Baryon bags in lattice QCD

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C. Gattringer, Physical Review D 97 (2018) [arXiv:1802.09417]

Introductory comments

- Often quantum field theories are formulated in terms of fundamental fields which are different from the physical degrees of freedom that are observable.
- Which degrees of freedom are relevant is usually a scale/energy dependent question.
- Here we show that one can exactly rewrite the QCD path integral such that the system can choose between different forms of degrees of freedom.
- Part of the motivation: For a efficient lattice Monte Carlo simulation, a good representation must capture the physically relevant degrees of freedom.
- The baryon bag picture I discuss here is part of a larger program of exactly mapping lattice field theories to dual representations in terms of worldlines and worldsheets.

What has been achieved with dual representations?

Dual representations were found for several interesting classes of models. Dual simulations have helped to understand finite density physics in quantum field theory non-perturbatively.

- Effective Polyakov loop models with chemical potential. PRL 2011, NPB 2011, CPC 2012, NPB 2012
- Charged ϕ^4 field at finite density (relativistic Bose gas). NPB 2013, PLB 2013
- \mathbb{Z}_3 and U(1) gauge-Higgs models with chemical potential. PRD 2012, PRL 2013, CPC 2013
- Scalar QED $_2$ and Schwinger model with heta and μ . NPB 2015, PRD 2015, NPB 2017
- O(N) and CP(N-1) models with chemical potential. PLB 2015, PRD 2016
- Heavy dense QCD at $N_c=2$ and 3. NPB 2017, PRD 2018
- Principal chiral model with chemical potential. PLB 2018
- Relation between low temperature condensation and scattering data. PRL 2015, PRL 2018

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Further work by: Shailesh Chandrasekharan, Ulli Wolff, Urs Wenger, Owe Philipsen

An important step would be combining sets of fermion worldlines into groups with real and positive weight. Maybe use Baryon Bags?

What are Baryon Bags?

- Baryon bags have nothing to do with the QCD bag model !!
- Fermion bags are domains in space-time where the degrees of freedom are free fermions. Shailesh Chandrasekharan, PRD 82 (2010)
- Outside fermion bags the dynamics is described by interaction terms.
- So far fermion bag formulations only for theories with 4-fermi interactions, but also lattice QCD allows for the introduction of fermion bags.
- Inside the fermion bags of lattice QCD three quarks propagate coherently like a free fermion ⇒ "Baryon Bags" with free fermion determinants for baryons inside the bag.
- Outside the baryon bags the relevant degrees of freedom are quarks and diquarks interacting with the gauge degrees of freedom.
- Baryon bags are an interesting tool for dual/worldline descriptions.

Framework

Lattice QCD with one flavor of staggered fermions:

$$Z = \int D[U] e^{S_G[U]} Z_F[U] \quad , \qquad Z_F[U] = \int D[\overline{\psi}, \psi] e^{S_F[\overline{\psi}, \psi, U]}$$

Action:

$$S_F[\overline{\psi},\psi,U] = \sum_x \left[2m \,\overline{\psi}_x \psi_x + \sum_\nu \gamma_{x,\nu} \left[e^{\mu \delta_{\nu 4}} \,\overline{\psi}_x U_{x,\nu} \psi_{x+\hat{\nu}} - e^{-\mu \delta_{\nu 4}} \,\overline{\psi}_{x+\hat{\nu}} U^{\dagger}_{x,\nu} \psi_x \right] \right]$$

Factorization of the Boltzmann factor:

$$e^{S_F[\overline{\psi},\psi,U]} = \prod_x e^{2m\,\overline{\psi}_x\psi_x} \prod_{x,\nu} e^{\gamma_{x,\nu}\,e^{\mu\delta_{\nu 4}}\,\overline{\psi}_x\,U_{x,\nu}\,\psi_{x+\hat{\nu}}} e^{-\gamma_{x,\nu}\,e^{-\mu\delta_{\nu 4}}\,\overline{\psi}_{x+\hat{\nu}}\,U_{x,\nu}^{\dagger}\,\psi_x}$$

Expansion of hopping terms

Taylor series:

$$e^{\gamma e^{\mu} \overline{\psi} U\psi} = 1 + \gamma e^{\mu} (\overline{\psi} U\psi) + \frac{e^{2\mu}}{2!} (\overline{\psi} U\psi)^2 + \frac{\gamma e^{3\mu}}{3!} (\overline{\psi} U\psi)^3$$
$$= \left[1 + \frac{\gamma e^{3\mu}}{3!} (\overline{\psi} U\psi)^3\right] \left[1 + \gamma e^{\mu} (\overline{\psi} U\psi) + \frac{e^{2\mu}}{2!} (\overline{\psi} U\psi)^2\right]$$
$$= e^{\frac{\gamma e^{3\mu}}{3!} (\overline{\psi} U\psi)^3} \sum_{k=0}^2 \frac{(\gamma e^{\mu} \overline{\psi} U\psi)^k}{k!}$$

Cubic term is independent of gauge fields:

$$\begin{split} (\overline{\psi}U\psi)^3 &= (\overline{\psi}_a U_{ab}\psi_b)^3 = \overline{\psi}_a \psi_b \overline{\psi}_{a'} \psi_{b'} \overline{\psi}_{a''} \psi_{b''} U_{ab} U_{a'b'} U_{a''b''} \\ &= -\overline{\psi}_a \overline{\psi}_{a'} \overline{\psi}_{a''} \psi_b \psi_{b'} \psi_{b''} U_{ab} U_{a'b'} U_{a''b''} \\ &= \overline{\psi}_3 \overline{\psi}_2 \overline{\psi}_1 \psi_1 \psi_2 \psi_3 \epsilon_{a \, a' a''} U_{ab} U_{a'b'} U_{a''b''} \epsilon_{b \, b'b''} \\ &= \overline{\psi}_3 \overline{\psi}_2 \overline{\psi}_1 \psi_1 \psi_2 \psi_3 \epsilon_{a \, a' a''} \epsilon_{a \, a' a''} \det U = 3! \overline{\psi}_3 \overline{\psi}_2 \overline{\psi}_1 \psi_1 \psi_2 \psi_3 \end{split}$$

Separation of baryon terms

Definition of baryon fields:

$$\overline{B}_{x} = \overline{\psi}_{x,3} \,\overline{\psi}_{x,2} \,\overline{\psi}_{x,1} \quad , \qquad B_{x} = \psi_{x,1} \,\psi_{x,2} \,\psi_{x,3} \qquad (= \epsilon_{abc} \,\psi_{x,a} \,\psi_{x,b} \,\psi_{x,c} \,/ \, 3! \,)$$

Hopping term with all indices:

$$e^{\gamma_{x,\nu} e^{\mu \delta_{\nu 4}} \overline{\psi}_x U_{x,\nu} \psi_{x+\hat{\nu}}} = e^{\gamma_{x,\nu} e^{3\mu \delta_{\nu 4}} \overline{B}_x B_{x+\hat{\nu}}} \sum_{k_{x,\nu}=0}^2 \frac{\left(\gamma_{x,\nu} e^{\mu \delta_{\nu 4}} \overline{\psi}_x U_{x,\nu} \psi_{x+\hat{\nu}}\right)^{k_{x,\nu}}}{k_{x,\nu}!}$$

Same for mass term:

$$e^{2m\,\overline{\psi}_x\psi_x} = e^{2M\,\overline{B}_xB_x} \sum_{s_x=0}^2 \frac{\left(2m\,\overline{\psi}_x\psi_x\right)^{s_x}}{s_x!}$$

 $M = 4m^3$

Factorized fermion integrand

Boltzmann factor:

$$e^{S_{F}[\overline{\psi},\psi,U]} = e^{S_{B}[\overline{B},B]} \sum_{\{s,k,\overline{k}\}} \prod_{x} \frac{\left(2m\,\overline{\psi}_{x}\psi_{x}\right)^{s_{x}}}{s_{x}!}$$
$$\times \prod_{x,\nu} \frac{\left[\gamma_{x,\nu}\,e^{\mu\,\delta_{\nu 4}}\,\overline{\psi}_{x}\,U_{x,\nu}\,\psi_{x+\hat{\nu}}\right]^{k_{x,\nu}}}{k_{x,\nu}!} \frac{\left[-\gamma_{x,\nu}\,e^{-\mu\,\delta_{\nu 4}}\,\overline{\psi}_{x+\hat{\nu}}\,U_{x,\nu}^{\dagger}\,\psi_{x}\right]^{\overline{k}_{x,\nu}}}{\overline{k}_{x,\nu}!}$$

Free staggered fermion action for baryon contributions:

$$S_B[\overline{B}, B] = \sum_x \left[2M \,\overline{B}_x \, B_x + \sum_\nu \gamma_{x,\nu} \left[e^{3\mu\delta_{\nu 4}} \,\overline{B}_x \, B_{x+\hat{\nu}} - e^{-3\mu\delta_{\nu 4}} \,\overline{B}_{x+\hat{\nu}} \, B_x \right] \right]$$

Chirally symmetric for m = 0, since $M = 4m^3$.

Partition sum with factorized baryon contributions

Partition sum:

$$Z = \int D[\overline{\psi}, \psi] e^{S_B[\overline{B}, B]} \int D[U] e^{S_G[U]} W_{QD}[\overline{\psi}, \psi, U]$$

Non-baryonic terms:

$$W_{QD}[\overline{\psi},\psi,U] = \sum_{\{s,k,l\}} \prod_{x} \frac{\left[2m \overline{\psi}_{x} \psi_{x}\right]^{s_{x}}}{s_{x}!} \prod_{x,\nu} \frac{(\gamma_{x,\nu})^{k_{x,\nu}+l_{x,\nu}} e^{\mu \delta_{\nu 4}(k_{x,\nu}-l_{x,\nu})}}{k_{x,\nu}! l_{x,\nu}!} \times \left[\overline{\psi}_{x} U_{x,\nu} \psi_{x+\hat{\nu}}\right]^{k_{x,\nu}} \left[-\overline{\psi}_{x+\hat{\nu}} U_{x,\nu}^{\dagger} \psi_{x}\right]^{l_{x,\nu}}$$

Sums over configurations of monomers quark- and diquark hops:

$$\sum_{\{s,k,l\}} = \left[\prod_{x} \sum_{s_{x}=0}^{2}\right] \left[\prod_{x,\nu} \sum_{k_{x,\nu}=0}^{2} \sum_{l_{x,\nu}=0}^{2}\right]$$

Evaluation in the strong coupling limit

Strong coupling \Rightarrow $S_G[U] = 0$

$$\begin{split} \int D[U] W_{QD}[\overline{\psi}, \psi, U] &= \sum_{\{s,k,l\}} \prod_{x} \frac{\left[2m \overline{\psi}_{x} \psi_{x}\right]^{s_{x}}}{s_{x}!} \prod_{x,\nu} \frac{(\gamma_{x,\nu})^{k_{x,\nu}+l_{x,\nu}} e^{\mu \delta_{\nu 4}(k_{x,\nu}-l_{x,\nu})}}{k_{x,\nu}! l_{x,\nu}!} \\ &\times \int_{\mathrm{SU}(3)} d \, \underline{U}_{x,\nu} \left[\overline{\psi}_{x} \, \underline{U}_{x,\nu} \, \psi_{x+\hat{\nu}}\right]^{k_{x,\nu}} \left[-\overline{\psi}_{x+\hat{\nu}} \, \underline{U}_{x,\nu}^{\dagger} \, \psi_{x}\right]^{l_{x,\nu}} \end{split}$$

Integrating the center group $\Rightarrow k_{x,\nu} = l_{x,\nu}$

Two SU(3) group integrals

$$\begin{split} &\int_{\rm SU(3)} dU \, U_{ab} \, U_{cd}^{\dagger} = \frac{1}{3} \, \delta_{ad} \, \delta_{bc} \\ &\int_{\rm SU(3)} dU \, U_{ab} \, U_{cd} \, U_{ef}^{\dagger} \, U_{gh}^{\dagger} = \frac{1}{8} [\delta_{af} \, \delta_{be} \, \delta_{ch} \, \delta_{dg} + \delta_{ah} \, \delta_{bg} \, \delta_{cf} \, \delta_{de}] - \frac{1}{24} [\delta_{af} \, \delta_{bg} \, \delta_{ch} \, \delta_{de} + \delta_{ah} \, \delta_{be} \, \delta_{cf} \, \delta_{dg}] \end{split}$$

Fully factorized strong coupling partition sum

At strong coupling the partition sum factorizes into baryon, quark- and diquark contributions:

$$Z = \int D[\overline{\psi}, \psi] e^{S_B[\overline{B}, B]} \sum_{\{s,k\}} \prod_x \frac{\left[2m \overline{\psi}_x \psi_x\right]^{s_x}}{s_x!} \prod_{x,\nu} \frac{(3-k_{x,\nu})!}{6 k_{x,\nu}!} [\overline{\psi}_x \psi_x \ \overline{\psi}_{x+\hat{\nu}} \psi_{x+\hat{\nu}}]^{k_{x,\nu}}$$

Baryon action:

$$S_B[\overline{B}, B] = \sum_x \left[2M \,\overline{B}_x \,B_x \,+\, \sum_\nu \gamma_{x,\nu} \left[e^{3\mu\delta_{\nu 4}} \,\overline{B}_x \,B_{x+\hat{\nu}} - e^{-3\mu\delta_{\nu 4}} \,\overline{B}_{x+\hat{\nu}} \,B_x \right] \right]$$

$$\overline{B}_x = \overline{\psi}_{x,3} \overline{\psi}_{x,2} \overline{\psi}_{x,1} \quad , \quad B_x = \psi_{x,1} \psi_{x,2} \psi_{x,3}$$

Saturating the Grassmann integral with baryon terms

Baryon action:

$$S_B[\overline{B}, B] = \sum_x \left[2M \,\overline{B}_x \,B_x \,+\, \sum_\nu \gamma_{x,\nu} \left[e^{3\mu\delta_{\nu 4}} \,\overline{B}_x \,B_{x+\hat{\nu}} - e^{-3\mu\delta_{\nu 4}} \,\overline{B}_{x+\hat{\nu}} \,B_x \right] \right]$$
$$\overline{B}_x \,=\, \overline{\psi}_{x,3} \,\overline{\psi}_{x,2} \,\overline{\psi}_{x,1} \quad, \quad B_x \,=\, \psi_{x,1} \,\psi_{x,2} \,\psi_{x,3}$$

Expanding $e^{S_B[\overline{B},B]}$ we may saturate the Grassmann integral in arbitrary subsets \mathcal{B}_i of space-time with the monomer, dimer and loop terms of the staggered action. Since \overline{B}_x and B_x are products of all three color components this completely saturates the Grassmann integral inside our chosen "Baryon Bags" \mathcal{B}_i .



Saturating the Grassmann integral in the complementary domain

Baryon bags: \mathcal{B}_i Configuration of all baryon bags: $\mathcal{B} = \bigcup_i \mathcal{B}_i$ (union of all bags) Complementary domain: $\overline{\mathcal{B}} = \Lambda/\mathcal{B}$

In the complementary domain we use the non-baryonic terms:

$$\sum_{\{s,k\}} \prod_{x} \frac{\left[2m \overline{\psi}_{x} \psi_{x}\right]^{s_{x}}}{s_{x}!} \prod_{x,\nu} \sum_{k_{x,\nu}=0}^{2} \frac{(3-k_{x,\nu})!}{6 k_{x,\nu}!} [\overline{\psi}_{x} \psi_{x} \ \overline{\psi}_{x+\hat{\nu}} \psi_{x+\hat{\nu}}]^{k_{x,\nu}}$$
Quark monomers: $\overline{\psi}_{x,a} \psi_{x,a}$
Diquark monomers: $\overline{\psi}_{x,a} \psi_{x,a} \overline{\psi}_{x,b} \psi_{x,b}$
Quark dimers: $\overline{\psi}_{x,a} \psi_{x,a} \overline{\psi}_{x+\hat{\nu},a'} \psi_{x+\hat{\nu},a'}$
Diquark dimers: $\overline{\psi}_{x,a} \psi_{x,a} \overline{\psi}_{x,b} \psi_{x,b} \overline{\psi}_{x+\hat{\nu},a'} \psi_{x+\hat{\nu},b'} \psi'_{x+\hat{\nu},b}$
 $\sum_{x} \sum_{x+\hat{v}} \sum_{x+\hat{v}}$

Baryon bag decomposition of the Grassmann measure

Theorem:

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Grassmann terms in the complementary domain $\overline{\mathcal{B}}$ do not mix with the terms in the baryon bags \mathcal{B}_i .

 \Rightarrow The Grassmann measure can be factorized into baryon bag terms and terms in the complementary domain:

$$\int D[\overline{\psi}, \psi] = \prod_{i} \int D_{\mathcal{B}_{i}}[\overline{\psi}, \psi] \times \int D_{\overline{\mathcal{B}}}[\overline{\psi}, \psi] \quad \text{with}$$

$$\int D_{\mathcal{B}_{i}}[\overline{\psi}, \psi] = \prod_{x \in \mathcal{B}_{i}} \int \prod_{a} d\psi_{x,a} \, d\overline{\psi}_{x,a} \,, \quad \int D_{\overline{\mathcal{B}}}[\overline{\psi}, \psi] = \int \prod_{x \in \overline{\mathcal{B}}} \int \prod_{a} d\psi_{x,a} \, d\overline{\psi}_{x,a}$$

 \Rightarrow Sum over quark and diquark terms is restricted to $\overline{\mathcal{B}}$:

$$\sum_{\{s,k \parallel \overline{\mathcal{B}}\}} = \prod_{x \in \overline{\mathcal{B}}} \sum_{s_x=0}^2 \prod_{(x,\nu) \in \overline{\mathcal{B}}} \sum_{k_{x,\nu}=0}^2$$

Quarks and diquarks in the complementary domain

Grassmann integral in the complementary domain

$$Z_{\overline{\mathcal{B}}} = \int D_{\overline{\mathcal{B}}}[\overline{\psi}, \psi] \sum_{\{s,k \parallel \overline{\mathcal{B}}\}} \prod_{x} \frac{\left[2m \overline{\psi}_{x} \psi_{x}\right]^{s_{x}}}{s_{x}!} \prod_{x,\nu} \sum_{k_{x,\nu}=0}^{2} \frac{(3-k_{x,\nu})!}{6 k_{x,\nu}!} [\overline{\psi}_{x} \psi_{x} \ \overline{\psi}_{x+\hat{\nu}} \psi_{x+\hat{\nu}}]^{k_{x,\nu}}$$

$$= \sum_{\{m,d \parallel \overline{\mathcal{B}}\}} (2m)^{\mathcal{N}(m_1) + \mathcal{N}(m_2)} \left(\frac{1}{3}\right)^{\mathcal{N}(d_1) + \mathcal{N}(d_2)}$$

- # quark monomers $= \mathcal{N}(m_1)$
- # diquark monomers $= \mathcal{N}(m_2)$
- # quark dimers $= \mathcal{N}(d_1)$
- # diquark hops $= \mathcal{N}(d_2)$



Fermion determinants for baryon bags

Contribution of a baryon bag \mathcal{B}_i :

$$Z_{\mathcal{B}_i} = \int D_{\mathcal{B}_i}[\overline{\psi}, \psi] e^{S_B^{(i)}[\overline{B}, B]} = \int D_{\mathcal{B}_i}[\overline{\psi}, \psi] e^{\sum_{x,y} \overline{B}_x D_{x,y}^{(i)} B_y}$$

Grassmann measure:

$$D_{\mathcal{B}_i}[\overline{\psi},\psi] = \prod_{x\in\mathcal{B}_i} \int \prod_a d\psi_{x,a} \, d\overline{\psi}_{x,a} = \prod_{x\in\mathcal{B}_i} \int d\psi_{x,3} \, d\psi_{x,2} \, d\psi_{x,1} \, d\overline{\psi}_{x,1} \, d\overline{\psi}_{x,2} \, d\overline{\psi}_{x,3}$$

Anticommuting baryon variables and measures form a new Grassmann algebra:

$$\overline{B}_x = \overline{\psi}_{x,3} \, \overline{\psi}_{x,2} \, \overline{\psi}_{x,1} \qquad , \qquad B_x = \psi_{x,1} \, \psi_{x,2} \, \psi_{x,3}$$

$$d\overline{B}_x \equiv d\overline{\psi}_{x,1} \, d\overline{\psi}_{x,2} \, d\overline{\psi}_{x,3} \quad , \quad dB_x \equiv d\psi_{x,3} \, d\psi_{x,2} \, d\psi_{x,1}$$

 \Rightarrow Gaussian integral gives rise to bag determinants:

$$Z_{\mathcal{B}_i} = \det D^{(i)}$$

Symmetries of the bag Dirac operator

Structure of the bag Dirac operators $D^{(i)}$ at $\mu = 0$: $D^{(i)} = 2M \mathbbm{1} + A^{(i)}$ with $A^{(i)}$ anti-hermitian

Staggered chiral transformation:

$$\Gamma_{5\ x,y}^{(i)} = \theta_x^{(i)} (-1)^{x_1 + x_2 + x_3 + x_4} \delta_{x,y}$$

Transformation of $A^{(i)}$:

$$\Gamma_5^{(i)} A^{(i)} \Gamma_5^{(i)} = -A^{(i)}$$

Eigenvalues come in complex conjugate pairs:

$$2M \pm i\lambda \quad \Rightarrow \quad \det D^{(i)} > 0$$

Bag determinants sum up large sets of contributions in the old Rossi/Wolff/Karsch/Mütter representation.

Final form of partition sum with baryon bags

Bag-factorized partition sum

$$Z = \sum_{\{\mathcal{B}\}} \prod_{i} \det D^{(i)} \times Z_{\overline{\mathcal{B}}}$$



- The partition function is a sum over configurations of baryon bags and the path integral is decomposed into baryon bag contributions and terms in the complementary domain.
- Inside the baryon bags \mathcal{B}_i the system chooses a description with freely propagating baryons as degrees of freedom.
- For $\mu = 0$ the bag determinants det $D^{(i)}$ are real and positive.
- In the complementary domain $\overline{\mathcal{B}}$ the relevant degrees of freedom are monomers and dimers for quarks and diquarks.
- The dynamics and scale of the fermion bags depends on the couplings. \Rightarrow MC update

Toy model for development: 2 fermion flavors coupled to U(1) gauge fields

(work together with Carlotta Marchis)

Bag-factorized partition sum

$$Z \;=\; \sum_{\{\mathcal{B}\}} \prod_i \operatorname{perm} H^{(i)} \times \; Z_{\overline{\mathcal{B}}}$$

- Again the partition function is a sum over configurations of bags and the path integral is decomposed into bag contributions and terms in the complementary domain.
- Inside the bags \mathcal{B}_i the system chooses a description with freely propagating composed bosons as degrees of freedom.
- However, the bosons are nilpotent and instead of bag determinants one obtains the permanent of a Helmholtz operator $H^{(i)}$ restricted to the corresponding bag \mathcal{B}_i .
- In the complementary domain $\overline{\mathcal{B}}$ the relevant degrees of freedom are monomers and dimers, as well as "filled fermion loops" that couple to the gauge dynamics.

- Baryon bags are a new way to organize the lattice path integral of QCD.
- The partition function is a sum over configurations of baryon bags and the path integral is decomposed into baryon bag contributions and terms in the complementary domain.
- Inside the baryon bags \mathcal{B}_i the system chooses a description with freely propagating baryons as degrees of freedom.
- In the complementary domain $\overline{\mathcal{B}}$ the relevant fermionic degrees of freedom are quarks and diquarks.
- Density, average size and other properties of the baryon bags B_i are expected to depend on the couplings. ⇒ MC update
- Baryon bags might be a key ingredient for a full dualization of QCD, e.g., based on the ACF approach. (C. Gattringer, C. Marchis, NPB 916, 2017 and PRD 97, 2018)