Recent Results on the confining string in Lattice Gauge Theories.

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Summary:

1. Introduction and motivation
2. Lorentz invariance and "universality".
3. Example 1: The boundary term.
4. Example 2: Interface in the 3d Ising model
5. Example 3: Rigid String.
6. Example 4: Thermodynamics of pure LGTs and the closed string spectrum

Lattice regularization and quark confinement.

Only a truly non-perturbative approach such as lattice regularization can describe the deconfinement transition and the confined phase of non-abelian gauge theories.
For SU(N) pure gauge theories on the lattice the dynamics are described by the standard Wilson action

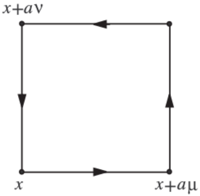
$$S_W = \beta \sum_{p=sp,tp} (1 - \frac{1}{N} \mathrm{ReTr} U_p)$$

where U_P is the product of four U_μ SU(N) variables on the space-like or time-like plaquette P and $\beta = \frac{2N}{g^2}$.
The partition function is

$$Z = \int \prod_{x,\mu} \mathrm{d} U_\mu(x) e^{-S_W}$$

the expectation value of an observable A

$$\langle A \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathrm{d} U_\mu(n)\, A(U_\mu(n))\, e^{-S_W}$$



Lattice determination of the interquark potential.

In pure lattice gauge theories the interquark potential is usually extracted from two (almost) equivalent observables

- Wilson loop expectation values $< W(R,T) >$ ("zero temperature potential")

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \log < W(R,T) >$$

- Polyakov loop correlators $< P(0)P(R)^\dagger >$ ("finite temperature potential")

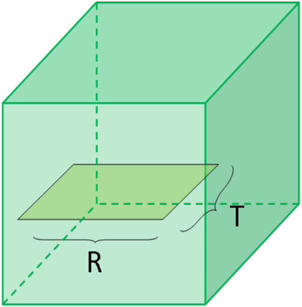
$$< P(0)P(R)^\dagger > \sim \sum_{n=0}^{\infty} c_n \; e^{-LE_n}$$

  where L is the inverse temperature, i.e. the length of the lattice in the compactified imaginary time direction

$$E_0 = V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^\dagger >$$

Wilson Loop.

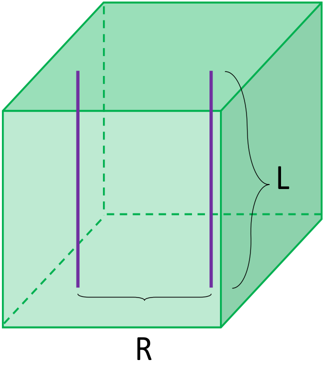
A Wilson loop of size $R \times T$



$$V(R) = \lim_{T \to \infty} -\frac{1}{T} \log < W(R, T) >$$

Polyakov loop correlator.

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^{\dagger} >$$

Wilson Loops.

In the Wilson loop framework confinement is equivalent to the well known area-perimeter-constant law:

$$< W(R,T) >= e^{-(\sigma RL + c(R+T)+k)}$$

which implies

$$V(R) = \sigma R + c \quad .$$

Confinement is usually associated to the creation of a thin flux tube joining the quark antiquark pair. (Nielsen-Olesen, 't Hooft, Wilson, Polyakov, Nambu ....) In this framework the "area law" represents the classical contribution to the interquark potential on top of which we expect to have quantum corrections. The theory which describes these quantum fluctuations is known as "effective string theory".

Effective string action

The simplest choice for the effective string action is to describe the quantum fluctuations of the flux tube as free massless bosonic degrees of freedom

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \, [\partial_\alpha X \cdot \partial^\alpha X] \ ,$$

where:

- S_{cl} describes the usual ("classical") area-perimeter term.
- $X_i(\xi_0, \xi_1)$ $(i = 1, \ldots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

The Lüscher term.

- The first quantum correction to the interquark potential is obtained summing over all the possible string configurations compatible with the Wilson loop (i.e. with Dirichlet boundary conditions along the Wilson loop).
- This is equivalent to the sum over all the possible surfaces borderd by the Wilson loop i.e. to the partition function

$$< W(R,T) >= \int e^{-\sigma RT - \frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i}$$

- The functional integration is a trivial gaussian integral, the result is

$$V(R) = \sigma R - \frac{(d-2)\pi}{24R} + c$$

- This quantum correction is known as "Lüscher term" and is universal i.e. it does not depend on the ultraviolet details of the gauge theory but only on the geometric properties of the flux tube.

The Lüscher term.

This correction is in remarkable agreement with numerical simulations. First high precision test in d=4 SU(3) LGT more than ten years ago. [1]

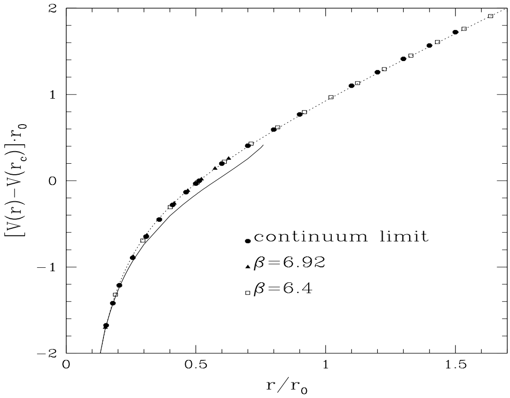


Figure : The static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory.

[1] S. Necco and R. Sommer, Nucl.Phys. B622 (2002) 328

The Lüscher term.

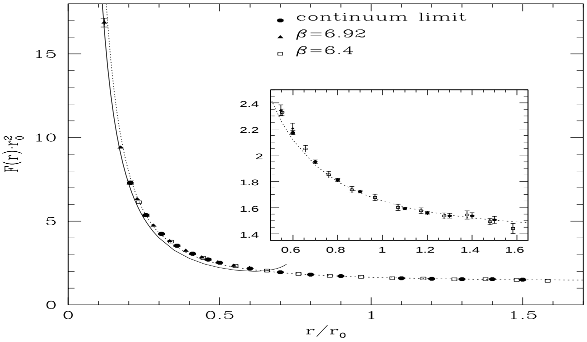


Figure : The force in the continuum limit and for finite resolution, where the discretization errors are estimated to be smaller than the statistical errors. The full line is the perturbative prediction. The dashed curve corresponds to the bosonic string model normalized by $r_0^2 F(r_0) = 1.65$.

The Nambu-Goto action.

- Evaluation of higher order quantum corrections requires further hypothesis on the nature of the flux tube. The simplest choice is the Nambu-Goto string in which quantum corrections are evaluated summing over all the possible surfaces bordered by the Wilson loop with a weight proportional to their area.

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right],$$

Interquark potential for the Nambu-Goto action.

- In the framework of the Nambu-Goto action one can evaluate exactly the energy of all the excited states of the flux tube:

$$E_n(R) = \sqrt{\sigma^2 R^2 + 2\pi\sigma\left(n - \frac{D-2}{24}\right)}$$

- In particular $E_0(R)$ corresponds to the interquark potential

$$V(R) = E_0(R) = \sqrt{\sigma^2 R^2 - 2\pi\sigma\frac{D-2}{24}}\,,$$

$$V(R) \quad \sim \quad \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 + O(1/R^5)\,,$$

The Nambu-Goto action.

High precision fit in the SU(2) case in 2+1 dimensions (A. Athenodorou, B. Bringoltz, M. Teper JHEP 1105:042 (2011))

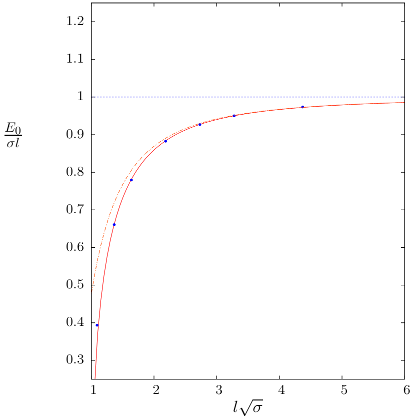
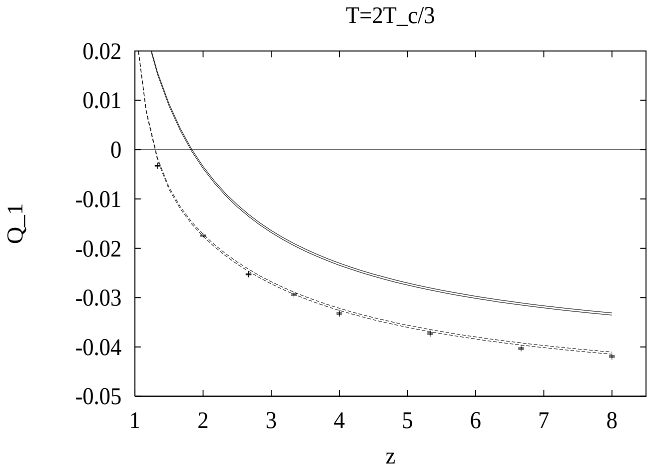


Figure 6: Energy of absolute ground state for SU(2) at $\beta = 5.6$. Compared to full Nambu-Goto (solid curve) and just the Lüscher correction (dashed curve).

The Nambu-Goto action.

High precision fit in the 2+1 dimensional Ising gauge model (M. Caselle, M. Hasenbusch, M. Panero JHEP 0301 (2003) 057)



Interquark potential via Polyakov Loop correlators and the deconfinement transition.

- In this case we have different boundary conditions in the two directions (space R and inverse temperature L).
- The novel feature of this observable is that by exchanging R and L (the so called "open-closed string transformation") we can study the finite temperature behaviour of the string tension.

$$V(R) = \sigma(T)R, \quad \sigma(T) = \sigma_0\sqrt{1 - \frac{(d-2)\pi T^2}{3\sigma_0}}$$

where T is now the temperature and σ_0 the zero temperature string tension

- From this expression we may deduce a "Nambu-Goto" prediction for the critical temperature:

$$\frac{T_c}{\sqrt{\sigma_0}} = \sqrt{\frac{3}{(d-2)\pi}}$$

which turns out to be in remarkable agreement with LGT results both in d=3 and d=4.

Why the Nambu-Goto action works so well?

- These results look nice, but they depend on a set of ad hoc assumptions on the behaviour of the flux tube. Why should we prefer the Nambu-Goto action to other possible choices for the flux tube action?
- They are "too universal" and show no dependence on the gauge group.
- It is somehow surprising that the Nambu-Goto model which looks so complex can be solved exactly at the quantum level (to all orders!!). How is it possible?
- Is there a "boundary" contribution due to the quarks at the flux tube boundaries?

In the past few years two important results changed our understanding of effective string theories and allowed us to answer to the above questions

Universality of effective string corrections.

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the action are universal and coincide with those of the Nambu-Goto action. This explains why N.-G. describes so well the infrared regime of Wilson loops or Polyakov Loop correlators.[1] [2] [3]
- The Nambu-Goto effective theory can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$ (where T and $\bar{T}$ are the two chiral components of the energy momentum tensor). This perturbation turns out to be quantum integrable and yields, using the Thermodynamic Bethe Ansatz (TBA), a spectrum which, in a suitable limit, coincides with the Nambu-Goto one. [4]

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[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065
[4] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Effective string action

The most general action for the effective string can be written as a low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2 (\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3 (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] + S_b \,,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- $X_i(\xi_0, \xi_1)$ $(i = 1, \dots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.
- In the Nambu-Goto case $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$

Effective string and spacetime symmetries.

- Symmetries of the action must hold in the low energy regime.
- String vacuum is not Poincaré invariant.

$\Longrightarrow$ Poincaré symmetry is broken spontaneously.

$$ISO(D-1,1) \rightarrow SO(D-2) \otimes ISO(1,1). \implies 3(D-2) \text{ Goldstone bosons?}$$

Only $D-2$ tranverse fluctuations of the string, where are the remaining Goldstone bosons?

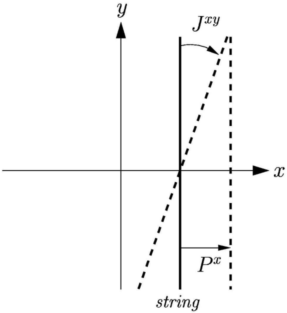
Goldstone's theorem states that there is a massless mode for each broken symmetry generator, but this counting cannot be naively extended to the case of spontaneously broken spacetime symmetries[1] .

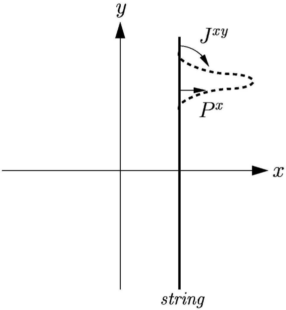
[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Effective string and spacetime symmetries.

The remaining $2(D-2)$ Lorentz transformations are realized non-linearly and induce a set of recurrence relations among different terms in the action.! [1]

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij}\xi_b - X_j \partial_b X_i \right)$$





[1] I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Non-linear realization and long-string expansion.

A few rules to construct the most general effective string action:

- Broken translations:
  $X^i \to X^i + a^i$. $\Longrightarrow$ Only field derivatives in the effective action.
- Broken rotation in the plane $(1, 2)$:

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij} \xi_b - X_j \partial_b X_i \right)$$

Number of derivatives minus number of fields (weight) preserved.

Fields and coordinates rescaling $\Longrightarrow$ Derivative expansion:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i .$$

Variations by broken rotation mix orders $\Longrightarrow$ Recurrence relations.

$ISO(1, 1)$ and $SO(D-2)$ invariance $\Longrightarrow$ Contraction of indices.

Effective string action is strongly constrained! [1] [2] [3]

- the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- In three dimensions the first allowed correction to the Nambu-Goto action turns out to be an eight derivatives term which gives a contribution to the interquark potential of the order $1/R^7$
- The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations a good agreement with the Nambu-Goto string was observed.
- The effective string action is much more predictive than typical effective models in particle physics!

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065

Beyond Nambu-Goto.

In order to find features associated to a particular LGT we must go beyond the Nambu-Goto approximation.

This effort was pursued mainly in four directions and led in the past few years some of most interesting and new results in the effective theory framework

- Boundary terms [1] [2]
- Interface free energy in the 3d Ising model[3]: Torus geometry, no boundary corrections
- Excited string states of SU(N) Yang-Mills theories[4]
- "Rigid string" and the interquark potential of the 3d U(1) gauge model [5]

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri and R. Pellegrini *JHEP05* **130** (2012), `arXiv:1202.1984`

[2] B. Brandt *JHEP07* **008** (2017), `arXiv:1705.03828`

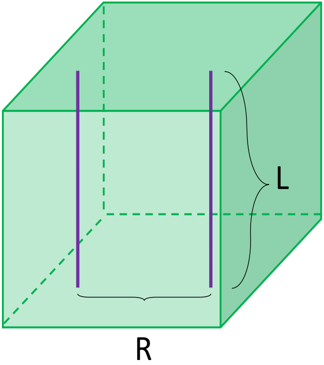
[3] M. Caselle, G. Costagliola, A.Nada, M. Panero and A. Toniato *Phys. Rev. D94* **034503** (2016), `arXiv:1604.05544`

[4] A. Athenodorou, B. Bringoltz and M. Teper, *JHEP* **02**, 030 (2011), `arXiv:1007.4720` .

[5] D. Vadacchino, M. Caselle, R. Pellegrini and M. Panero, arXiv:1311.4071 .

Example 1: The boundary term .

Expectation value of two Polyakov loops at distance R and Temperature $T = 1/L$



$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log < P(0)P(R)^{\dagger} >$$

The boundary term of the effective action: Constraints imposed by the Lorentz invariance

If the boundary is a Polyakov line in the ξ_0 direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the most general boundary action should be of this type

$$S_b = \int d\xi_0 \left[b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \dots \right] .$$

Imposing Lorentz invariance one finds that $b_1 = 0$ and that the b_2 term is only the first term of a Lorentz invariant expression[1] :

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right] .$$

which is the analogous in the case of the boundary action of the Nambu-Goto action for the "bulk" effective action.

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

The boundary contribution to the interquark potential

Following the above discussion, the leading correction coming from the boundary turns out to be:

$$S_{b,2}^{(1)} = \int d\xi_0 \left[b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X \right] .$$

Its contribution to the interquark potential can be evaluated performing a simple gaussian functional integration[1]

$$\langle S_{b,2}^{(1)} \rangle \; = \; -b_2 \frac{\pi^3 L}{60 R^4} E_4 (\mathrm{i} \frac{L}{2R}) .$$

where the Eisenstein function E_4, is defined as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n ,$$

where $q = e\ 2\pi i \tau$ and $\sigma_p(n)$ is the sum of the p-th powers of the divisors of n:

$$\sigma_p(n) = \sum_{m|n} m^p .$$

[1] O. Aharony and M. Field JHEP01(2011)065

The boundary contribution to the interquark potential

- We end up with the following expression for the interquark potential

$$V(R) = \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 - b_2 \frac{\pi^3(D-2)}{60R^4} + O(1/R^5) \ ,$$

  where b_2 is a new physical parameter, similar to the string tension σ, which depends on the theory that we study and should be determined by simulations and comparison with experiments.

- To test this picture we performed a set of high precision simulations in the case of the 3d gauge Ising model, which is the simplest possible confining gauge theory.

Simulation I: Polyakov loops

- In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators $P(R, L)$ (where L is the length of the two loops and R their distance) we measured the following ratio:

$$R_P(R, L) = \frac{P(R+1, L)}{P(R, L)} \ .$$

- Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of R and L with very high precision.

Simulation settings

- We performed our simulations in the 3d gauge Ising model, using a dual algorithm

data set	β	L	σ	$1/T_c$
1	0.743543	68	0.0228068(15)	5
2	0.751805	100	0.0105255(11)	8
3	0.754700	125	0.0067269(17)	10

Table : Some information on the data sample

Results

- The values of b_2 extracted from the data show the expected scaling behaviour $b_2 \sim \frac{1}{\sqrt{\sigma}^3}$

data set	b_2	$b_2\sqrt{\sigma}^3$	χ^2
1	7.25(15)	- 0.0250(5)	1.2
2	26.8(8)	- 0.0289(9)	1.8
3	57.9(12)	- 0.0319(7)	1.3

Table : Values of b_2 as a function of β

Wilson loops

As a check of our analysis we performed the same simulation for the Wilson loops fixing the value of b_2 obtained above. In this case there is no more parameter to fit and we can directly compare our predictions with the results of the simulations. To eliminate all the non-universal parameters we constructed the following combination:

$$R_W^{'}(L, Lu) = \frac{W(L,R)}{W(L+1, R-1)} - \exp\{-\sigma(1 + L(1-u))\} \; , \qquad u = R/L$$

Wilson loops

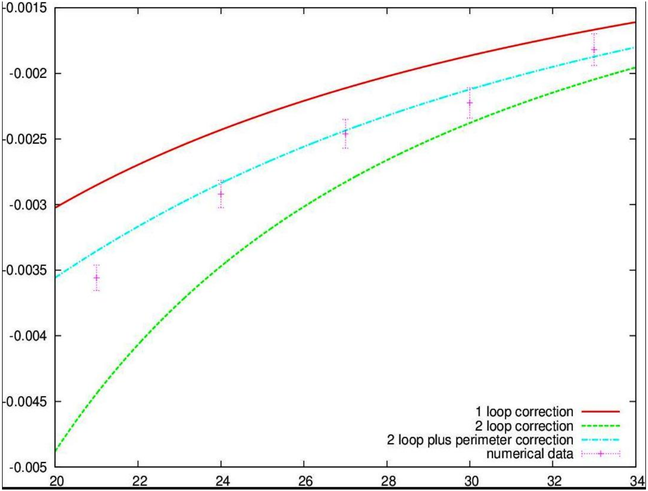


Figure : $R'_W(L, L\frac{4}{3})$ at $\beta = 0.754700$.

SU(2) and SU(3) in 2+1 dimensions

Recently B. Brandt performed the same analysis in the $SU(2)$ and $SU(3)$ LGTs[1] in three dimensions. He found the following values for the non-universal parameter b_2

$$b_2\sqrt{\sigma}^3 = -0.0257(3)(38)(17)(3) \quad \textit{for} \quad SU(2)$$

and

$$b_2\sqrt{\sigma}^3 = -0.0187(2)(13)(4)(2) \quad \textit{for} \quad SU(3)$$

[1] B. Brandt *JHEP07* **008** (2017), `arXiv:1705.03828`

Example 2: The interface free energy in the 3d Ising model.

- To identify terms beyond the Nambu-Goto action in the "bulk" effective string one must somehow screen the boundary term. A perfect choice is to study the interface free energy in the 3d Ising spin model. Due to the periodic boundary conditions in both longitudinal directions there are no boundary corrections in this observable [1].

- Evaluating the interface free energy amounts to evaluate the ratio Z_a/Z_p of the partition function with antiperiodic boundary conditions in the transverse direction Z_a with respect to the standard partition function Z_p

- Evaluating these partition functions with Montecarlo methods is highly non trivial, we proposed for this task a new "non equilibrium" algorithm based on the Jarzynski identity.

[1] M. Caselle, G. Costagliola, A.Nada, M. Panero and A. Toniato *Phys. Rev. D94* **034503** (2016), `arXiv:1604.05544`

The interface free energy

The Nambu-Goto effective action for the interface free energy can be calculated analytically[1] : for a system in D spacetime dimensions one find:

$$\frac{Z_\mathrm{a}}{Z_\mathrm{p}} = 2\left(\frac{\sigma}{2\pi}\right)^{\frac{D-2}{2}} V_\mathrm{T}\sqrt{\sigma L_1 L_2 u}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty} c_k c_{k'}\left(\frac{\mathcal{E}_{k,k'}}{u}\right)^{\frac{D-1}{2}} K_{\frac{D-1}{2}}\left(\sigma L_1 L_2 \mathcal{E}_{k,k'}\right),$$

where $u = L_2/L_1$, V_T denotes the "volume" of the system along the dimensions transverse to the interface (so $V_\mathrm{T} = L_0$ in our case), $K_\nu(z)$ denotes the modified Bessel function of the second kind of order ν and argument z, while c_k and $c_{k'}$ are coefficients appearing in the expansion of an inverse power of Dedekind's η function:

$$\frac{1}{\eta(iu)^{D-2}} = \sum_{k=0}^{\infty} c_k q^{k-\frac{D-2}{24}}, \qquad \text{with } q = \exp(-2\pi u)$$

(so that, for the $D = 3$, c_k equals the number of partitions of k) and

$$\mathcal{E}_{k,k'} = \sqrt{1 + \frac{4\pi u}{\sigma L_1 L_2}\left(k + k' - \frac{D-2}{12}\right) + \left[\frac{2\pi u(k-k')}{\sigma L_1 L_2}\right]^2}.$$

[1] M.Billo', M. Caselle and L. Ferro *JHEP* **0602** (2006) 070, arXiv:hep-th/0601191

Let us define

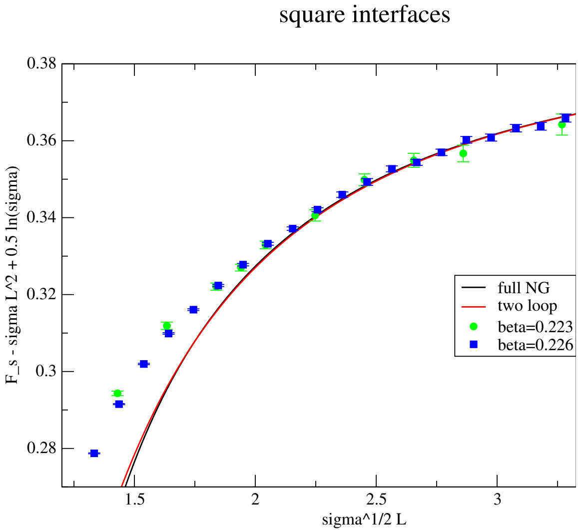
$$y(L_1, L_2) = F_{int} - F_{NG}$$

and let us study the regime $L_2 >> L_1$. We tried to fit y with different powers of $1/L_1$. We could obtain good values of the reduced χ^2 only with the following combination:

$$y = \frac{1}{(L_1\sqrt{\sigma})^7}\left[k_{-7} + \frac{k_{-9}}{(L_1\sqrt{\sigma})^2}\right],$$

N_2	k_{-7}	k_{-9}	χ_r^2
48	0.389(1)	0.03(3)	1.09
64	0.432(2)	0.22(3)	1.06
80	0.593(2)	0.25(3)	1.47
96	0.650(5)	0.410(7)	0.07

Table : Results of the fits of the difference between numerical results for the interface free energy and the corresponding Nambu-Goto prediction


square interfaces
0.38
0.36
0.34
0.32
0.3
0.28
F_s - sigma L^2 + 0.5 ln(sigma)
1.5
2
2.5
3
sigma^1/2 L
full NG
two loop
beta=0.223
beta=0.226

Example 3: Rigid String.

Another imortant path to go beyond the Nambu-Goto approximation is to add to the effective string a term proportional to the square of the extrinsic curvature [1]

Thus, the effective string action up to term proportional to $1/R^4$ is

$$S = S_{\rm NG} + S_{2,K} + S_{\rm b}$$

with:

$$S_{\rm NG} \simeq S_{\rm cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X - \frac{1}{4} (\partial_\alpha X \cdot \partial^\alpha X)^2 \right] ,$$

$$S_{2,K} \simeq \alpha \int d^2\xi (\Delta X)^2,$$

$$S_{\rm b} \simeq b_2 \int d\xi_0 \left[\partial_1 \partial_0 X \cdot \partial_1 \partial_0 X \right] .$$

Thus we are left with three free parameters (σ, α and b_2) which will be fitted comparing with the numerical data.

[1] M. Caselle, M.Panero, R. Pellegrini, D. Vadacchino, JHEP 1501 (2015) 105

Zeta-function regularization of the extrinsic curvature action

the Gaussian part of the action is

$$S = \sigma \int_0^{N_t} dt \int_0^R dr \left[1 + \frac{1}{2} \partial_\alpha X \cdot \partial^\alpha X \right] + \alpha \int_0^{N_t} dt \int_0^R dr \, (\Delta X)^2,$$

where R denotes the interquark distance, N_t is the system size in the Euclidean time direction and Δ is the two-dimensional Laplace operator $\Delta = \partial^2/\partial t^2 + \partial^2/\partial r^2$.
The interquark potential is defined as

$$V(R) = - \lim_{N_t \to \infty} \frac{1}{N_t} \ln \left\{ \int [DX] e^{-S[X]} \right\},$$

Zeta-function regularization

The Gaussian part of the action can be rewritten as

$$S = \sigma \int_0^{N_t} dt \int_0^R dr \left[1 + \frac{1}{2} X \left(1 - \frac{2\alpha}{\sigma}\Delta\right)(-\Delta) X\right].$$

Carrying out the Gaussian integration, one obtains

$$V(R) = \lim_{N_t \to \infty} \left\{ \sigma R + \frac{1}{2N_t} \mathrm{Tr}\, \ln(-\Delta) + \frac{1}{2N_t} \mathrm{Tr}\, \ln\left(1 - \frac{1}{m^2}\Delta\right) \right\},$$

The parameter $m = \frac{\sigma}{2\alpha}$, which has dimensions of a mass, encodes the contribution due to the extrinsic curvature.

Zeta-function regularization

The operator traces are singular but can be evaluated using the zeta-function regularization:

$$V(R) = \sigma R + V_{\text{NG}}(R) + V_{\text{ext}}(R, m),$$

where $V_{\text{NG}}(R)$ and $V_{\text{ext}}(R, m)$ are the Gaussian limit of the Nambu-Goto and of the extrinsic curvature contributions respectively:

$$V_{\text{NG}}(R) \equiv \lim_{N_t \to \infty} \frac{1}{2N_t} \operatorname{Tr} \ln(-\Delta) = -\frac{\pi}{24R},$$

$$V_{\text{ext}}(R, m) \equiv \lim_{N_t \to \infty} \frac{1}{2N_t} \operatorname{Tr} \ln \left(1 - \frac{1}{m}\Delta\right) = -\frac{m}{2\pi} \sum_{n=1}^{\infty} \frac{K_1\left(2nmR\right)}{n},$$

where $K_\alpha(z)$ denotes a modified Bessel function of the second kind.

Analytical properties of $V_{\rm ext}(R,m)$

$V_{\rm ext}(R,m)$ has a logarithmic branching point at $R=0$ and a set of square-root singularities for negative values of $(mR)^2$. The first is located at $(mR)^2=-\pi^2$, and defines the radius of convergence of the small mR expansion.

$$V_{\rm ext}(R,m) = -\frac{\pi}{24R}+\frac{m}{4}+\frac{m^2R}{4\pi}\left(\ln\frac{mR}{2\pi}+\gamma_{\rm E}-\frac{1}{2}\right) + \frac{m^2R}{2\pi}\sum_{n=1}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)\zeta(2n+1)}{\Gamma(n+2)\Gamma\left(n-\frac{1}{2}\right)}\left(\frac{mR}{\pi}\right)^{2n},$$

where $\gamma_{\rm E}=0.5772156649\ldots$ is the Euler-Mascheroni constant and $\zeta(x)$ denotes the Riemann zeta function.

In the large-R limit $V_{\rm ext}(R,m)$ decreases exponentially. Its behavior is dominated by the lowest-index Bessel function appearing in the sum:

$$V_{\rm ext}(R,m)\simeq-\sqrt{\frac{m}{16\pi R}}e^{-2mR}\qquad \text{for}\ \ R\gg\frac{1}{m}.$$

This is the typical behavior expected for a massive perturbation of a 2d CFT,

Analytical properties of $V_{\rm ext}(R,m)$

- $V_{\rm ext}(R,m)$ can be understood as a massive perturbation of the $c=1$ free bosonic theory. In fact, the combination

$$c_0(mR) = -\frac{24R}{\pi} V_{\rm ext}(R,m)$$

  coincides with the ground state scaling function $c_0(mR)$ describing this perturbation.
- $c_0(mR)$ is a monotonically decreasing function of its argument and interpolates between 1 for $mR=0$ and 0 for $mR \to \infty$.
- Notice the analogy with the Nambu-Goto case: while the Nambu-Goto model can be described as an irrelevant *massless* perturbation of the $c=1$ free bosonic 2d CFT, the rigid string is described by a relevant *massive* perturbation of the same CFT.
- In the $mR \to 0$ limit, the free bosonic theory is recovered: thus we find a second "Lüscher" term, in addition to the one from $V_{\rm NG}(R)$.

Main differences between the NG and rigid strings

- The field density profile around the string is gaussian in the case of the Nambu-Goto string, while it decreases exponentially in the rigid string case.
- This exponential defines a new scale, knwon as London penetration length in condensed matter and as intrinsic width in confining gauge theories.
- While in NG the string width increases logarithmically with the interquark distance, the intrinsic width of the rigid string is constant
- At very short distances the Lüscher term is doubled

3d U(1) model

- The contribution due to rigidity turns out to be very small in typical confining gauge theories except the 3d U(1) gauge model in which it has dramatic effects on all the observables from the interquark potential to the flux tube width.
- A possible reason is that in the 3d U(1) model the glueball mass has a non trivial scaling and the ratio $m_0/\sqrt{\sigma}$ vanishes in the continuum limit.
  From Polyakov's semiclassical solution[1] we know that

$$\frac{m_0}{\sqrt{\sigma}} = C(2\pi\beta)^{3/4} e^{-\pi^2 v(0)\beta/2}$$

with C and $v(0)$ constants whose value can be evaluated exactly in the semiclassical approximation.

[1] A. Polyakov, Nucl. Phys. B120 (1977) 429

3d U(1) model and the rigid string

The identification of the rigidity parameter with the glueball mass is supported also by another proposal of Polyakov[1] who gave a heuristic dual string description for the 3d $U(1)$ model in terms of a rigid string.
Polyakov was also able to compute the dependence of their coupling constants on the electric charge and the glueball mass of the original $\mathrm{U}(1)$ theory

$$S_{\text{Pol}} = c_1 e^2 m_0 \int d^2\xi \sqrt{g} \; + c_2 \frac{e^2}{m_0} \int d^2\xi \sqrt{g} K^2$$

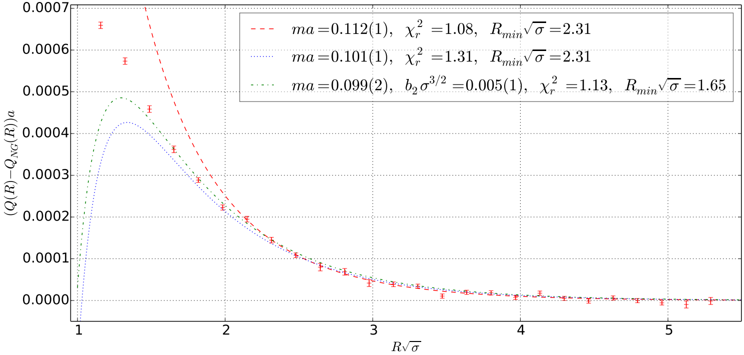
where c_1 and c_2 are two undetermined constants. If we identify these coupling constants with σ and α defined above we find (apart from an undetermined constant) $\sqrt{\sigma/\alpha} \sim m_0$.

[1] A. Polyakov, Nucl. Phys. B486 (1997) 23

3d U(1) model and the rigid string

We tesetd this conjecture performing a set of simulations for different values of β in the 3d U(1) gauge model, measuring Polyakov loop correlators for different values of the interquark distance and fitting them with the rigid string prediction, using σ, b_2 and m as free parameters. Including in the fits also the rigidity term dramatically improves the quality of the fits, moreover the rigidity parameter scales exactly as predicted by Polyakov. Here are the best-fit results for m

β	ma	$m_0 a$	m/m_0
1.7	0.28(9)	0.88(1)	0.32(10)
1.9	0.25(4)	0.56(1)	0.45(7)
2.0	0.17(2)	0.44(1)	0.39(4)
2.2	0.11(1)	0.27(1)	0.41(4)
2.4	0.06(2)	0.20(1)	0.30(10)


$ma = 0.112(1),\ \chi_r^2 = 1.08,\ R_{min}\sqrt{\sigma} = 2.31$
$ma = 0.101(1),\ \chi_r^2 = 1.31,\ R_{min}\sqrt{\sigma} = 2.31$
$ma = 0.099(2),\ b_2\sigma^{3/2} = 0.005(1),\ \chi_r^2 = 1.13,\ R_{min}\sqrt{\sigma} = 1.65$
$(Q(R) - Q_{NG}(R))a$
$R\sqrt{\sigma}$
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0.0000
1
2
3
4
5

Take home message for this part

- In the case of the 3d U(1) model, including also a rigidity term allows to perfectly fit the data.
- While the NG action was shown to be described by a massless perturbation of the $c = 1$ free field theory (perturbed by the irrelevant operator $T\bar{T}$), the rigid string correction can be described as a massive perturbation of the $c = 1$ free field theory.
- The 3d $U(1)$ lattice model turns out to be a perfect laboratory to study the cross-over from a purely Nambu-Goto string at low β to a purely rigid string at large β.
- The main differences between the NG and rigid strings are:
  - The field density profile around the string is (almost) a Gaussian in the case of a Nambu-Goto string, while it decreases exponentially in the rigid string case. This exponential defines a new scale, known as the London penetration length in condensed matter theory, and as intrinsic width in confining gauge theories
  - While in the Nambu-Goto case the string width increases logarithmically with the interquark distance at zero temperature and linearly at high temperature, the intrinsic width of the rigid string is constant
  - At very short distances the coefficient of the Lüscher term is doubled.

Example 4: Thermodynamics of pure LGTs and the closed string spectrum

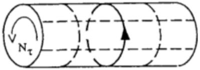
- One of the main features of SU(N) non-abelian gauge theories is the existence of a deconfinement phase transition, i.e. a temperature above which gluons are "deconfined", like the quark-gluon plasma (QGP) in Quantum Chromodynamics.

- If the confining theory is described by string-like objects, we should expect stronger and stronger string effects, due to string excitations, as the temperature increases. To identify these signatures we must study the the thermodynamics of gauge theories still in the confining phase but as near as possible to the deconfinement transition.

- Since we study pure gauge theories in the confining phase the only degrees of freedom are the so-called **"glueballs"**. Looking at the thermodynamics of the theory in the confining phase we have a tool to explore the glueball spectrum of the theory and, possibly, its string-like features.

Thermodynamic quantities

On a $N_t \times N_s^3$ lattice the volume is $V = (aN_s)^3$ (where a is the lattice spacing), while the temperature is determined by the inverse of the temporal extent (with periodic boundary conditions): $T = (aN_t)^{-1}$.



The thermodynamic quantities taken into account will be:

- the pressure p, that in the thermodynamic limit (i.e. for large and homogenous systems) can be written as

$$p \simeq \frac{T}{V} \log Z(T, V)$$

- the trace of the energy-momentum tensor Δ, that in units of T^4 is

$$\frac{\Delta}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right)$$

Energy density $\epsilon = \Delta + 3p$ and entropy density $s = \frac{\epsilon + p}{T} = \frac{\Delta + 4p}{T}$ can be easily calculated.

Ideal glueball gas

The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs.

The prediction of an ideal relativistic Bose gas can be used to describe the thermodynamics of such gas. Its partition function for 3 spatial dimensions is

$$\log Z = (2J+1)\frac{2V}{T}\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

where m is the mass of the glueball, J is its spin and K_2 is the modified Bessel function of the second kind of index 2.

Observables such as Δ and p thus can be easily derived:

$$p = \frac{T}{V}\log Z = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right)^2 K_2\left(k\frac{m}{T}\right)$$

$$\Delta = \epsilon - 3p = 2(2J+1)\left(\frac{m^2}{2\pi}\right)^2 \sum_{k=1}^{\infty}\left(\frac{T}{km}\right) K_1\left(k\frac{m}{T}\right)$$

Test with the SU(2) model

The SU(2) model is a perfect laboratory to test these results.

- It is easy to simulate: very precise results may be obtained with a reasonable amount of computing power
- The deconfinement transition is of second order and thus it is expected to coincide with the Hagedorn temperature
- The masses of several states of the glueball spectrum are known with remarkable accuracy
- The infrared physics of the model is very similar to that of the SU(3) theory, with one important difference: the gauge group is real and thus only C=1 glueball exist. The glueball spectrum contains only half of the states with respect to SU(3).

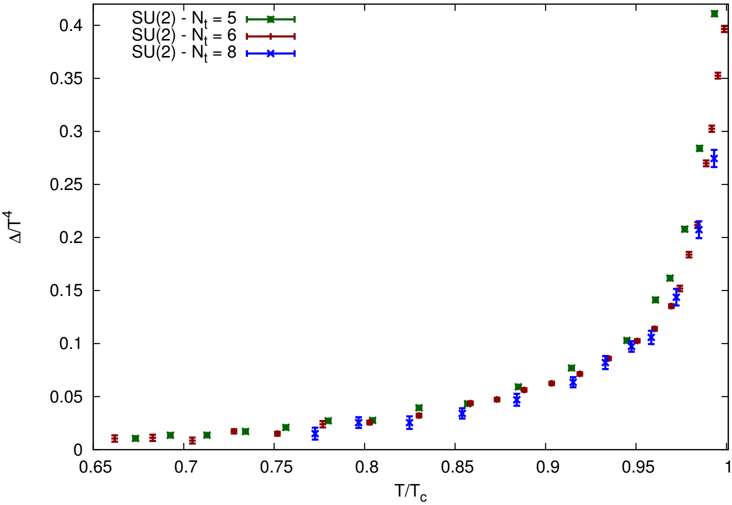
Lattice setup

N_s^4 at $T=0$	$N_s^3 \times N_t$ at $T \neq 0$	n_β	β-range	$n_{\rm conf}$
32^4	$60^3 \times 5$	17	$[2.25, 2.3725]$	1.5×10^5
40^4	$72^3 \times 6$	25	$[2.3059, 2.431]$	1.5×10^5
40^4	$72^3 \times 8$	12	$[2.439, 2.5124]$	10^5

Table : *

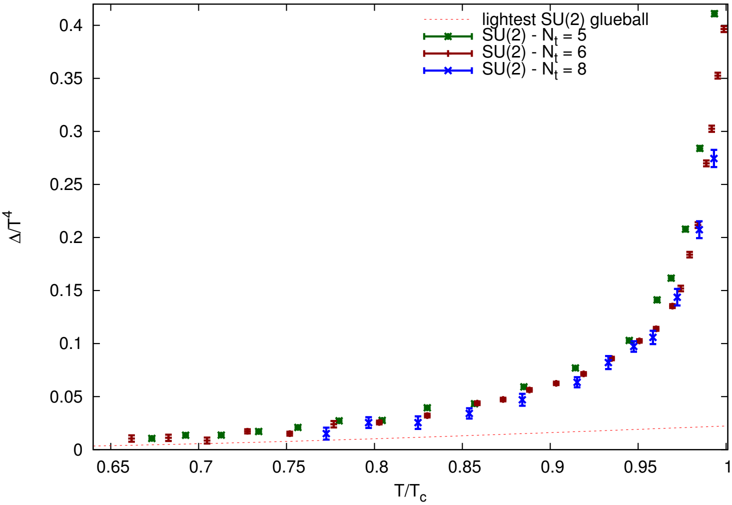
Setup of our simulations. The first two columns show the lattice sizes (in units of the lattice spacing a) for the $T=0$ and finite-temperature simulations, respectively. In the third column, n_β denotes the number of β-values simulated within the β-range indicated in the fourth column. Finally, in the fifth column we report the cardinality $n_{\rm conf}$ of the configuration set for the $T=0$ and finite-T simulations.

SU(2): trace of energy-momentum tensor



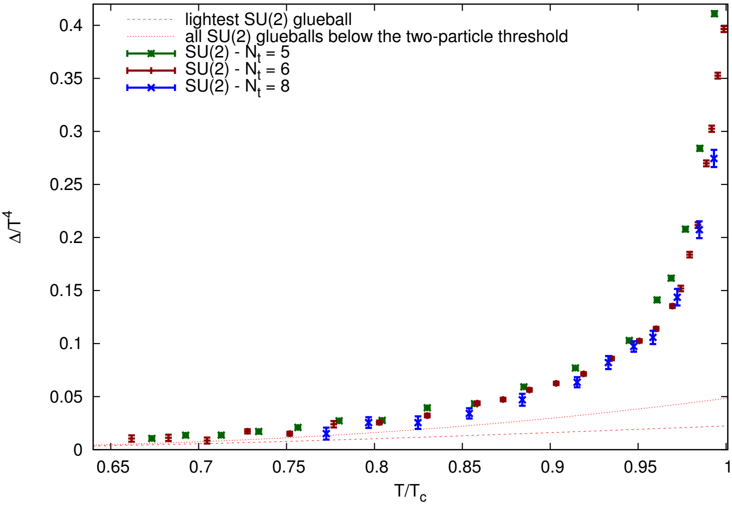
Despite the small values of N_t the data scale reasonably well.

SU(2): trace of energy-momentum tensor



Plot of the contribution of lowest glueball state $m_{0^{++}}$ compared with the data .

SU(2): trace of energy-momentum tensor



The contribution of all glueball states with mass $m < 2m_{0^{++}}$.

A few important observations

Usually the thermodynamics of the system is saturated by the first state (or, in some cases, the few lowest states) of the spectrum due to the exponential dependence on the mass.

The large gap between the $m_{0^{++}}$ and the $m < 2m_{0^{++}}$ curves and those between them and the data show that the spectrum must be of the Hagedorn type, i.e. that the number of states increases exponentially with the mass.

A Hagedorn spectrum is typically the signature of a string like origin of the spectrum.

The thermal behaviour of the model in the confining phase is thus a perfect laboratory to study the nature of this spectrum and of the underlying string model.

Effective string theory suggests that, with a very good approximation, this model should be a Nambu-Goto string. Let us see the consequences of this assumption.

Glueballs as rings of glue

A closed string model for the full glueball spectrum that follows the original work of Isgur and Paton[1,2] can be introduced to account for the values of thermodynamic quantities near the transition. In the closed-string approach glueballs are described in the limit of large masses as **"rings of glue"**, that is closed tubes of flux modelled by closed bosonic string states.

The **mass spectrum** of a closed strings gas in D spacetime dimensions is given by

$$m^2 = 4\pi\sigma \left(n_L + n_R - \frac{D-2}{12} \right)$$

where $n_L = n_R = n$ are the total contribution of left- and right-moving phonons on the string.

Every glueball state corresponds to a given phonon configuration, but associated to each fixed n there are multiple different states whose number is given by $\pi(n)$, i.e. the **partitions** of n.

[1] N. Isgur and J. Paton, *A Flux Tube Model for Hadrons in QCD*, 1985

[2] R. Johnson and M. Teper, *String models of glueballs and the spectrum of SU(N) gauge theories in (2+1)-dimensions*, 2002

The **density of states** $\rho(n)$ is expressed through the square of $\pi(n)$:

$$\rho(n) = \pi(n_L)\pi(n_R) = \pi(n)^2 \simeq 12\,(D-2)^{\frac{D-1}{2}} \left(\frac{1}{24n}\right)^{\frac{D+1}{2}} \exp\left(2\pi\sqrt{\frac{2(D-2)n}{3}}\right)$$

in D spacetime dimensions.

Spectral density

The **Hagedorn temperature**[1] is defined as

$$T_H = \sqrt{\frac{3\sigma}{\pi(D-2)}}$$

The spectral density as a function of the mass (i.e. $\hat{\rho}(m)\mathrm{d}m = \rho(n)\mathrm{d}n$) can be expressed as

$$\hat{\rho}(m) = \frac{(D-2)^{D-1}}{m}\left(\frac{\pi T_H}{3m}\right)^{D-1} e^{m/T_H}$$

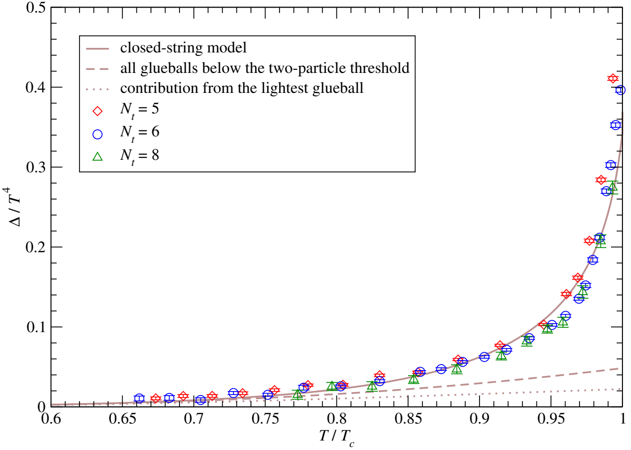
where the characteristic Hagedorn-like exponential spectrum appears and can be used to describe the glueball spectrum above an arbitrary mass threshold.

The trace of the energy-stress tensor can be integrated on masses bigger than $2m_{0^{++}}$ with the degeneracy $\hat{\rho}(m)$ and summed to the contribution of the mass states computed on the lattice.

$$\Delta = \sum_{m<2m_{0^{++}}} (2J+1)\Delta(m,T) + \int_{2m_{0^{++}}}^{\infty} \mathrm{d}m\,\hat{\rho}(m)\,\Delta(m,T)$$

[1] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147 (1965)

SU(2): trace of energy-momentum tensor



SU(2) *vs.* SU(3)

The SU(3) case was studied for the first time in 2009 in the pioneering work of Meyer[1]. Now, using the high precision lattice data for SU(3) of [2] we are in the position to test the Hagedorn behaviour in a very stringent way.
There are two main diffeences between SU(2) and SU(3):

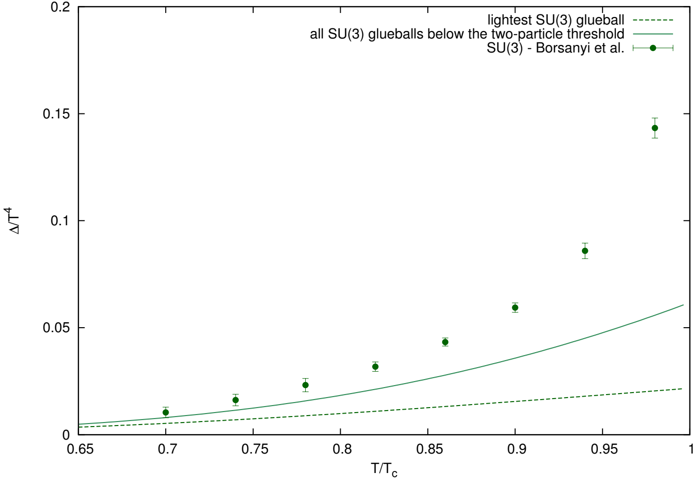
- SU(3) has a first order deconfining transition, so $T_c < T_H$.
- SU(3) has complex representations, thus glueballs have an additional quantum number C and the glueball spectrum contains twice the number of glueballs than in the SU(2) case

In principle we could consider in this case T_H as a free parameter, but in the effective string framework we may safely fix it at the expected Nambu-Goto value $T_H = \sqrt{3\sigma/2\pi} = 0.691..\sqrt{\sigma}$. Lorentz invariance of the effective string tells us that this should be a very good approximation of the exact result and that we should expect only small deviations from this value.
The relation between T_H and T_c is:

$$\frac{T_H}{T_c} = 1.098$$

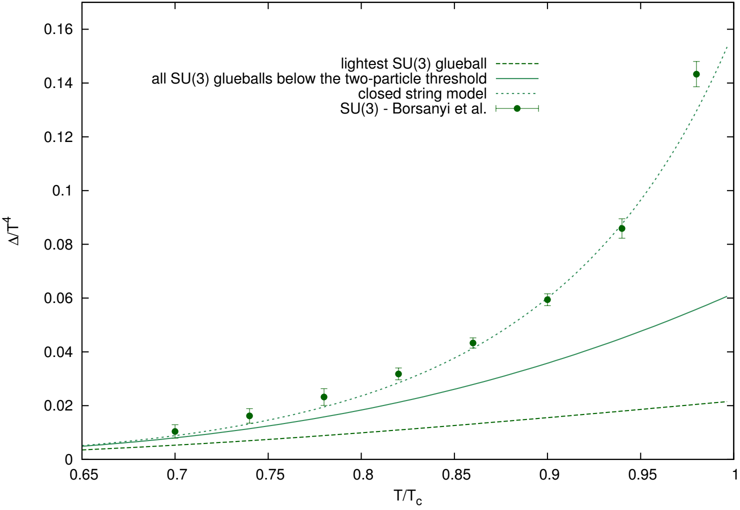
[1] H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009
[2] Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

SU(3): trace of energy-momentum tensor



Also in this case the $m < 2m_{0^{++}}$ sector of the glueball spectrum is not enough to fit the behaviour of Δ/T^4 .

SU(3): trace of energy-momentum tensor



While including the whole Hagedorn spectrum we find again a remarkable agreement with no free parameter!

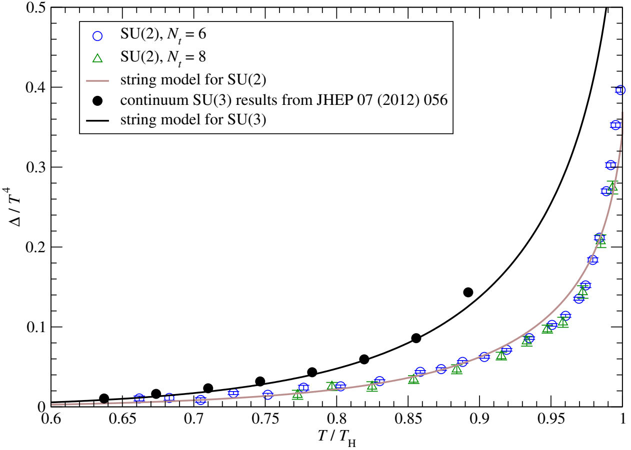
SU(2) *vs.* SU(3)

It is instructive to compare the SU(2) and SU(3) data

For $N = 3$ the closed flux tube has two possible orientations that account for the $\mathcal{C} = +1/-1$ sectors. Thus a further twofold degeneracy must be introduced in the string spectrum.

This doubling of the Hagedorn spectrum is clearly visible in the data

SU(2) *vs.* SU(3): results for trace of energy-momentum tensor



SU(N) Yang-Mills theories in $(2+1)$ dimensions

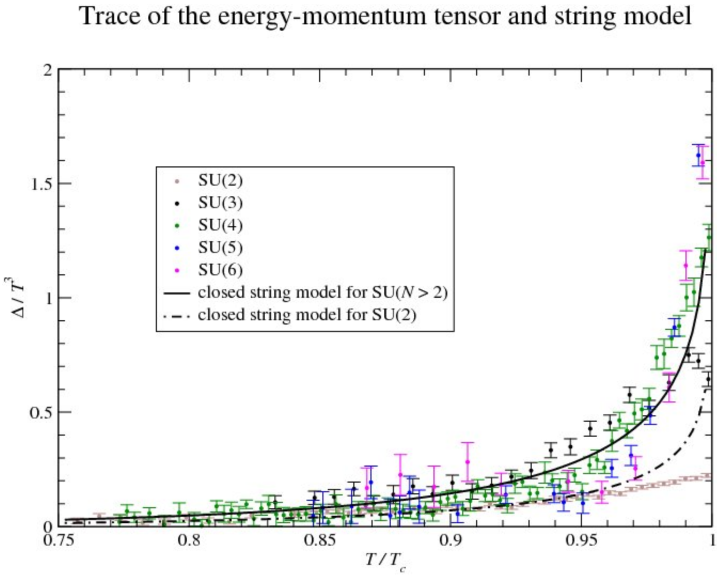
The same picture is confirmed by a study we performed a few years ago[1] in (2+1) dimensional SU(N) Yang-Mills theories for $N = 2, 3, 4, 6$.
Also in (2+1) dimensions we found that:

- a Hagedorn spectrum was mandatory to fit the thermodynamic data
- there was a jump between the SU(2) and the SU($N > 2$) case due to the doubling of the spectrum
- we had to fix the Hagedorn temperature to the Nambu-Goto value which, due to the different number of trensverse degrees of freedom is different from the (3+1) dimensional one: $T_H = \sqrt{3\sigma/\pi} = 0.977..\sqrt{\sigma}$

Moreover we found that in the vicinity of the critical point there was an excess of Δ/T^4 with respect to our predictions for $N = 4, 5$ and 6 and that this excess increases with N. This could be understood as due to the k-string glueballs

[1] M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

SU(N) Yang-Mills theories in $(2+1)$ dimensions



Take home message:

- The thermodynamics of SU(2) and SU(3) Yang-Mills theories in $d = (3 + 1)$ is well described by a gas of non-interacting glueballs
- The agreement is obtained only assuming a Hagedorn spectrum for the glueballs
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the Nambu-Goto effective string.
- The results agree with previous findings in $d = (2 + 1)$ SU(N) Yang Mills theories with $N = 2, 3, 4, 5, 6$
- As N increases the data suggest the presence of extra states in the spectrum which could be k-glueballs states, which could be described by a k-string spectrum

Conclusions

Effective string theories are a powerful tools to describe the infrared begaviour of confining gauge theories

- They are strongly constrained by Lorentz invariance
- The simplest option: the Nambu-Goto action plays the role of a mean field approximation
- Higher order terms are now accessible to modern Montecarlo simulations and will open a window to distinguish the different "confining habits" of different gauge models
- For instance the rigidity term has relevant consequences in the confining string:
  - The field density profile around the string is (almost) a Gaussian in the case of a Nambu-Goto string, while it decreases exponentially in the rigid string case. This exponential defines a new scale, known as the London penetration length in condensed matter theory, and as intrinsic width in confining gauge theories
  - While in the Nambu-Goto case the string width increases logarithmically with the interquark distance at zero temperature and linearly at high temperature, the intrinsic width of the rigid string is constant
  - At very short distances the coefficient of the Lüscher term is doubled.
- Starting from these ideas lot of interesting applications are possible, for instance it is possible to show the "Hagedorn nature" of the deconfinement transition

Backup Slides

Evaluation of the Lüscher term.

- The gaussain integration gives:

$$\int e^{-\frac{\sigma}{2}\int d^2\xi X^i(-\partial^2)X^i} \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \quad .$$

- The determinant must be evaluated with Dirichlet boundary conditions. The spectrum of $-\partial^2$ with Dirichlet boundary conditions is:

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{T^2} + \frac{n^2}{R^2} \right)$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin \frac{m\pi\tau}{T} \sin \frac{n\pi\varsigma}{R} \quad .$$

Evaluation of the Lüscher term.

- The determinant can be regularized with the ζ-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s = 0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad .$$

- The result is

$$\left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} = \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad .$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1 - q^n)$$

with $q \equiv e^{2\pi i\tau}$ and $\tau = iT/R$.

Derivation of the Nambu-Goto action.

- The Nambu-Goto action is given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad ,$$

where g is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in R^d:

$$g = \det(g_{\alpha\beta}) = \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \ldots, d)$$

- Choosing the "physical gauge"

$$X^1 = \tau \quad X^2 = \varsigma$$

g may be expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \ (i = 3, \ldots, d) \ . \end{aligned}$$

- Expanding we find:

$$S \sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8} (\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4} (\partial_\alpha X \cdot \partial_\beta X)^2 + \ldots \right] ,$$

Application: the boundary term of the effective action: Constraints imposed by the Lorentz invariance

If the boundary is a Polyakov line in the ξ_0 direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the most general boundary action should be of this type

$$S_b = \int d\xi_0 \left[b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \dots \right] .$$

Imposing Lorentz invariance one finds that $b_1 = 0$ and that the b_2 term is only the first term of a Lorentz invariant expression[1] :

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right] .$$

which is the analogous in the case of the boundary action of the Nambu-Goto action for the "bulk" effective action.

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

The boundary contribution to the interquark potential

Following the above discussion, the leading correction coming from the boundary turns out to be:

$$S_{b,2}^{(1)} = \int d\xi_0 \left[b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X \right] .$$

Its contribution to the interquark potential can be evaluated performing a simple gaussian functional integration[1]

$$\langle S_{b,2}^{(1)} \rangle \;=\; -b_2 \frac{\pi^3 L}{60 R^4} E_4 (\mathrm{i} \frac{L}{2R}) .$$

where the Eisenstein function E_4, is defined as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n ,$$

where $q = e\ 2\pi i \tau$ and $\sigma_p(n)$ is the sum of the p-th powers of the divisors of n:

$$\sigma_p(n) = \sum_{m|n} m^p .$$

[1] O. Aharony and M. Field JHEP01(2011)065

The boundary contribution to the interquark potential

- We end up with the following expression for the interquark potential

$$V(R) = \sigma R - \frac{\pi(D-2)}{24R} - \frac{1}{2\sigma R^3}\left(\frac{\pi(D-2)}{24}\right)^2 - b_2 \frac{\pi^3(D-2)}{60R^4} + O(1/R^5) \ ,$$

  where b_2 is a new physical parameter, similar to the string tension σ, which depends on the theory that we study and should be determined by simulations and comparison with experiments.
- To test this picture we performed a set of high precision simulations in the case of the 3d gauge Ising model, which is the simplest possible confining gauge theory.

Simulation I: Polyakov loops

- In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators $P(R, L)$ (where L is the length of the two loops and R their distance) we measured the following ratio:

$$R_P(R, L) = \frac{P(R+1, L)}{P(R, L)} \ .$$

- Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of R and L with very high precision.

Simulation settings

- We performed our simulations in the 3d gauge Ising model, using a dual algorithm

data set	β	L	σ	$1/T_c$
1	0.743543	68	0.0228068(15)	5
2	0.751805	100	0.0105255(11)	8
3	0.754700	125	0.0067269(17)	10

Table : Some information on the data sample

Results

- The values of b_2 extracted from the data show the expected scaling behaviour $b_2 \sim \frac{1}{\sqrt{\sigma}^3}$

data set	b_2	$b_2\sqrt{\sigma}^3$	χ^2
1	7.25(15)	0.0250(5)	1.2
2	26.8(8)	0.0289(9)	1.8
3	57.9(12)	0.0319(7)	1.3

Table : Values of b_2 as a function of β

Simulation II: Wilson loops

As a check of our analysis we performed the same simulation for the Wilson loops fixing the value of b_2 obtained above. In this case there is no more parameter to fit and we can directly compare our predictions with the results of the simulations. To eliminate all the non-universal parameters we constructed the following combination:

$$R_W^{'}(L, Lu) = \frac{W(L,R)}{W(L+1,R-1)} - \exp\{-\sigma(1+L(1-u))\} \ , \qquad u = R/L$$

Simulation II: Wilson loops

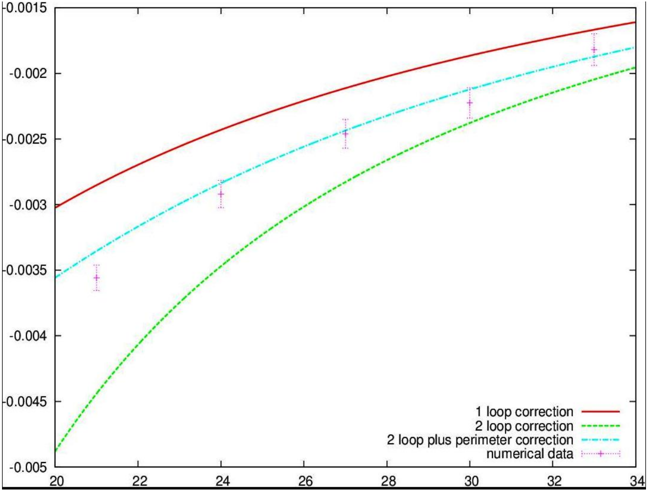


Figure : $R'_W(L, L\frac{4}{3})$ at $\beta = 0.754700$.

Scale setting

The SU(2) scale setting is fixed by calculating the string tension via the computation of Polyakov loop correlators with the multilevel algorithm.

The range of the parameter β which has been considered ($\beta \in [2.27, 2.6]$) covers approximately the temperature region analyzed in the finite temperature simulations.

The string tension is obtained with a two-parameter fit of potential

$$V = -\frac{1}{N_t} \log \langle PP \rangle$$

with the first order effective string prediction for the potential

$$V = \sigma r + V_0 - \frac{\pi}{12r}$$

Higher order effective string corrections turned out to be negligible within the precision of our data.

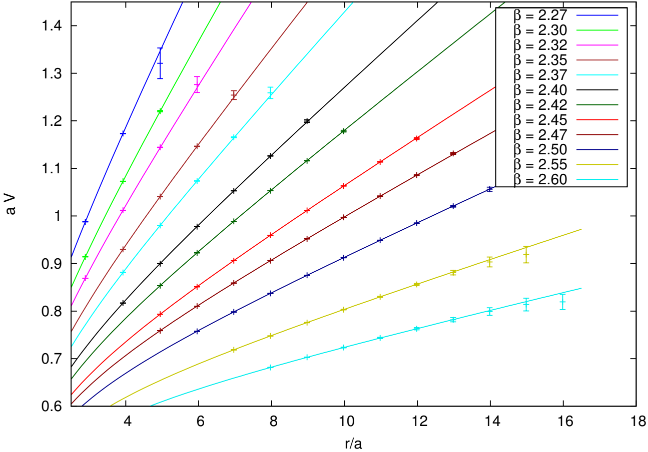
Scale setting

β	$r_{\rm min}/a$	σa^2	aV_0	$\chi^2_{\rm red}$
2.27	2.889	0.157(8)	0.626(14)	0.6
2.30	2.889	0.131(4)	0.627(30)	0.1
2.32	3.922	0.115(6)	0.627(32)	2.3
2.35	3.922	0.095(3)	0.623(20)	0.2
2.37	3.922	0.083(3)	0.621(18)	1.0
2.40	4.942	0.068(1)	0.617(10)	1.4
2.42	4.942	0.0593(4)	0.613(5)	0.1
2.45	4.942	0.0482(2)	0.608(4)	0.4
2.47	4.942	0.0420(4)	0.604(5)	0.3
2.50	5.954	0.0341(2)	0.599(2)	0.1
2.55	6.963	0.0243(13)	0.587(11)	0.2
2.60	7.967	0.0175(16)	0.575(16)	0.3

Table : *

Results for the string tension in units of the inverse squared lattice spacing at different values of the Wilson action parameter β (first column), calculated by fitting the potential V as a function of the tree-level improved interquark distance r to the Cornell form. V was extracted from Polyakov loop correlators on lattices of temporal extent $L_t = 32a$.

Scale setting

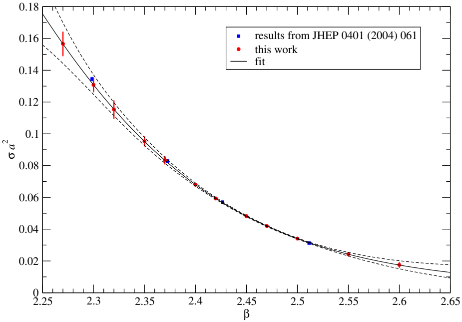


Scale setting

The values of the string tension are interpolated by a fit to

$$\log(\sigma a^2) = \sum_{j=0}^{n_{\mathrm{par}}-1} a_j(\beta - \beta_0)^j \qquad \text{with } \beta_0 = 2.35$$

which yields a χ^2_{red} of 0.01. It is presented below along with older data[1].



[1] B. Lucini, M. Teper, U. Wenger, *The high temperature phase transition in SU(N) gauge theories*, 2003

Thermodynamics on the lattice

The pressure can be estimated by the means of the so-called "integral method"[1]:

$$p(T) \simeq \frac{T}{V} \log Z(T,V) = \frac{1}{a^4} \frac{1}{N_t N_s^3} \int_0^{\beta(T)} d\beta' \frac{\partial \log Z}{\partial \beta'}.$$

It can be written (relative to its $T=0$ vacuum contribution) as

$$\frac{p(T)}{T^4} = -{N_t}^4 \int_0^{\beta} d\beta' \left[3(P_\sigma + P_\tau) - 6P_0\right]$$

where P_σ and P_τ are the expectation values of spacelike and timelike plaquettes respectively and P_0 is the expectation value at zero T.
The trace of energy-momentum tensor is simply

$$\frac{\Delta(T)}{T^4} = T \frac{\partial}{\partial T}\left(\frac{p}{T^4}\right) = -{N_t}^4 T \frac{\partial \beta}{\partial T}\left[3(P_\sigma + P_\tau) - 6P_0\right].$$

ϵ and s can be obtained indirectly as linear combinations.

[1] J. Engels *et al.*, *Nonperturbative thermodynamics of SU(N) gauge theories*, 1990

Geometrical description.

A more intuitive geometrical description of this result is obtained using the original string action, without fixing the physical gauge.
The effective action is given by the most general mapping:

$$X^\mu : \mathcal{M} \to \mathbb{R}^D, \qquad \mu = 0, \cdots, D-1$$

- $\mathcal{M}$: two-dimensional surface describing the worldsheet of the string
- $\mathbb{R}^D$: (flat) D dimensional target space $\mathbb{R}^D$ of the gauge theory.

Main Result [1] :

- The first few terms of the action compatible with Poincaré and parity invariance are suitable combinations of geometric invariants constructed from the induced metric $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$.
- These terms can be classified according to their weight, i.e. the difference between the number of derivatives minus the number of fields X^μ

[1] O. Aharony and Z. Komargodski, JHEP **1305** (2013) 118

Geometrical description.

- The only term of weight zero is the Nambu-Goto action

$$S_{\rm NG} = \sigma \int d^2\xi \sqrt{g} \ ,$$

where $g \equiv \det(g_{\alpha\beta})$.

- This term has a natural geometric interpretation: it measures the area swept out by the worldsheet in space-time.
- Fixing the physical gauge one finds (choosing an euclidean metric)

$$S = \sigma \int d^2\xi \sqrt{det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8}(\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4}(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] ,$$

Geometrical description.

- At weight two, two new contributions appear:

$$S_{2,\mathcal{R}} = \gamma \int d^2\xi \sqrt{g}\mathcal{R} \ ,$$

$$S_{2,K} = \alpha \int d^2\xi \sqrt{g} K^2,$$

where $\mathcal{R}$ denotes the Ricci scalar constructed from the induced metric, and $K \equiv \Delta(g)X$ is the extrinsic curvature, where $\Delta(g)$ is the Laplacian in the space with metric $g_{\alpha\beta}$.

However both these terms can be neglected!

- $\mathcal{R}$ is topological in two dimensions and, since in the long string limit in which we are interested we do not expect topologically changing fluctuations, its contribution is constant and can be neglected.
- K^2 is proportional to the equation of motion of the Nambu-Goto Lagrangian and can be eliminated by a suitable field redefinition.

Geometrical description.

Thus the first non trivial terms appear at level four and contribute to the interquark potential with terms proportional to $1/R^7$ in agreement with the derivation in the physical gauge.

However something must be missing in the picture since high precision simulations of various 3d gauge models show large deviations with respect to the Nambu-Goto prediction, which turn out to be much stronger than the expected $1/R^7$ corrections!

- Interface free energy in the 3d Ising model[1]: Torus geometry, no boundary corrections
- Excited string states of SU(N) Yang-Mills theories[2]
- Interquark Potential in the 3d U(1) gauge model [3]

[1] M. Caselle, M. Hasenbusch and M. Panero, *JHEP* **0709** (2007) 117, arXiv:0707.0055

[2] A. Athenodorou, B. Bringoltz and M. Teper, *JHEP* **02**, 030 (2011), arXiv:1007.4720 .

[3] D. Vadacchino, M. Caselle, R. Pellegrini and M. Panero, arXiv:1311.4071 .