Radiative Corrections from medium to high energy experiments

Radiative corrections and form factors in Dalitz decays of lightest pseudoscalar mesons and Σ^0

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Overview

Precision frontier
↪ together with theoretical predictions small uncertainties
↪ low-energy sector of strong interactions remains a challenge

Hadronic parameters from experiment

- QED corrections left out entirely
- some of the relevant terms neglected
- approximative results (leading logs, soft-photon approximation)

Artificial discrepancies between theory and experiment
↪ measured observables or related hadronic parameters may include unsubtracted QED part

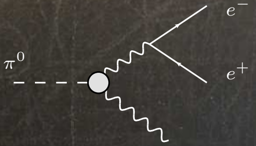
QED radiative corrections in the low-energy QCD sector
↪ π^0, $\eta^{(\prime)}$, Σ^0, K^+, ...
↪ direct application to experiment

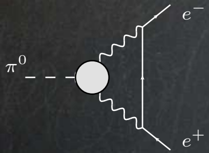
Radiative corrections for π^0 decays

Rare decay of π^0

Decay modes of the neutral pion:

Process	Branching ratio
$\pi^0 \to \gamma\gamma$	$(98.823 \pm 0.034)\,\%$
$\pi^0 \to e^+e^-\gamma$	$(1.174 \pm 0.035)\,\%$
$\pi^0 \to e^+e^+e^-e^-$	$(3.34 \pm 0.16) \times 10^{-5}$
$\pi^0 \to e^+e^-$	$(6.46 \pm 0.33) \times 10^{-8}$





Rare decay $\pi^0 \to e^+e^-$

- interesting way to study low-energy (long-distance) dynamics in the SM
- systematic theoretical treatment dates back to *Drell*, NC (1959)
- suppressed compared to the decay $\pi^0 \to \gamma\gamma$ by a factor of $2(\alpha m_e/M_\pi)^2$
  - ↪ one-loop structure + helicity suppression
  - ↪ may be sensitive to possible effects of new physics

Rare decay of π^0

KTeV measurement

KTeV-E799-II experiment at Fermilab (*Abouzaid et al.*, PRD 75 (2007))
↪ precise measurements of branching ratio $\pi^0 \to e^+e^-$ (794 candidates)

$$\frac{\Gamma(\pi^0 \to e^+e^-(\gamma),\, x > 0.95)}{\Gamma(\pi^0 \to e^+e^-\gamma,\, x > 0.232)} = (1.685 \pm 0.064 \pm 0.027) \times 10^{-4}$$

↪ final result for lowest order (no final state radiation)

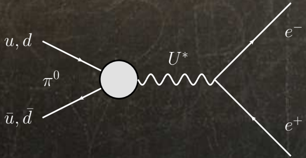
$$B^{\text{no-rad}}_{\text{KTeV}}(\pi^0 \to e^+e^-) = (7.48 \pm 0.29 \pm 0.25) \times 10^{-8}$$

Comparison with SM prediction (*Dorokhov and Ivanov*, PRD 75 (2007))

$$B^{\text{no-rad}}_{\text{SM}}(\pi^0 \to e^+e^-) = (6.23 \pm 0.09) \times 10^{-8}$$

↪ interpreted as $3.3\,\sigma$ discrepancy between theory and experiment

New physics?



- very fashionable to ascribe eventual discrepancies to effects of new physics

BUT

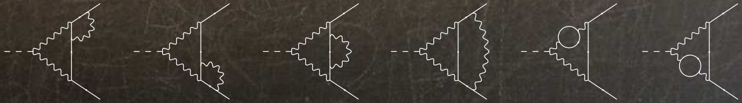
- first, look for more conventional solution (i.e. within SM)

  $\hookrightarrow$ radiative corrections

  $\hookrightarrow$ electromagnetic-transition-form-factor modeling

Radiative corrections for $\pi^0 \to e^+e^-$

Final results



Vaško and Novotný, JHEP 1110 (2011)

Size of the radiative corrections (newly calculated)

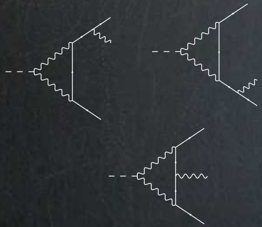
$$\delta^{\text{NLO}}(0.95) \equiv \delta^{\text{virt.}} + \delta^{\text{BS}}(0.95) = (-5.5 \pm 0.2)\,\%$$

- can be thought as model-independent
- differs significantly from previous approximate calculations

Bergström, Z.Ph.C 20 (1983): $\delta(0.95) = -13.8\,\%$
Dorokhov et al., EPJC 55 (2008): $\delta(0.95) = -13.3\,\%$

- original KTeV vs. SM discrepancy reduced to the 2σ level
- contact interaction coupling finite part set to

$$\chi^{(\text{r})}_{\text{LMD}}(M_\rho) = 2.2 \pm 0.9$$



TH, Kampf and Novotný,
EPJC 74 (2014)

Dalitz decay of the neutral pion

Introduction

Quantity really measured by KTeV

$$\left.\frac{\Gamma(\pi^0\to e^+e^-(\gamma)\,,\ x>0.95)}{\Gamma(\pi^0\to e^+e^-\gamma(\gamma)\,,\ x>0.2319)}\right|_{\text{KTeV}} = (1.685\pm 0.064\pm 0.027)\times 10^{-4}$$

↪ Dalitz decay comes into play

- second most important decay channel of the neutral pion
  ↪ branching ratio $(1.174\pm 0.035)\,\%$
- first studied by *Richard H. Dalitz,* PPSA 64 (1951)
- experimental data of this process provide the information about singly-virtual pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,q^2)$
  ↪ in particular about its slope parameter a_π

$$\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,M^2x)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)} \simeq 1 + a_\pi x$$

$$x = \frac{(p_{e^+}+p_{e^-})^2}{M_{\pi^0}^2}\,,\qquad y = -\frac{2}{M_{\pi^0}^2}\frac{p_\gamma\cdot(p_{e^-}-p_{e^+})}{1-x}$$

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Introduction

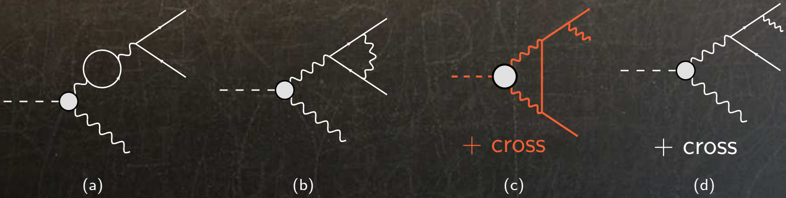
- radiative corrections to the total decay rate of the Dalitz decay
  $\hookrightarrow$ first addressed (numerically) by *Joseph*, NC 16 (1960)

- pioneering study of corrections to the differential decay rate
  $\hookrightarrow$ *Lautrup and Smith*, PRD 3 (1971)
  $\hookrightarrow$ soft-photon approximation

- extended by *Mikaelian and Smith*, PRD 5 (1972)
  $\hookrightarrow$ hard-photon corrections
  $\hookrightarrow$ whole range of bremsstrahlung photon energy
  $\hookrightarrow$ table of values

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Novel approach



- new calculations motivated by needs of NA48/NA62 experiments at CERN
  $\hookrightarrow$ measure the slope of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, q^2)$: *Lazzeroni et al.*, PLB 768 (2017)

$$a_\pi^{\text{NA62}} = 3.68(57)\,\%$$

- unlike before no approximation was used
  $\hookrightarrow$ can be used also for related decays $\eta \to \ell^+\ell^-\gamma$ etc.
- C++ code returns the correction for any given x and y
  $\hookrightarrow$ propagated into MC generator of NA62 experiment
- *TH, Kampf and Novotný*, PRD 92 (2015)

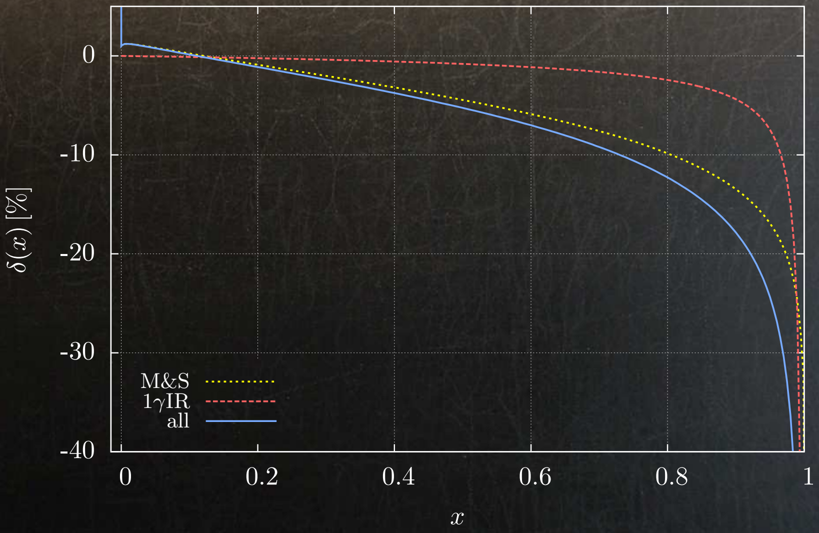
Radiative corrections for $\pi^0 \to e^+e^-\gamma$

The overall NLO correction $\delta(x,y)$ given in percent (Dalitz-plot corrections)

y x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	2.761	2.714	2.599	2.449	2.273	2.061	1.786	1.402	0.803	-0.357	-5.657
0.02	2.756	2.720	2.622	2.480	2.300	2.073	1.774	1.355	0.703	-0.546	-5.859
0.03	2.669	2.639	2.552	2.419	2.242	2.012	1.704	1.267	0.586	-0.716	-6.125
0.04	2.558	2.531	2.452	2.327	2.155	1.925	1.611	1.164	0.464	-0.874	-6.372
0.05	2.437	2.412	2.340	2.221	2.053	1.824	1.509	1.054	0.341	-1.025	-6.601
0.06	2.311	2.288	2.221	2.108	1.944	1.717	1.400	0.940	0.216	-1.172	-6.815
0.07	2.184	2.163	2.099	1.990	1.830	1.605	1.288	0.824	0.092	-1.315	-7.017
0.08	2.056	2.036	1.975	1.870	1.714	1.491	1.173	0.707	-0.033	-1.455	-7.211
0.09	1.928	1.909	1.851	1.749	1.596	1.374	1.057	0.588	-0.157	-1.593	-7.397
0.10	1.801	1.783	1.726	1.628	1.477	1.257	0.940	0.469	-0.281	-1.729	-7.578
0.15	1.170	1.154	1.105	1.016	0.874	0.661	0.345	-0.131	-0.900	-2.394	-8.424
0.20	0.546	0.532	0.486	0.402	0.266	0.057	-0.258	-0.738	-1.520	-3.048	-9.219
0.25	-0.079	-0.092	-0.135	-0.217	-0.350	-0.556	-0.871	-1.355	-2.148	-3.704	-9.995
0.30	-0.713	-0.726	-0.768	-0.847	-0.978	-1.184	-1.499	-1.988	-2.790	-4.372	-10.77
0.35	-1.366	-1.378	-1.419	-1.497	-1.627	-1.833	-2.149	-2.641	-3.454	-5.058	-11.56
0.40	-2.044	-2.056	-2.097	-2.174	-2.304	-2.509	-2.827	-3.324	-4.146	-5.773	-12.37
0.45	-2.759	-2.771	-2.811	-2.887	-3.017	-3.222	-3.543	-4.044	-4.875	-6.525	-13.22
0.50	-3.521	-3.533	-3.572	-3.648	-3.777	-3.983	-4.306	-4.811	-5.653	-7.324	-14.12
0.55	-4.344	-4.356	-4.395	-4.470	-4.599	-4.806	-5.130	-5.640	-6.492	-8.186	-15.08
0.60	-5.249	-5.261	-5.299	-5.373	-5.501	-5.708	-6.034	-6.549	-7.410	-9.128	-16.12
0.65	-6.262	-6.273	-6.310	-6.383	-6.510	-6.717	-7.044	-7.563	-8.435	-10.18	-17.28
0.70	-7.425	-7.435	-7.470	-7.541	-7.666	-7.871	-8.198	-8.721	-9.603	-11.37	-18.60
0.75	-8.802	-8.811	-8.844	-8.910	-9.031	-9.232	-9.558	-10.08	-10.98	-12.77	-20.14
0.80	-10.51	-10.52	-10.54	-10.60	-10.72	-10.91	-11.23	-11.76	-12.66	-14.49	-22.02
0.85	-12.78	-12.78	-12.80	-12.85	-12.95	-13.13	-13.44	-13.96	-14.86	-16.72	-24.47
0.90	-16.21	-16.21	-16.21	-16.23	-16.29	-16.43	-16.71	-17.21	-18.11	-20.00	-28.00
0.95	-23.17	-23.14	-23.08	-23.01	-22.96	-22.98	-23.14	-23.53	-24.36	-26.26	-34.45
0.99	-54.29	-54.07	-53.44	-52.50	-51.35	-50.15	-49.03	-48.16	-47.76	-48.47	-55.83

Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Results



Radiative corrections for $\pi^0 \to e^+e^-\gamma$

Determination of ratio R

Precise and reliable determination of $R \equiv \frac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)}$

↪ for small slope and up to NLO radiative corrections

$$R \simeq \frac{\alpha}{\pi} \iint (1 + a_\pi x)^2 (1 + \delta(x,y)) \frac{(1-x)^3}{4x} \left[1 + y^2 + \frac{4m_e^2}{M_\pi^2 x}\right] \mathrm{d}x\,\mathrm{d}y$$

Conservative estimate for uncertainty (a_π, NNLO): $R = 1.1978(5)(3)\,\%$

↪ chosen $a_\pi^{\text{univ}} = 3.55(70)\,\%$, covers

source	VMD	LMD	THS	dispers.	Padé aps.	NA62	A2	PDG
a_π [%]	3.00	2.45	2.92(4)	3.15(9)	3.21(19)	3.68(57)	3.0(1.0)	3.35(31)
b_π $[10^{-3}]$	0.90	0.74	0.87(2)	1.14(4)	1.04(22)	×	×	×

Constraint: $1 \simeq \mathcal{B}(\pi^0 \to \gamma\gamma) + \mathcal{B}(\pi^0 \to e^+e^-\gamma(\gamma)) + \mathcal{B}(\pi^0 \to e^+e^-e^+e^-)$

$$\mathcal{B}(\pi^0 \to \gamma\gamma) = 98.8131(6)\,\%\,, \quad \mathcal{B}(\pi^0 \to e^+e^-\gamma(\gamma)) = 1.1836(6)\,\%$$

TH, Goudzovski and Kampf, PRL 122 (2018)

↪ could be used in future exp. analysis of $K^+ \to \pi^+e^+e^-$

PDG

$$R = 1.188(35)\,\%\,, \quad \mathcal{B}(\pi^0 \to \gamma\gamma) = 98.823(34)\,\%\,, \quad \mathcal{B}(\pi^0 \to e^+e^-\gamma) = 1.174(35)\,\%$$

Determination of ratio R

Why it works?

TFF normalization $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$ drops out in R
↪ TFF dependence solely represented by its shape

for π^0, transfered momentum significantly (kinematically) limited
↪ linear expansion very good approximation
↪ slope a_π constitutes the only parameter of the low-energy QCD sector

allowing for 20 % uncertainty on a_π^{univ} due to:

- smallness of the slope
- strong suppression of the region $x \approx 1$ where the $a_\pi x$ term matters

$$R \simeq \frac{\alpha}{\pi} \iint (1 + a_\pi x)^2 (1 + \delta(x,y)) \frac{(1-x)^3}{4x} \left[1 + y^2 + \frac{4m_e^2}{M_\pi^2 x} \right] \mathrm{d}x\, \mathrm{d}y$$

‘Recent’ KTeV measurement

Abouzaid et al., PRD 100 (2019)
$\hookrightarrow$ based on 1999 data and *E. Abouzaid*, Ph.D. thesis (2007)

$$\frac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = (1.1559 \pm 0.0046 \pm 0.0106)\,\%$$

TH, Goudzovski and Kampf, PRL 122 (2018)
Conservative estimate for uncertainty (a_π, NNLO): $R = 1.1978(5)(3)\,\%$
$\hookrightarrow$ chosen $a_\pi^{\text{univ}} = 3.55(70)\,\%$

$\Longrightarrow$ $3.6\,\sigma$ discrepancy between theory and experiment

PDG average: $R = 1.188(35)\,\%$
$\hookrightarrow$ most recent (archived ALEPH data) *Beddall and Beddall*, EPJC 54 (2008)

Need for new measurements
$\hookrightarrow$ R, improvement on a_π (maybe b_π) welcome

Radiative corrections for $\eta^{(\prime)}$ Dalitz decays

Dalitz decays of $\eta^{(\prime)}$

Introduction

$\eta^{(\prime)}$ Dalitz decays

- small branching ratios
  $\hookrightarrow$ hadronic decay modes are open
- access to electromagnetic transition form factors
  $\hookrightarrow$ $\eta^{(\prime)}$-meson structure
  $\hookrightarrow$ valuable input for other quantities and e.g. $g-2$ of the muon
  $\hookrightarrow$ radiative corrections crucial to extract relevant information from data

naive rad. corrections for $\eta \to e^+e^-\gamma$: *Mikaelian and Smith*, PRD 5 2890 (1972)

- numerical values correspond to simple change $M_{\pi^0} \to M_\eta$
  $\hookrightarrow$ π^0 case: *Mikaelian and Smith*, PRD 5 1763 (1972)

inclusive radiative corrections
$\hookrightarrow$ no momentum or angular cuts on the bremsstrahlung photon applied

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays
What should be taken into account?

The $\eta^{(\prime)}$ case compared to π^0

- larger rest mass

  $\hookrightarrow$ M_η above muon-pair threshold: $M_\eta > 2m_\mu$
  $\hookrightarrow$ $M_{\eta'}$ above lowest-lying resonances: $M_{\eta'} > M_\rho, M_\omega$
    $\hookrightarrow$ sensitive to the widths of resonances
    $\hookrightarrow$ ω narrow, ρ broad resonance in $\pi\pi$ scattering

- strange-flavor content

  $\hookrightarrow$ quark-flavor basis
  Feldmann et al., PLB 449 (1999), *Escribano et al.*, JHEP 06 (2005)

$$j^\ell \equiv \frac{i}{2}\left[\bar{u}\gamma_5 u + \bar{d}\gamma_5 d\right], \quad j^{\mathsf{s}} \equiv \frac{i}{\sqrt{2}}\left[\bar{s}\gamma_5 s\right]$$

- η-η' mixing: $\langle 0|j^A|\eta^B\rangle = B_0 F_\pi f_A \delta^{AB}$, $\langle \eta^A|\eta^B\rangle = \delta^{AB}$, $A, B \in \{\ell, \mathsf{s}\}$

$$|\eta\rangle = \cos\phi\,|\eta^\ell\rangle - \sin\phi\,|\eta^{\mathsf{s}}\rangle$$
$$|\eta'\rangle = \sin\phi\,|\eta^\ell\rangle + \cos\phi\,|\eta^{\mathsf{s}}\rangle$$

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays

Novel approach

Full set of NLO QED radiative corrections:
TH, Kampf, Leupold and Novotný, PRD 97 (2018)

- compared to previous approach:
  $\hookrightarrow$ muon loops + hadronic VP
  $\hookrightarrow$ 1γIR at one-loop level
  $\hookrightarrow$ form-factor effects (also in BS)
  $\hookrightarrow$ higher orders in the final-state-lepton mass not neglected
- general framework: three additional processes
  $\hookrightarrow$ also muon decay modes

η case: most of the ingredients in *TH, Kampf and Novotný,* PRD 92 (2015)

η' case: real challenge

$\hookrightarrow$ resulting framework also applicable to the π^0 case (numerically compatible)
  $\hookrightarrow$ overkill (correction to the correction of order 1 %)

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays
Virtual corrections

Photon self-energy in the form $\Pi(s) = \Pi_{\mathsf{L}}(s) + \Pi_{\mathsf{H}}(s)$

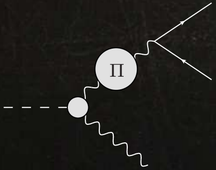
- lepton loops (electrons and as well muons)

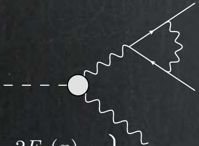
$$\Pi_{\mathsf{L}}(M_P^2 x) = \frac{\alpha}{\pi} \sum_{\ell'=e,\mu} \left\{ \frac{8}{9} - \frac{\beta_{\ell'}^2}{3} + \left(1 - \frac{\beta_{\ell'}^2}{3}\right) \frac{\beta_{\ell'}}{2} \log\left[-\gamma_{\ell'} + i\epsilon\right] \right\}$$

- hadronic contribution

  ↪ *Jegerlehner*, Z.Ph.C 32 (1986)

$$\Pi_{\mathsf{H}}(s) = -\frac{s}{4\pi^2\alpha} \int_{4m_\pi^2}^{\infty} \frac{\sigma_{\mathsf{H}}(s')\,\mathrm{d}s'}{s - s' + i\epsilon}$$

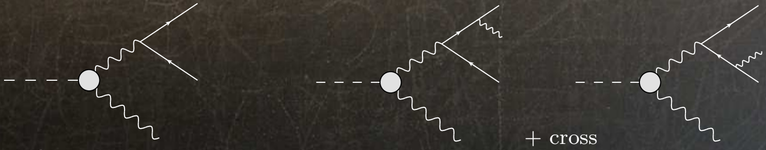




$$\delta^{\mathsf{virt}}(x,y) = \frac{1}{|1 + \Pi(M_P^2 x)|^2} - 1 + 2\,\mathrm{Re}\left\{ F_1(x) + \frac{2F_2(x)}{1 + y^2 + \frac{\nu^2}{x}} \right\}$$

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays

Form-factor dependence

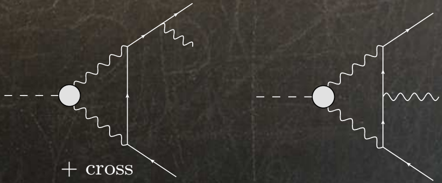


Form factor effects: two distinct cases

- singly-virtual TFF: LO and BS
  $\hookrightarrow$ integration over BS-photon angles and energies → widths become crucial
- doubly-virtual TFF: 1γIR correction at one-loop level

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays

1γIR contribution at one-loop level



1γIR contribution at one-loop level

- beyond effective approach
- we don't expect substantial model dependence → VMD-inspired model
  ↪ strange-flavor content and η-η' mixing

$$e^2\mathcal{F}^{\mathsf{VMD}}_{\eta\gamma^*\gamma^*}(p^2,q^2) = -\frac{N_c}{8\pi^2F_\pi}\frac{2e^2}{3}$$

$$\times\left[\frac{5}{3}\frac{\cos\phi}{f_\ell}\frac{M^4_{\omega/\rho}}{(p^2-M^2_{\omega/\rho})(q^2-M^2_{\omega/\rho})} - \frac{\sqrt{2}}{3}\frac{\sin\phi}{f_s}\frac{M^4_\phi}{(p^2-M^2_\phi)(q^2-M^2_\phi)}\right]$$

Radiative corrections for $\eta^{(\prime)} \to \ell^+\ell^-\gamma$ decays

Bremsstrahlung

- slope not negligible
- for η: expansion in slope a would be still (somewhat) suitable

$$\mathcal{F}\big((p_\gamma + p_{e^+} + p_{e^-})^2\big) \simeq \mathcal{F}(M_P^2 x)\left[1 + a\,\frac{2p_\gamma \cdot (p_{e^+} + p_{e^-})}{M_P^2}\right]$$

- for η': such an expansion not applicable anymore
  $\hookrightarrow$ BS necessarily depends on the form-factor model

sensitivity to width of ρ meson → recent dispersive calculations used
Hanhart et al., EPJC 73 (2013), EPJC 77 (2017)

Källén–Lehmann spectral representation → common spectral density function

$$\frac{\mathcal{F}(q^2)}{\mathcal{F}(0)} \simeq 1 + q^2 \int_{4m_\pi^2}^{\Lambda^2} \frac{\mathcal{A}(s)\,\mathrm{d}s}{q^2 - s + i\epsilon}$$

$$\mathcal{A}(s) = w_\omega \mathcal{A}_\omega(s) + w_\phi \mathcal{A}_\phi(s) - \frac{\kappa}{96\pi^2 F_\pi^2}\left[1 - \frac{4m_\pi^2}{s}\right]^{3/2} P(s)\,R(s)\,|\Omega(s)|^2$$

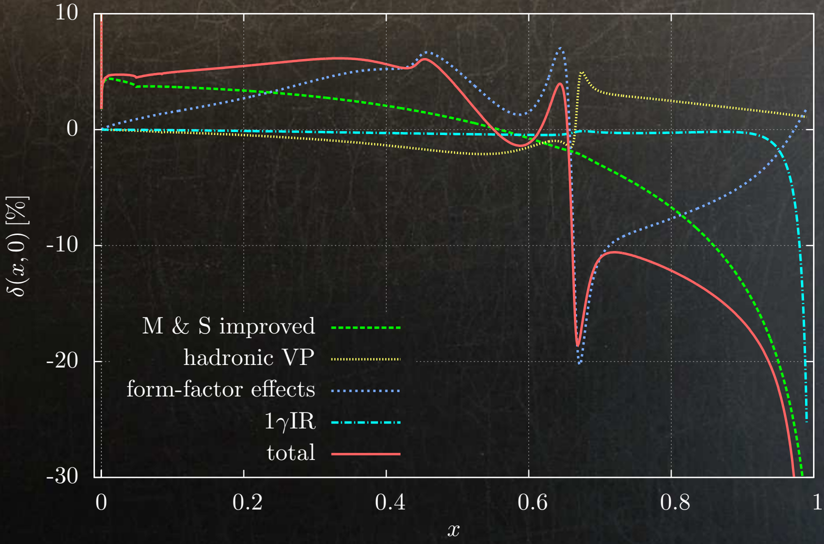
Radiative corrections for $\eta' \to e^+e^-\gamma$ decays

The overall NLO correction $\delta(x,y)$ given in percent (Dalitz-plot corrections)

y x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	4.66	4.56	4.33	4.02	3.67	3.26	2.77	2.11	1.13	-0.64	-7.44
0.05	4.51	4.45	4.28	4.02	3.68	3.23	2.67	1.89	0.72	-1.31	-9.04
0.10	4.95	4.90	4.74	4.49	4.15	3.68	3.13	2.14	0.87	-1.29	-9.51
0.15	5.22	5.17	5.01	4.75	4.37	3.86	3.17	2.09	0.75	-1.50	-10.1
0.20	5.49	5.43	5.26	4.95	4.48	4.06	3.03	1.91	0.51	-1.83	-10.7
0.25	5.79	5.72	5.49	5.08	4.82	3.87	2.78	1.63	0.18	-2.26	-11.3
0.30	6.07	5.96	5.62	5.30	4.68	3.55	2.43	1.24	-0.26	-2.79	-12.1
0.35	6.12	5.97	5.75	5.37	4.21	3.07	1.94	0.71	-0.86	-3.49	-13.0
0.40	5.64	5.84	5.66	4.59	3.46	2.35	1.21	-0.06	-1.69	-4.42	-14.1
0.45	6.02	4.96	4.11	3.20	2.23	1.20	0.10	-1.19	-2.88	-5.69	-15.4
0.50	3.60	3.49	2.00	1.00	0.22	-0.62	-1.61	-2.85	-4.53	-7.35	-17.1
0.55	0.05	0.07	0.11	-0.09	-1.92	-2.93	-3.80	-4.89	-6.45	-9.18	-18.7
0.60	-1.17	-1.13	-1.02	-0.88	-0.79	-0.97	-3.17	-5.37	-7.09	-9.78	-19.2
0.70	-11.1	-11.1	-11.1	-11.0	-11.0	-11.1	-11.2	-11.5	-12.2	-13.7	-21.1
0.75	-10.9	-10.9	-10.9	-10.9	-10.9	-11.0	-11.2	-11.6	-12.4	-14.2	-22.0
0.80	-12.2	-12.2	-12.1	-12.1	-12.2	-12.3	-12.5	-13.0	-13.8	-15.7	-23.6
0.85	-13.9	-13.9	-13.9	-13.9	-14.0	-14.1	-14.4	-14.8	-15.7	-17.6	-25.5
0.90	-16.5	-16.5	-16.5	-16.5	-16.6	-16.8	-17.0	-17.5	-18.4	-20.3	-28.1
0.95	-22.2	-22.2	-22.2	-22.2	-22.2	-22.3	-22.6	-23.0	-23.9	-25.7	-33.4
0.99	-55.8	-55.6	-55.0	-54.0	-52.9	-51.8	-50.7	-49.9	-49.5	-50.2	-56.6

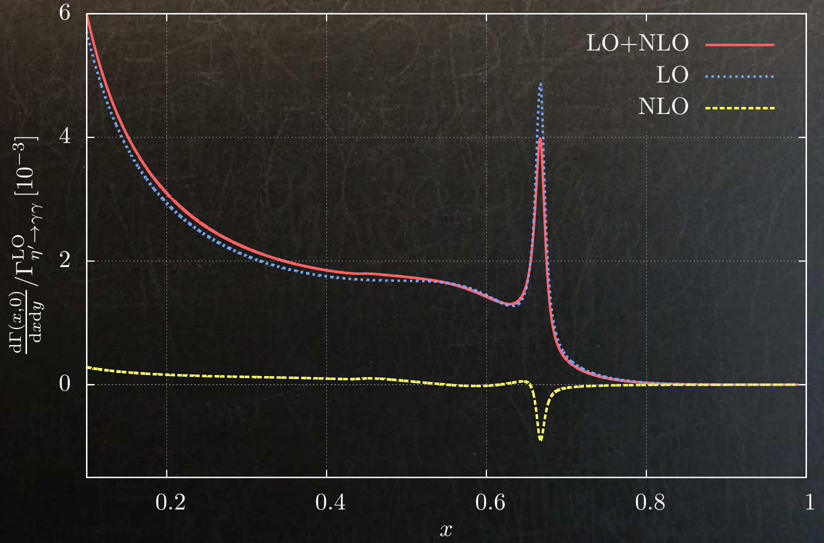
Radiative corrections for $\eta' \to e^+e^-\gamma$ decays

The overall NLO correction $\delta(x, 0)$ in comparison to its constituents



Radiative corrections for $\eta' \to e^+e^-\gamma$ decays

The two-fold differential decay width $\mathrm{d}\Gamma(x,0)$ at NLO



Radiative corrections for Σ^0 Dalitz decay

Dalitz decay of Σ^0

Introduction

Experimental knowledge of hyperons rather limited
$\hookrightarrow$ unstable (electron scattering rather difficult)
$\hookrightarrow$ magnetic moments (and electric charges) known

Form factors at high energies
$\hookrightarrow$ e^+e^- scattering to hyperon + antihyperon
$\hookrightarrow$ direct and transition form factors accessible

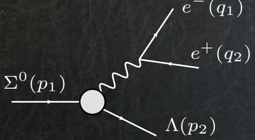
At low energies
$\hookrightarrow$ Dalitz decay $Y \to Y'e^+e^-$
$\hookrightarrow$ possibly high statistics in future at FAIR (PANDA: $p\bar{p}$, HADES: pp)

Dalitz decays in baryon-octet sector?

Dalitz decay of Σ^0
Introduction

Dalitz decay $\Sigma^0 \to \Lambda e^+ e^-$

- e^+e^- invariant mass only up to $M_\Sigma - M_\Lambda \simeq 77\,\text{MeV}$
- provides electric and magnetic transition form factors of $\Sigma^0 \to \Lambda$ transition
- extracting transition radii challenging
  - high-precision measurement required
  - competing with QED radiative corrections



Predictions of electric and magnetic radii

- *Kubis and Meissner*, EPJC 18 (2001)
- *Granados, Leupold and Perotti*, EPJA 53 (2017)

Dalitz decay of Σ^0

Leading order

$\Sigma^0\Lambda\gamma$ vertex: $\langle 0|j^\mu|\Sigma^0\bar{\Lambda}\rangle = e\bar{v}_\Lambda(\vec{p}_2)\,G^\mu(p_1+p_2)\,u_\Sigma(\vec{p}_1)$, with

$$G^\mu(q) \equiv \left[\gamma^\mu - (M_\Sigma - M_\Lambda)\frac{q^\mu}{q^2}\right]G_1(q^2) - \frac{i\sigma^{\mu\nu}q_\nu}{M_\Sigma + M_\Lambda}G_2(q^2)$$

Define magnetic and electric form factors

$$G_{\mathrm{M}}(q^2) \equiv G_1(q^2) + G_2(q^2) \qquad = \kappa\left(1 + \frac{1}{6}\langle r_{\mathrm{M}}^2\rangle q^2 + \mathcal{O}(q^4)\right)$$

$$G_{\mathrm{E}}(q^2) \equiv G_1(q^2) + \frac{q^2}{(M_\Sigma + M_\Lambda)^2}G_2(q^2) = \frac{1}{6}\langle r_{\mathrm{E}}^2\rangle q^2 + \mathcal{O}(q^4)$$

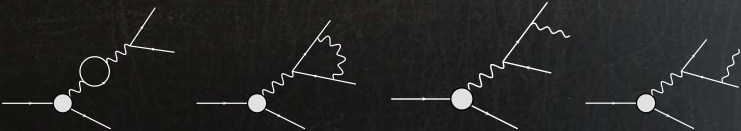
Matrix element squared dominated by the magnetic part

$$\overline{|\mathcal{M}^{\mathrm{LO}}(x,y)|^2} \simeq 2e^4|G_{\mathrm{M}}(\Delta_M^2 x)|^2\frac{(1-x)}{x}\left(1 + y^2 + \frac{\nu^2}{x}\right)$$

Radiative corrections for $\Sigma^0 \to \Lambda e^+e^-$

Introduction

Radiative corrections to the differential decay width in soft-photon approximation
↪ *Sidhu and Smith*, PRD 4 3344 (1971)

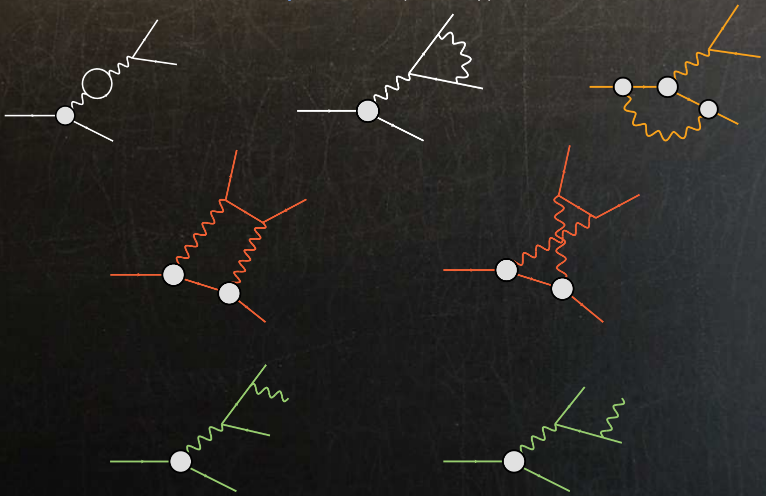


Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Diagrams

TH and Leupold, EPJC 80 (2020)

$\hookrightarrow$ inclusive radiative corrections beyond the soft-photon approximation



Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Bremsstrahlung

Low-energy expansion of the form factors:

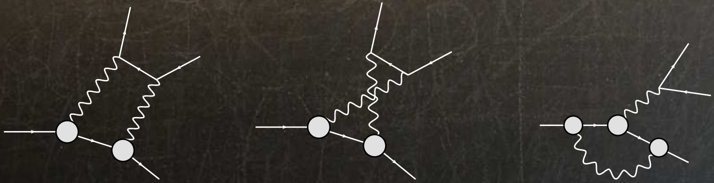
$$G_\mathsf{M}\big((k+q_1+q_2)^2\big) \simeq G_\mathsf{M}\big((q_1+q_2)^2\big)\left\{1+\frac{1}{6}\langle r_\mathsf{M}^2\rangle[2k\cdot(q_1+q_2)]\right\},$$

$$G_\mathsf{E}\big((k+q_1+q_2)^2\big) \simeq G_\mathsf{E}\big((q_1+q_2)^2\big)\left\{1+\frac{2k\cdot(q_1+q_2)}{(q_1+q_2)^2}\right\}.$$

Subsequently, integrate over the energy and emission angle of bremsstrahlung photon
↪ radiative corrections for inclusive process

Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Virtual corrections



By loop-momenta-power counting, FFs required to regulate the UV region

1γIR: UV-convergence already achieved in the simplest case with constant FFs
$\hookrightarrow$ $G_\mathsf{E}(q^2) = G_\mathsf{E}(0) = 0$ and $G_\mathsf{M}(q^2) = G_\mathsf{M}(0) = \kappa$

$$G_1(q^2) = \kappa \frac{q^2}{q^2 - M_V^2}\,, \quad G_2(q^2) = -\kappa \frac{M_V^2}{q^2 - M_V^2}$$

Ansatz satisfying high-energy constraints

$$G_1(q^2) = \kappa \left(3 - \frac{M_V^2 \langle r_\mathsf{M}^2 \rangle}{6}\right) \frac{q^2 M_V^4}{(q^2 - M_V^2)^3}\,, \quad G_2(q^2) = -\kappa \frac{M_V^6}{(q^2 - M_V^2)^3}$$

These contributions to the NLO decay width are found to be negligible

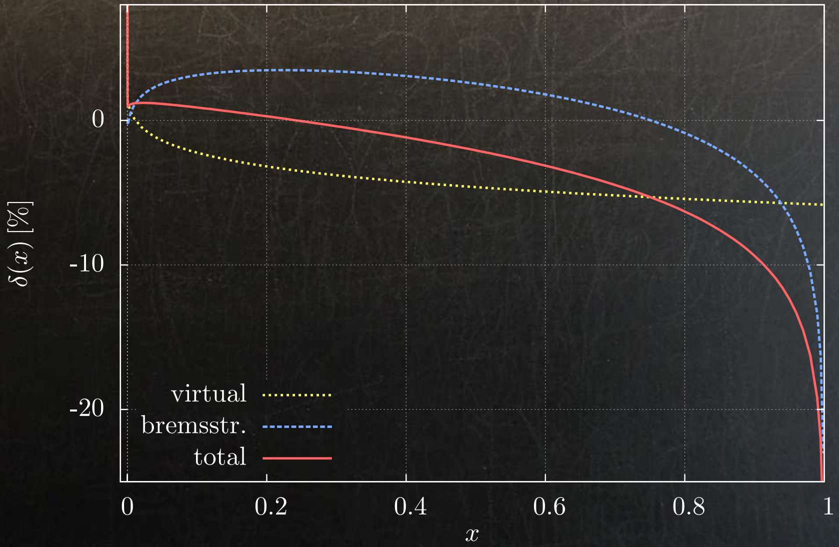
Radiative corrections for $\Sigma^0 \to \Lambda e^+e^-$

Results

x \ y	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	2.50	2.44	2.31	2.18	2.03	1.87	1.65	1.33	0.83	-0.20	-8.26
0.02	2.67	2.61	2.49	2.34	2.19	2.00	1.75	1.40	0.84	-0.26	-5.84
0.03	2.71	2.66	2.55	2.41	2.24	2.04	1.78	1.41	0.83	-0.33	-5.75
0.04	2.71	2.67	2.56	2.42	2.26	2.06	1.79	1.41	0.80	-0.40	-5.84
0.05	2.69	2.65	2.55	2.42	2.26	2.05	1.78	1.39	0.77	-0.47	-5.98
0.06	2.66	2.62	2.53	2.40	2.24	2.04	1.76	1.37	0.73	-0.53	-6.13
0.07	2.61	2.58	2.49	2.37	2.21	2.01	1.74	1.34	0.69	-0.60	-6.29
0.08	2.56	2.53	2.45	2.34	2.18	1.98	1.71	1.30	0.65	-0.66	-6.44
0.09	2.51	2.48	2.41	2.29	2.15	1.95	1.68	1.27	0.60	-0.73	-6.60
0.10	2.45	2.43	2.36	2.25	2.11	1.91	1.64	1.23	0.56	-0.79	-6.75
0.15	2.14	2.12	2.07	1.99	1.87	1.69	1.42	1.01	0.31	-1.12	-7.47
0.20	1.79	1.78	1.75	1.69	1.59	1.43	1.17	0.75	0.04	-1.46	-8.14
0.25	1.43	1.42	1.40	1.36	1.28	1.14	0.89	0.48	-0.26	-1.81	-8.78
0.30	1.05	1.05	1.04	1.01	0.95	0.82	0.59	0.17	-0.57	-2.18	-9.40
0.35	0.65	0.65	0.65	0.64	0.59	0.48	0.26	-0.15	-0.91	-2.57	-10.0
0.40	0.23	0.23	0.24	0.24	0.21	0.11	-0.10	-0.51	-1.28	-2.99	-10.6
0.45	-0.22	-0.22	-0.20	-0.18	-0.20	-0.29	-0.49	-0.89	-1.68	-3.43	-11.3
0.50	-0.71	-0.70	-0.67	-0.64	-0.65	-0.72	-0.91	-1.31	-2.11	-3.91	-11.9
0.55	-1.23	-1.22	-1.19	-1.15	-1.14	-1.20	-1.38	-1.78	-2.59	-4.43	-12.6
0.60	-1.81	-1.79	-1.75	-1.70	-1.68	-1.73	-1.90	-2.30	-3.12	-5.01	-13.3
0.65	-2.46	-2.44	-2.38	-2.32	-2.29	-2.32	-2.49	-2.89	-3.72	-5.65	-14.1
0.70	-3.19	-3.16	-3.11	-3.03	-2.99	-3.01	-3.17	-3.56	-4.41	-6.38	-14.9
0.75	-4.04	-4.01	-3.94	-3.86	-3.80	-3.81	-3.96	-4.36	-5.22	-7.23	-15.9
0.80	-5.06	-5.03	-4.96	-4.86	-4.79	-4.79	-4.93	-5.33	-6.21	-8.26	-17.0
0.85	-6.36	-6.33	-6.24	-6.14	-6.05	-6.04	-6.18	-6.58	-7.47	-9.56	-18.4
0.90	-8.16	-8.12	-8.03	-7.91	-7.81	-7.79	-7.92	-8.32	-9.24	-11.4	-20.3
0.95	-11.2	-11.1	-11.0	-10.9	-10.8	-10.8	-10.9	-11.3	-12.2	-14.4	-23.4
0.99	-18.0	-18.0	-17.9	-17.7	-17.6	-17.6	-17.7	-18.1	-19.0	-21.2	-30.3

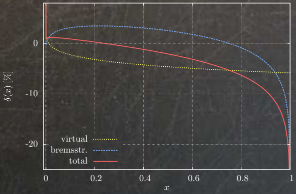
Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Results



Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Correction to the form-factor slope



Estimate of size of correction to the magnetic form-factor slope a

- take half of the slope of the curve in the low-x region
- farther from the threshold ($\nu^2 \ll x_0 \ll 1$):

$$\Delta a \equiv a_{(+\text{QED})} - a \simeq \frac{1}{2}\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}\bigg|_{x=x_0}$$

$a_{(+\text{QED})}$: measured value implicitly containing the QED radiative correction

$$\frac{1}{2}\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}\bigg|_{x=x_0} \approx -3.5\,\%$$

- bigger than the estimate on the slope $a \equiv \frac{1}{6}\langle r_\text{M}^2\rangle\Delta_M^2$ itself ($a \approx 1.8(3)\,\%$)

Using no radiative corrections in the experimental analysis

- one expects "measured" radius $\langle r_\text{M}^2\rangle_{(+\text{QED})}$ to be negative

$$\langle r_\text{M}^2\rangle_{(+\text{QED})} = \langle r_\text{M}^2\rangle + \frac{6}{\Delta_M^2}\Delta a\,, \quad \text{with } \frac{6}{\Delta_M^2}\Delta a \approx -35\,\text{GeV}^{-2}$$

$(\chi\text{PT}:\ \langle r_\text{M}^2\rangle = 18.5(2.6)\,\text{GeV}^{-2})$; in general for hadronic radii: $\langle r^2\rangle \leq (1\,\text{fm})^2 \approx 25\,\text{GeV}^{-2}$

Radiative corrections for $\Sigma^0 \to \Lambda e^+ e^-$

Integrated decay width

Integrate over the Dalitz plot (values from *Kubis and Meissner,* EPJC 18 (2001))

$$R \equiv \frac{\Gamma(\Sigma^0 \to \Lambda e^+ e^-)}{\Gamma(\Sigma^0 \to \Lambda \gamma)} = 5.541(2) \times 10^{-3}$$

$\hookrightarrow$ neglect electric form factor → expansion in magnetic form-factor slope $a \equiv \frac{1}{6}\langle r_{\mathrm{M}}^2 \rangle \Delta_M^2$

$$R = R_0 + aR_1 + \mathcal{O}(a^2)$$

+ higher order corrections as additional uncertainty

$$R = [5.530(3) + 0.626(2)a] \times 10^{-3}$$

$\hookrightarrow$ consistent with the NLO result in *Sidhu and Smith,* PRD 4 3344 (1971)

$$R_{\mathrm{S\&S}} = (5.532 + 0.627a) \times 10^{-3} \, [\approx 5.544 \times 10^{-3}]$$

From $\mathcal{B}(\Sigma^0 \to \Lambda\gamma) + \mathcal{B}(\Sigma^0 \to \Lambda e^+ e^-) + \mathcal{B}(\Sigma^0 \to \Lambda\gamma\gamma) \simeq 1$

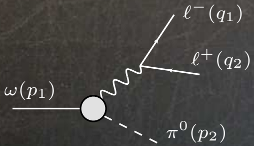
$$\mathcal{B}(\Sigma^0 \to \Lambda\gamma) = [99.4501(3) - 0.0619(2)a]\,\%$$

$$\mathcal{B}(\Sigma^0 \to \Lambda e^+ e^-) = [0.5499(3) + 0.0619(2)a]\,\%$$

$a = 0.02(2)$: $\mathcal{B}(\Sigma^0 \to \Lambda\gamma) = 99.449(2)\,\%$, $\mathcal{B}(\Sigma^0 \to \Lambda e^+ e^-) = 0.551(2)\,\%$

Radiative corrections for $\omega \to \pi^0 \ell^+ \ell^-$ decays

Radiative corrections for $\omega \to \pi^0 \ell^+ \ell^-$ decays



Dalitz decay $\omega \to \pi^0 \ell^+ \ell^-$

- $\pi\omega V$ correlator measured in the past
  ↪ NA60: Arnaldi et al., PLB 677 (2009)
  ↪ A2: Adlarson et al., PRC 95 (2017)
- radiative corrections not included (not available)

Inclusive NLO QED radiative corrections beyond the soft-photon approximation

Bachelor project: *Jacob Lindahl* (2021)

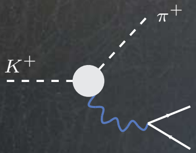
Radiative corrections for $K^+ \to \pi^+ \ell^+ \ell^-$ decays

Rare decays $K^+ \to \pi^+\ell^+\ell^-$

Parametrization

LO appears at $\mathcal{O}(p^4)$ + unitarity loop correction from $\pi\pi$ rescattering ↓
↪ universally used parametrization for the fit: $V_+(x) = a_+ + b_+ x + V_+^{\pi\pi}(x)$
↪ *Ecker et al.*, NPB 291 (1987), *D'Ambrosio et al.*, JHEP 08 (1998)

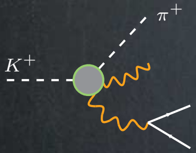
$$\frac{\mathrm{d}\Gamma_+}{\mathrm{d}x} = \frac{G_\mathsf{F}^2\alpha^2 M_K^5}{3(4\pi)^5}\lambda^{3/2}(x)\sqrt{1-\frac{4r_\ell^2}{x}}\left(1+\frac{2r_\ell^2}{x}\right)|V_+(x)|^2$$



LFU
↪ a_+s and b_+s should be the same for both (e and μ) channels
↪ discrepancy due to NP via SD effects
Moreover, the ratio deviates significantly from the VMD ansatz

$$\text{VMD:}\quad \frac{b_+}{a_+} = \frac{M_K^2}{M_\rho^2} \approx 0.4\,, \qquad \text{exp.:}\quad \frac{b_+}{a_+} \approx 1.4$$

Measurement of quadratic term $c_+ x^2$ may further test the VMD hypothesis



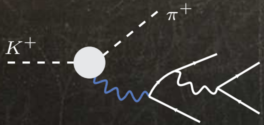
ℓ	a_+	b_+	exp.
e	$-0.587(10)$	$-0.655(44)$	E865
e	$-0.578(16)$	$-0.779(66)$	NA48/2
μ	$-0.575(39)$	$-0.813(145)$	NA48/2
μ	$-0.592(15)$	$-0.699(58)$	NA62(2021)

to improve precision → radiative corrections, see also *Kubis and Schmidt*, EPJC 70 (2010)

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Feynman diagrams

One-photon-exchange topology

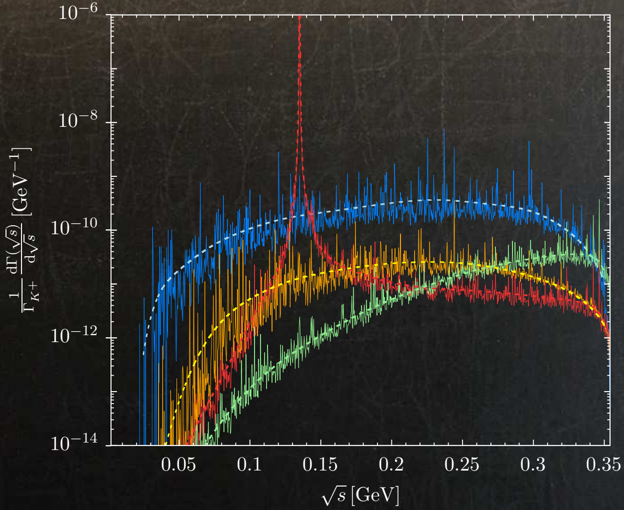


Two-photon-exchange topology

TH, arXiv:2207.02234

$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Contributions to the branching ratio



$K^+ \to \pi^+ e^+ e^- e^+ e^-$

Contributions to the branching ratio

Branching ratio calculated using Monte Carlo event generator technique:

$$B = \frac{1}{\Gamma_0} \frac{1}{4} \frac{1}{2M_K} \Phi_5 \frac{1}{N} \sum_{N \text{ events}} \overline{|\mathcal{M}|^2}$$

	$B(\sqrt{s} < 120\,\text{MeV})$	$B(\sqrt{s} > 150\,\text{MeV})$	B
(1)	5.60×10^{-12}	5.44×10^{-11}	6.70×10^{-11}
(2a)	3.11×10^{-13}	3.85×10^{-12}	4.60×10^{-12}
(2b)	1.40×10^{-13}	1.97×10^{-12}	$7.0(3) \times 10^{-6}$
κ	7.08×10^{-15}	3.69×10^{-12}	3.72×10^{-12}
$\sum$	$6.1(4) \times 10^{-12}$	$6.0(6) \times 10^{-11}$	$7.2(7) \times 10^{-11}$

$B(K^+ \to \pi^+ 4e) \simeq B(K^+ \to \pi^+\pi^0) B(\pi^0 \to 4e)$
$\hookrightarrow B(K^+ \to \pi^+\pi^0) = 20.67(8)\,\%$ and $B(\pi^0 \to 4e) = 3.38(16) \times 10^{-5}$

Summary

NLO QED radiative corrections for discussed decays are now available

Meson sector

- $\pi^0 \to e^+e^-$
  Vaško and Novotný, JHEP 1110 (2011)
  TH, Kampf and Novotný, EPJC 74 (2014)
  $\hookrightarrow$ THS model: *TH and S. Leupold*, EPJC 75 (2015)
  $\hookrightarrow$ measure $B(\pi^0 \to e^+e^-)$, extract $\chi^{(r)}(M_\rho)$
- $\pi^0 \to e^+e^-\gamma$
  TH, Kampf and Novotný, PRD 92 (2015)
  $\hookrightarrow$ precise determination of R: *TH, Goudzovski and Kampf*, PRL 122 (2018)
  $\hookrightarrow$ could be used in future exp. analysis of $K^+ \to \pi^+e^+e^-$
- $\eta^{(\prime)} \to \ell^+\ell^-\gamma$
  TH, Kampf, Leupold and Novotný, PRD 97 (2018)
- $K^+ \to \pi^+\ell^+\ell^-$
  TH, in preparation
  $\hookrightarrow$ SM estimate of $B(K^+ \to \pi^+4e)$: *TH*, arXiv:2207.02234

Baryon sector

- $\Sigma^0 \to \Lambda e^+e^-$
  TH and Leupold, EPJC 80 (2020)

Summary

NLO QED radiative corrections for discussed decays are now available

Meson sector

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Baryon sector

- $\Sigma^0 \to \Lambda e^+e^-$
  TH and Leupold, EPJC 80 (2020)

Thank you for listening!