Two-Photon Exchange in Chiral Perturbation Theory. [1]

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[1] Presented at Workshop in Trento July 2022

The MUSE precision experiment at PSI has
beam momenta of the order of the muon mass.

Furthermore, we know that at MUSE energies the muons are not radiating the bremsstrahlung photons according to the "peaking" approximation.

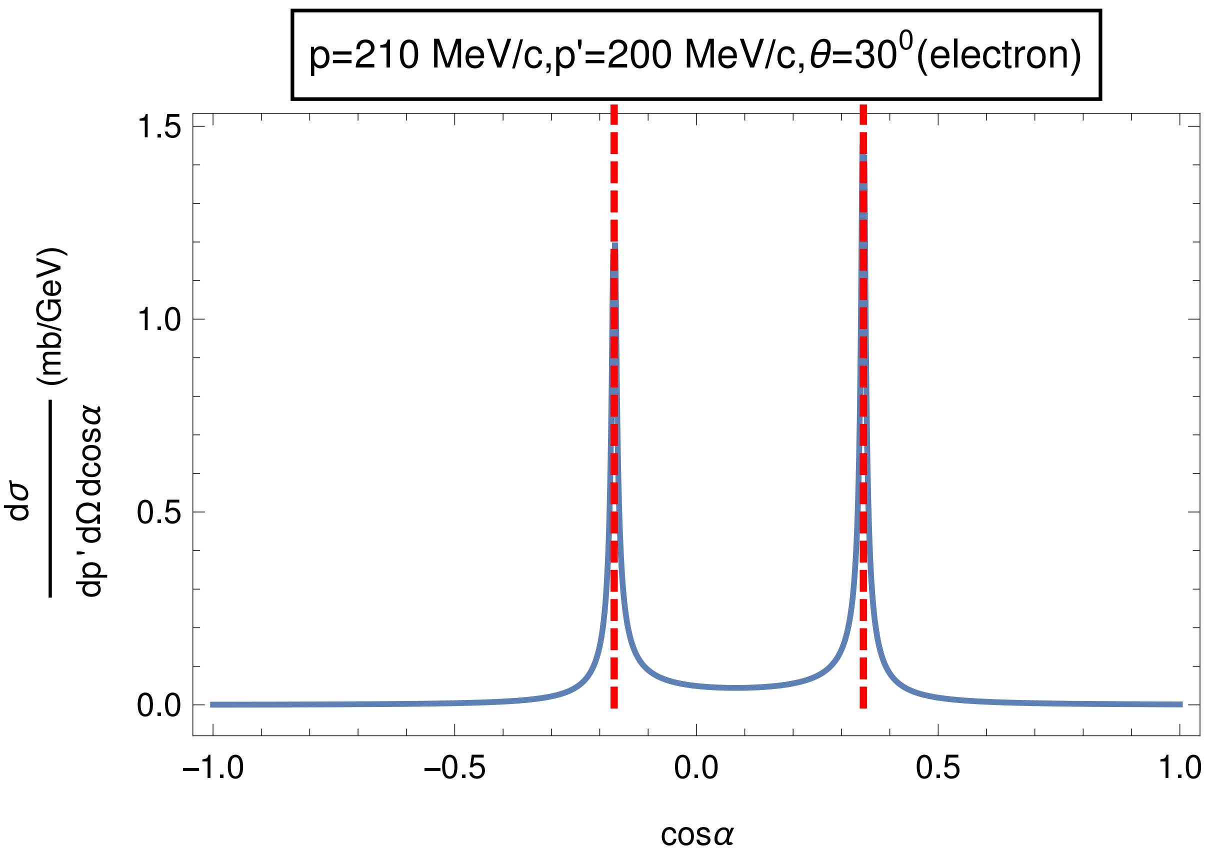
The low-energy lepton proton elastic scattering studied by MUSE require the masses of the relativistic leptons to be non-zero.

Leptons are treated relativistically
Nucleon is a point particle

Needed radiation corrections $\delta_{2\gamma}$ have been completed and published: for soft photon approximation.

$$\left[\frac{\mathrm{d}\sigma_{el}^{(\mathrm{NLO})}(Q^2)}{\mathrm{d}\Omega_l'}\right]_\gamma = \left[\frac{\mathrm{d}\sigma_{el}(Q^2)}{\mathrm{d}\Omega_l'}\right](1+\delta_{2\gamma})$$

All evaluations are done in the lab. frame, i.e. $\vec{p}_p = 0$.



The <u>electron</u> beam shows clear "peaking" radiation.
The two radiation-peaks occur for the photon angles α which <u>coincide</u> with the <u>incoming</u> or <u>outgoing</u> electron direction.
The momentum transfer $Q = p - p'$ is <u>well defined</u> for <u>electron</u> scattering.

Why Heavy Baryon Chiral Perturbation Theory?

(1): At the relatively small MUSE momenta the recoil of the point-like proton is moving non-relativistically.

(2): At these low energies we keep the lepton mass in all our expressions.

(3): The proton mass, $m_p \sim 1$ GeV is large compared to lepton momenta, and $m_p \sim \Lambda_\chi \sim 4\pi f_\pi$, which is the chiral scale.

Standard heavy baryon chiral perturbation theory (χPT) has the expansion of the Lagrangian in powers of m_p^{-1} counted with the chiral expansion in p/Λ_χ.

χPT is a low-energy model-independent theory where gauge invariance is naturally included.

We evaluate all radiative corrections for elastic lepton-proton scattering using χPT, which is an effective low-energy field theory of QCD.

The effective hadronic Lagrangian $\mathcal{L}_\chi$ is expanded in increasing chiral order.

$$\mathcal{L}_\chi = \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(2)}_{\pi\pi} + \cdots \ ,$$

The leading order (LO) Lagrangian is:

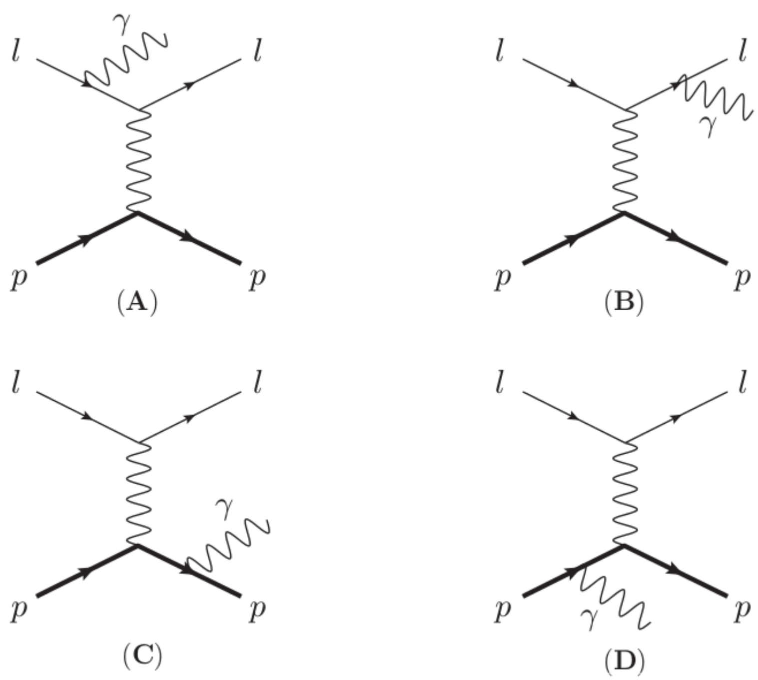
$$\mathcal{L}^{(1)}_{\pi N} \ = \ \bar{\psi}_N (iv \cdot D + g_A S \cdot u)\psi_N$$

Our nucleon velocity is $v^\mu = (1, \vec{0})$,
which gives the covariant nucleon spin $S^\mu = (0, \frac{1}{2}\vec{\sigma})$.

We evaluate the LO and next-to-leading order (NLO) contributions.

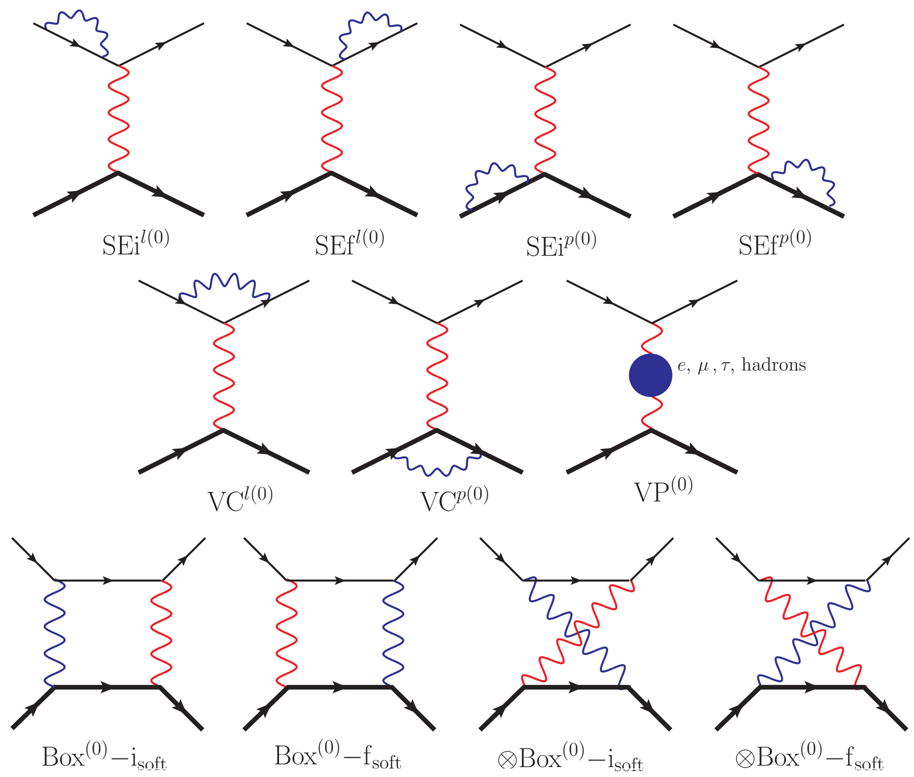
The pion degrees of freedom <u>do not enter</u> at this order,
i.e., the pion Lagrangian $\mathcal{L}^{(2)}_{\pi\pi}$ enter only at NNLO.

In χPT the pion-loops build-up the proton form factor
(the charge r.m.s. radius!)
at NNLO.



Only diagrams (A) and (B) contribute to the LO bremsstrahlung cross section in the Coulomb gauge.

At leading order (LO) the proton does not radiate.



Radiative corrections to elastic lepton proton scattering at LO.
The vacuum polarization blob diagram has leptonic and hadronic contributions.

(1): At LO there are essentially no radiative proton corrections at LO (more later).
(2): The Two-Photon-Exchange (TPE) contribution is zero at LO in soft photon approximation.

In essence, at LO in χPT
the lepton scatters off a static Coulomb potential.

A few comments on the LO evaluations of $\delta_{2\gamma}^{(0)}$:

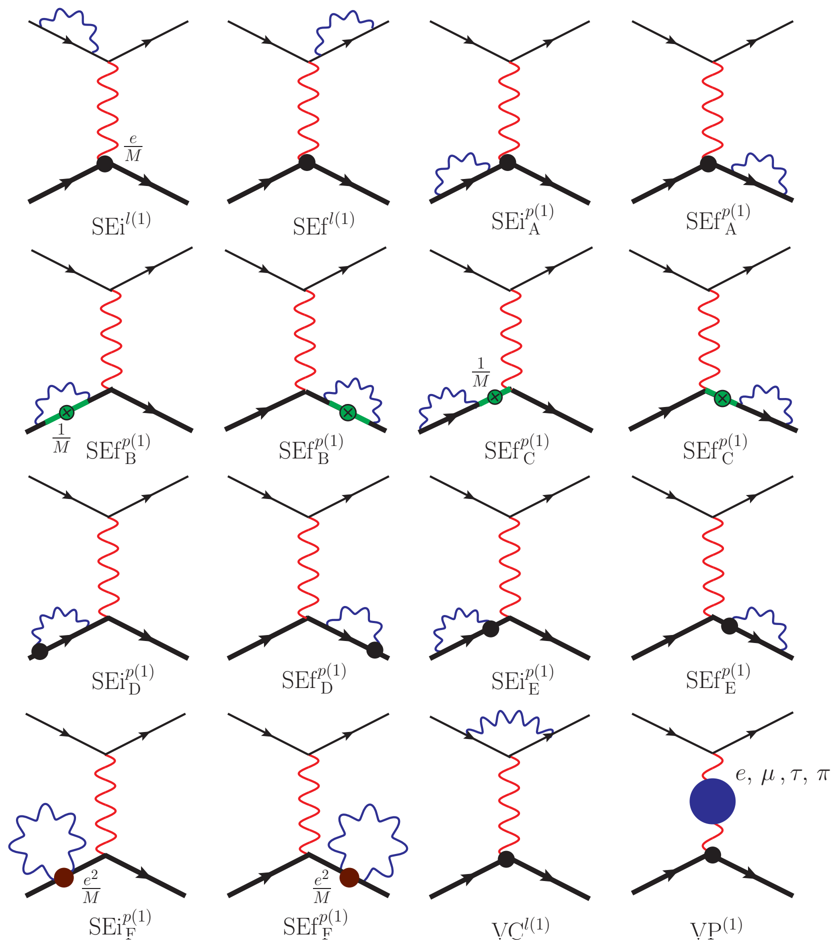
- **All radiative corrections vanish in the limit $Q^2 \to 0$.**
- **The lepton Dirac form factor, F_1, is almost independent of the beam momentum.**
- **The lepton Pauli form factor, F_2, depends strongly on the beam momentum.**
- **Whereas the bremsstrahlung correction is negative, the vacuum polarization and the vertex corrections are positive.**

Next we will present the NLO radiative corrections.

Some parts of the NLO Lagrangian $\mathcal{L}_{\pi N}^{(2)}$ will contribute.

$$\mathcal{L}_{\pi N}^{(2)} \;=\; \frac{1}{m_N}\bar{\psi}_N[(v\cdot\mathcal{D})^2 - \mathcal{D}\cdot\mathcal{D} + \cdots]\psi_N\,,$$

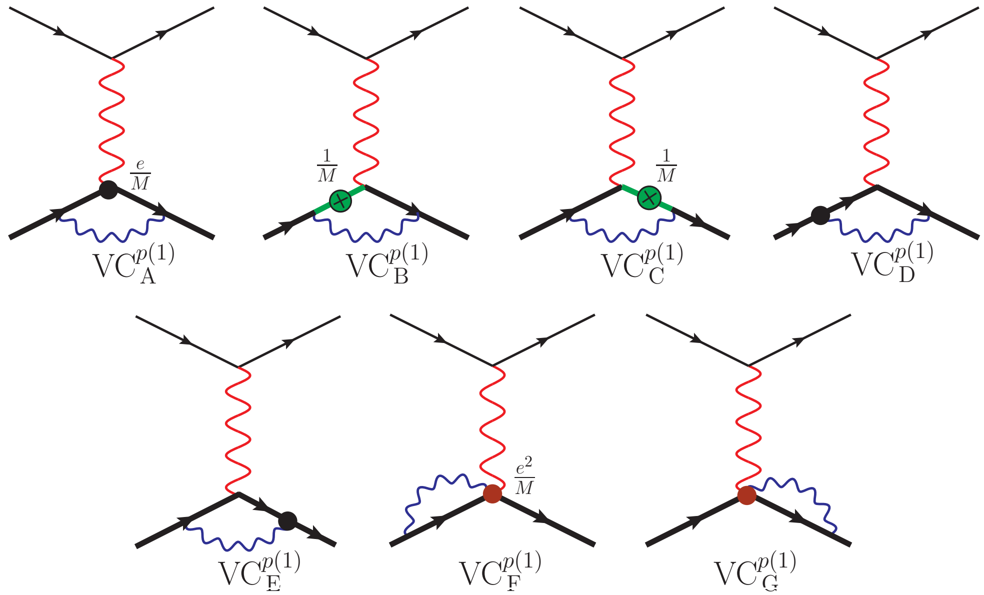
where for the non-relativistic proton $\mathcal{D}_\mu = \partial_\mu + eA_\mu$.



Some NLO contributions to lepton proton elastic scattering.

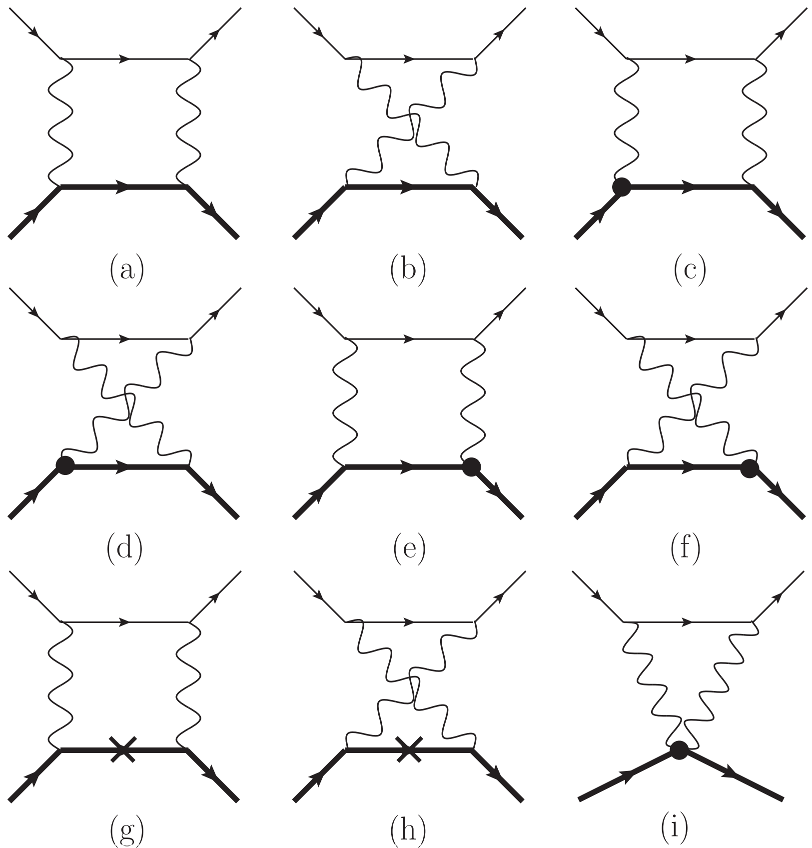
The NLO photon-nucleon vertices (filled circles) are from $\mathcal{L}^{(2)}_{\pi N}$.

<u>Only</u> the four lepton diagrams in *dimensional regularizations* are <u>non-zero</u>. However, since they <u>interfere</u> with the LO elastic lepton-proton scattering amplitude, they are $\mathcal{O}(m_p^{-2})$.



The proton vertex corrections at NLO are displayed.

In *dimensional regularizations* **we show that these diagrams give** **zero** **contributions.**



The two-photon exchange contributions.

Diagrams (a) and (b) contain both LO and NLO contributions since the proton propagator has, e.g., a $v \cdot p_p' \approx (\vec{p}_p'){}^2/(2m_p)$ term.

The other seven diagrams give NLO contributions.

The TPE diagrams at LO and NLO are evaluated.

We do not use the soft photon approximation.

First, we derive the LO χPT result from diagrams (a) and (b).

We find the LO Feshbach McKinley term to be

$$\delta_a + \delta_b \;=\; -\frac{\pi\alpha Q^2}{2E\sqrt{-Q^2}}\left[\frac{1}{1+\frac{Q^2}{4E^2}}\right]$$

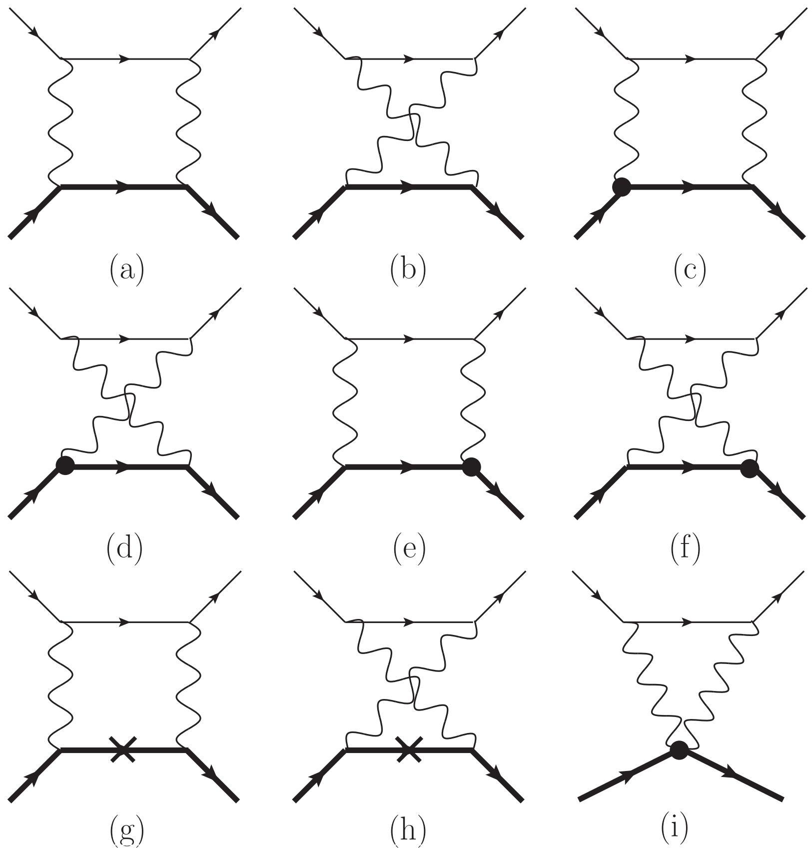
We expand a la Feshbach and McKinley when $Q \ll 2E$ and $Q^2 \approx -\vec{Q}^{\,2}$ to find:

$$\delta_a + \delta_b \;=\; \frac{\pi\alpha|\vec{Q}|}{2E}\left[1+\frac{\vec{Q}^{\,2}}{4E^2}+\cdots\right] = \delta_{Feshbach} + \cdots$$

Note that $v \cdot Q = E - E' = \mathcal{O}(\frac{1}{m_p})$.

Second, regarding the NLO contributions a few observations regarding the divergent parts in ***dimensional regularization*** **are pertinent.**

At NLO we consider the IR-divergences of four diagrams (a), (b), (g) and (h).



We <u>do not</u> use the soft photon approximation.

Diagram (i) was evaluated exactly (no approximations).
It has no divergences.
This diagram gives a finite contribution to the two-photon-exchange correction.

By adding the divergent parts of diagrams (a), (b), (g) and (h) , we find:

$$(\delta_a+\delta_b+\delta_h+\delta_g)_{div} = \frac{8\pi\alpha E}{m_p}\left[\frac{Q^2}{Q^2+4E^2}\right](2p-Q)\cdot\left\{I_1^{(-)}(p',0|1,0,1,2)_{div} - I_1^{(+)}(p,0|1,0,1,2)_{div}\right\} + \mathcal{O}(m_p^{-2}) \,.$$

We only consider the divergences of the real parts of the integrals.

The difference of the two TPE integrals above is:

$$\begin{aligned}(2p-Q)_\mu\cdot I_T^\mu &= (2p-Q)_\mu\cdot\left\{\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu}{[k^2+i\eta][(k-p')^2-m_l^2+i\eta][v\cdot k+i\eta]^2}\right. \\ &- \left.\frac{1}{i}\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu}{[k^2+i\eta][(k+p)^2-m_l^2+i\eta][v\cdot k+i\eta]^2}\right\}\end{aligned}$$

Observe!

In the soft-photon approximation when either $k\to 0$ or $k\to Q$ the factor $(2p-Q)\cdot k$ is zero.

Provided the bremsstrahlung process will cancel this IR divergent terms when the soft photon approximation is not implemented, we have derived an analytic expression for two-photon exchange in χPT.

This analytic expression include both soft and hard photon exchanges.

$$(2p-Q)_\mu \cdot I_T^\mu = \frac{(1-\epsilon)}{(4\pi)^2}\frac{\Gamma(-\epsilon)}{(4\pi\mu^2)^\epsilon}\Big[4E\,X_{even} + \left[4E^3(1+\beta^2) - E(4m_l^2 - Q^2)\right]\,Y_{even} + \\ + \left[8E^2 - 4m_l^2 + Q^2\right]\,\left[Z_{odd} + Z'_{odd}\right]\Big]\,.$$

The factor in front of the square bracket gives the IR-divergence, where the spacial dimension is $d = 4 - 2\epsilon$.

The preliminary real parts are as follows (where $p = |\vec{p}\,|$):

$$\begin{aligned} X_{even} &= -\frac{2}{(1-\epsilon)p} ln\sqrt{\frac{1+\beta}{1-\beta}} + \frac{2\epsilon}{(1-\epsilon)p}\Big[-\frac{\pi^2}{2} - Li_2\left(\frac{2\beta}{1+\beta}\right) + ln\sqrt{\frac{1+\beta}{1-\beta}} - ln^2\sqrt{\frac{1+\beta}{1-\beta}} \\ &- ln(m_l^2) ln\sqrt{\frac{1+\beta}{1-\beta}}\Big] + \mathcal{O}(\epsilon^2) \\ Y_{even} &= -\frac{1}{(1-\epsilon)p^3}\left[\frac{\beta}{1-\beta^2} - ln\sqrt{\frac{1+\beta}{1-\beta}}\right] - \frac{\epsilon}{(1-\epsilon)p^3}\Big[\frac{2\beta}{1-\beta^2} - ln^2\sqrt{\frac{1+\beta}{1-\beta}} \\ &- Li_2\left(\frac{2\beta}{1+\beta}\right) + ln(m_l^2)\left(\frac{\beta}{1-\beta^2} - ln\sqrt{\frac{1+\beta}{1-\beta}}\right) + \frac{\pi^2}{6}\Big] + \mathcal{O}(\epsilon^2) \\ Z'_{odd} &= \frac{1}{(1-\epsilon)m_l^2} + \frac{\epsilon}{(1-\epsilon)m_l^2} ln(m_l^2) + \mathcal{O}(\epsilon^2) \quad \text{and} \quad Z_{odd} = 0 + \mathcal{O}(\epsilon^2)\,, \end{aligned}$$

We subtract the following IR-divergent part from the above expression to obtain
the finite NLO two-photon-exchange correction
from diagrams (a), (b), (g) and (h):

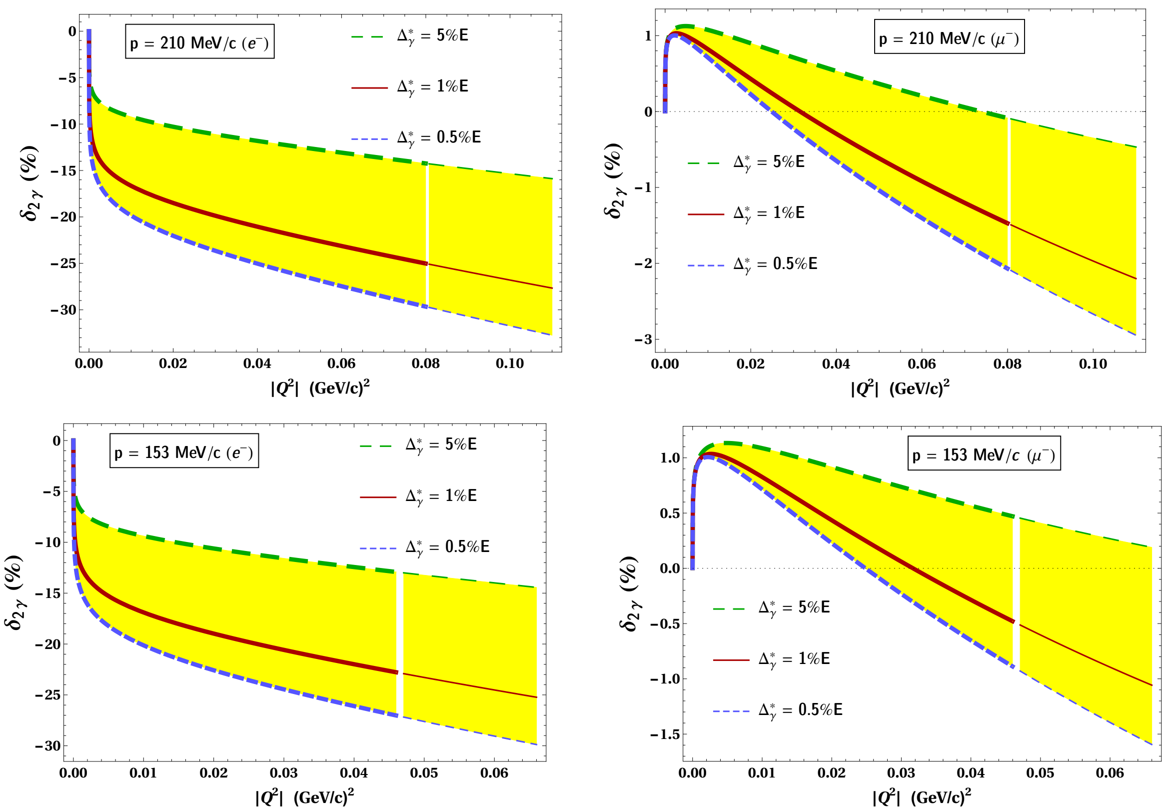
$$[(2p-Q)_\mu I_T^\mu]_{div} \; = \; \frac{1}{(4\pi)^2}\frac{\Gamma(-\epsilon)}{(4\pi\mu^2)^\epsilon}\left[\frac{Q^2}{\beta^3 E^2} ln\sqrt{\frac{1+\beta}{1-\beta}} - \frac{4\beta^3E^2+Q^2}{\beta^2E^2}\right]$$

In addition, we include the finite contribution from the "seagull" diagram, diagram (i),
which has no IR-divergence.

Finally, we have analytic finite contributions from the other four two-photon-exchange diagrams.

<u>If</u> we use the <u>soft photon approximation</u>
the NLO corrections $\delta_{2\gamma}^{(1)}$ in χPT, we showed in PRD <u>104</u> that:

- **The NLO TPE <u>does depend</u> on the lepton charge.**
  The total correction $\delta_{2\gamma} = \delta_{2\gamma}^{(0)} + \delta_{2\gamma}^{(1)}$ is smaller for ℓ^+-p than the ℓ^--p scattering.
- **Both the TPE and the bremsstrahlung contributions are roughly linear with increasing Q^2.**
- **Radiative corrections are evaluated at LO and NLO.**
- **We have only one free parameter in our evaluations:**
  the upper limit of the *soft (undetected) photons* **in the lab frame is $\sim \Delta_\gamma^*$.**
- **The parameter Δ_γ^* depends on the detector sensitivity.**



The lower limit of the photon energy detection $\sim \Delta_\gamma^*$.

Results are found in the soft photon approximation.

The (yellow) bands are the variation of the *lab*-frame detector resolution when $0.5\,\% < \Delta_\gamma^* < 5\,\%$ of the incident lepton energy E.

The evaluated lepton vertex corrections give the lowest order QED Dirac form factors, $F_1(q^2)$ and $F_2(q^2)$, at small q^2 including the lepton mass dependence.

We use the Ward identity for F_1, and at $Q^2 = 0$, we find the expected lepton gyromagnetic moment

$$g \; = \; 2 + 2F_2(Q^2 = 0) = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)$$

For $q^2 < 0$ and $-q^2 \gg m_l^2$ we find the usual expression for $F_2^{loop}(q^2)$. In the soft photon approximation the IR divergences of all the radiative photon-loop corrections do cancel the IR divergences of the bremsstrahlung diagrams.

At $\mathcal{L}_{\pi N}^{(2)}$ two photons can couple to the heavy proton in one vertex, i.e., we have a TPE triangle diagram and a similar bremsstrahlung diagram. Both can be evaluated exactly and give finite contributions to the radiative cross section.

All the UV divergences require the usual counter-terms in the Lagrangian.

Summary

We have derived within χPT the analytic expression for the two-photon exchange correction.
It includes both soft and hard photon exchanges.

The electron behaves relativistically.
The "peaking approximation" and radiative corrections applies to electron scattering, except.

The exception occur when p' is of the order of 20 − 40 MeV/c.
The cross section increases rapidly with small p' values below 30 MeV/c.
This increase is due to the small electron mass.

Theoretically, as $m_e \to 0$ one could have a singularity in the diff. cross section at a finite p' value.

The PRD 104 article does not included the "hard" photon components in the TPE evaluations nor the "hard" bremsstrahlung contributions.

In χPT at NNLO the pion-loops, meaning the hadronic proton form factors including the proton r.m.s. radius, will contribute, as will all $\mathcal{O}(m_p^{-2})$ terms.