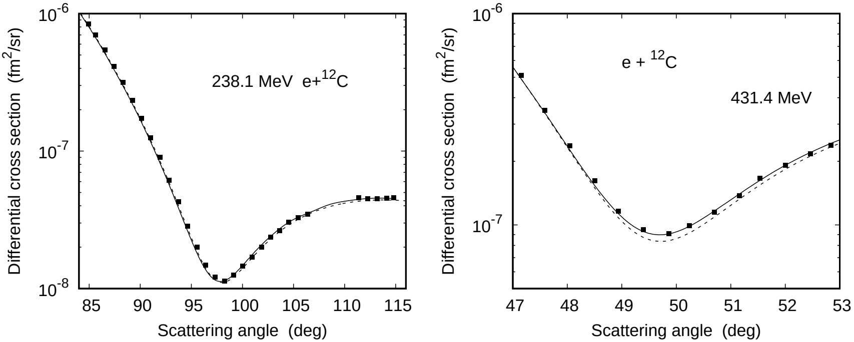
$$M^{te3}_{fi}(LM\omega_L) \sim \int \frac{d\boldsymbol{p}}{(q_2^2 + i\epsilon)(q_1^2 + i\epsilon)}\sum_{m,k=1}^{3}(t_{mk}\,T_{mk} \ + \cdots)$$

Change variables from $\boldsymbol{p}$ to $\boldsymbol{q}_2 = \boldsymbol{p} - \boldsymbol{k}_f$: $\quad (p_0 = \frac{E_p - E_f}{c})$

$$M^{te3}_{fi}(LM,\omega_L) \sim \int_0^\infty \frac{\boldsymbol{q}_2^2 d|\boldsymbol{q}_2|}{\boldsymbol{q}_2^2 - p_0^2 - i\epsilon}\int_{-1}^{1} d(\cos\vartheta_q)$$

$$\times \int_0^{2\pi} d\varphi_q\ \frac{g(q_1,q_2)}{\underbrace{(D - E\cos\varphi_q + i\epsilon)}_{(\sim t_{mk})}\underbrace{(A + B\cos\varphi_q - i\epsilon)}_{(\sim q_1^2 + i\epsilon)}}$$

Cross section including dispersion for $e+^{12}$C collisions



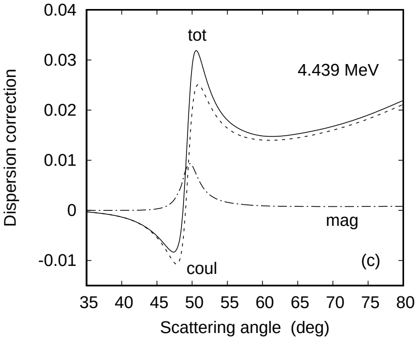
Experiment: Offermann et al (1991)

Cross section increase in first diffraction minimum:
238.1 MeV: 5%, 431.4 MeV: 8%

Dispersion correction

$$\Delta\sigma^{\mathrm{box}} = \frac{d\sigma^{\mathrm{box}}/d\Omega - d\sigma_{\mathrm{coul}}/d\Omega}{d\sigma_{\mathrm{coul}}/d\Omega}$$

Additivity: $\Delta\sigma^{\mathrm{box}} = \sum_{L,\omega_L} \Delta\sigma^{\mathrm{box}}(L,\omega_L)$

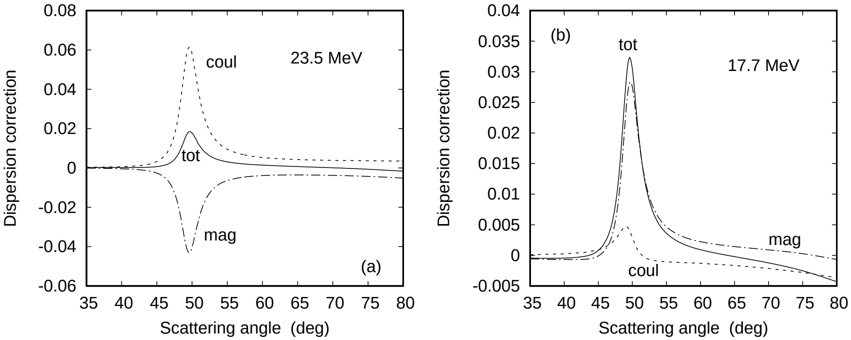


431.4 MeV $e+^{12}$C

$L^\pi, \omega_L = 2^+,\ 4.439$ MeV

Effect of magnetic scattering near first diffraction minimum

Intermediate dipole excitation for 431.1 MeV $e+{}^{12}$C

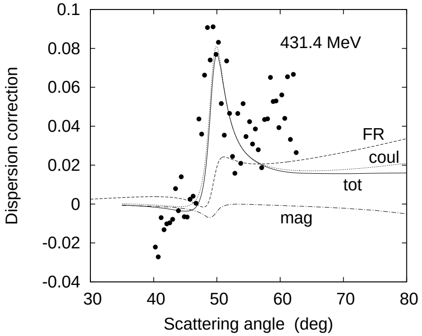


Result:

Magnetic scattering important for dipole excitation
but negligible for quadrupole excitation

Comparison of the dispersion correction with experiment:

$$\Delta\sigma_{\text{exp}} = \frac{d\sigma_{\text{exp}}/d\Omega}{d\sigma_{\text{coul}}/d\Omega} - 1$$

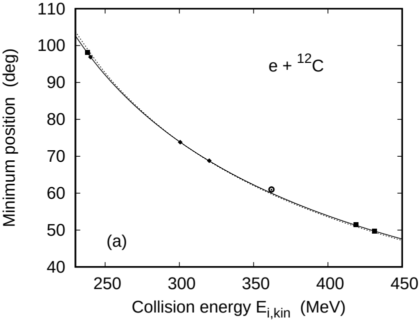


Result:

Peak enhancement by factor 4 compared to Friar/Rosen
Small influence of magnetic scattering

Systematic investigation of 1$^{\mathrm{st}}$ diffraction minimum

Energy-angle dependence from phase-shift theory



Momentum transfer

$$q \approx 2\,\frac{E_i}{c}\,\sin(\theta/2)$$

$\cdots \quad q = 1.82\ \mathrm{fm}^{-1}$ fix

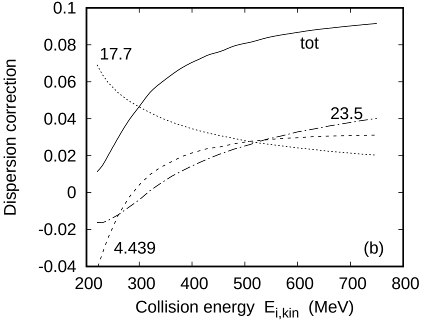
Experiment: ■ Offermann et al

♦ Reuter et al

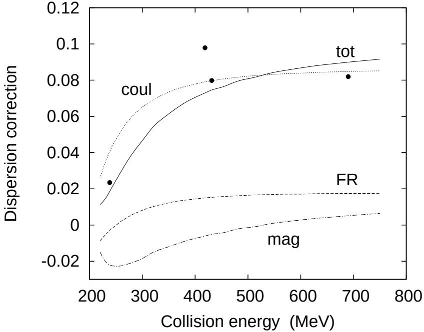
⊙ Jefferson Lab (362 MeV, $\theta = 61^\circ$)

Dispersion correction in the position of the 1^{st} minimum:

Multipole contributions:



Comparison with experiment:



Experiment: Offermann et al

Increase of dispersion correction with collision energy

Reduction of $\Delta\sigma^{\mathrm{box}}$ by magnetic scattering at low energies

Approximations and accuracy when calculating dispersion:

$$A_{fi}^{\rm box} = \frac{\sqrt{E_i E_f}}{\pi^2\ c^3} \sum_{LM,\omega_L} \int d\boldsymbol{p} \frac{1}{(q_2^2 + i\epsilon)(q_1^2 + i\epsilon)} \sum_{\mu,\nu=0}^{3} t_{\mu\nu}\ T^{\mu\nu}$$

(a) No consideration of crossed box diagram

(b) Only dominant pole contribution in Theorem of Residues

(c) $E_p = E_i - \omega_L$ in M_{fi}^{te2} and M_{fi}^{te3} $(M = \infty)$

(d) Numerical accuracy: at most 5%

(e) No consideration of further dipole excitations (at most 10%)

(f) No consideration of further multipole states with $L \geq 2$ (??)

Spin asymmetry including dispersion

Projectiles with spin polarization $\zeta_i = (1, \alpha_s, \varphi_s)$:

Mixing of helicity eigenstates

$$u_{k_i}^{(\sigma_i)} = e^{-i\varphi_s/2} \cos(\alpha_s/2)\, u_{k_i}^{(+)} + e^{i\varphi_s/2} \sin(\alpha_s/2)\, u_{k_i}^{(-)}$$

Cross section for electrons polarized along $\boldsymbol{n} \sim \boldsymbol{k}_i \times \boldsymbol{k}_f$:

$$\frac{d\sigma}{d\Omega}(\zeta_i) = \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{unpol}} [1 + S\,(\zeta_i \cdot \boldsymbol{n})]$$

Sherman function $$S = \frac{d\sigma/d\Omega(\zeta_i) - d\sigma/d\Omega(-\zeta_i)}{(d\sigma/d\Omega)_{\mathrm{unpol}}}$$

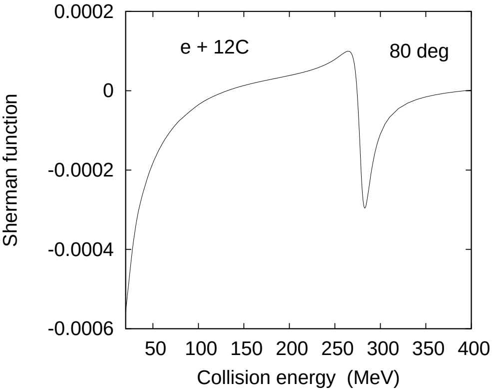
Potential scattering: $$S_{\mathrm{coul}} = \frac{2\ \mathrm{Re}\ (AB^*)}{|A|^2 + |B|^2}$$

Motivation: S sensitive to phases

$\Longrightarrow$ stringent test of theoretical models

Control measurement of beam polarization

Background in parity violation experiments



Resonant behaviour
near cross section minima

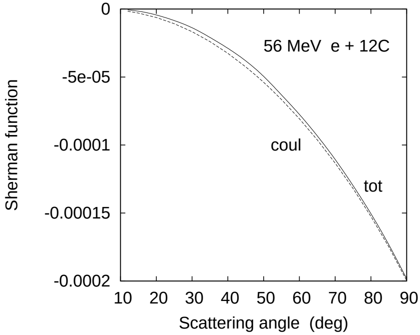
Sherman function including dispersion

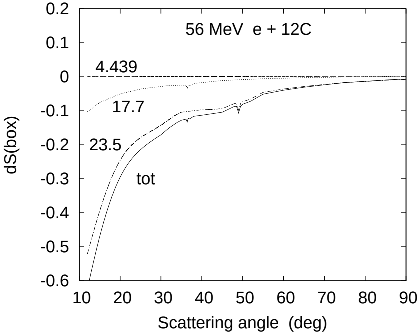
$$S^{\rm box} = \frac{d\sigma^{\rm box}/d\Omega(\zeta_i) - d\sigma^{\rm box}/d\Omega(-\zeta_i)}{(d\sigma^{\rm box}/d\Omega)_{\rm unpol}}$$

Dispersion correction:

$$S^{\rm box} = S_{\rm coul}(1+\Delta S) \quad \Longrightarrow \quad \Delta S = \frac{S^{\rm box}}{S_{\rm coul}} - 1$$

Low momentum transfer ($q \in [0.06, 0.4]$ fm^{-1}):





Anomalous behaviour of S at fixed (E_i, θ) pair

Fourth term of dispersion amplitude:

$$M_{fi}^{te3}(LM, \omega_L) \sim \int_0^\infty \frac{\boldsymbol{q}_2^2 d|\boldsymbol{q}_2|}{\boldsymbol{q}_2^2 - p_0^2 - i\epsilon} \int_{-1}^{1} d(\cos\vartheta_q)$$

$$\times \int_0^{2\pi} d\varphi_q \, \frac{g(q_1, q_2)}{(D - E\cos\varphi_q + i\epsilon)\,(A + B\cos\varphi_q - i\epsilon)}$$

Singularities at $|\boldsymbol{q}_2| = p_0$

and $|D| = E$ for p_1, p_2, $|A| = B$ for p_3, p_4

Colliding singularities: $p_3 = p_1 \approx p_0$ for $q \approx 2\omega_L/c$

——xx——-x————-x——$\longrightarrow$ $|\boldsymbol{q}_2|$

$p_0 p_3 p_1$ p_4 p_2

$A_{fi}^{\rm box} \sim \ln|p_3 - p_1|$ (but S is finite)

Remedy: Higher-order theory

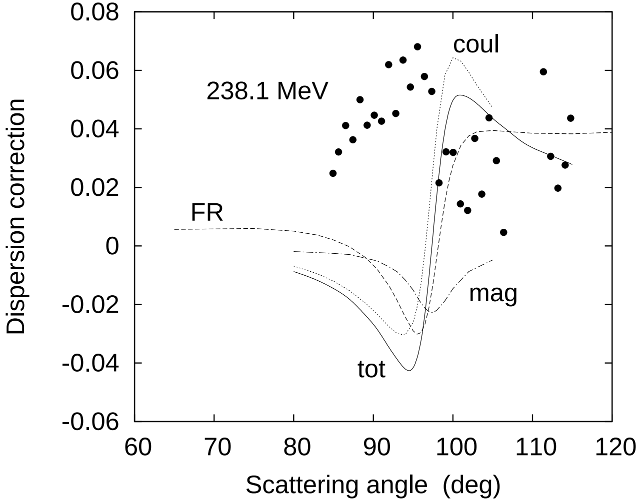
Summary

Importance of magnetic contribution to dispersion:
Strong influence in excitation of specific nuclear states
Mutual cancellation in sum over excited states

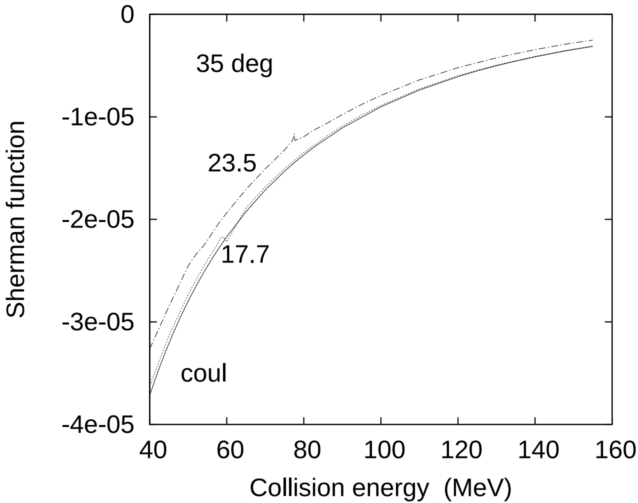
Energy dependence of dispersion in first diffraction minimum:
Increase with collision energy
Induced predominantly by Coulombic excitation (at $E_i \gtrsim 350$ MeV)

Test of dispersion theory by experimental data:
Closure approximation: Underestimates dispersion particularly at high energies
Restriction to three prominent excited states: Qualitative agreement with data

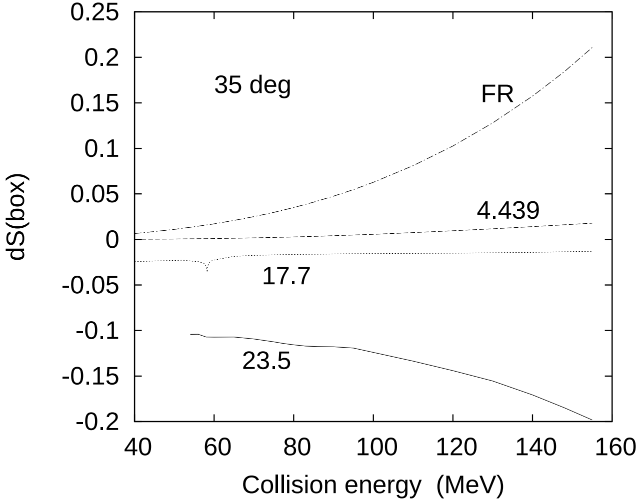
Thank you!



0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
60
70
80
90
100
110
120
Dispersion correction
Scattering angle (deg)
238.1 MeV
coul
FR
mag
tot



Sherman function
Collision energy (MeV)
0
-1e-05
-2e-05
-3e-05
-4e-05
40
60
80
100
120
140
160
35 deg
23.5
17.7
coul


35 deg
FR
4.439
17.7
23.5
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
40
60
80
100
120
140
160
Collision energy (MeV)
dS(box)