Resurgence, Large N and Phase Transitions

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Gauge Topology 3: From Lattice to Colliders
ECT* Trento, May, 2018

G. Basar, GD, & M. Ünsal: arXiv:1501.05671, 1701.06572;

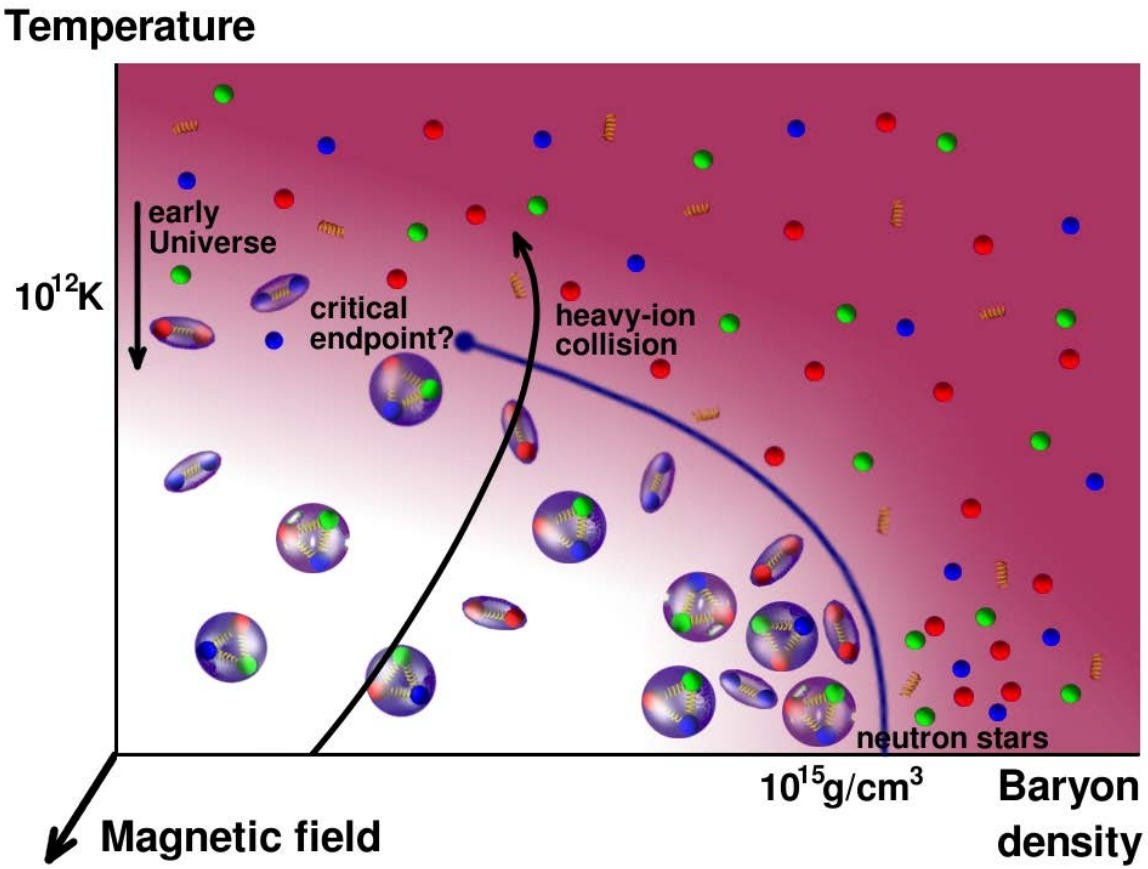
A. Ahmed & GD: arXiv:1710.01812; C. Coger & GD: to appear

support: UConn & DOE Division of High Energy Physics

"Big Picture" Physical Motivation

- non-perturbative definition of QFT in the continuum
- analytic continuation of path integrals
- "sign problem" in finite density QFT
- dynamical & non-equilibrium physics in path integrals
- new physical understanding of phase transitions

Physical Motivation



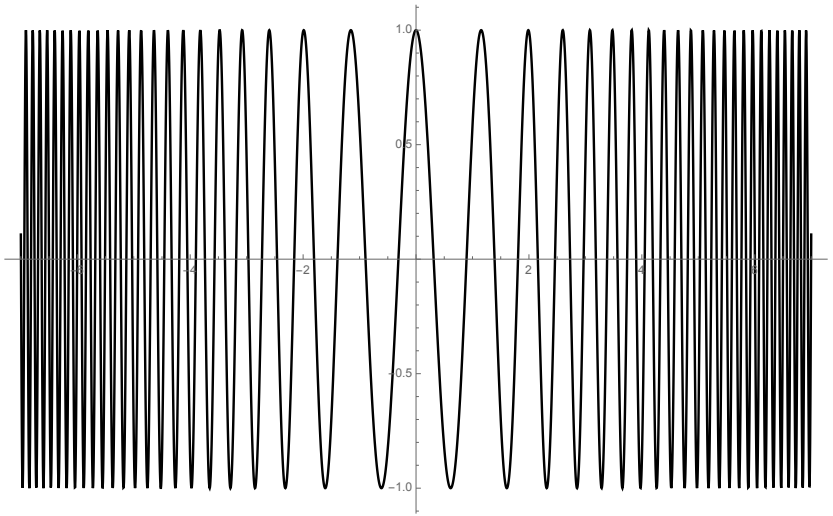
- phase transitions and Lee-Yang & Fisher zeroes
- can resurgence add anything new to this approach ?

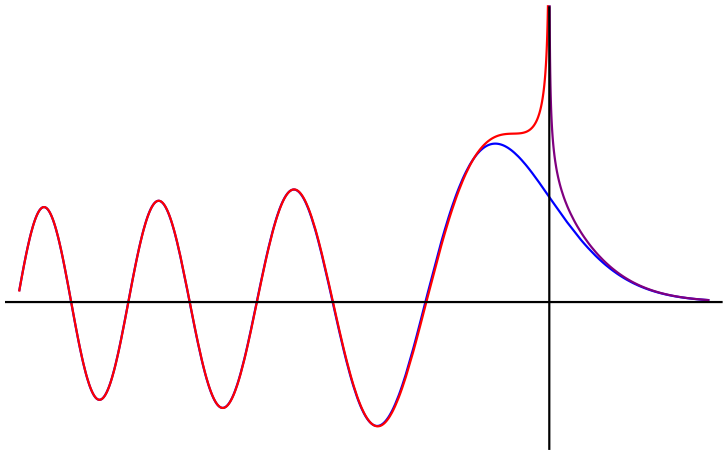
- probe intermediate regions using asymptotic information?

Physical Motivation

what does a Minkowski path integral mean, computationally?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$





$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3 + x\,t\right)}\, dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

- massive cancellations $\Rightarrow$ $\quad \mathrm{Ai}(+5) \approx 10^{-4}$

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

- since we need complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to expect that similar tools would be useful also for path integrals

- idea: phase transition = change of dominant saddle(s)

Resurgence

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

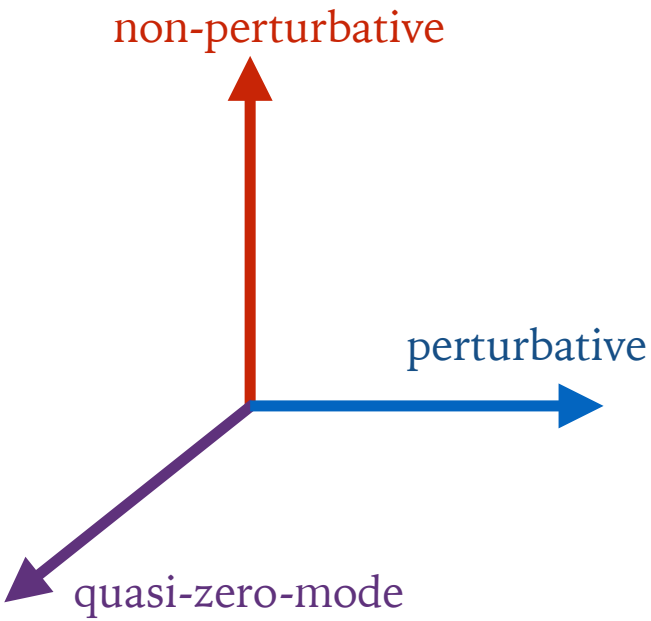
- perturbative series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'

$\Rightarrow$ well adapted for phase transition analysis

- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, difference equations, fluid mechanics, QM, Matrix Models, QFT, String Theory, ...
- define the path integral constructively as a trans-series

Decoding a Resurgent Trans-series in QFT

$$\int \mathcal{D}A\, e^{-\frac{1}{\hbar}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{\hbar}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$



- expansions in different directions are <u>quantitatively</u> related
- expansions about different saddles are <u>quantitatively</u> related

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \cdots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓

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- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$ ✓
- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

"raw" asymptotics is <u>inconsistent</u> with analytic continuation

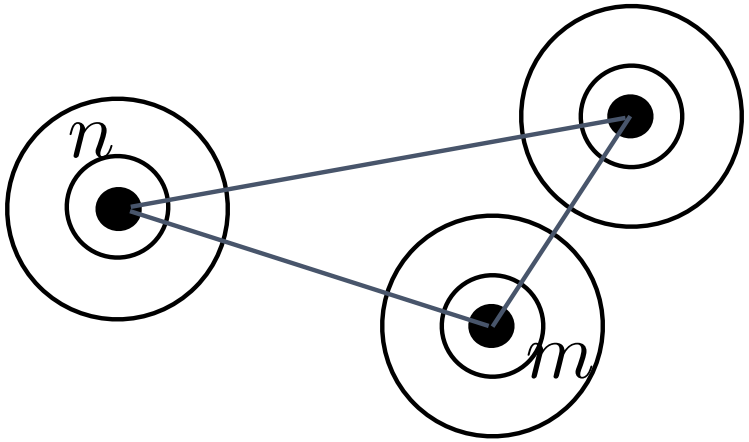
- resurgence: add infinite series of non-perturbative terms

"non-perturbative completion"

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



resurgence = global complex analysis (also with divergent series)

Resurgence of Airy function: "perturbation theory"

- formal large x perturbative series solution to ODE:

$$y'' = x\,y \;\Rightarrow\; \begin{Bmatrix} 2\,\mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} \sim \frac{e^{\mp\frac{2}{3}x^{3/2}}}{\sqrt{\pi}\,x^{1/4}} \sum_{n=0}^{\infty} (\mp 1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{n!\,\left(\frac{2}{3}\right)^n\,x^{3n/2}}$$

- non-perturbative connection formula:

$$\mathrm{Ai}\left(e^{\mp\frac{2\pi i}{3}}\,x\right) = \pm\frac{i}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Bi}\left(x\right) + \frac{1}{2}e^{\mp\frac{\pi i}{3}}\mathrm{Ai}\left(x\right)$$

- Borel-Écalle sum: cut along neg. t axis: $t \in (-\infty, -1]$

$$Z(x) = \sum_{n=0}^{\infty} \frac{(-1)^n |a_n|}{x^{3n/2}} = \frac{4}{3}x^{3/2}\int_0^{\infty} dt\, e^{-\frac{4}{3}x^{3/2}t}\;{}_2F_1\left(\frac{1}{6},\frac{5}{6},1;-t\right)$$

- discontinuity across cut $\Rightarrow$ correct non-pert. connection

$$Z\left(e^{\frac{2\pi i}{3}}\,x\right) - Z\left(e^{-\frac{2\pi i}{3}}\,x\right) = i\,e^{-\frac{4}{3}x^{3/2}}Z\left(x\right)$$

Resurgence of Airy function: "path integral"

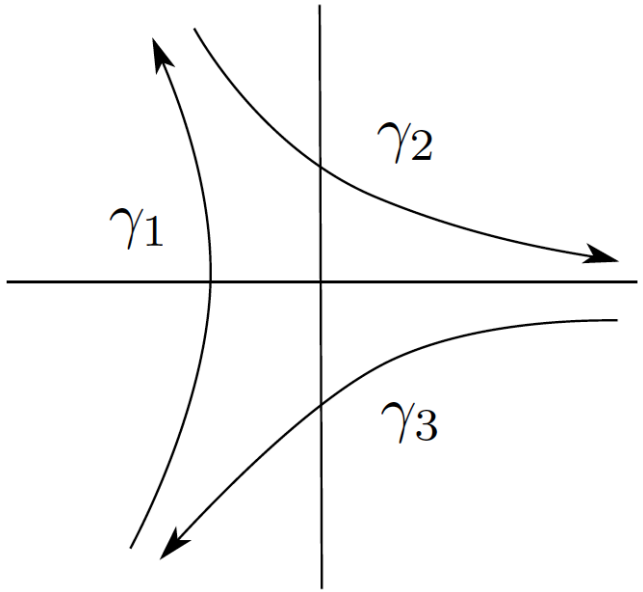
- saddle analysis

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\left(x\,t+\frac{t^3}{3}\right)}$$

- write $x \equiv r\,e^{i\theta}$, $t \equiv -i\sqrt{r}z$:

$$\mathrm{Ai}(x) = \frac{\sqrt{r}}{2\pi i}\int_{\gamma_k} dz\, e^{r^{3/2}\left(e^{i\theta}\,z-\frac{z^3}{3}\right)}$$

allowed z integration contours



- saddles at $z = \pm e^{i\theta/2}$
- saddle exponent ($\equiv$ "action") $= \pm\frac{2}{3}r^{3/2}e^{3i\theta/2}$

$x > 0 \Rightarrow \theta = 0 \Rightarrow$ contour through only 1 saddle ($z = -1$)
$\Rightarrow$ action $= -\frac{2}{3}r^{3/2} = -\frac{2}{3}x^{3/2}$

$x < 0 \Rightarrow \theta = \pm\pi \Rightarrow$ contour through 2 saddles ($z = \pm i$)
$\Rightarrow$ action $= \pm i\frac{2}{3}r^{3/2} = \pm i\frac{2}{3}(-x)^{3/2}$

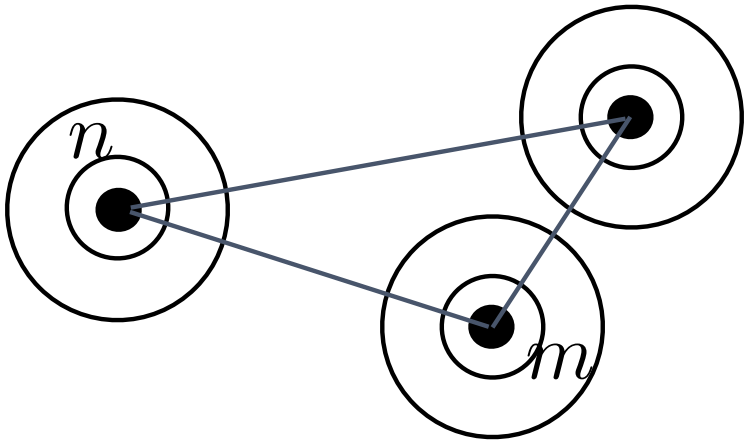
Resurgence: large-order/low-order relations

- fluctuations about the two saddles are explicitly related

$$a_n^{(+)} = \left\{1, -\frac{5}{72}, \frac{385}{10368}, -\frac{85085}{2239488}, \frac{37182145}{644972544}, -\frac{5391411025}{46438023168}, \dots\right\}$$

- large order/low order relation:

$$a_n^{(-)} \sim \frac{(n-1)!}{2^n}\left(1 - \frac{5}{72}\frac{2}{(n-1)} + \frac{385}{10368}\frac{2^2}{(n-1)(n-2)} - \dots\right)$$



- these resurgence relations are generic !

(Başar, GD, Ünsal)

Resurgence and Phase Transitions

Can we learn anything new about phase transitions ?

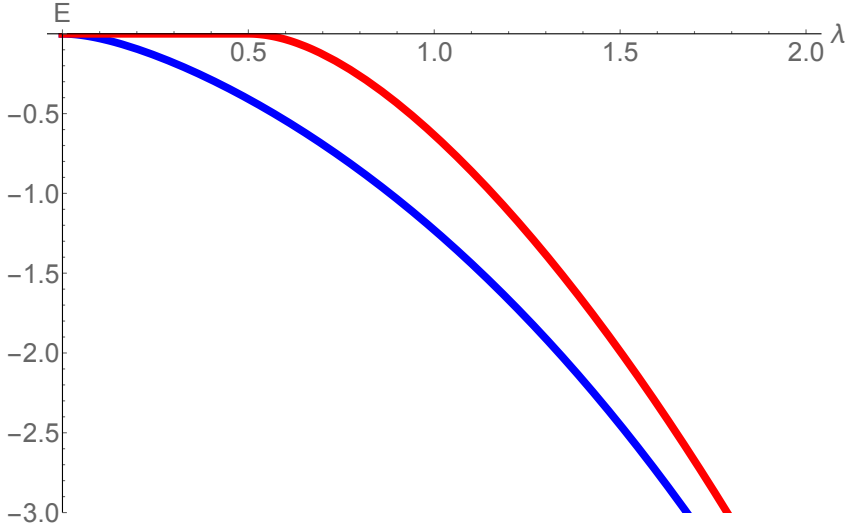
Bound States Crossing from the Continuum

- simple example: double-delta function potential well

$$V(x) = -\lambda\left(\delta(x-1) + \delta(x+1)\right)$$

- exact quantization condition: $\xi = \lambda\left(1 \pm e^{-2\xi}\right)$
- <u>convergent trans-series instanton expansion</u>:

$$\xi_{\pm} = \lambda - \frac{1}{2}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\left(\pm 2\,\lambda\, e^{-2\lambda}\right)^{k}$$



quantum phase transition at $\lambda = \frac{1}{2}$

Tunneling Ionization vs. Multiphoton Ionization Keldysh (1964)

- atomic ionization in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{\omega\sqrt{2mE_b}}{e\mathcal{E}}$
- time-dep. WKB $\Rightarrow \Gamma_{\text{ionization}} \sim \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\, g(\gamma)\right]$

$$\Gamma_{\text{ionization}} \sim \begin{cases} \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\mathcal{E}}{2\omega\sqrt{2mE_b}}\right)^{2E_b/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- phase transition: tunneling vs. multi-photon ionization
- phase transition: real vs. complex instantons

Tunneling vs. Multiphoton Pair Production in QED

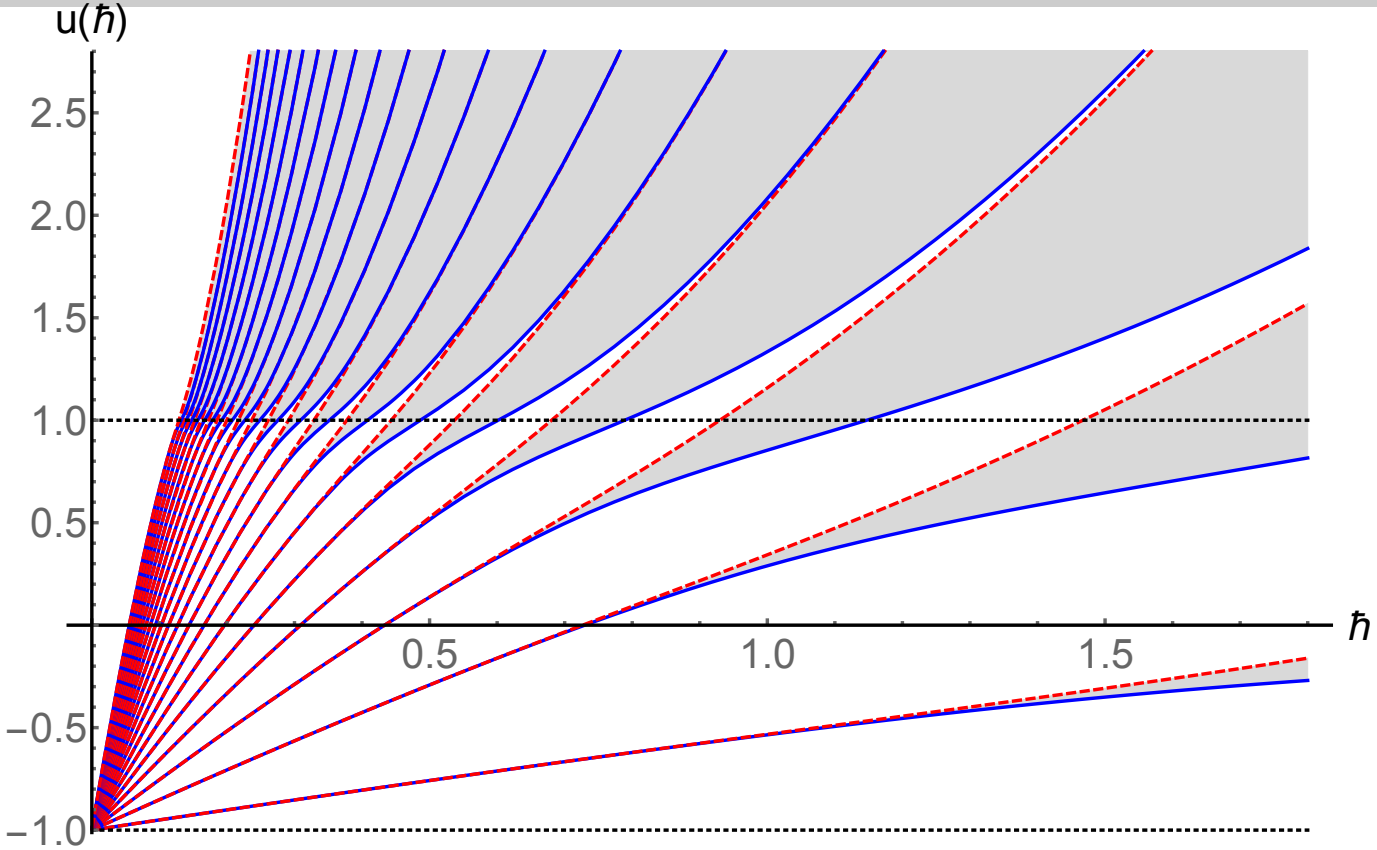
Brézin/Itzykson, 1970; Popov, 1971

- "Schwinger effect" with $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB $\quad\Rightarrow\quad \Gamma_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,\tilde{g}(\gamma)\right]$

$$\Gamma_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\,\mathcal{E}}{\omega\,m\,c}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- phase transition: tunneling vs. multi-photon pair production
- phase transition: real/complex instantons (Dumlu, GD, 2011)
- momentum spectrum: quantum interference effects

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = E\,\psi$



- phase transition: narrow bands vs. narrow gaps
- real vs. complex instantons (Dykhne, 1961)
- phase transition = "instanton condensation"
- exact mapping to $\mathcal{N} = 2$ SU(2) SUSY QFT (Nekrasov et al)

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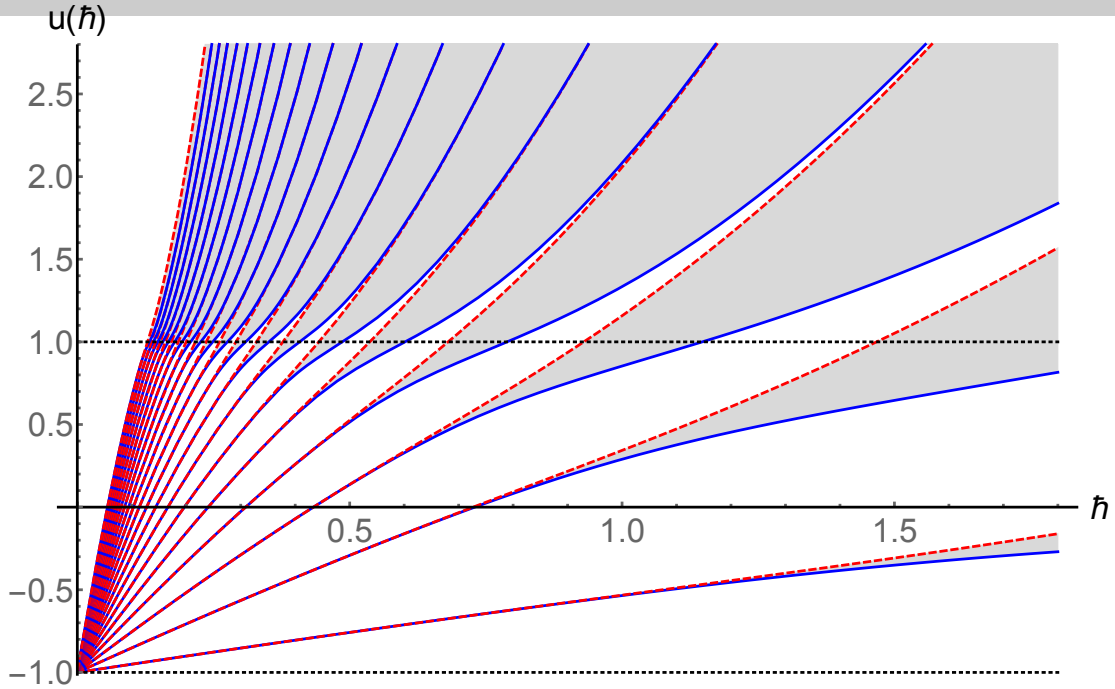


$$E_\pm(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\rm inst}(\hbar, N) + \ldots$$

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S\int_0^\hbar \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right.$$

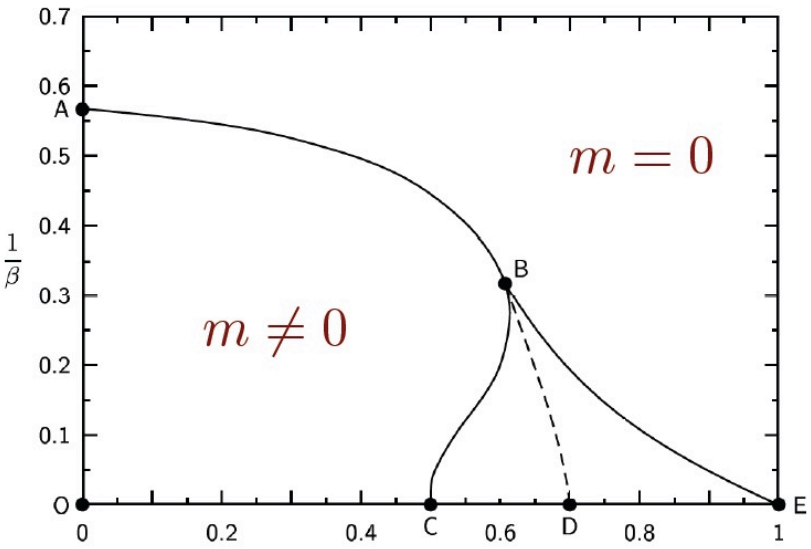
all non-perturbative effects encoded in perturbative expansion

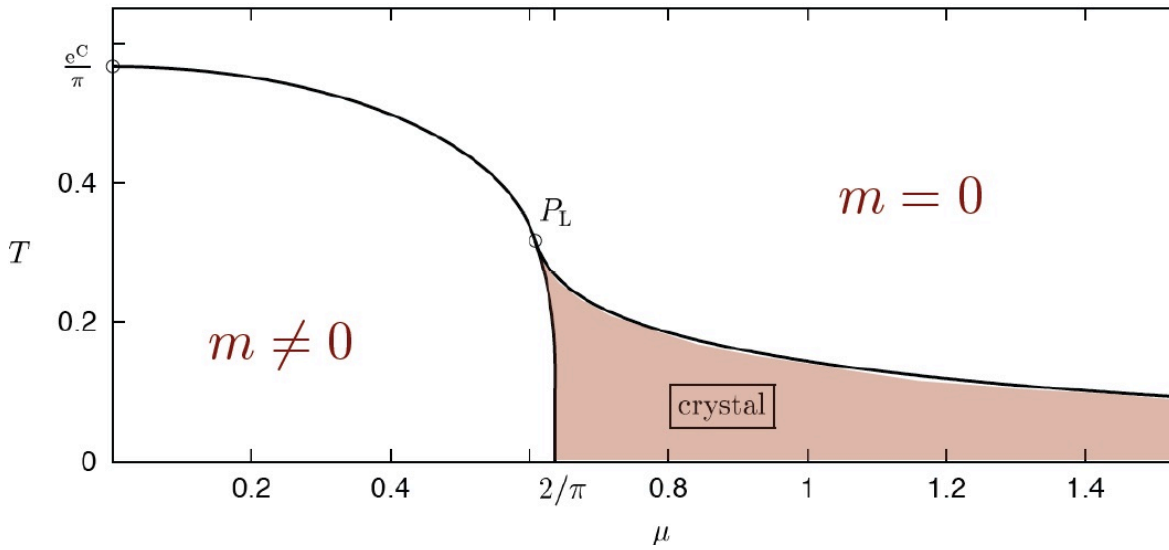
GD & Ünsal (2013); Başar, GD & Ünsal (2017): applies to bands & gaps

Phase Transition in 1+1 dim. Gross-Neveu Model

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2} \left(\bar{\psi}\psi\right)^2$$

- large N_f chiral symmetry breaking phase transition





- saddles: exact solution of inhomogeneous gap equation

$$\sigma(x;T,\mu) = \frac{\delta}{\delta\sigma(x;T,\mu)} \mathrm{Tr}_{T,\mu} \ln\left(i\,\partial\!\!\!/ - \sigma(x;T,\mu)\right)$$

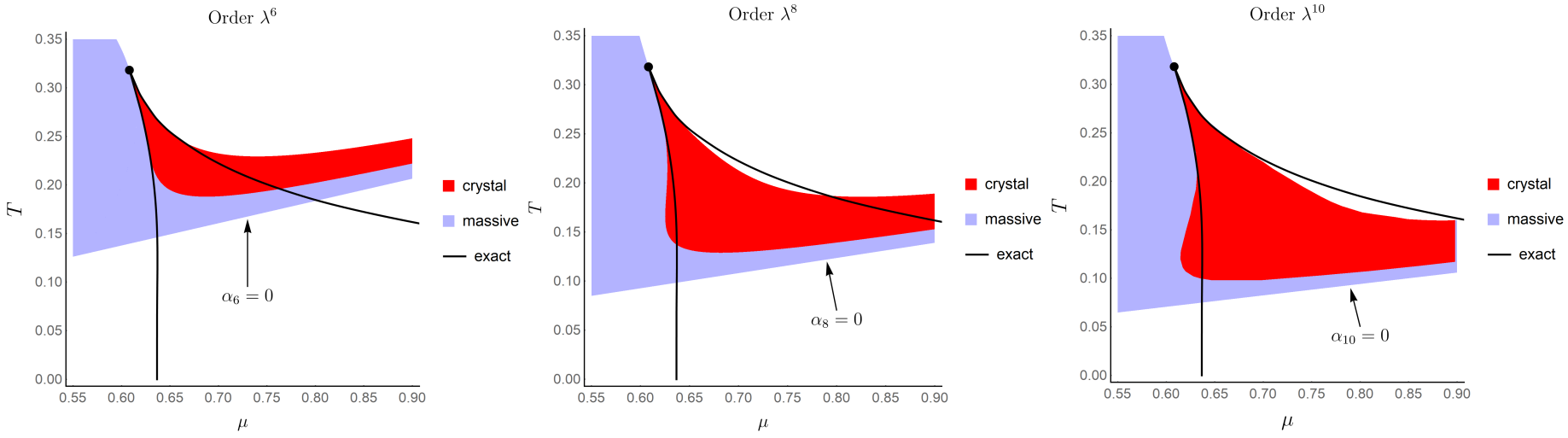
Phase Transition in 1+1 dim. Gross-Neveu Model

- tricritical point: divergent Ginzburg-Landau expansion

$$\Psi(T,\mu) = \sum_n \alpha_n(T,\mu) f_n[\sigma(x;T,\mu)]$$

- successive orders of GL expansion reveal the full crystal phase

(Basar, GD, Thies, 2011; Ahmed, 2018)



- large μ expansion $\Rightarrow$ location of critical point $\mu_c = \frac{2}{\pi}$
- non-perturb. $e^{-\frac{1}{\rho}}$ effects at phase transition at $\mu_c = \frac{2}{\pi}$

Phase Transition in 1+1 dim. Gross-Neveu Model

- most difficult point: $\mu_c = \frac{2}{\pi}$, $T = 0$
- high density (convergent !)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2 \left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots\right)$$

- low density (non-perturbative !)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} \mathcal{F}_{k-1}(\rho)$$

- analogous for μ expansion

2d Yang-Mills: Douglas-Kazakov Large N Phase Transition

- 2d Yang-Mills on a sphere
- "spectral sum" for partition function

$$Z(a,N) = \sum_R (\dim R)^2 \, e^{-\frac{a}{2N} C_2(R)}$$

- large N phase transition at $a_c = \pi^2$ (Douglas-Kazakov)
- "instanton condensation" (Gross-Matytsin)
- large a: saddles = monopole solutions: $A_\mu = \vec{n}\, \mathcal{A}_\mu$

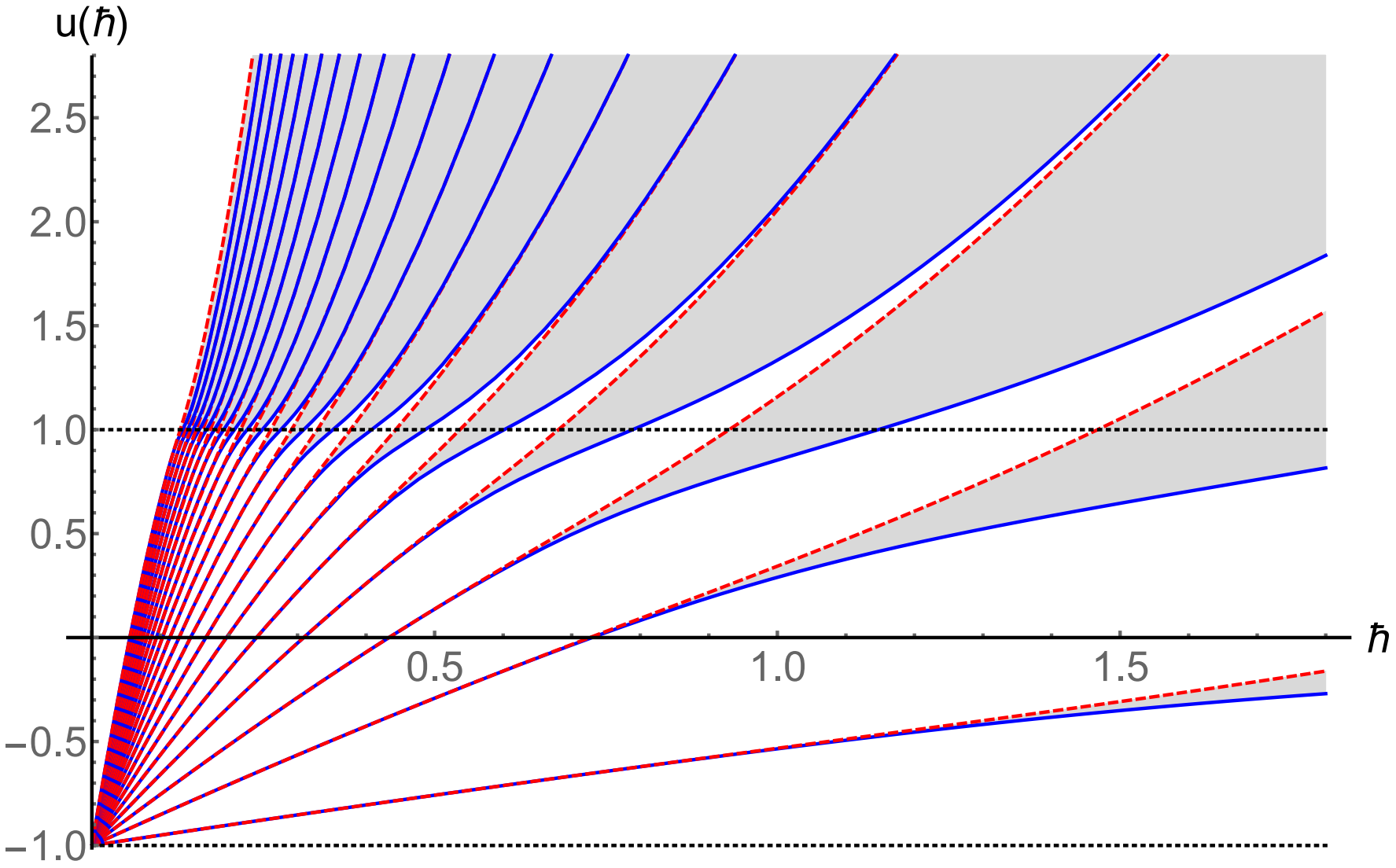
$$Z(a,N) = \sum_{\vec{n}} \mathcal{F}(\vec{n}) \, e^{-\frac{2\pi^2 N}{a} \vec{n}^2}$$

- dual descriptions: generalized Poisson duality
- phase transition = change of saddles

Other Examples: Phase Transitions

- particle-on-circle (Schulman PhD thesis 1968):
sum over spectrum versus sum over winding (saddles)
- Bose gas (Cristoforetti et al)
- Thirring model (Alexandru et al)
- Hubbard model (Tanizaki et al; ...)
- Ising model (Coger, GD, to appear)
- Hydrodynamics: short time/late time (Heller et al; Basar, GD)
- Large N matrix models (Mariño, Schiappa, Couso, Russo, ...)
- Gross-Witten-Wadia model (Ahmed, GD, 2017)
- Painlevé systems (Costin, GD, to appear)
- ...

Phase Transitions at Large N



- phase transition at $t_c = \frac{8}{\pi} \quad (t \equiv N\hbar/2)$

"Parametric Resurgence": Both N and g^2

• trans-series expansion is a double-expansion: can be organized in different ways

$$
\begin{aligned}
F(N,g^2) &\sim \sum_n g^{2n} p_n^{(0)}(N) + e^{-\frac{S}{g^2}} \sum_n g^{2n} p_n^{(1)}(N) + \dots \\
&\sim \sum_k \frac{1}{g^{2k}} c_k(N) \quad + \quad ??? \\
&\sim \sum_h \frac{1}{N^{2h-2}} f_h^{(0)}(N\,g^2) + e^{-S\,N} \sum_h \frac{1}{N^{2h-2}} f_h^{(1)}(N\,g^2) + \dots
\end{aligned}
$$

• how does a divergent trans-series at weak coupling turn into a convergent series at strong-coupling?

• what happens to the resurgent structure?

• what about the 't Hooft limit? $N \to \infty; g^2 \to 0; N g^2 = t$

• separated by a phase transition: "instantons condense"

Uniform Large N for 't Hooft limit: the basic paradigm

- new "uniform" large N approximation
- 't Hooft limit: $\lambda \equiv N\, g^2$ fixed
- e.g. Bessel functions:

$$I_N\left(\frac{1}{g^2}\right) \sim \begin{cases} \sqrt{\frac{g^2}{2\pi}}\, e^{1/g^2} & , \quad g \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}} \left(\frac{e}{2Ng^2}\right)^N & , \quad N \to \infty,\ g^2 \text{ fixed} \end{cases}$$

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- uniform asymptotics:

$$I_N\left(N\frac{1}{Ng^2}\right) \sim \frac{\exp\left[\sqrt{N^2 + \frac{1}{g^4}}\right]}{\sqrt{2\pi}\left(N^2 + \frac{1}{g^4}\right)^{\frac{1}{4}}} \left(\frac{\frac{1}{N\,g^2}}{1 + \sqrt{1 + \frac{1}{(N\,g^2)^2}}}\right)^N$$

- phase transition: $Ng^2 \sim 1$: coalescence of saddles ("bion")

Resurgence in Matrix Models: Mariño: 0805.3033, Ahmed & GD: 1710.01812

Gross-Witten-Wadia Unitary Matrix Model

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2}\mathrm{tr}\left(U + U^\dagger\right)\right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables: g^2 and N ('t Hooft coupling: $t \equiv g^2 N/2$)
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal!)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- similar to 2d Yang-Mills on sphere and disc

Gross-Witten-Wadia Unitary Matrix Model

P. Buividovich, GD, S. Valgushev, 1512.09021

- integrate over $N \times N$ unitary matrices

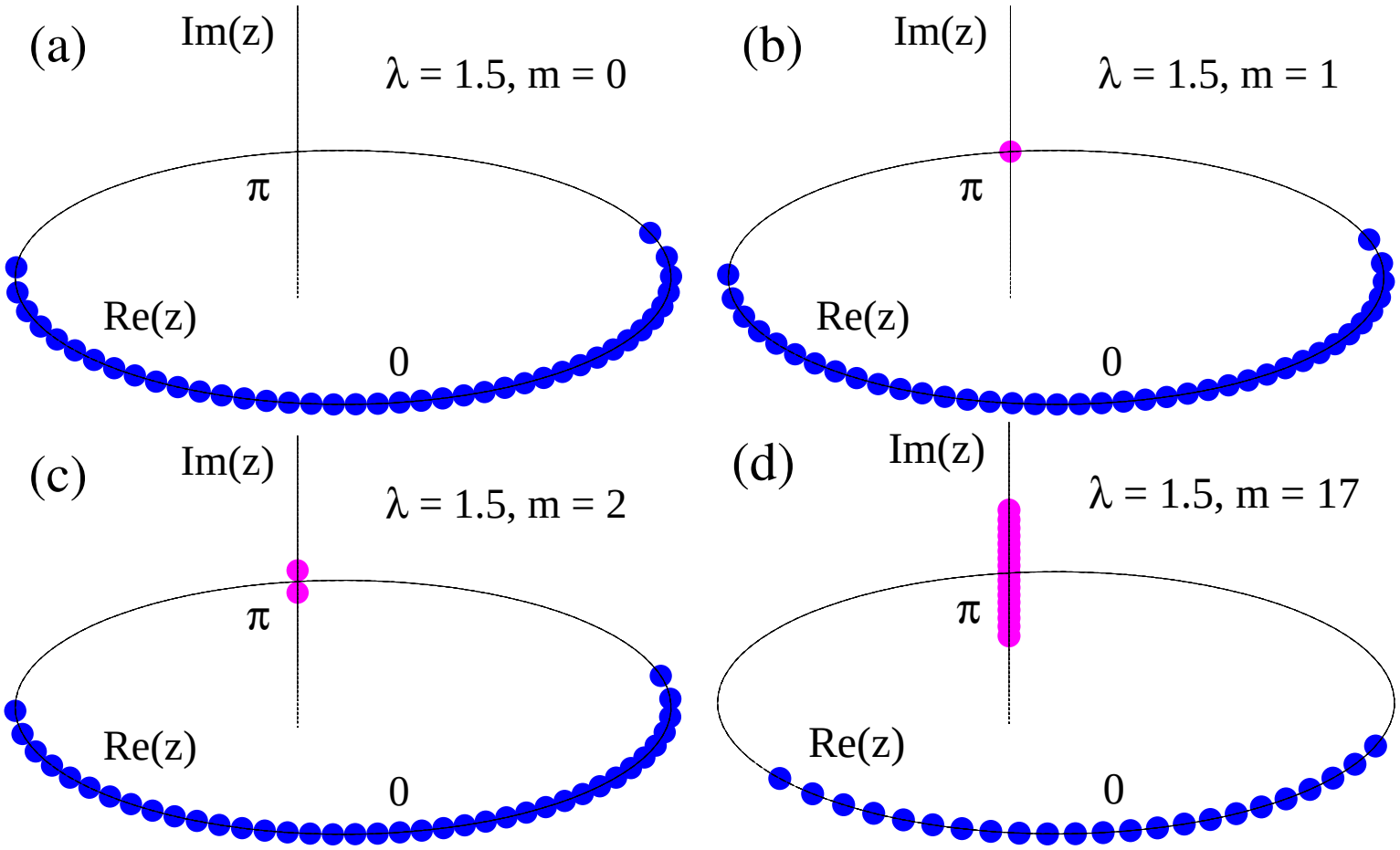
$$Z = \int \mathcal{D}U \, \exp\left[\frac{N}{\lambda}\mathrm{Tr}\left(U + U^\dagger\right)\right]$$

- in terms of eigenvalues e^{iz_j}:

$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} dz_i \exp\left[-\frac{2N}{\lambda}\sum_i \cos(z_i) + \ln \prod_{i<j} \sin^2\left(\frac{z_i - z_j}{2}\right)\right]$$

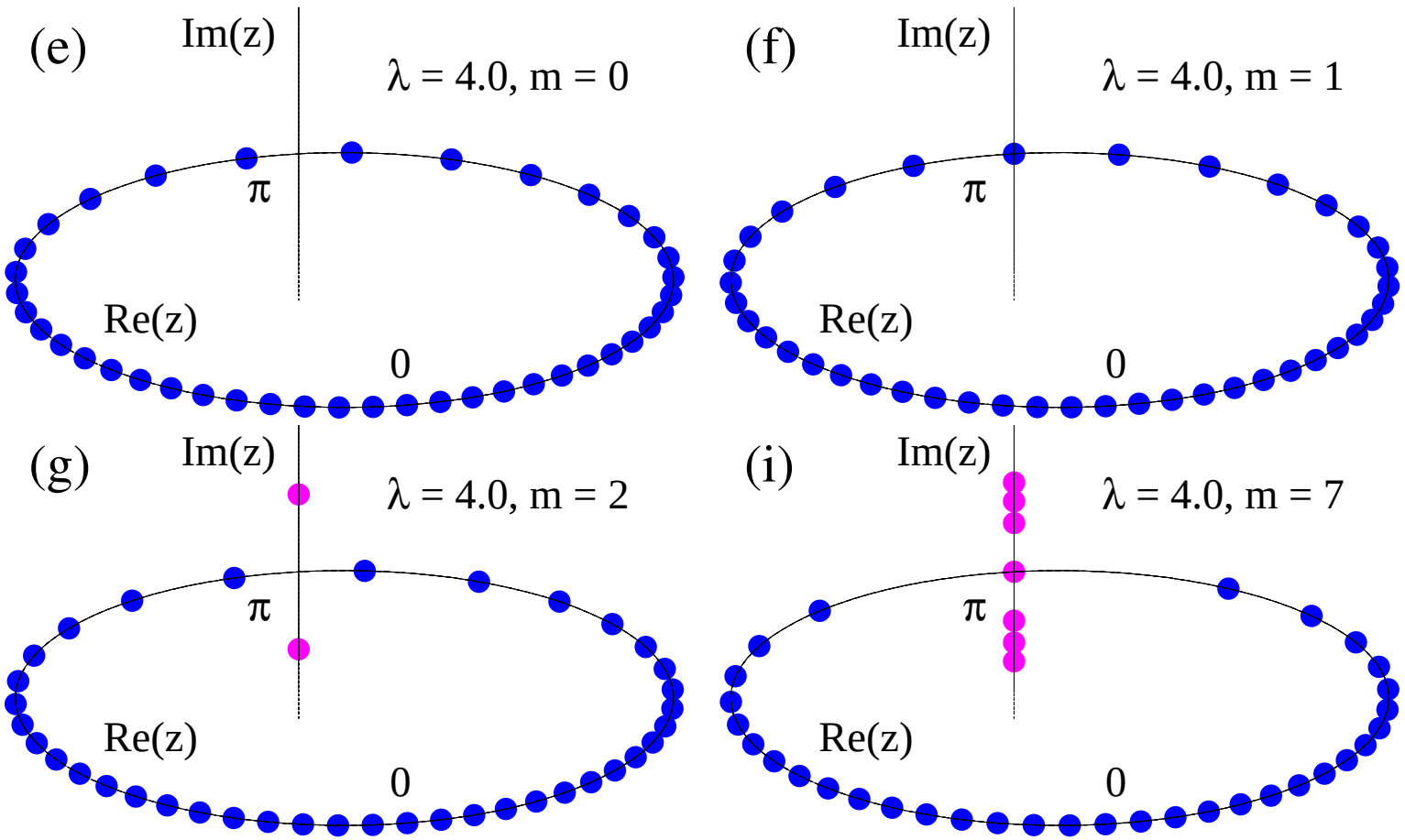
- saddle point approach: $\partial S / \partial z_i = 0$
- which saddles (real/complex?) govern large N behavior?
- how to see the phase transition at finite N?

Gross-Witten-Wadia Model: weak coupling: $\lambda < 2$



- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number
- dominant non-perturbative saddle has $m = 1$

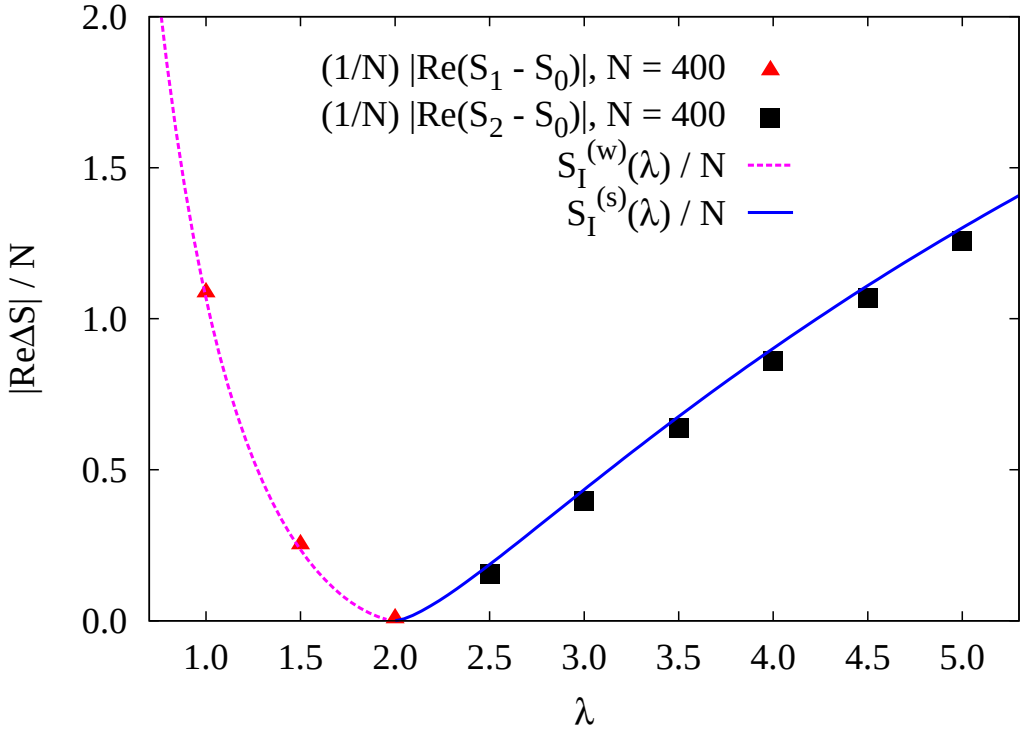
Gross-Witten-Wadia Model: strong coupling: $\lambda > 2$



- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number
- dominant non-perturbative saddle has $m = 2$

Gross-Witten-Wadia Model: non-vacuum saddles

- weak coupling ($\lambda < 2$): $m = 1$ dominant
- strong coupling ($\lambda > 2$): $m = 2$ dominant



$$\lambda < 2: \quad S_I^{(weak)} = 4/\lambda\sqrt{1-\lambda/2} - \operatorname{arccosh}\left((4-\lambda)/\lambda\right)$$

$$\lambda > 2: \quad S_I^{(strong)} = 2\operatorname{arccosh}\left(\lambda/2\right) - 2\sqrt{1-4/\lambda^2}$$

- microscopic view of strong-coupling "instanton/saddle"

Resurgence in Gross-Witten-Wadia Model: Transmutation of the Trans-series

Ahmed & GD: 1710.01812

- partition function $= N \times N$ Toeplitz determinant

$$Z(g^2, N) = \det\left(I_{j-k}(x)\right)_{j,k=1,\ldots N} \qquad , \quad x \equiv \frac{2}{g^2}$$

- so $Z(g^2, N)$ satisfies $(N+1)^{\text{th}}$ order **linear** ODE, $\forall N$

$\Rightarrow$ weak-coupling resurgent trans-series "guaranteed"

$$Z(x,N) \quad \sim \quad Z_0(x,N)\left[\sum_{n=0}^{\infty}\frac{a_n^{(0)}(N)}{x^n} + i\frac{(4x)^{N-1}}{\Gamma(N)}e^{-2x}\sum_{n=0}^{\infty}\frac{a_n^{(1)}(N)}{x^n} + \right.$$
$$\left. \ldots + \frac{G(N+1)}{\prod_{i=0}^{N-1}\Gamma(N-i)}e^{-2Nx}\sum_{n=0}^{\infty}\frac{a_n^{(N)}(N)}{x^n}\right]$$

- but strong-coupling expansion is **convergent**!

$$Z(x,N) \sim e^{x^2/4}\left[1 - \left(\frac{(x/2)^{N+1}}{(N+1)!}\right)^2\left(1 - \frac{1}{2}\frac{(N+1)\,x^2}{(N+2)^2} + \ldots\right) + \ldots\right]$$

Gross-Witten-Wadia Model: Large N

to study large N properly we need to let N become complex

Resurgence in Gross-Witten-Wadia Model

Ahmed & GD: 1710.01812

- key idea for large N: map to a Painlevé function (P III)

$$\Delta(x,N) \equiv \langle \det U \rangle = \frac{\det\left[I_{j-k+1}\left(x\right)\right]_{j,k=1,\ldots,N}}{\det\left[I_{j-k}\left(x\right)\right]_{j,k=1,\ldots,N}}$$

- for any N, $\Delta(x,N)$ satisfies a PIII-type equation:

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

$\Rightarrow$ generate trans-series solutions: weak- & strong-coupling

- N is a parameter ! & $\Rightarrow$ large N limit by rescaling
- direct relation to the partition function:

$$Z(x,N) = \exp\left[\frac{1}{2}\int_0^x x\,dx\left(1-\Delta^2(x,N)\right)\left(1+\Delta(x,N-1)\Delta(x,N+1)\right)\right]$$

Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a divergent series:
  $\rightarrow$ trans-series non-perturbative completion
- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !

Resurgence in Gross-Witten-Wadia Model

- weak-coupling expansion is a divergent series:
  $\rightarrow$ trans-series non-perturbative completion
- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !
- Δ small $\Rightarrow$ linearize $\rightarrow$ Bessel equation

$$\Delta'' + \frac{1}{x}\Delta' + \Delta\left(1-\Delta^2\right) + \frac{\Delta}{(1-\Delta^2)}\left[\left(\Delta'\right)^2 - \frac{N^2}{x^2}\right] = 0$$

$$\Rightarrow \Delta(x,N)\Big]_{\text{strong}} \approx \sigma\, J_N(x)$$

- strong-coupling expansion ($x \equiv \frac{2}{g^2}$) is clearly convergent
- but full solution is a non-perturbative trans-series:

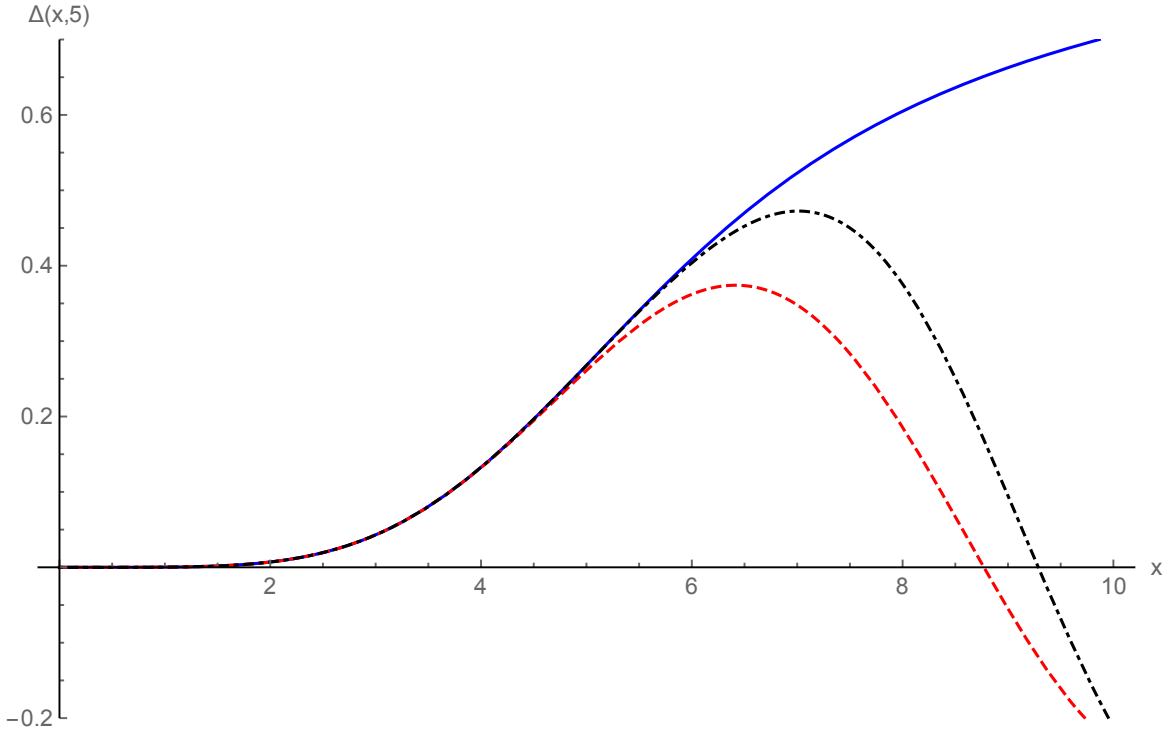
$$\Delta(x,N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(x,N)$$

- higher instanton terms $\Delta_{(k)}(x,N) \sim (J_N(x))^k$

Resurgence in Gross-Witten-Wadia Model

- (convergent) strong-coupling trans-series expansion

$$\Delta(x,N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(x,N)$$

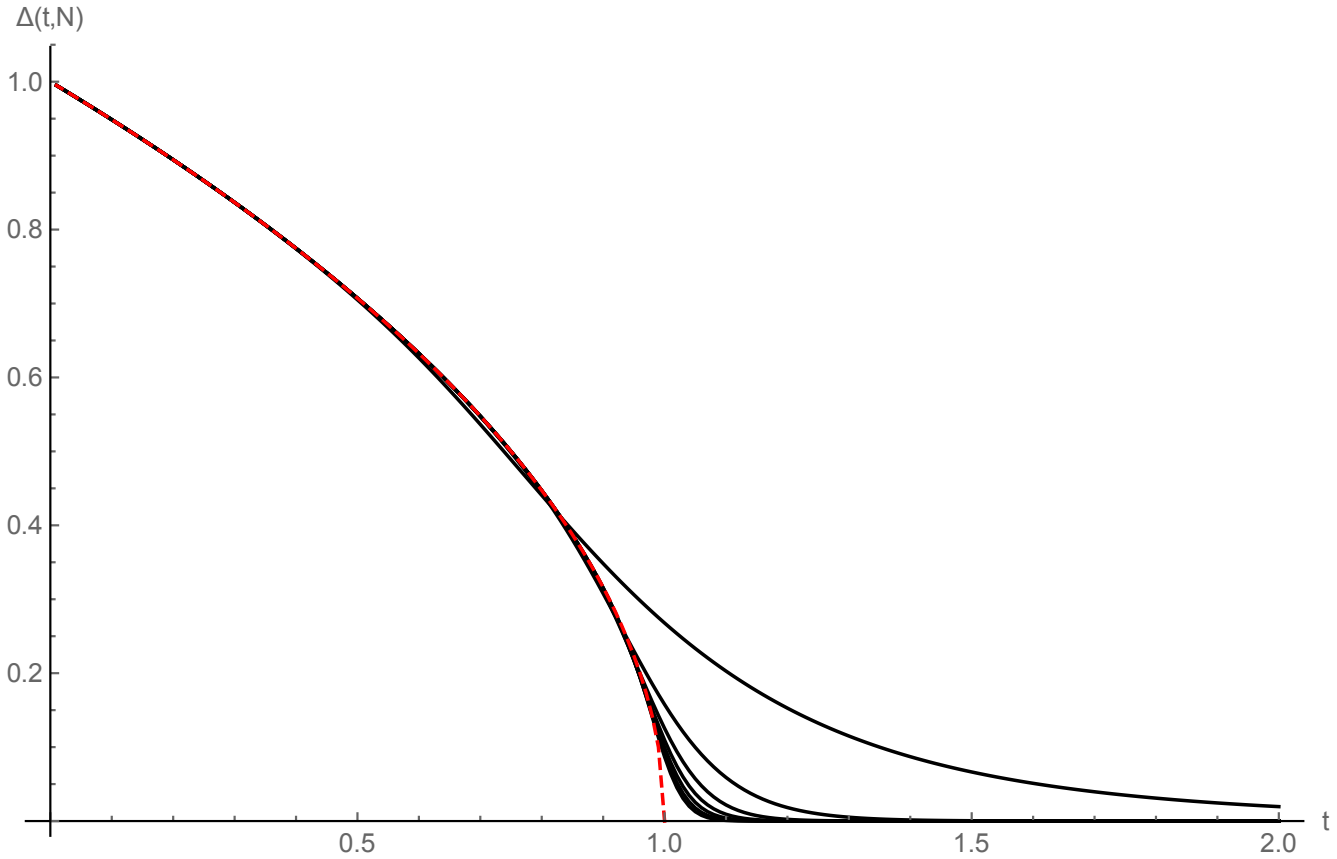


blue: exact $N = 5$, red: $\Delta_{(1)} = J_5(x)$, black: includes $\Delta_{(3)}$

Resurgence in GWW: 't Hooft limit and phase transition

- Gross-Witten-Wadia $N = \infty$ phase transition:

$$\Delta(t,N) \xrightarrow{N\to\infty} \begin{cases} 0 & , \quad t \geq 1 \quad \text{(strong coupling)} \\ \sqrt{1-t} & , \quad t \leq 1 \quad \text{(weak coupling)} \end{cases}$$



$t \equiv \frac{N}{x} \equiv \frac{Ng^2}{2}$

black lines: increasing N

red dashed line: $\Delta = \sqrt{1-t}$

Resurgence in GWW: 't Hooft limit and phase transition

- full large N trans-series at weak-coupling:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}}\,\sigma_{\text{weak}}\,\frac{t\,e^{-NS_{\text{weak}}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \ldots$$

- large N weak-coupling action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

- large-order growth of perturbative coefficients ($\forall t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\,\frac{\Gamma(2n-\frac{5}{2})}{(S_{\text{weak}}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{S_{\text{weak}}(t)}{(2n-\frac{7}{2})} + .\right.$$

- confirm (parametric!) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{1}{N} + \ldots$$

Resurgence in GWW: 't Hooft limit and phase transition

- large N transseries at strong-coupling: $\Delta(t,N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t,N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t,N)$$

  - "Debye expansion" for Bessel function: $J_N(N/t)$

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \frac{1}{4(t^2-1)} \left(\frac{\sqrt{t} e^{-NS_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

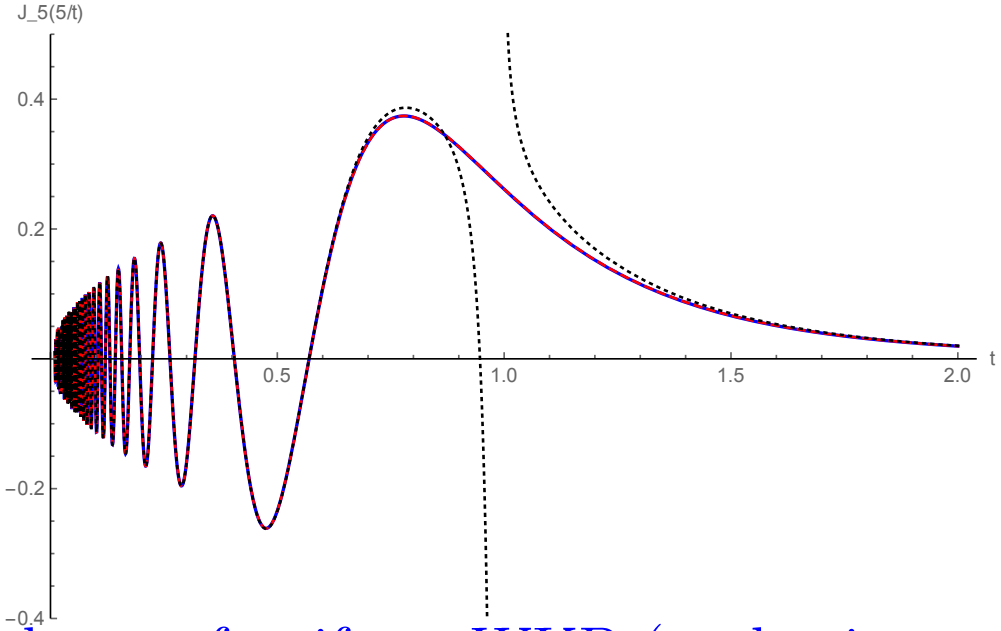
  - large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi (2S_{\text{strong}}(t))^n} \left(1 + U_1(t) \frac{(2S_{\text{strong}}(t))}{(n-1)} + U_2(t) \frac{(2S_{\text{strong}}(t))^2}{(n-1)(n-2)} + \right.$$

Resurgence in GWW: 't Hooft limit and phase transition

- problem with conventional large N approximation
- Debye expansion has unphysical divergence at $t = 1$
- uniform asymptotic expansion:

$$J_N\left(\frac{N}{t}\right) \sim \left(\frac{4\left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3}}{1-1/t^2}\right)^{\frac{1}{4}} \frac{\text{Ai}\left(N^{\frac{2}{3}}\left(\frac{3}{2}S_{\text{strong}}(t)\right)^{2/3}\right)}{N^{\frac{1}{3}}}$$



nonlinear analogue of uniform WKB (coalescing saddles: bions)

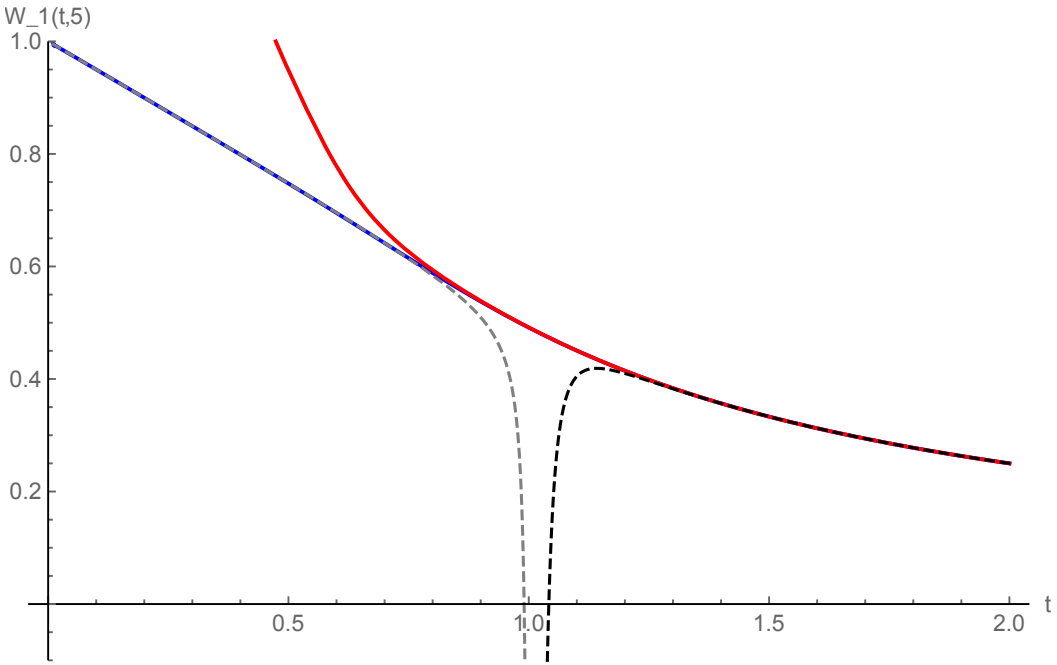
Resurgence in GWW: 't Hooft limit and phase transition

- Wilson loop: $\mathcal{W} \equiv \frac{1}{N}\frac{\partial \ln Z}{\partial x}$

$$\mathcal{W}(t,N) = \frac{1}{2t}\left(1-\Delta^2(t,N)\right)\left(1+\Delta(t,N-1)\Delta(t,N+1)\right)$$

- uniform large N approximation at strong-coupling:

$$\mathcal{W}(t,N)\Big|^{\text{strong}} \approx \frac{1}{2t}\left(1-J_N^2(N/t)\right)\left(1+J_{N-1}(N/t)J_{N+1}(N/t)\right)$$



blue: exact
red: uniform large N
dashed: usual large N

uniform resummation of instantons & fluctuations

Gross-Witten: beta function at large N

- beta function has full trans-series expansion
- leading large N : $\beta(\lambda) = \begin{cases} -2\lambda \log \lambda \,, \lambda \geq 2 \\ 2(\lambda - 4) \log \frac{4}{4-\lambda} \,, \lambda \leq 2 \end{cases}$
- for any N: $\beta(\lambda) = \frac{2}{\frac{d}{d\lambda} \log \log W_N(\lambda)}$ $\rightarrow$ <u>trans-series</u> for β

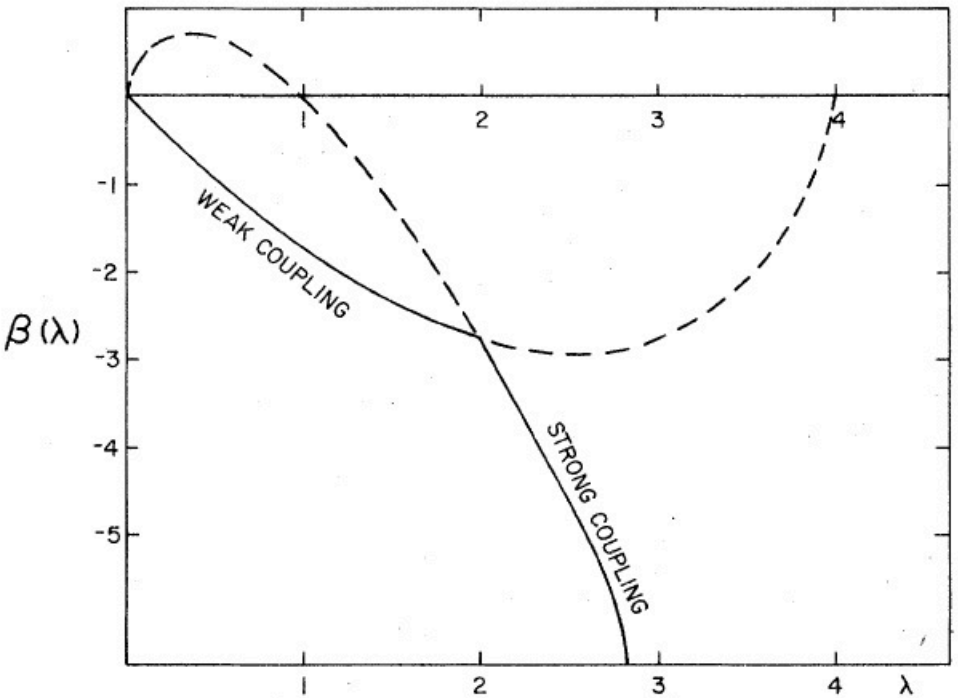


FIG. 1. The β function as a function of λ. The dashed lines are the (invalid) extrapolation of the weak- and strong-coupling results beyond the phase transition at $\lambda = 2$.

- naive large N incorrectly predicts new fixed points at $\lambda = 1$ and $\lambda = 4$
- <u>uniform large N</u> smoothly passes from weak-coupling to strong-coupling curve, developing a kink at $N = \infty$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- phase transitions $\leftrightarrow$ Stokes phenomenon
- non-perturbative effects even for convergent series !

- ionization & Schwinger effect
- Gross-Neveu: chiral symmetry breaking & crystals
- 3rd order phase transition at $N = \infty$ in Gross-Witten-Wadia unitary matrix model
- physics: instanton condensation
- physics: uniform large N "instantons" (*cf.* "bions")