Topological properties of CP^{N-1} models in the large-N limit

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Gauge Topology 3: from Lattice to Colliders
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Role of $\mathbb{C}P^{N-1}$ in non-perturbative QCD

The $2d$ CP^{N-1} models share many fundamental properties with QCD: confinement, asymptotic freedom, topologically-stable instantons, θ-vacua...

These theories admit an analytic solution in the large-N limit. They have been employed as a theoretical laboratory for the study of non-perturbative features of QCD (e. g. Witten, 1979).

The CP^{N-1} have been also extensively studied numerically through Monte Carlo simulations:

- lattice CP^{N-1} simulations need low numerical effort,
- CP^{N-1} models are ideal test-bed for new algorithms to solve LQCD non-trivial computational problems,
- possibility of a comparison between numerical and analytic large-N results.

Topology and θ-dependence

In the CP^{N-1} models one can introduce a topological charge Q and a corresponding θ-term in the action.

This work focuses on the study of the θ-dependence of the vacuum energy (density):

$$f(\theta) \equiv -\frac{1}{V}\log Z(\theta) = \frac{1}{2}\chi\theta^2\left(1 + \sum_{n=1}^{\infty} b_{2n}\theta^{2n}\right).$$

The coefficients of the θ-expansion are related to the cumulants k_m of the probability distribution of Q:

$$\left.\frac{d^m f}{d\theta^m}\right|_{\theta=0} = -\frac{i^m}{V}k_m \implies \begin{cases} \chi = \dfrac{\langle Q^2\rangle|_{\theta=0}}{V} \\ b_2 = \left.\dfrac{-\langle Q^4\rangle + 3\langle Q^2\rangle^2}{12\langle Q^2\rangle}\right|_{\theta=0} \end{cases}$$

θ-dependence and phenomenology

The study of $f(\theta)$ is of particular relevance in QCD and in $SU(N)$ gauge theories:

- θ-dependence of pure Yang-Mills enters η' physics,
- $f_{QCD}(\theta)$ enters axion phenomenology and, thus, the resolution of the strong-CP problem.

In QCD and Yang-Mills, numerical MC simulations on the lattice are one of the most reliable tools to measure χ and the b_{2n} coefficients.

This constitutes a strong motivation to perform a similar numerical study for the lattice CP^{N-1} models.

Analytic predictions for $f(\theta)$ for CP^{N-1}

Unlike in QCD, the θ-dependence of the vacuum energy can be calculated analytically in the large-N limit.

Some quantitative predictions obtained for the CP^{N-1} models are:

$$\xi^2\chi = \frac{1}{2\pi}\frac{1}{N} - \frac{0.0606}{N^2} + O\left(\frac{1}{N^2}\right) \quad \text{(Campostrini and Rossi; 1992)}$$

$$b_2 = -\frac{27}{5}\frac{1}{N^2} + O\left(\frac{1}{N^3}\right) \quad \text{(Del Debbio et al.; 2006)}$$

$$b_4 = -\frac{25338}{175}\frac{1}{N^4} + O\left(\frac{1}{N^5}\right) \quad \text{(Bonati, D'Elia, Rossi and Vicari; 2016)}$$

$$b_{2n} = \bar{b}_{2n}\frac{1}{N^{2n}} + O\left(\frac{1}{N^{2n+1}}\right)$$

Numerical study of $f(\theta)$ for CP^{N-1}

Lattice measures of $f(\theta)$ for the CP^{N-1} are quite limited compared to the one of QCD and $SU(N)$ gauge theories.

- The susceptibility has been measured and its large-N limit has been checked, but large uncertainties exist about the sign of the first non-trivial $1/N$ correction,
- b_2 has never been measured for CP^{N-1}, contrary to the case of Yang-Mills and QCD,
- determinations of b_4 and higher coefficients have never been reported, neither for CP^{N-1} nor for Yang-Mills/QCD.

The goals of this work are:

- extension of the lattice measure of the vacuum energy $f(\theta)$ to higher orders in θ;
- extension of the study of the large-N limit of $f(\theta)$ and comparison with analytic predictions.

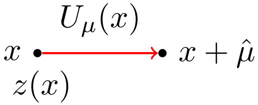
Lattice action

The chosen continuum Euclidean action is:

$$S = N\beta \int d^2x \, \bar{D}_\mu \bar{z}(x) D_\mu z(x), \quad |z|^2 = 1, \quad \beta \equiv 1/g.$$

We discretized it adopting the $O(a)$ Symanzik-improved lattice action:

$$\begin{aligned} S_L = & -\frac{8}{3} N\beta_L \sum_{x,\mu} \Re[\bar{U}_\mu(x)\bar{z}(x+\hat{\mu})z(x)] \\ & +\frac{1}{6} N\beta_L \sum_{x,\mu} \Re[\bar{U}_\mu(x+\hat{\mu})\bar{U}_\mu(x)\bar{z}(x+2\hat{\mu})z(x)] \\ & \equiv \beta_L E_L. \end{aligned}$$



$$U_\mu(x) \sim \exp\{iaA_\mu(x)\}$$

Lattice topological charge

The definition of topological charge we chose is:

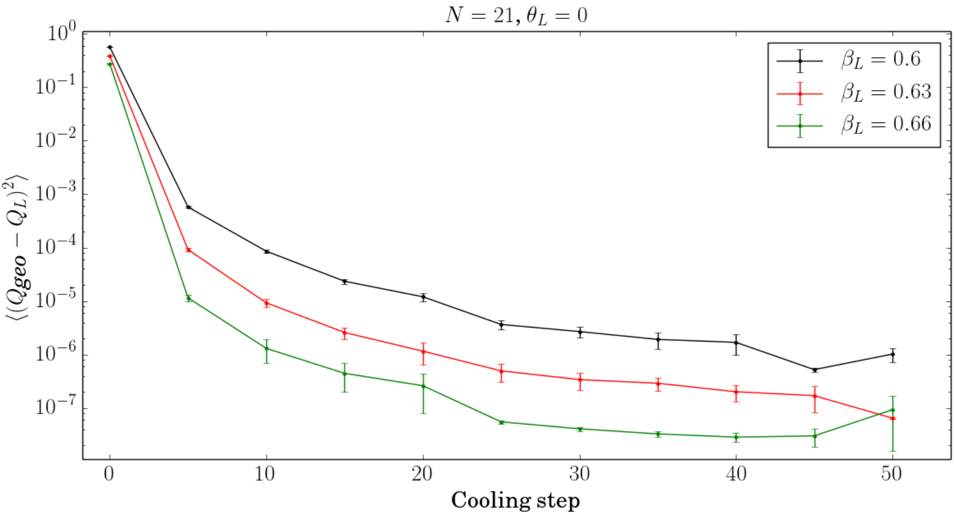
$$Q = \frac{1}{4\pi}\varepsilon_{\mu\nu}\int d^2x\, F_{\mu\nu}(x) = \frac{1}{2\pi}\int d^2x\, F_{12}(x)$$

Several discretizations are possible, for example:

- $Q_L = \frac{1}{2\pi}\sum_x \Im\{\Pi_{12}(x)\},$ (Non-geometric)
- $Q_{geo} = \frac{1}{2\pi}\sum_x \Im\left[\log\left(\Pi_{12}(x)\right)\right].$ (Geometric)

$$\Pi_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+\hat{\mu})\bar{U}_\mu(x+\hat{\nu})\bar{U}_\nu(x)$$

Cooling



$N = 21, \theta_L = 0$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$\langle (Qgeo - Q_L)^2 \rangle$
0
10
20
30
40
50
Cooling step
$\beta_L = 0.6$
$\beta_L = 0.63$
$\beta_L = 0.66$

Computational problems

This set-up suffers from two computational problems:

- Critical Slowing Down (CSD) of topological modes,
- difficulties in measuring high-order cumulants of Q.

1) When approaching the continuum limit ($\xi_L \to \infty$) and the large-N limit, the machine time needed to change the charge of a configuration exponentially grows with ξ_L and N.

This is due to the impossibility of changing the winding number of a configuration with a continuum deformation.

2) The measure of high-order cumulants of Q becomes very noisy for large lattice sizes.

This happens because the Gaussian behaviour is dominant in the thermodynamic limit for the central limit theorem.

To obtain a precise measure of $f(\theta)$ we need to adopt numerical strategies to improve the measure of topological observables.

In this work we applied:

- imaginary-θ method to improve measure accuracy of cumulants; (Panagopoulos and Vicari, 2011)
- simulated tempering algorithm to dampen the CSD of topological modes. (Marinari and Parisi, 1992; Vicari, 1993)

Analytic continuation

Being the θ-dependence of the theory analytic around $\theta = 0$, one can continue the path integral to imaginary angles:

$$\theta \equiv -i\theta_I \implies S_{top} = -i\theta Q = -\theta_I Q \in \mathbb{R}.$$

The topological term was discretized using the non-geometric definition.

The continuation of the vacuum energy is:

$$f(\theta_I) = f(\theta = -i\theta_I) = -\frac{1}{2}\chi\theta_I^2 \left(1 + \sum_{n=1}^{\infty}(-1)^n b_{2n}\theta_I^{2n}\right).$$

$\implies$ The measure of χ and of the b_{2n} coefficients can be extracted from $f(\theta_I)$.

The θ_I-dependence of the cumulants of Q is related to $f(\theta_I)$:

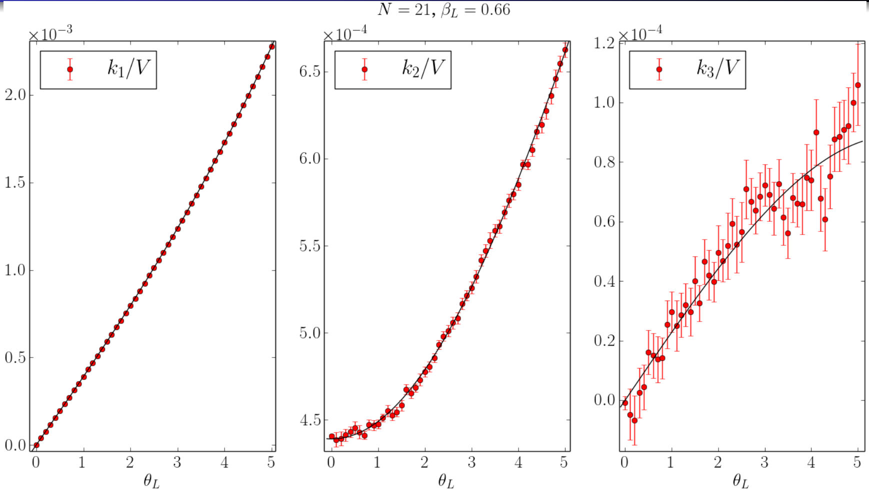
$$\frac{d^m f(\theta_I)}{d\theta_I^m} = -\frac{1}{V} k_m(\theta_I),$$

A global fit of the θ_I-dependence of the cumulants, which can be measured on the lattice, leads to an improved measure of χ and the b_{2n}:

$$\frac{k_1(\theta_I)}{V} = \chi\theta_I\big[1 - 2b_2\theta_I^2 + 3b_4\theta_I^4 + O(\theta_I^5)\big],$$
$$\frac{k_2(\theta_I)}{V} = \chi\big[1 - 6b_2\theta_I^2 + 15b_4\theta_I^4 + O(\theta_I^5)\big],$$
$$\frac{k_3(\theta_I)}{V} = \chi\big[-12b_2\theta_I + 60b_4\theta_I^3 + O(\theta_I^4)\big]...$$

On the lattice: $\theta_I = Z_\theta \theta_L$.

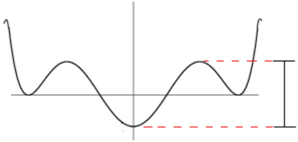
Imaginary-θ fit results



$\mathcal{O}(\beta = 0.66, N = 21)$	Standard method	Imaginary-θ	Gain
$\chi \cdot 10^4$	$4.401(11)$	$4.3908(58)$	~ 2
$b_2 \cdot 10^3$	$-5.36(40)$	$-4.958(76)$	~ 5
$b_4 \cdot 10^5$	-11 ± 21	$-1.27(20)$	~ 100

The simulated tempering algorithm

The simulated tempering consists in promoting the temperature T as a dynamical variable.



The system heats up during its evolution and can escape from the local minima in which it is trapped.

In the case of the CP^{N-1} models, one can promote both β and θ_I to dynamical variables:

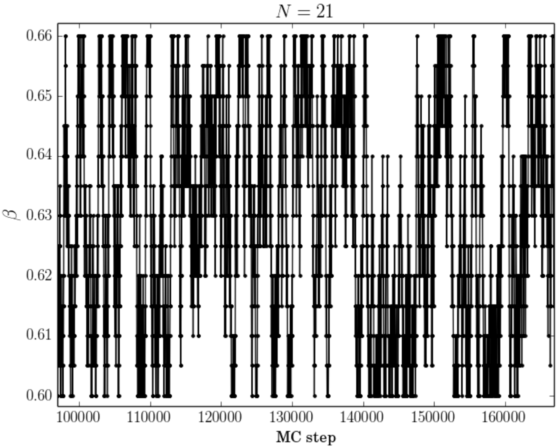
$$P \propto \exp\{-S_L + \theta_L Q_L\} = \exp\{-\beta_L E_L + \theta_L Q_L\}.$$

- When β decreases, the algorithm changes Q more easily.
- When θ_I increases, higher-charge configurations are more probable to realize.

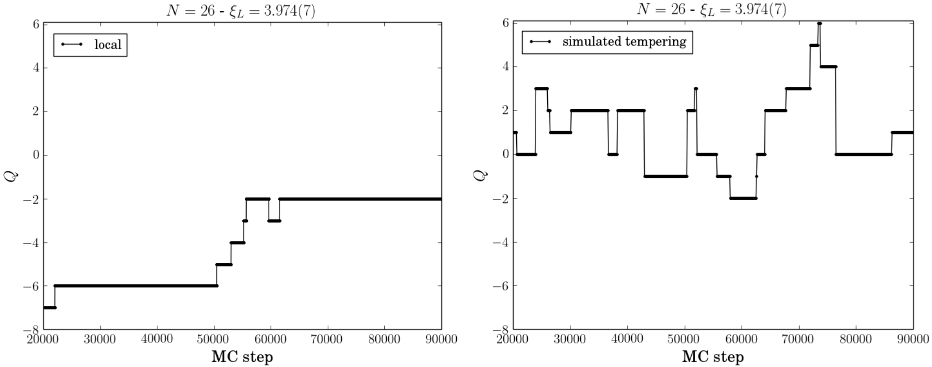
Simulated tempering set-up

To obtain an ergodic algorithm, the β interval needs to be chosen properly.

- $\beta_{min} \to$ local algorithm decorrelates fast,
- $\beta_{max} \to$ close to the continuum limit,
- $\delta\beta \to$ reasonable acceptance of change of β.

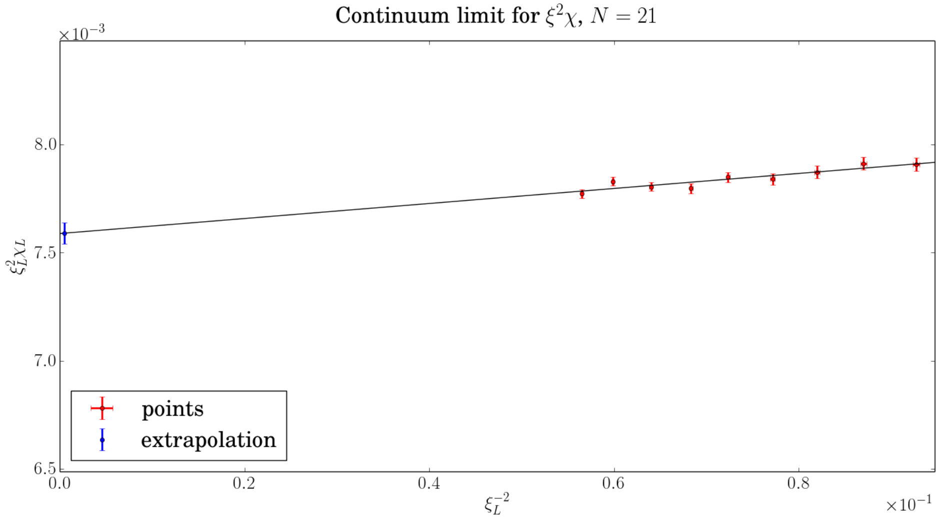


Simulated tempering provides an improving in the autocorrelation time at equal machine time.

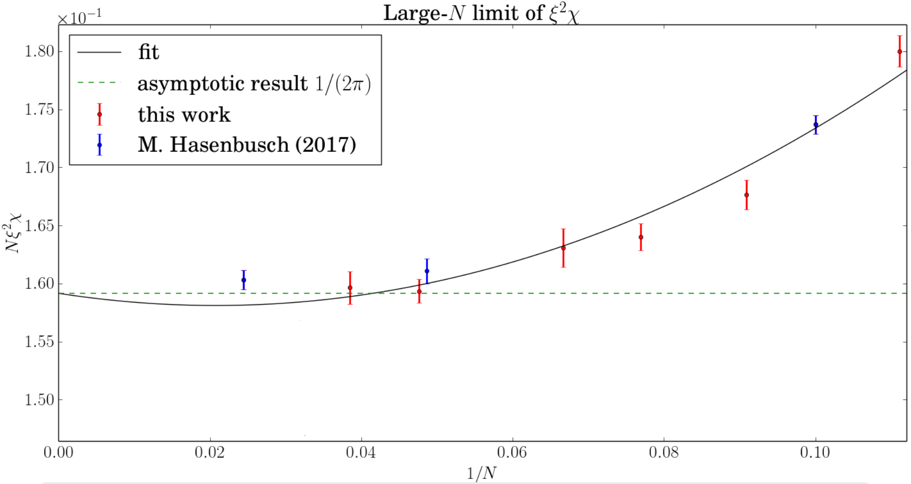


Continuum limit

Linear corrections in the lattice spacing ($\sim \xi_L^{-1}$) are killed by the Symanzik action.



Large-N limit of topological susceptibility



$$(N\xi^2\chi)_{theo} \simeq 0.1592 - 0.0606/N + O(1/N^2),$$

$$(N\xi^2\chi)_{MC} = 0.1600(8) + e_2/N + e_3/N^2,$$

$$\text{Fit: } |e_2| \lesssim 10^{-1}, \quad e_3 = O(1), \quad \chi^2_{red} = 17/7.$$

$N-2$: ansatz

The CP^1 model is equivalent to the $O(3)$ σ model.
Many theoretical works (e. g. Richard et al., 1983), supported by numerical evidence (e. g. D'Elia et al., 1997), show that the latter has divergent topological susceptibility.

$\implies$ Ansatz:

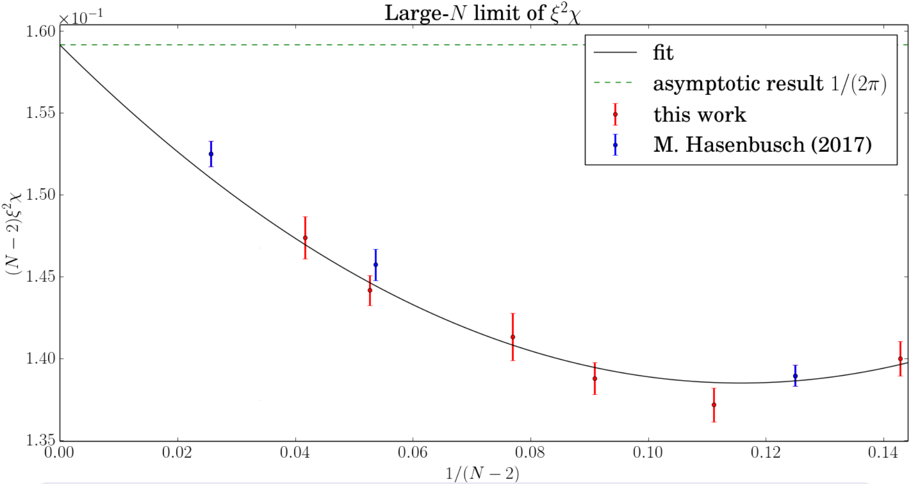
$$\xi^2\chi = \frac{1}{2\pi}\frac{1}{N-2} + e_2'\frac{1}{(N-2)^2} + O\left[\frac{1}{(N-2)^3}\right],$$

$$(e_2')_{theo} = (e_2)_{theo} - \frac{1}{\pi} \simeq -0.379...$$

It can be seen as a partial resummation of the $1/N$ series:

$$\frac{1}{N-2} = \frac{1}{N}\left(1-\frac{2}{N}\right)^{-1} = \frac{1}{N}\left[1+\frac{2}{N}+O\left(\frac{1}{N^2}\right)\right].$$

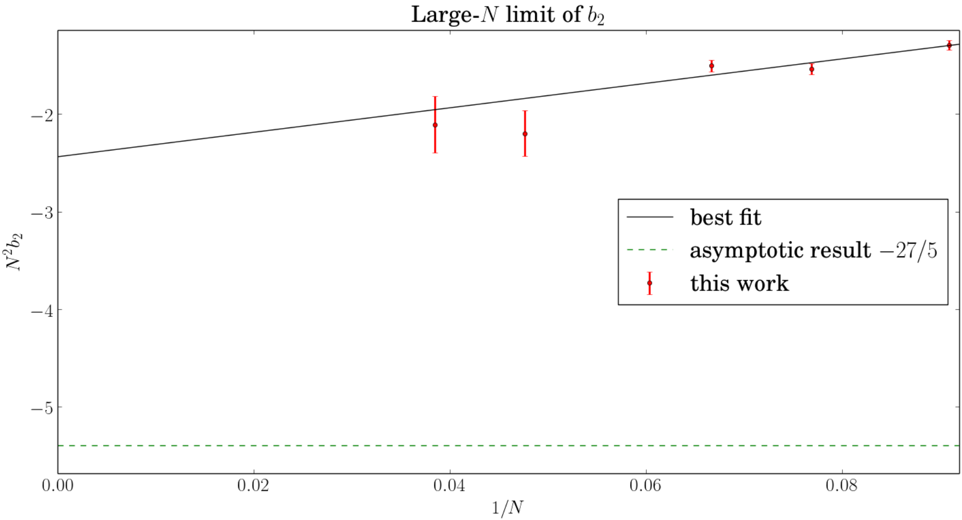
$N-2$: results



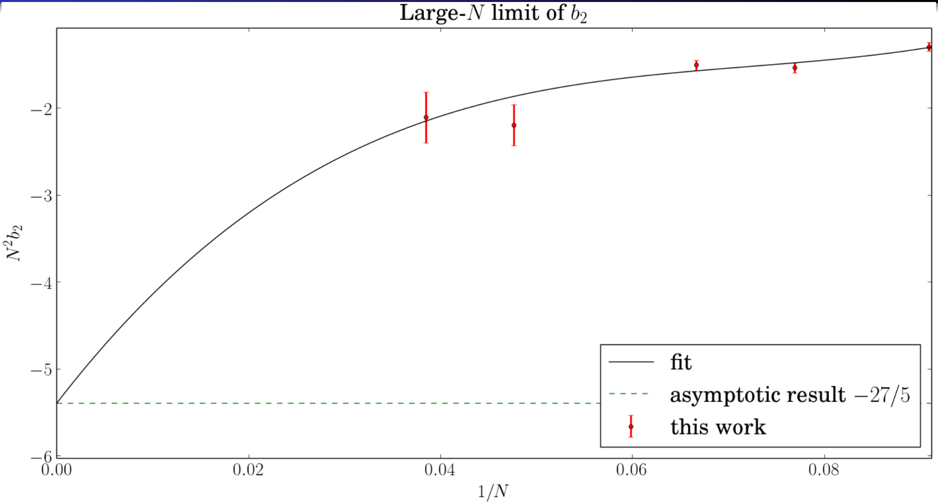
$$[(N-2)\xi^2\chi]_{theo} \simeq 0.1592 - 0.379/(N-2) + O[1/(N-2)^2],$$

$$[(N-2)\xi^2\chi]_{MC} = 0.160(1) + e_2'/(N-2) + e_3'/(N-2)^3,$$

Fit: $e_2' = -0.35(2) \quad (e_2 = -0.03(2)), \quad e_3' = O(1), \; \chi^2_{red} = 7.5/7$



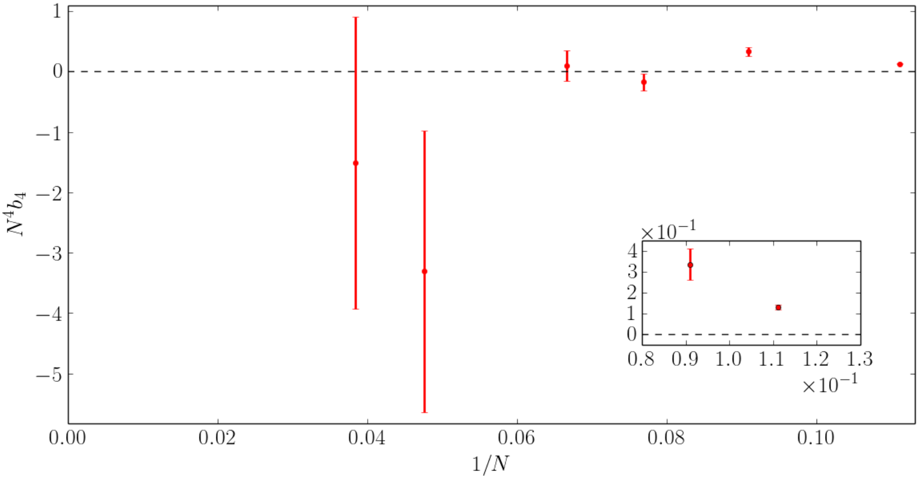
$$(N^2 b_2)_{theo} = -5.4 + O(1/N)$$
$$(N^2 b_2)_{MC} = -2.4(3) + O(1/N), \quad \chi^2_{red} = 6/3$$



$$N^2 b_2 = -27/5 + a/N + b/N^2 + c/N^3, \quad \chi^2_{red} = 4/2$$

This suggests that the apparent discrepancy is due to large corrections to the predicted asymptotic limit.

Results for b_4



Summary and future perspective

Summarizing, this work consists in:

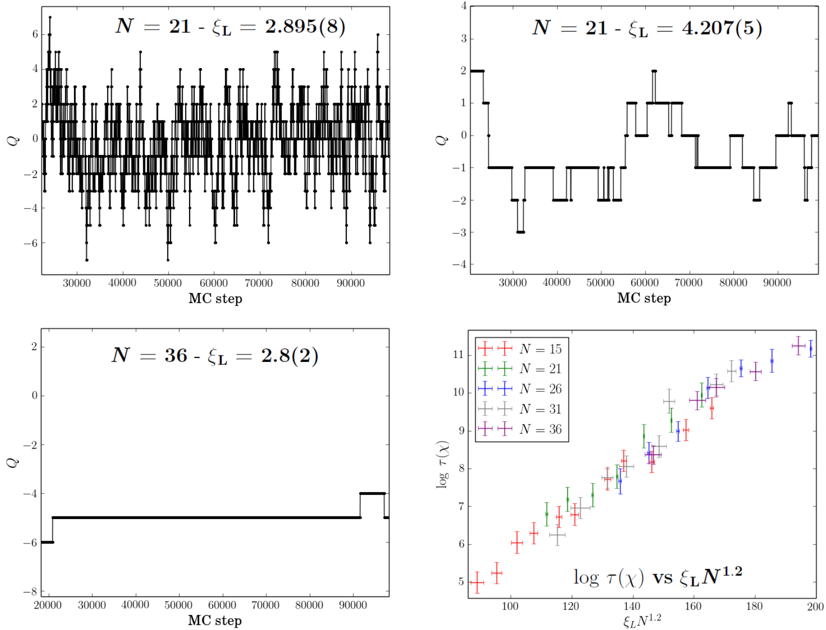
- application of imaginary-θ method and of simulated tempering algorithm to lattice CP^{N-1} models to improve measure accuracy of topological observables,
- lattice determination of χ, b_2 and b_4 for $N \in [9, 31]$,
- numerical study of the large-N limit of χ and b_2 and comparison with analytic predictions.

In the next future it would be interesting to:

- improve the study of the large-N limit of χ and b_2 including larger Ns and improving measure accuracy,
- further investigation of the $N-2$ ansatz for the susceptibility,
- try other proposed algorithm to improve this analysis.

Thank you for your attention!

Topological Charge Freezing



Estimation of the free energy

To obtain an ergodic algorithm, it is of utmost importance that the (β_L, θ_L) exploration is uniform.

$$P[\phi, \beta_L, \theta_L] \to P'[\phi, \beta_L, \theta_L] = e^{-\beta_L E_L[\phi] + \theta_L Q_L[\phi] + F_L(\beta_L, \theta_L)}$$

The Metropolis probability of changing parameter is:

$$\mathcal{P}(\beta_{old} \to \beta_{new}) = e^{-\Delta\beta E_L + \Delta F_L}, \quad \mathcal{P}(\theta_{old} \to \theta_{new}) = e^{\Delta\theta Q_L + \Delta F_L}.$$

The free energy as a function of (β_L, θ_L) can be estimated on the lattice with a numerical integration:

$$\frac{\partial F_L}{\partial \beta_L} = \langle E_L \rangle \, , \quad \frac{\partial F_L}{\partial \theta_L} = - \langle Q_L \rangle \, .$$