ECT* 18th–22nd July 2022

Radiative corrections from medium to high energy experiments

The McMule framework for NNLO QED calculations

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McMule

Monte Carlo for MUons and other LEptons

https://mule-tools.gitlab.io

P. Banerjee, A. Coutinho, T. Engel, A. Gurgone, F. Hagelstein, S. Kollatzsch, L. Naterop, A. Proust, M. Rocco, N. Schalch, V. Sharkovska, A. Signer, Y. Ulrich

- fixed-order QED corrections for (leptonic) scattering and decay processes

  some NLO, most NNLO, toying with N^{3}LO, but no TPE and no parton shower (yet)
- fully differential Monte Carlo integrator
- really NNLO, i.e. make no approximation (nearly possible ...)
- whenever possible make use of progress made for QCD @ LHC

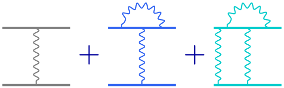
QCD @ LHC	$\Leftrightarrow$	QED @ low & medium energy	
non-abelian	$\gtrsim$	abelian	matrix elements somewhat easier
non-abelian	$\gg$	abelian	IR structure much easier ①
massless fermions	$\ll$	massive fermions	loop amplitudes much harder ②
jets	$<$	exclusive w.r.t. collinear radiation	numerics harder $\supset \log(m^2/Q^2) \equiv L$ much harder for small masses ③

stealing from QCD

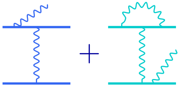
- master integrals (reduction and computation), automated tools, EFT methods
- use dimensional regularisation for IR singularities, not photon mass
- use subtraction method for phase-space integration, not slicing method
- for the future: match fixed-order result to parton shower

physical $(2 \to 2)$ cross section (e.g. Møller)

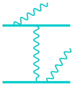
$$\sigma = \int d\Phi_2 \Big| \quad + \quad + \quad + \dots \Big|^2$$



$$+ \int d\Phi_3 \Big| \quad + \quad + \dots \Big|^2$$



$$+ \int d\Phi_4 \Big| \quad + \dots \Big|^2$$



$$+ \dots$$

challenges

1. fully differential phase-space integration

⇒ FKS$^\ell$

2. virtual amplitudes with massive particles

⇒ one-loop: OpenLoops

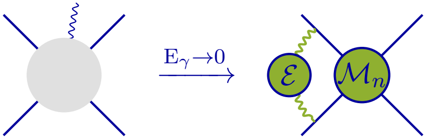
⇒ two-loop: massification

3. numerical instabilities due to pseudo-singularities

⇒ next-to-soft stabilisation

① FKS$^\ell$

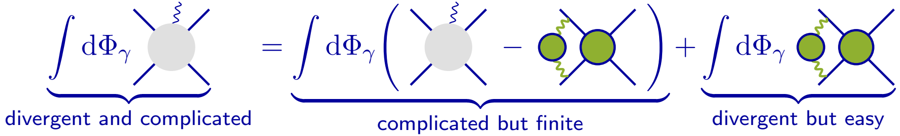
only soft singularities



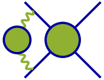
$$\mathcal{M}^{(\ell)}_{n+1} = \mathcal{E}\,\mathcal{M}^{(\ell)}_{n} \;+\; \mathcal{O}(E_\gamma^{-1})$$

$$\text{eikonal } \mathcal{E} = \sum_{i,j} \frac{p_i\cdot p_j}{(p_\gamma\cdot p_i)\;(p_\gamma\cdot p_j)} \sim \mathcal{O}(E_\gamma^{-2})$$

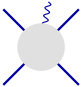
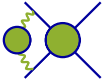
$\Rightarrow$ subtraction scheme (FKS$^\ell$)



divergent but easy $\quad E_\gamma \simeq \xi_1\, Q$; auxiliary unphysical parameter $0 < \xi_c \le 1$

$\int \mathrm{d}\Phi_\gamma$  $\sim \int_0^{\xi_c} \mathrm{d}\xi_1\, \xi_1\, \frac{1}{\xi_1^2} \xrightarrow{\text{dim. reg.}} \int_0^{\xi_c} \mathrm{d}\xi_1\, \xi_1^{-1-\epsilon} \Rightarrow \underbrace{\hat{\mathcal{E}}(\xi_c)}_{1/\epsilon}\, \mathcal{M}_n^{(0)}$

complicated but finite

$\int \mathrm{d}\Phi_\gamma \Bigg($  $-$  $\Bigg) = \int \mathrm{d}\Phi_{n+1}^{d=4} \left(\frac{1}{\xi_1}\right)_c \left(\xi_1\, \mathcal{M}_{n+1}^{(0)}\right)$

$$\int_0^1 \mathrm{d}\xi_1 \left(\frac{1}{\xi_1}\right)_c f(\xi_1) \equiv \int_0^1 \mathrm{d}\xi_1\, \frac{f(\xi_1) - f(0)\theta(\xi_c - \xi_1)}{\xi_1}$$

subtraction scheme

we do not write $\sigma_n^{(1)} = \sigma_n^{(v)}(\lambda) + \sigma_n^{(s)}(\lambda,\omega) + \sigma_{n+1}^{(h)}(\omega)$ photon mass λ, resolution ω

we do write $\sigma_n^{(1)} = \sigma_n^{(1)}(\xi_c) + \sigma_{n+1}^{(1)}(\xi_c)$

$$\sigma_n^{(1)}(\xi_c) = \int \mathrm{d}\Phi_n^{d=4} \Bigg(\underbrace{\mathcal{M}_n^{(1)}}_{1/\epsilon} + \underbrace{\hat{\mathcal{E}}(\xi_c)}_{1/\epsilon} \mathcal{M}_n^{(0)} \Bigg) = \int \mathrm{d}\Phi_n^{d=4} \underbrace{\mathcal{M}_n^{(1)f}(\xi_c)}_{\text{finite}}$$

$$\sigma_{n+1}^{(1)}(\xi_c) = \int \mathrm{d}\Phi_{n+1}^{d=4} \left(\frac{1}{\xi_1}\right)_c \left(\xi_1 \mathcal{M}_{n+1}^{(0)f}\right)$$

the ξ_c dependence cancels between the two terms (implementation/stability check)

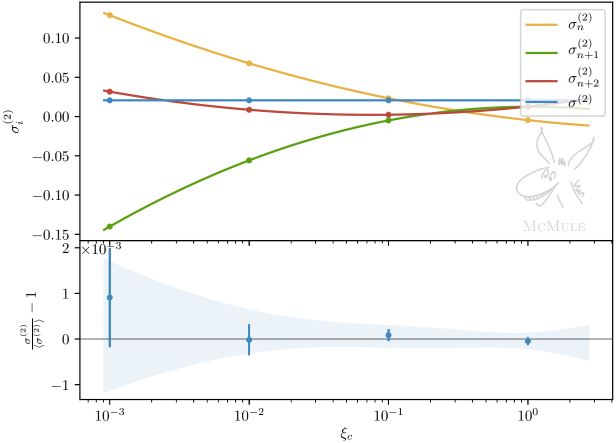
FKS$^{\ell=2}$ "double virtual", "real-virtual' ", "double real"

$$\sigma_n^{(2)}(\xi_c) = \int \, \mathrm{d}\Phi_n^{d=4} \Big(\underbrace{\mathcal{M}_n^{(2)}}_{1/\epsilon^2} + \underbrace{\hat{\mathcal{E}}(\xi_c)}_{1/\epsilon} \underbrace{\mathcal{M}_n^{(1)}}_{1/\epsilon} + \frac{1}{2!}\mathcal{M}_n^{(0)} \underbrace{\hat{\mathcal{E}}(\xi_c)^2}_{1/\epsilon^2} \Big) = \int \, \mathrm{d}\Phi_n^{d=4} \underbrace{\mathcal{M}_n^{(2)f}(\xi_c)}_{\text{finite}}$$

$$\sigma_{n+1}^{(2)}(\xi_c) = \int \, \mathrm{d}\Phi_{n+1}^{d=4} \left(\frac{1}{\xi_1}\right)_c \left(\xi_1 \, \mathcal{M}_{n+1}^{(1)f}(\xi_c)\right),$$

$$\sigma_{n+2}^{(2)}(\xi_c) = \int \, \mathrm{d}\Phi_{n+2}^{d=4} \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \left(\xi_1 \xi_2 \, \mathcal{M}_{n+2}^{(0)f}\right)$$

simple generalisation [Engel,AS,Ulrich,1909.10244], QED eikonal does not get further corrections consistency/implementation/stability check here e.g. for $\mu\, e \to \mu\, e$ ("mixed" = TPE)



- ξ_c (in)dependence
- no approximations made (subtraction not slicing)
- in principle any ξ_c is ok
- use 'good' $\xi_c \sim 10^{-1}$ for actual runs (no large cancellations)

FKS$^\ell$ for N$^\ell$LO

YFS: $$e^{\hat{\mathcal{E}}} \sum_{\ell=0}^{\infty} \mathcal{M}_n^{(\ell)} = \sum_{\ell=0}^{\infty} \mathcal{M}_n^{(\ell)f}$$

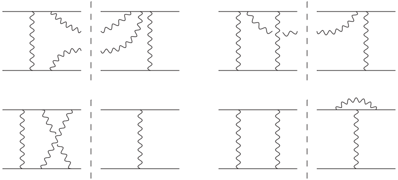
$$\sigma^{(\ell)} = \sum_{j=0}^{\ell} \sigma_{n+j}^{(\ell)}(\xi_c) \quad \text{with} \quad \sigma_{n+j}^{(\ell)}(\xi_c) = \int \mathrm{d}\Phi_{n+j}^{d=4} \left(\prod_{i=1}^{j} \left(\frac{1}{\xi_i}\right)_c \xi_i \right) \mathcal{M}_{n+j}^{(\ell-j)f}(\xi_c)$$

in principle: if we have all matrix elements, we can get fully differential cross sections at any order in α → ①

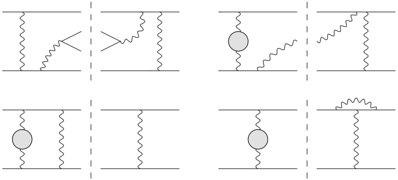
in practice: wishful thinking ... → ② and ③

- ready for a full (e.g. Møller) NNLO calculation photonic and fermionic contributions
- compute double-real amplitudes
- use OpenLoops [Buccioni, Pozzorini, Zoller] for real-virtual amplitudes, numerical stability → ③
- ③ → apply next-to-soft stabilisation
- massive two-loop integrals not all known → ②
- ② → massify massless two-loop amplitudes [Bern,Dixon,Ghinculov] (and one-loop squared)
- use FKS^2 (open e^+e^- production not yet included)
- let the mule trot [McMule, 2107.12311]

photonic



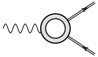
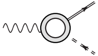
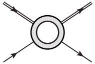
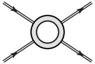
fermionic/hadronic



② loops with masses

- scales (e.g. masses) are the enemy of loop-integral calculators
- for one-loop amplitudes we use OpenLoops, remarkable numerical stability
- but massive two-loop integrals for $2 \to 2$ are not all known

[here should go a list of an army of loop-calculating theoreticians … sorry]

$\gamma^* \to ee$		✔ full m dependence	harmonic (M)PL	~ 2004
$\mu \to e\nu\bar{\nu}$		✔ full m_1 and m_2	generalised (M)PL	2018
$\mu e \to \mu e$		✔ full m_1, $m_2 = 0$	generalised (M)PL	2021
$ee \to ee$		✗ only planar	elliptic MPL	2022

multiple polylogarithms (MPL)

simple loop integrals, one scale $z \Rightarrow$ logs and dilogs = polylogs

$$\mathrm{Li}_1(z) = -\log(1-z) \quad \text{and} \quad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(z)$$

more complicated loop integrals, many scales $a_1 \dots a_n, z \Rightarrow$ multiple polylogs

$$G(a_1 \dots a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1}\, G(a_2 \dots a_n; z)$$

for $a_i \in \{-1, 0, 1\}$: harmonic polylogs (HPL) [Remiddi, Vermaseren]

for generic a_i: generalised polylogs (GPL) [Goncharov]

for Monte Carlo need fast (Fortran) numerical evaluation [handyG 1909.01656]

sadly, this is not the end $\Rightarrow$ elliptic integrals ...

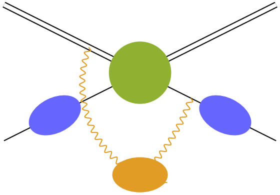
collinear factorization ⇒ massification [Penin; Becher, Melnikov; Engel, Gnendiger, AS, Ulrich]

get leading mass effects based on massless loops, miss terms $\lim_{m\to 0} = 0$ e.g. m^2/Q^2

process e.g $\mu e \to \mu e$ at NNLO $\mathcal{A}(m) = \mathcal{S} \times Z \times Z \times \mathcal{A}(0) + \mathcal{O}(m) \supset \{1/\epsilon^2, L^2\}$

based on factorisation, SCET, and method of regions [Beneke, Smirnov]

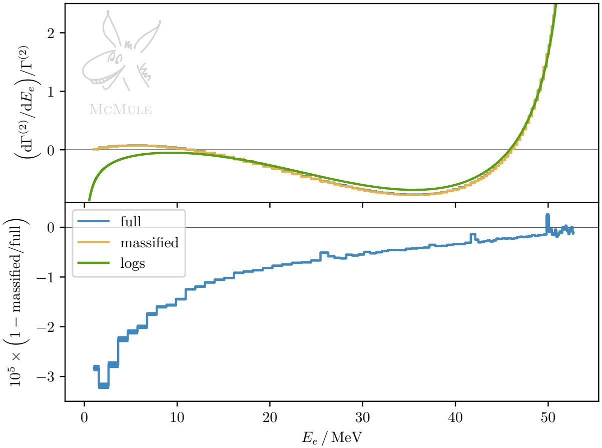
soft: process dependent
in QED $S = 1 +$ fermion loops
$\to$ compute separately with full m dependence



collinear: universal
'converts' $1/\epsilon \to L$

hard: massless electron $\mathcal{A}(0) \sim 1/\epsilon^4$

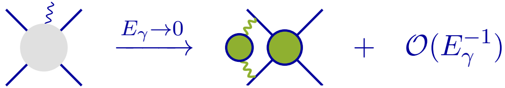
test massification α^2 coefficient of positron energy spectrum in muon decay



- exact result available
  [McMule 1909.10244]
- error in massification
  few % × NNLO
  few $\cdot\, 10^{-2} \times (\alpha/\pi)^2 L^2$
- in agreement with estimate
  $\sim (\alpha/\pi)^2\, m_e^2/m_\mu^2 \times L^2$
- about same size as α^3 contribution

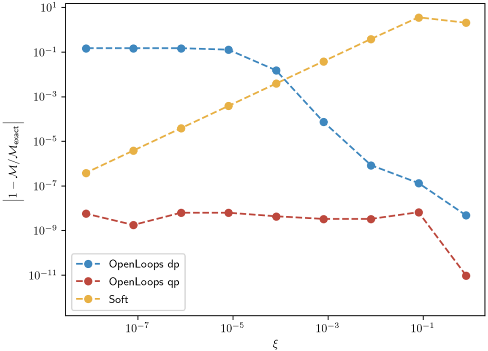
③ next-to-soft stabilisation

real-virtual corrections trivial in principle, extremely delicate numerically



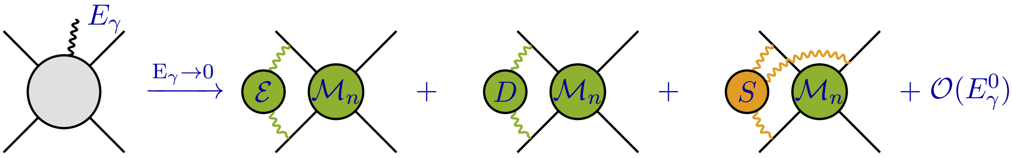
- soft limit (of collinear emission) $E_\gamma = \xi\,\sqrt{s}/2$
- Bhabha scattering (as example)

  [McMule, 2106.07469]
- M_{exact} Mathematica expression
- full M vs soft limit
- stability problem



③ next-to-soft limit

extend LBK theorem [Low 1958; Burnett, Kroll 1968] to one-loop [Engel, AS, Ulrich, 2112.07570]



eikonal approx.
$1/E_\gamma^2$
leading soft
$\exists$ @ tree level

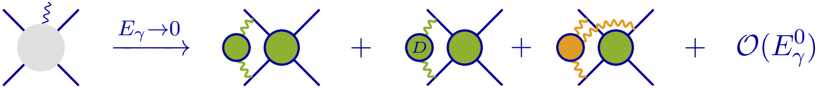
LBK diff. op.
$1/E_\gamma$
$\exists$ @ tree level

soft contribution
$1/E_\gamma$
new at one loop

next-to-soft approximation (NTS)

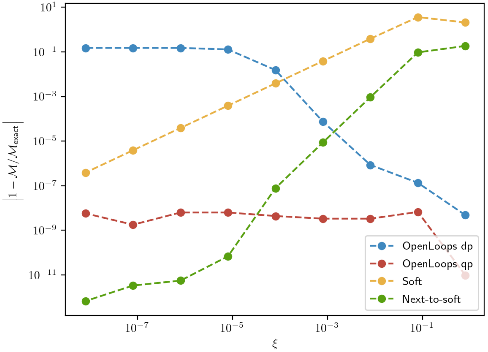
use again EFT (HQEFT) and method of region, hard and soft

can be extended beyond NLO (almost certainly) but not (universally) beyond NTS real-virtual corrections trivial in principle, extremely delicate numerically

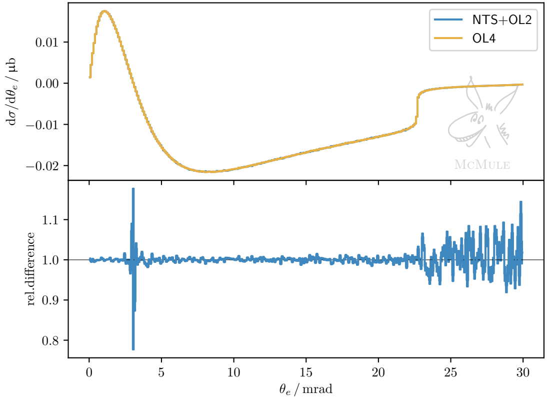


- soft limit (of collinear emission)
- Bhabha scattering (as example)

  [McMule, 2106.07469]
- M_{exact} Mathematica expression
- full M vs next-to-soft limit
- stability problem solved



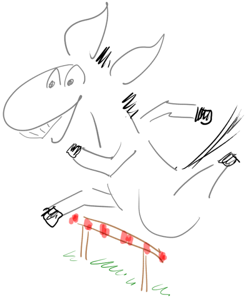
test next-to-soft stabilisation vs OL4 (OpenLoops quad) for $\mu e \to \mu e$ real-virtual



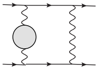
- e.g. $\xi_c = 10^{-4}$
- for $\theta_e > 23\,\mathrm{mrad}$ few events
- same statistics, same result
- 70 days vs 4 days
- integrated results for $\xi_c = 10^{-4/5/6}$:

NTS	OL4
$-0.29268(4)$	$-0.29267(4)$
$-0.44789(6)$	$-0.44778(6)$
$-0.64662(9)$	$-0.64649(9)$

final hurdles



a few more hurdles to jump over

computation of VP e.g.  $\mathcal{M}_{\rm vp} = \int [{\rm d}\ell] \frac{\Pi(\ell^2)}{\ell^2} \tilde{\mathcal{M}}_{\rm vp}$

dispersive $\mathcal{M}_{\rm vp}^{\rm had} = \frac{\alpha}{3\pi} \int_{4m_\pi^2}^{\infty} \frac{{\rm d}z}{z} R^{\rm had}(z) \int [{\rm d}\ell] \frac{\tilde{\mathcal{M}}_{\rm vp}}{\ell^2 - z}$

[Cabbibo, Gatto 1961; Ritbergen et al, Davydychev et al, Actis et al, Kuhn et al, Carloni Calame et al. . . .]

hyperspherical $\mathcal{M}_{\rm vp} = -\int {\rm d}Q^2 \Pi(-Q^2) \int \frac{{\rm d}\Omega}{2\pi^2} \tilde{\mathcal{M}}_{\rm vp}$

[Levine, Roskies 1974; Laporta, Fael]

collinear pseudo-singularities $\lim_{\to 0} \sphericalangle(p_\gamma, p_i) \ \Rightarrow \ L$

partitioning of phase space $\Rightarrow$ at most one small angle per region

tune phase space $\Rightarrow$ [Vegas] $x_i \ \leftrightarrow \ \cos\sphericalangle$

Outlook

the NNLO era is here, not only for QCD, also for QED

to tackle state-of-the-art problems, a modern approach to QED is required

future steps

- add TPE for $\ell p \to \ell p$ (done for pointlike p)
- NNNLO 'form factor' contributions $\gamma^* \to \ell^+ \ell^-$ (MUonE)
- combine fixed-order QED with (YFS?) parton shower
- combine fixed-order QED with electroweak
- polarised leptons
- generally towards higher energies (mainly numerical)
- integrator $\to$ generator (desired ??)

McMule