Radiative Corrections to Elastic Muon-Proton Scattering

Norbert Kaiser (TUM)

Radiative Corrections from medium to high energy experiments,
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- AMBER experiment (at CERN) to measure proton charge-radius
- Cross section $d\sigma/dt$ including radiative corrections of order α/π
- γ-loop form factors of muon and proton (including proton structure)
- Two-photon exchange box diagrams (including proton structure)
- Soft photon bremsstrahlung off muon and proton
- Results for AMBER and MUSE kinematics

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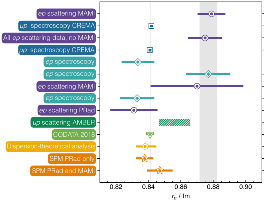
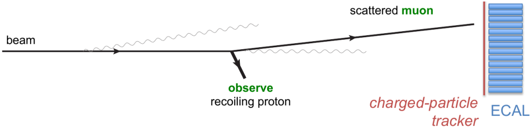
arXiv:nucl-th/2202.04409]

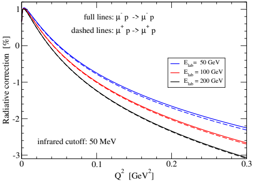
- Discrepancies in measurements of proton charge-radius $r_p = \sqrt{\langle r^2 \rangle_{ch}}$

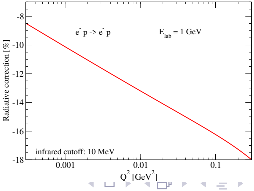


- AMBER: instead of electrons, high-energy muons scattering off protons



- Advantage of muons over light electrons: smaller radiative corrections





Differential cross section

- Introduce for process $\mu^-(k_1) + p(p_1) \to \mu^-(k_2) + p(p_2)$ dimensionless Mandelstam variables, $M = 938.272\,\text{MeV}$ proton mass

$$s = (p_1 + k_1)^2/M^2, \quad t = (k_1 - k_2)^2/M^2, \quad u = (p_1 - k_2)/M^2,$$
$$s + t + u = 2 + 2r, \quad r = (m_\mu/M)^2 = 1.2681 \cdot 10^{-2}$$

- Unpolarized differential cross section including corrections $\sim \alpha/\pi$

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha^2}{M^2 t^2 P}\Big\{H_0(1 + 2\Pi_{\rm vp} + \delta_{\rm soft}) + H_1 + H_2\Big\}, \qquad P = s^2 - 2s(1+r) + (1-r)^2$$

- H_0 generalized Rosenbluth formula: $G_{E,M}$ electric/magnetic form factors

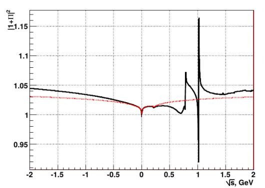
$$H_0 = \left[\frac{(s+1-r)^2}{4-t} + r - s\right](4G_E^2 - tG_M^2) + t\left(r + \frac{t}{2}\right)G_M^2$$

- $\Pi_{\rm vp}$ vacuum polarization in one-photon exchange
- $\delta_{\rm soft}$ correction factor from (undetected) soft photon radiation
- H_1 interference of 1γ-exch. with photon-loop form fact. of muon & proton
- H_2 interference of 1γ-exch. with planar & crossed 2γ-exch. box-diagram
- Cross section for $\mu^+(k_1) + p(p_1) \to \mu^+(k_2) + p(p_2)$ through $s \leftrightarrow u$ crossing
- H_0 and H_1 are even under $s \leftrightarrow u$, whereas H_2 is odd

Differential cross section

- Vacuum polarization due to two lightest leptons sufficient, $Q = \sqrt{-t}M$

$$\Pi_{\rm vp} = \frac{\alpha}{3\pi}\sum_{e,\mu}\left[\frac{1}{\tau^2} - \frac{5}{3} + \frac{2\tau^2-1}{\tau^3}\sqrt{1+\tau^2}\ln\left(\tau+\sqrt{1+\tau^2}\right)\right], \quad \tau = \frac{Q}{2m_{\rm lept}}$$



- Interference term H_1: lepton $\times$ hadron tensor with form factors in each

$$H_1 = 2F_1^\gamma H_0 + F_2^\gamma t(2G_E^2 + tG_M^2)$$
$$+8G_E^\gamma\left[\frac{(s+1-r)^2}{4-t} + r - s\right]G_E + G_M^\gamma\left[2s + t - \frac{2(s+1-r)^2}{4-t}\right]tG_M$$

- Photon-loop form factors of muon $F_{1,2}^\gamma$: standard QED calculation

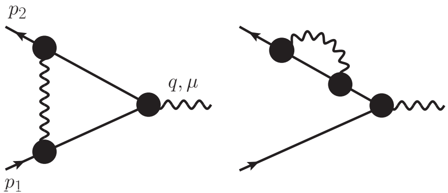
$$F_1^\gamma(t) = \frac{\alpha t}{2\pi}\int_4^\infty dx\,\frac{1}{x(xr-t)\sqrt{x^2-4x}}\left\{\left[2\xi_{\rm ir} + \ln r + \ln(x-4)\right](x-2) + 4 - \frac{3x}{2}\right\}$$

$$F_2^\gamma(t) = \frac{2\alpha}{\pi}\frac{r}{\sqrt{t^2-4rt}}\ln\frac{\sqrt{4r-t}+\sqrt{-t}}{2\sqrt{r}}, \qquad \text{infrared divergence } \xi_{\rm ir} = \ln\frac{M}{m_\gamma}$$

Photon-loop form factors

- In case of proton include structure: its (strong) electromagnetic form factors $G_{E,M}$ and Δ^+ resonance to model inelastic contributions

$$G^{\gamma}_{E,M} = \frac{\alpha}{\pi}\xi_{\rm ir}\left(1 + \frac{2t-4}{\sqrt{t^2-4t}} \ln \frac{\sqrt{4-t}+\sqrt{-t}}{2}\right) G_{E,M} + G^{\gamma-{\rm fin}}_{E,M} + G^{\gamma\Delta}_{E,M} + G^{\gamma\Delta\Delta}_{E,M}$$



- Wave function renormalization factor $Z_2\alpha/\pi$ from selfenergy diagram

$$Z_2 = \int_{m_\gamma/M}^{\infty} dx \left\{ \left[\frac{1}{x} - \frac{3x^3}{4} + \frac{3(x^4+2x^2-4)}{4\sqrt{4+x^2}} \right] F_1^2(x) + [\ldots] F_1(x)F_2(x) + [\ldots] F_2^2(x) \right\}$$

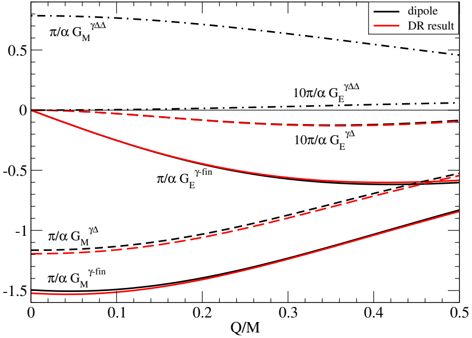
- Triangle diagrams: project on $G^{\gamma}_{E,M}$ via Dirac-trace, choose Breit-frame $p_{1,2} = \frac{M}{2}(\sqrt{4+q^2}, \mp\vec{q})$, Wick-rotation to euclidean space, 4-dimensional spherical coordinates, input (strong) form factors depend only on radius

Photon-loop form factors

- Photon-loop form factors $G^{\gamma}_{E,M}$ include infrared singular part $\sim \frac{\alpha}{\pi} G_{E,M}$

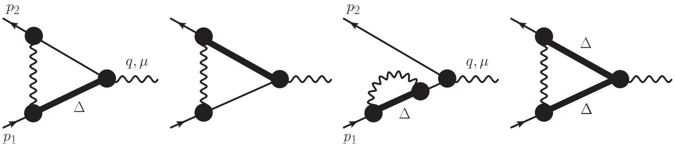
$$\int_{m_\gamma/M}^{\infty} dx \int_{-1}^{1} dz \frac{-(2+q^2)F_1^2(x)}{\pi q x (x + iz\sqrt{4+q^2})} \ln \frac{x + iz\sqrt{4+q^2} + q\sqrt{1-z^2}}{x + iz\sqrt{4+q^2} - q\sqrt{1-z^2}}$$

$$= \xi_{\rm ir} \left[\frac{-2q^2-4}{q\sqrt{4+q^2}} \ln \frac{q+\sqrt{4+q^2}}{2} \right] + \text{finite}, \quad \text{Taylor expansion in } q^2 = -t$$



- Black lines: dipole fit, red lines: most recent DR result, note $G_E^{\gamma}(0) = 0$

$\Delta^+(1232)$ resonance excitation



- Lorentz-covariant description of Δ-isobar by Rarita-Schwinger spinor Ψ_α
- Commonly used minimal $\Delta^+ p\gamma$-vertex: transition magn. moment $\kappa^* \simeq 5.0$

$$V_1^{\mu\alpha} = \frac{ie\kappa^*}{\sqrt{6}M}(g^{\mu\alpha}\gamma\cdot l - \gamma^\mu l^\alpha)\gamma_5\, G_\Delta(\sqrt{-l^2})\,, \qquad l_\mu V_1^{\mu\alpha} = 0$$

- Phenomenological transition form factor from π-electroprod. in Δ-region

$$G_\Delta(Q) = \left(1 + \frac{Q^2}{\Lambda^2}\right)^{-2} \exp\left(-\frac{Q^2}{7\Lambda^2}\right), \qquad \Lambda = 843\,\text{MeV}$$

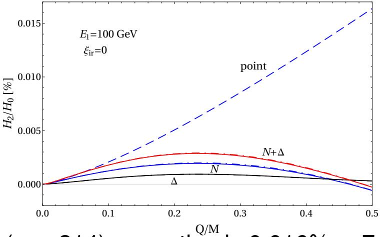
- Common form of Rarita-Schwinger propagator: index $\beta \to$ index α

$$\frac{i}{3}\,\frac{\gamma\cdot P + M_\Delta}{M_\Delta^2 - P^2}\left(3g_{\alpha\beta} - \gamma_\alpha\gamma_\beta - \frac{2P_\alpha P_\beta}{M_\Delta^2} + \frac{P_\alpha\gamma_\beta - \gamma_\alpha P_\beta}{M_\Delta}\right)$$

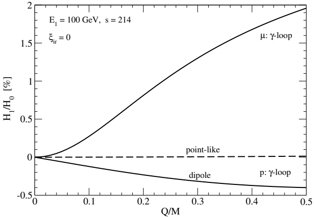
- $\Delta^+\Delta^+\gamma$-vertex from gauging kinetic term of free RS-Lagrangian

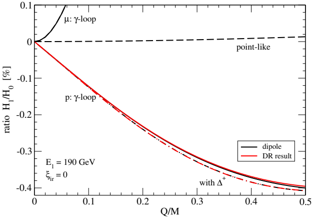
$$ie\left(-\gamma^\mu g_{\alpha\beta} + \gamma_\alpha g^\mu_\beta + \gamma_\beta g^\mu_\alpha - \gamma_\alpha\gamma^\mu\gamma_\beta\right) \times \text{dipole} \times \exp(-2Q^2/7\Lambda^2)$$

- For a point-like proton, 2γ-exchange amplitude H_2 is analytically known
- Form factors as sums of monopoles, Δ^+ state with dispersive approach

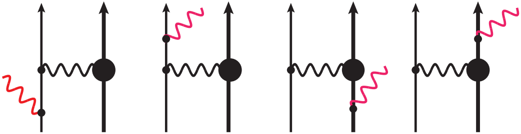


- For $E_1 = 100\,\text{GeV}$ ($s = 214$) correction is $0.016\% = 7.743\% - 7.727\%$
- H_2/H_0 does not exceed 0.003% for low momentum transfers





- H_1/H_0: muon γ-loop dominated $2F_1^\gamma$, effect of proton γ-loop changes significantly by proton structure, up to -0.4% while Δ^+ is negligible



- μ^- and p in initial and final state radiate a photon (ϵ, ℓ): soft amplitude

$$e\left(\frac{\epsilon\cdot k_1}{\ell\cdot k_1} - \frac{\epsilon\cdot k_2}{\ell\cdot k_2} - \frac{\epsilon\cdot p_1}{\ell\cdot p_1} + \frac{\epsilon\cdot p_2}{\ell\cdot p_2}\right) \qquad \ell \text{ in numerator neglected}$$

- Amplitude squared and summed over polarizations $\sum_{pol} \epsilon^\mu \epsilon^\nu = -g^{\mu\nu}$

$$4\pi\alpha\Big\{ -\frac{m_\mu^2}{(\ell\cdot k_1)^2} - \frac{m_\mu^2}{(\ell\cdot k_2)^2} - \frac{M^2}{(\ell\cdot p_1)^2} - \frac{M^2}{(\ell\cdot p_2)^2} + \frac{2k_1\cdot k_2}{\ell\cdot k_1\,\ell\cdot k_2} + \frac{2k_1\cdot p_1}{\ell\cdot k_1\,\ell\cdot p_1} - \frac{2k_1\cdot p_2}{\ell\cdot k_1\,\ell\cdot p_2} - \frac{2k_2\cdot p_1}{\ell\cdot k_2\,\ell\cdot p_1} + \frac{2k_2\cdot p_2}{\ell\cdot k_2\,\ell\cdot p_2} + \frac{2p_1\cdot p_2}{\ell\cdot p_1\,\ell\cdot p_2}\Big\}$$

- Photons with $|\vec{\ell}| < \lambda$ undetectable in experiment: master integral for $\delta_{\rm soft}$

$$\int_0^\lambda d\ell \frac{\ell^2}{2\sqrt{m_\gamma^2+\ell^2}} \int_{-1}^1 dz \frac{-1}{\left(E\sqrt{m_\gamma^2+\ell^2} - p\ell z\right)^2} = \frac{1}{E^2-p^2}\left[\ln\frac{m_\gamma}{2\lambda} + \frac{E}{2p}\ln\frac{E+p}{E-p}\right]$$

- Applies to 4 terms with squares in denominator, for 6 terms with products in denominator use Feynman trick: $(AB)^{-1} = \int_0^1 dx[Ax + B(1-x)]^{-2}$

Soft photon correction factor

- Universal part cancels infrared divergences from virtual photon-loops, remainder depends logarithmically on infrared cutoff λ for soft photons

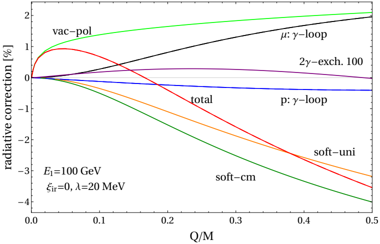
$$\delta_{\text{soft}}^{(\text{uni})} = \frac{4\alpha}{\pi}\left(\ln\frac{M}{2\lambda} - \xi_{\text{ir}}\right)\left\{1 + \frac{t-2r}{\sqrt{t^2-4rt}}\ln\frac{\sqrt{4r-t}+\sqrt{-t}}{2\sqrt{r}} + \frac{t-2}{\sqrt{t^2-4t}}\ln\frac{\sqrt{4-t}+\sqrt{-t}}{2}\right.$$
$$\left. + \frac{2(1+r-s)}{\sqrt{s-\rho_+}\sqrt{s-\rho_-}}\ln\frac{\sqrt{s-\rho_+}+\sqrt{s-\rho_-}}{2r^{1/4}} + \frac{2(1+r-u)}{\sqrt{\rho_+-u}\sqrt{\rho_--u}}\ln\frac{\sqrt{\rho_+-u}+\sqrt{\rho_--u}}{2r^{1/4}}\right\}$$

- $\rho_\pm = 1 + r \pm 2\sqrt{r}$, last two terms odd under $s \leftrightarrow u$ refer to 2γ- exch. boxes
- Part specific for assuming in center-of-mass frame a small momentum sphere $|\vec{\ell}\,| < \lambda$ for undetected soft radiation, $P = s^2 - 2s(1+r) + (1-r)^2$

$$\delta_{\text{soft}}^{(\text{cm})} = \frac{\alpha}{\pi}\left\{\frac{2}{\sqrt{P}}\left[(s-1+r)\ln\frac{s-1+r+\sqrt{P}}{2\sqrt{sr}} + (s+1-r)\ln\frac{s+1-r+\sqrt{P}}{2\sqrt{s}}\right]\right.$$
$$+ \int_0^{1/2} dx\left[\frac{(t-2)(s+1-r)}{[1-tx(1-x)]\sqrt{R_t}}\ln\frac{s+1-r+\sqrt{R_t}}{s+1-r-\sqrt{R_t}} + \frac{(t-2r)(s-1+r)}{[r-tx(1-x)]\sqrt{R_t}}\ln\frac{s-1+r+\sqrt{R_t}}{s-1+r-\sqrt{R_t}}\right]$$
$$+ \int_0^1 dx\left[\frac{(1+r-s)[s+(1-r)(1-2x)]}{(1-2x)[sx(1-x)+(1-2x)(1-x-rx)]\sqrt{P}}\ln\frac{s+(1-2x)(1-r+\sqrt{P})}{s+(1-2x)(1-r-\sqrt{P})}\right.$$
$$\left.\left. + \frac{(1+r-u)[s+(1-r)(1-2x)]}{[1+(r-1)x-ux(1-x)]\sqrt{R_u}}\ln\frac{s+(1-r)(1-2x)+\sqrt{R_u}}{s+(1-r)(1-2x)-\sqrt{R_u}}\right]\right\}$$

- with polynomials $R_t = P + 4stx(1-x)$ and $R_u = P + 4x(1-x)[su-(1-r)^2]$
- For positive muons $\mu^+ p \to \mu^+ p$, last term $\int_0^1 dx \ldots$ changes sign.

- Individual radiative corrections to $\mu^- p \to \mu^- p$ including proton structure



- Photon-loop around proton and 2γ-exchange are suppressed
- Major role played by vacuum polarization and soft photon radiation
- Requires calculation of (hard) bremsstrahlung incl. detector acceptance