Topological Charge in Lattice QCD and the Chiral Condensate

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Gauge Topology 3, ECT*

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Topological Charge in Lattice QCD and the Chiral Condensate

- **Based on the articles:**
  - Topological susceptibility from twisted mass fermions using spectral projectors and the gradient flow, [arXiv:1709.06596].
  - Comparison of topological charge definitions in Lattice QCD, [arXiv:1708.00696].
  - Topological charge using cooling and the gradient flow, [arXiv:1509.04259].
  - Gluon Green functions free of Quantum fluctuation, [arXiv:1604.08887].
  - Neutron electric dipole moment using $N_f = 2 + 1 + 1$ twisted mass fermions, [arXiv:1510.05823].
  - Work in progress with P. Boucaud, F. De Soto, J. Rodríguez-Quinterio and S. Zafeiropoulos
- **In collaboration with :**
  - Constantia Alexandrou (University of Cyprus, The Cyprus Institute)
  - Philippe Boucaud (CNRS, Orsay)
  - Krzysztof Cichy (Goethe-Universität Frankfurt am Main, Adam Mickiewicz University)
  - Martha Constantinou (Temple University)
  - Feliciano De Soto (University Pablo de Olavide)
  - Arthur Dromard (Regensburg University)
  - Derek P. Horkel (Temple University)
  - Karl Jansen (NIC, DESY Zeuthen)
  - Giannis Koutsou (The Cyprus Institute)
  - José Rodríguez-Quinterio (Universidad de Huelva)
  - Urs Wenger (University of Bern)
  - Falk Zimmermann (Universität Bonn)
  - Savvas Zafeiropoulos (University of Heidelberg)

Overview of Topics

- Why Studying Topological Charge
  - → The Neutron Electric Dipole Moment "depends" on the Topological Charge
  - → Instanton dependence of $\alpha_{\mathrm{MOM}}(k)$ (Feliciano's presentation)
  - → Interested in the QCD Vaccuum
- There is not only one unique way of extracting the topological charge (corrections of $\mathcal{O}(a)$)
- Each different method on the Lattice is accompanied with pros and cons
- To better understand the different methods:
  - → Comparison of Topological charge definitions which belong to the **two categories**
    - (g) Gluonic
    - (f) Fermionic
  - → Comparison of Topological Susceptibility: gluonic Vs. fermionic
  - → Pion mass dependence of the Top. Susceptibility provides **The Chiral Condensate**
  - → What we learn? – Is there Universality?

Investigation

- Several definitions of the topological charge:
  - (f) fermionic (Index, Spectral flow, Spectral Projectors).
  - (g) gluonic with UV fluctuations removed via smoothing (gradient flow, cooling, smearing,...).

? How are these definitions numerically related?

- Gradient Flow is a well defined smoothing scheme with good renormalizability properties.

  M. Lüscher [arXiv:1006.4518]

! Gradient Flow is numerically equivalent to cooling!

C. Bonati and M. D'Elia [arXiv:1401.2441] and C. Alexandrou, AA and K. Jansen, [arXiv:1509.0425]

? Can this be applied to other smoothing schemes?

C. Alexandrou, *el al*, [arXiv:1708.00696]

- Comparison of different definitions presented by Krzysztof Cichy *et. al* ...

  K. Cichy *et. al*, [arXiv:1411.1205]

  ! Most definitions are highly correlated.

  ! The topological susceptibilities are in the same region.

  → Not meaningfull comparison.

Gluonic Definition of the Topological Charge

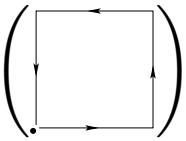
g Topological charge can be defined as:

$$\mathcal{Q} = \int d^4x\, q(x)\,, \quad \text{with} \quad q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left\{F_{\mu\nu}F_{\rho\sigma}\right\}\,.$$

g Discretizations of $q(x)$ on the lattice:

- Plaquette (leading correction term of $\mathcal{O}(a^2)$)

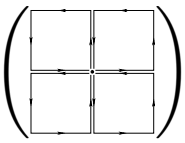
$$q_L^{\mathrm{plaq}}(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{plaq}}C_{\rho\sigma}^{\mathrm{plaq}}\right)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{plaq}}(x) = \mathrm{Im}\Bigg($$



$$\Bigg)\,.$$

- Clover (leading correction term of $\mathcal{O}(a^2)$)

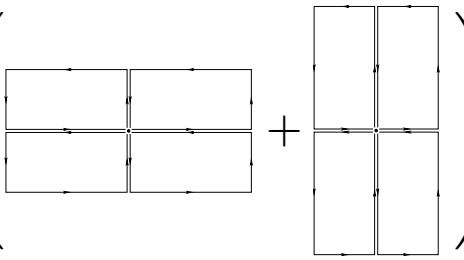
$$q_L^{\mathrm{clov}}(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{clov}}C_{\rho\sigma}^{\mathrm{clov}}\right)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{clov}}(x) = \frac{1}{4}\mathrm{Im}\Bigg($$



$$\Bigg)\,.$$

- Improved (leading correction term of $\mathcal{O}(a^4)$)

$$q_L^{\mathrm{imp}}(x) = b_0 q_L^{\mathrm{clov}}(x) + b_1 2 q_L^{\mathrm{rect}}(x)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{rect}}(x) = \frac{1}{8}\mathrm{Im}\Bigg($$



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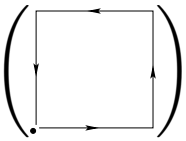

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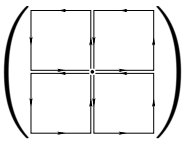

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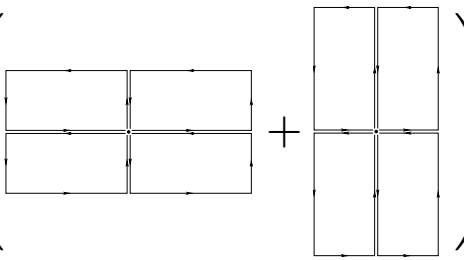

Smoothing $\rightarrow$ Remove the ultraviolet fluctuations

Smoothing Schemes for Gluonic Topological Charge

- (g) The Gradient Flow

  M. Lüscher, *JHEP* 1008 (2010) 071

- (g) Cooling.

  M. Teper, *Phys. Lett. B* 162 (1985) 357.

- (g) APE smearing.

  M. Albanese et al. [APE Coll.], *Phys. Lett. B* 192 (1987) 163.

- (g) Stout smearing.

  C. Morningstar and M. J. Peardon, *Phys. Rev.* D69 (2004) 05450

- (g) HYP smearing.

  A. Hasenfratz and F. Knechtli, *Phys. Rev.* D64 (2001) 034504

- (g) Several other modified versions of smearing.

Credits to Ed Bennett (2013 Royal Society Award)
"Finding needles in four-dimensional haystacks"

Smoothing Schemes for Gluonic Topological Charge

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- Cooling.

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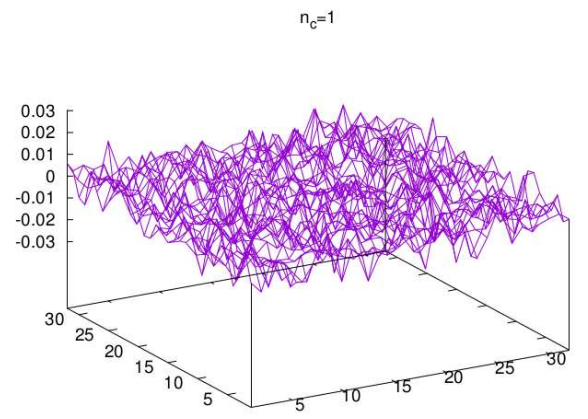
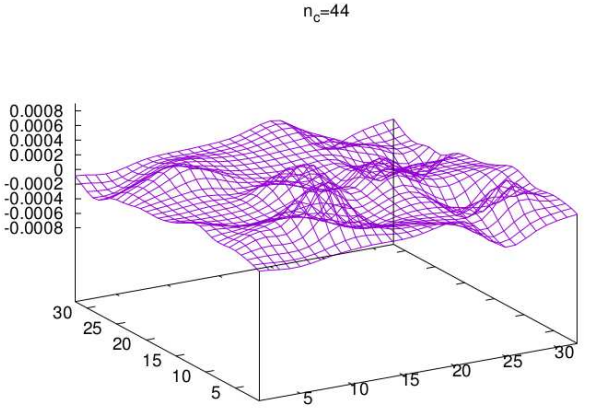




C.f De Soto's talk

The Flagship smoother: The Gradient Flow

- Defined by a local diffusion equation that evolves the Gauge field as a function of the flow time
- Gauge field generated by the gradient flow does not require renormalization
- Solution of the evolution equations:

$$\begin{aligned} \dot{V}_\mu(x,\tau) &= -g_0^2\left[\partial_{x,\mu} S_G(V(\tau))\right] V_\mu(x,\tau) \\ V_\mu(x,0) &= U_\mu(x)\ , \end{aligned}$$

- With link derivative defined as:

$$\partial_{x,\mu} S_G(U) = i\sum_a T^a \frac{\mathrm{d}}{\mathrm{d}s} S_G\left(e^{isY^a}U\right)\Big|_{s=0} \equiv i\sum_a T^a \partial^{(a)}_{x,\mu} S_G(U)\,,$$

- The flown fields are given by:

$$B_\mu(\tau,x) = \int d^4y \frac{e^{-\frac{(x-y)^2}{4\tau}}}{(4\pi\tau)^2} A_\mu(x) \qquad \text{smoothing radius} : \sqrt{8\tau}$$

- Iterative process for gradient flow time τ
- One can define a reference flow time t_0 such that

$$t^2\langle E(t)\rangle|_{t=t_0} = 0.3$$

with $t = a^2\tau$ and

$$E(t) = -\frac{1}{2V}\sum_x \mathrm{Tr}\left\{F_{\mu\nu}(x,t)F_{\mu\nu}(x,t)\right\}$$

The Flagship smoother: The Gradient Flow

- Gradient flow depends on the detais of the Smoothing Action ($c_0 + 8c_1 = 1$) :

$$S_g = \frac{\beta}{N} \sum_x \left(c_0 \sum_{\substack{\mu,\nu=1 \\ 1 \leq \mu < \nu}}^{4} \left\{ 1 - \mathrm{ReTr}(U^{1\times 1}_{x,\mu,\nu}) \right\} + c_1 \sum_{\substack{\mu,\nu=1 \\ \mu \neq \nu}}^{4} \left\{ 1 - \mathrm{ReTr}(U^{1\times 2}_{x,\mu,\nu}) \right\} \right),$$

  - Wilson: $c_1 = 0$,
  - Symanzik tree-level improved: $c_1 = -1/12$,
  - Iwasaki: $c_1 = -0.331$,

- Different Actions are expected to lead to different Instanton (with scale λ) behaviour

$$S_{\mathrm{Lat}}(a, \lambda) = S_{\mathrm{cont}} \left\{ 1 + (a/\lambda)^2 a_2 + (a/\lambda)^4 a_4 + \mathcal{O}\left(a/\lambda\right)^6 \right\}$$

  - Wilson: $a_2 = -1/5$
  - Symanzik tree-level improved: $a_2 = 0, a_4 = -17/210$
  - Iwasaki: $a_2 = +2.972/5$

- Stable instanton solutions require a lattice action which increases by decreasing the scale parameter λ
- Only for Iwasaki we expect stable solutions
- However, the most-frequently used action for smoothing is the Wilson

The Flagship smoother: The Gradient Flow

- Gradient Flow provides a concrete theoretical framework for smoothing
- The evolution of the fields is processing in gradient flow time steps ϵ (typically $\epsilon \leq 0.1$)
- This means that it requires many iterations to reach a given gradient flow time ($\tau = n_{\rm int}\epsilon$)
- Other "theoretically weaker" smoothing techniques
  - provide similar results?
  - how fast are they?

The Wilson flow Vs. Cooling

Gradient Flow

- Solution of the evolution equations:

$$\begin{aligned} \dot{V}_\mu\left(x,\tau\right) &= -g_0^2\left[\partial_{x,\mu} S_G(V(\tau))\right] V_\mu\left(x,\tau\right) \\ V_\mu\left(x,0\right) &= U_\mu\left(x\right)\ , \end{aligned}$$

- With link derivative defined as:

$$\begin{aligned} \partial_{x,\mu} S_G(U) &= i\sum_a T^a \frac{\mathrm{d}}{\mathrm{d}s} S_G\left(e^{isY^a}U\right)\Big|_{s=0} \\ &\equiv i\sum_a T^a \partial^{(a)}_{x,\mu} S_G(U)\ , \end{aligned}$$

- Total gradient flow time: τ

Cooling

- Cooling $U_\mu(x) \in SU(N)$: $U_\mu^{\mathrm{old}}(x) \to U_\mu^{\mathrm{new}}(x)$ with

$$P(U) \propto e^{(\lim_{\beta\to\infty} \beta \frac{1}{N} \mathrm{ReTr} X_\mu{}^\dagger U_\mu)}.$$

- Choose a $U_\mu^{\mathrm{new}}(x)$ that maximizes:

$$\mathrm{ReTr}\{\, U_\mu^{\mathrm{new}}(x) X_\mu^\dagger(x)\}.$$

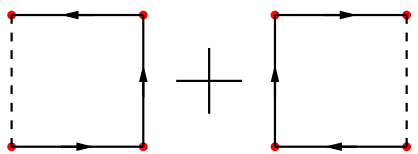
- One full cooling iteration $n_c = 1$

Perturbative expansion of links

- A link variable which has been smoothed can been written as:

$$U_\mu(x, j_{\rm sm}) \simeq \mathbb{1} + i \sum_a u^a_\mu(x, j_{\rm sm}) T^a \,.$$

- Simple staples are written as:



per space-time slice, thus:

$$X_\mu(x, j_{\rm sm}) \simeq 6 \cdot \mathbb{1} + i \sum_a w^a_\mu(x, j_{\rm sm}) T^a \,.$$

- For the Wilson flow with $\Omega_\mu(x) = U_\mu(x) X^\dagger_\mu(x)$

$$g_0^2 \partial_{x,\mu} S_G(U)(x) = \frac{1}{2}\left(\Omega_\mu(x) - \Omega^\dagger_\mu(x)\right) - \frac{1}{6}{\rm Tr}\left(\Omega_\mu(x) - \Omega^\dagger_\mu(x)\right) .$$

where

$$g_0^2 \partial_{x,\mu} S_G(U) \simeq i \sum_a \left[6 u^a_\mu(x,\tau) - w^a_\mu(x,\tau)\right] T^a \,.$$

Wilson flow Vs. Cooling

- Evolution of the Wilson flow by an infinitesimally small flow time ϵ:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right] .$$

  where $U_\mu(x,\tau+\epsilon) \simeq \mathbb{1} + i\sum_a u_\mu^a(x,\tau+\epsilon)T^a$

- After a cooling step:

$$u_\mu^a(x,n_c+1) \simeq \frac{w_\mu^a(x,n_c)}{6} .$$

- Wilson flow would evolve the same as cooling if $\epsilon = 1/6$.

\+ Cooling has an additional speed up of two.

! As we saw before, cooling has the same effect as the Wilson flow if:

$$\tau \simeq \frac{n_c}{3} .$$

C. Bonati and M. D'Elia, *Phys. Rev.* **D89** (2014), 105005 [arXiv:1401.2441]

- We generalized of this result for smoothing actions with rectangular terms (b_1):

$$\tau \simeq \frac{n_c}{3-15b_1} .$$

C. Alexandrou, AA and K. Jansen, *Phys. Rev.* **D92** (2015), 125014 [arXiv:1509.0425]

- What happens for other smoothing schemes??

Gradient flow Vs. Cooling

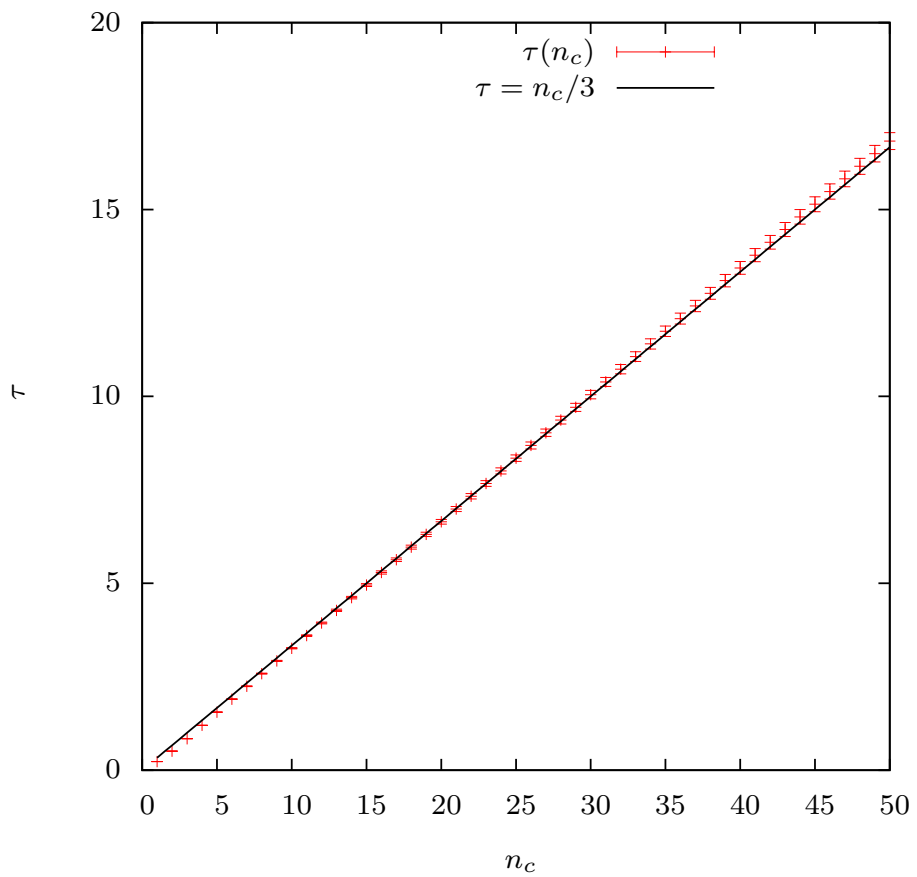
Test the matching conditions for

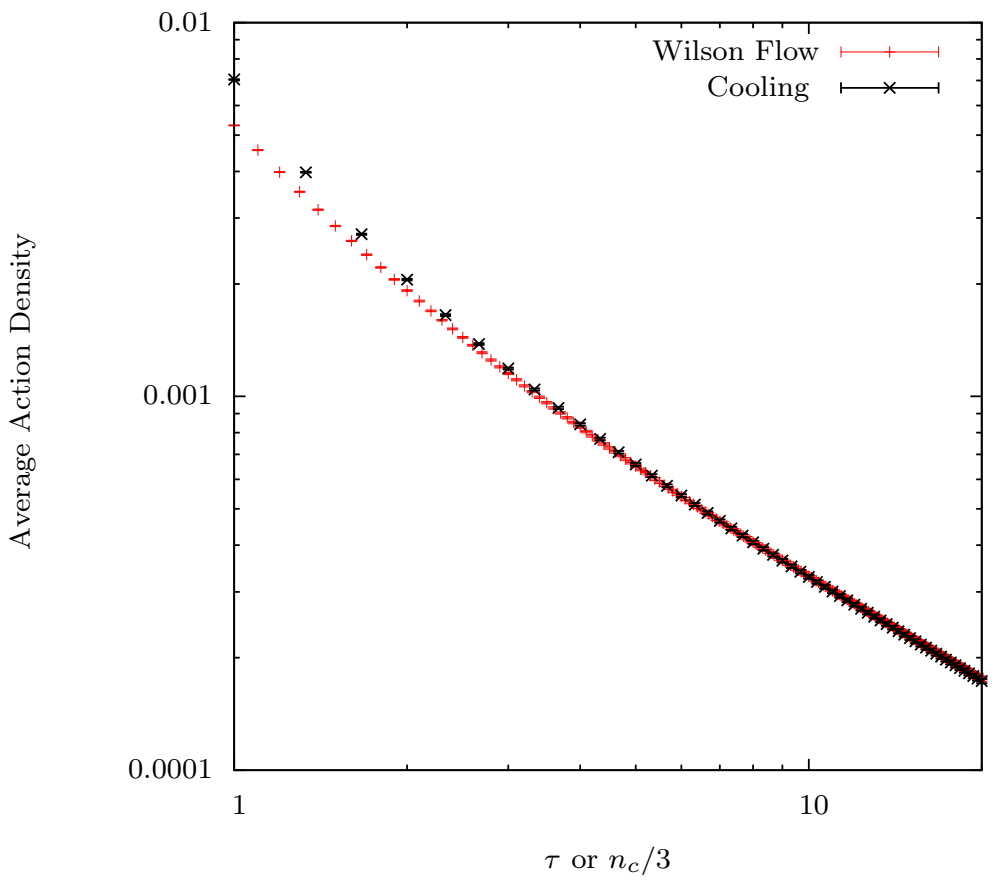
- $N_f = 2$ twisted mass fermions
  - Produced with the Iwasaki Gluonic Action
  - Lattice spacing $a \simeq 0.085$fm ($r_0/a = 5.35(4)$)
  - Pion Mass $m_\pi \simeq 340$ MeV
  - Lattice size $L/a = 16$
- $N_f = 2 + 1 + 1$ twisted mass fermions,
  - Produced with the Iwasaki Gluonic Action
  - Lattice spacing $a = 0.0823(10)$ fm ($r_0/a = 5.710(41)$)
  - Pion Mass $m_\pi \simeq 370$ MeV
  - Lattice size $L/a = 32$
- $N_f = 0$ Pure Gauge,
  - Produced with the Symanzik tree-level improved
  - Lattice spacing $a = 0.285(10)$ fm
  - Lattice size $L/a = 32$

Gradient flow Vs. Cooling

Matching condition: $\tau \simeq \frac{n_c}{3}$.

Define function $\tau(n_c)$ such as τ and n_c change the action by the same amount.



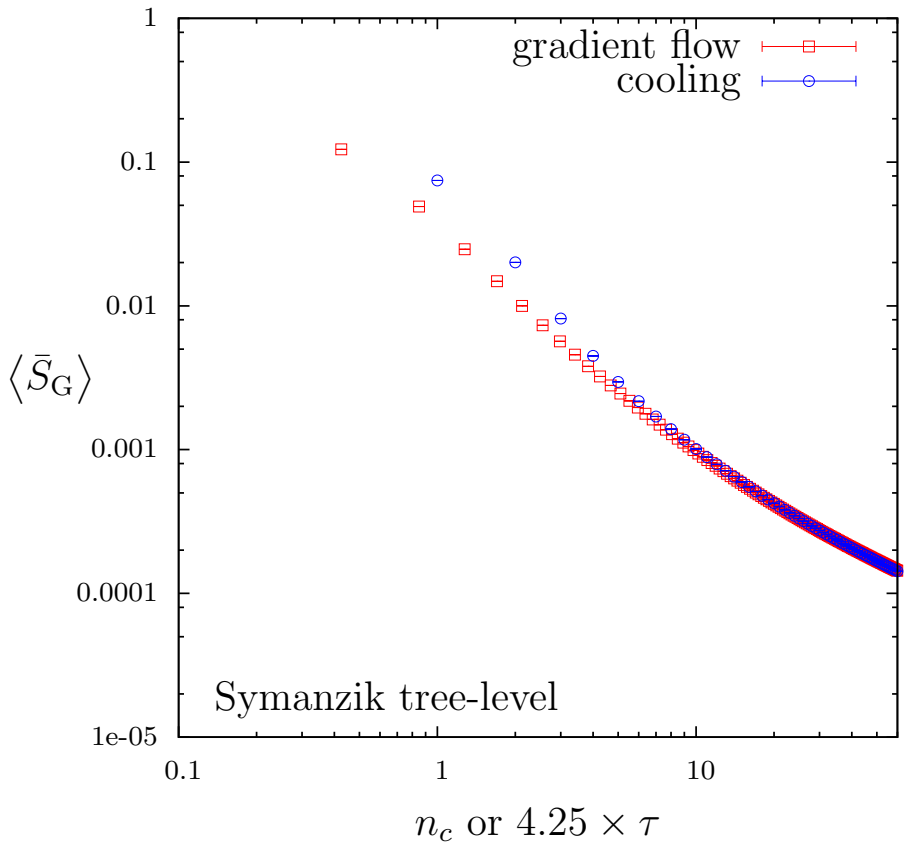


Topological Charge: Wilson flow Vs. Cooling

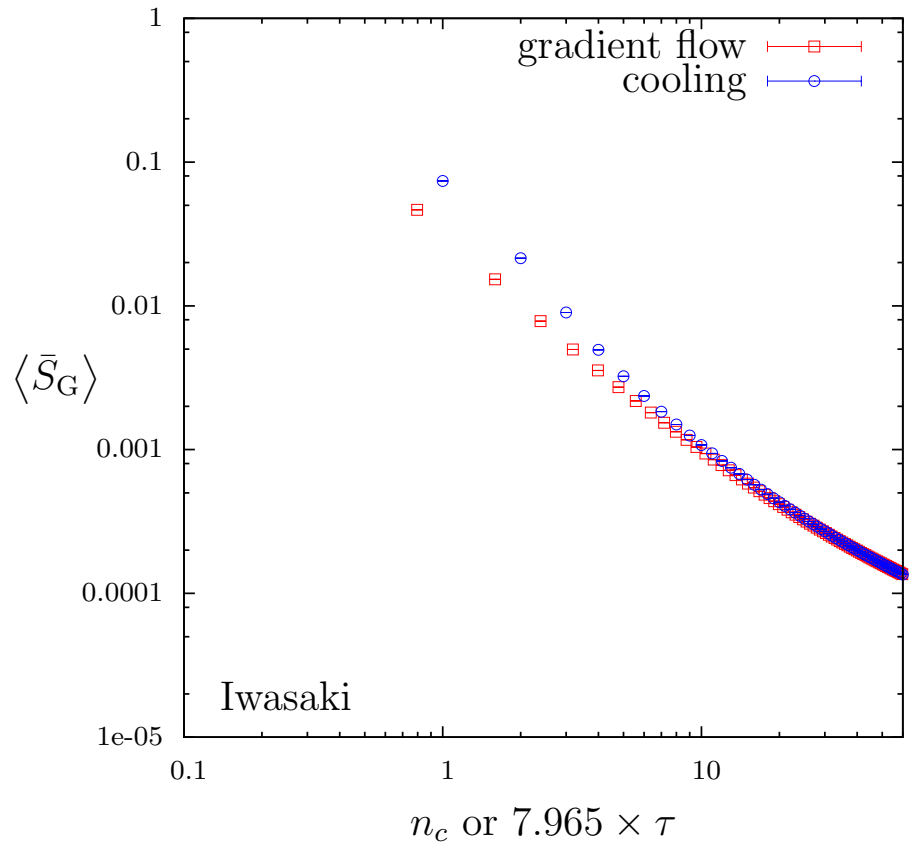
- For smoothing actions with rectangular terms:

$$\tau \simeq \frac{n_c}{3 - 15b_1} \, .$$

Symanzik tree-level: $n_c = 4.25\tau$



Iwasaki: $n_c = 7.965\tau$



Wilson flow Vs. APE

- According to the APE operation:

$$U_\mu^{(n_{\rm APE}+1)}(x) = Proj_{SU(3)}\left[(1-\alpha_{\rm APE})\,U_\mu^{(n_{\rm APE})}(x) + \frac{\alpha_{\rm APE}}{6}X_\mu^{(n_{\rm APE})}(x)\right] .$$

- Evolution of the Wilson flow by an infinitesimally small flow time ϵ is expressed as:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right] .$$

- Evolution of the APE smearing with parameter $\alpha_{\rm APE}$ is expressed as:

$$u_\mu^a(x,n_{\rm APE}+1) \simeq u_\mu^a(x,n_{\rm APE}) - \frac{\alpha_{\rm APE}}{6}\left[6u_\mu^a(x,n_{\rm APE}) - w_\mu^a(x,n_{\rm APE})\right] .$$

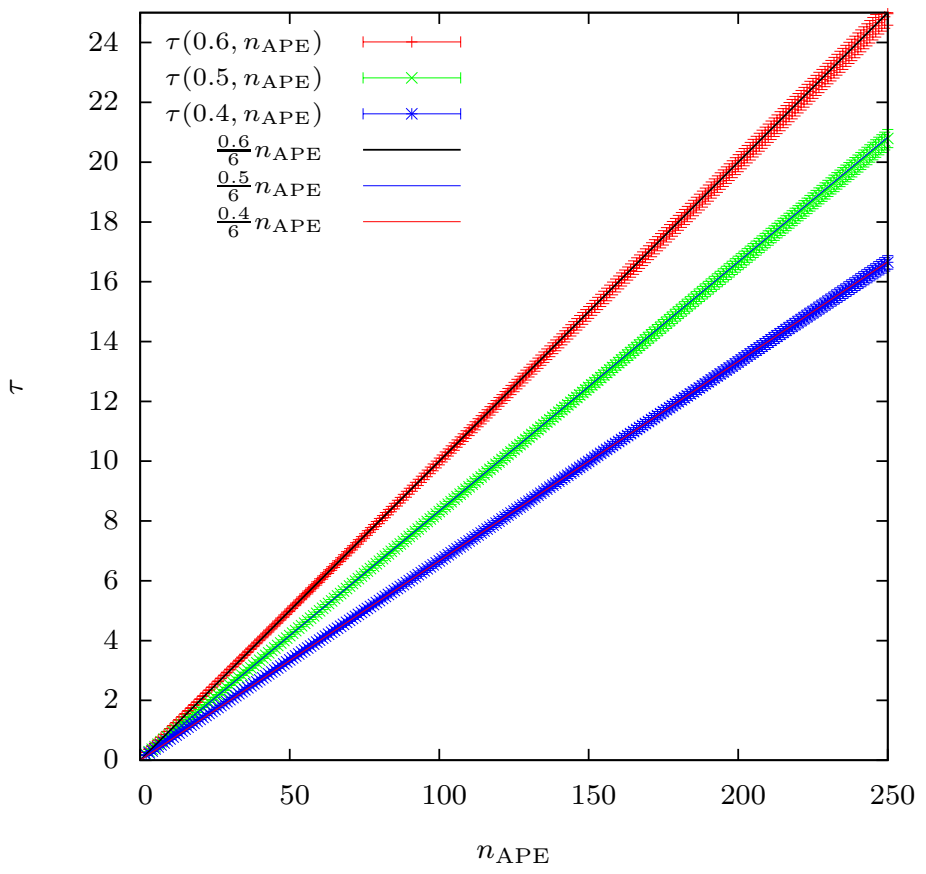
- Hence, APE has the same effect as the Wilson flow if:

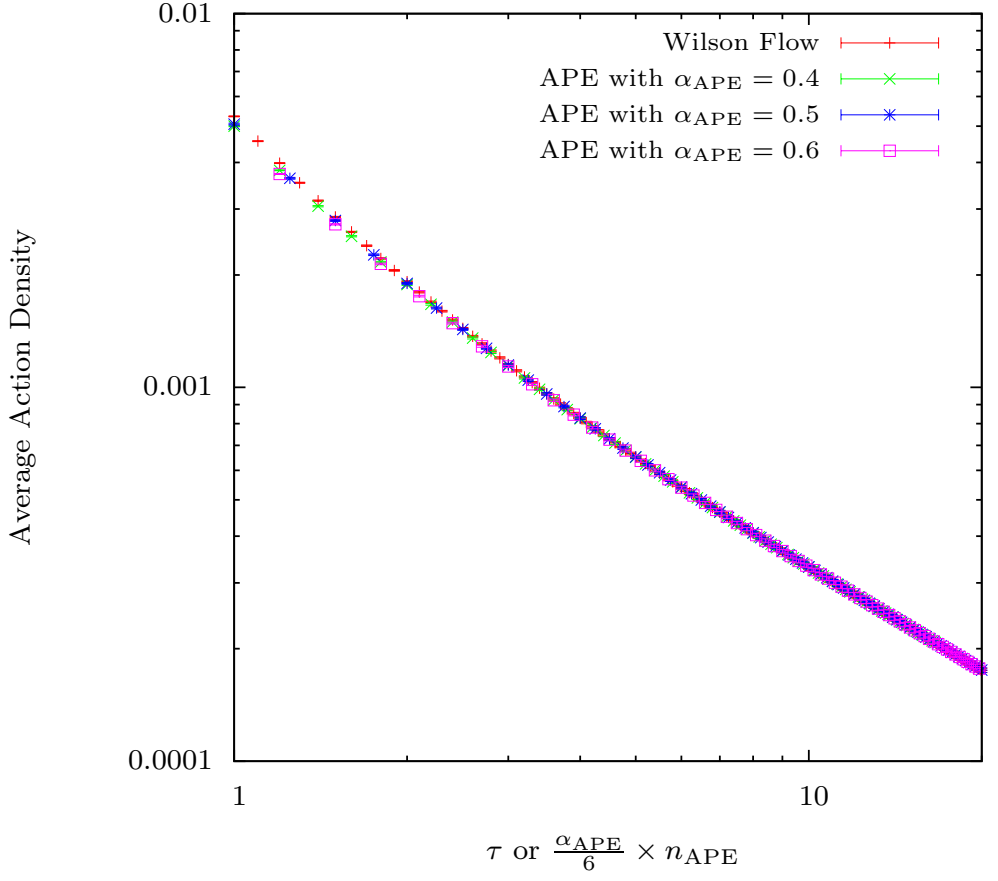
$$\tau \simeq \frac{\alpha_{\rm APE}}{6}n_{\rm APE} .$$

Wilson flow Vs. APE

Matching condition: $\tau \simeq \frac{\alpha_{\rm APE}}{6} n_{\rm APE}$.

Define function $\tau(\alpha_{\rm APE}, n_{\rm APE})$ such as τ and $n_{\rm APE}$ changes action by the same amount





Wilson flow Vs. stout

- According to the stout smearing operation:

$$U_\mu^{(n_{\rm st}+1)}(x) = \exp\left(iQ_\mu^{n_{\rm st}}(x)\right) U_\mu^{(n_{\rm st})}(x)\,.$$

with

$$Q_\mu(x) = \frac{i}{2}\left(\Xi_\mu^\dagger(x) - \Xi_\mu(x)\right) - \frac{i}{6}{\rm Tr}\left(\Xi_\mu^\dagger(x) - \Xi_\mu(x)\right)\,, \quad \text{with} \quad \Xi_\mu(x) = \rho_{\rm st} X_\mu(x) U_\mu^\dagger(x)$$

- Evolution of the Wilson flow by an infinitesimally small flow time ϵ is expressed as:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right]\,.$$

- Evolution of the stout smearing with parameter $\rho_{\rm st}$ is expressed as:

$$u_\mu^a(x,n_{\rm st}+1) \simeq u_\mu^a(x,n_{\rm st}) - \rho_{\rm st}\left[6u_\mu^a(x,n_{\rm st}) - w_\mu^a(x,n_{\rm st})\right]\,.$$

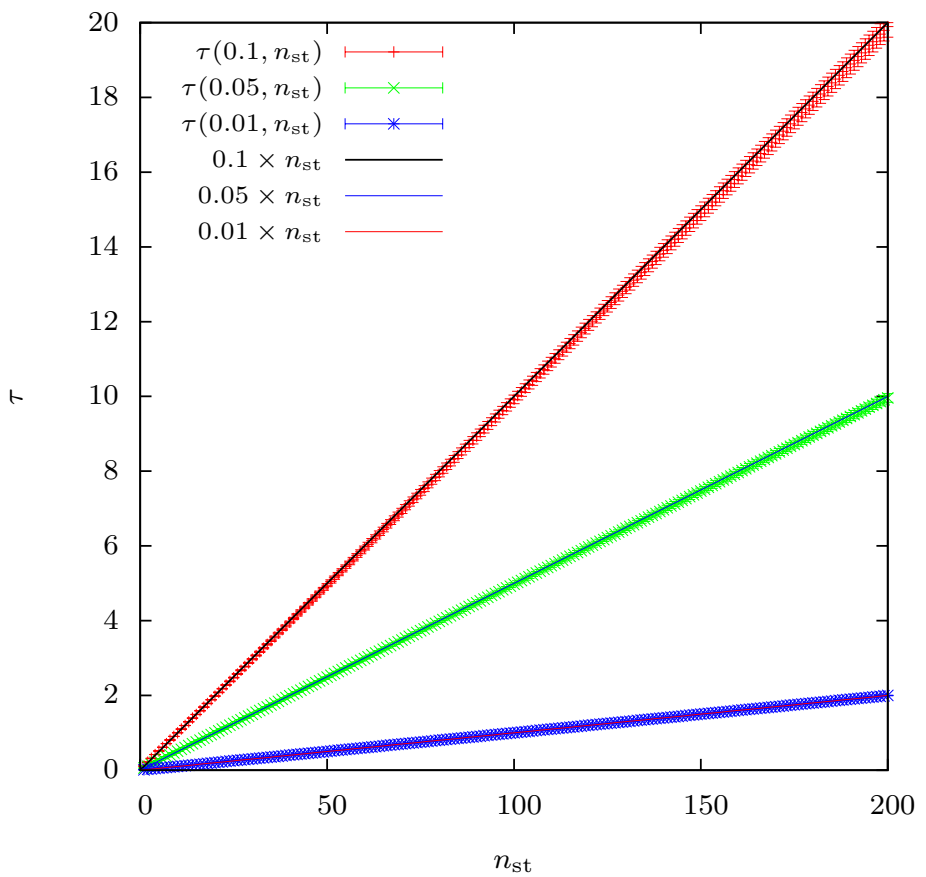
- Hence, stout smearing has the same effect as the Wilson flow if

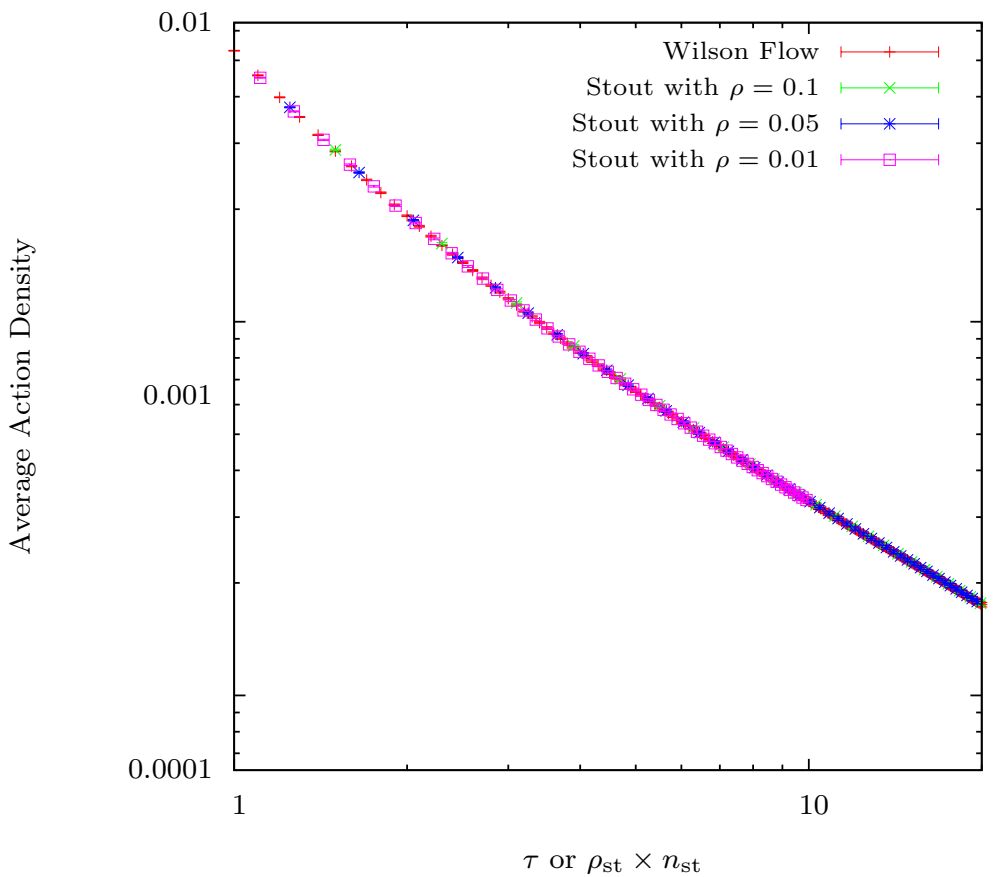
$$\tau \simeq \rho_{\rm st} n_{\rm st}\,.$$

Wilson flow Vs. stout

Matching condition: $\tau \simeq \rho_{\rm st} n_{\rm st}$.

Define function $\tau(\rho_{\rm st}, n_{\rm st})$ such as τ and $n_{\rm st}$ changes action by the same amount



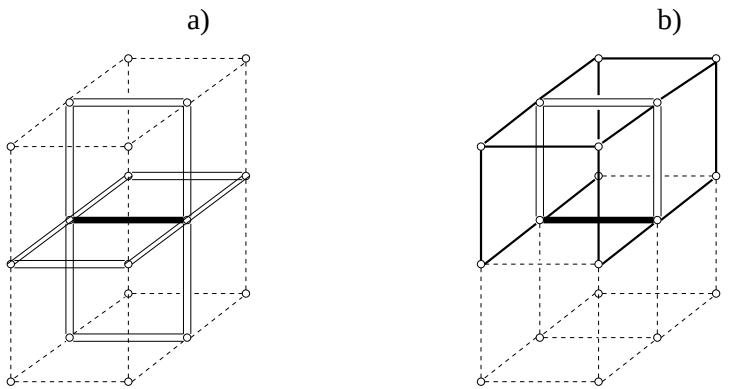


Wilson flow Vs. HYP

- We considered HYP smearing with parameters:

$$\alpha_{\mathrm{HYP1}} = 0.75, \quad \alpha_{\mathrm{HYP2}} = 0.6 \quad \alpha_{\mathrm{HYP3}} = 0.3$$

- HYP staples not the same as $X_\mu(x)$ (A. Hasenfratz and F. Knechtli, *Phys. Rev.* D64 (2001) 034504).



- Define function $\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}})$ and fit using the ansatz:

$$\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}}) = A\ n_{\mathrm{HYP}} + B\ n_{\mathrm{HYP}}^2 + C\ n_{\mathrm{HYP}}^3\ .$$

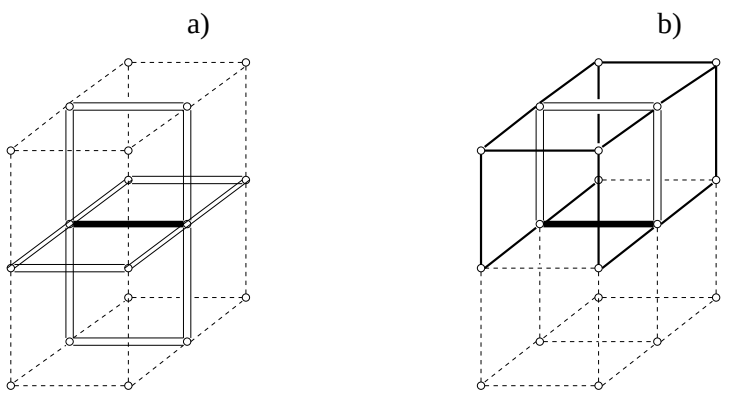
with $A = 0.25447(32)$, $B = -0.001312(90)$, $C = 1.217(91) \times 10^{-5}$

Wilson flow Vs. HYP

- We considered HYP smearing with parameters:

$$\alpha_{\mathrm{HYP1}} = 0.75, \quad \alpha_{\mathrm{HYP2}} = 0.6 \quad \alpha_{\mathrm{HYP3}} = 0.3$$

- HYP staples not the same as $X_\mu(x)$ (A. Hasenfratz and F. Knechtli, *Phys. Rev.* D64 (2001) 034504).



- Define function $\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}})$ and fit using the ansatz:

$$\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}}) = A\ n_{\mathrm{HYP}} + B\ n_{\mathrm{HYP}}^2 + C\ n_{\mathrm{HYP}}^3\,.$$

with $A = 0.25447(32)$, $B = -0.001312(90)$, $C = 1.217(91) \times 10^{-5}$

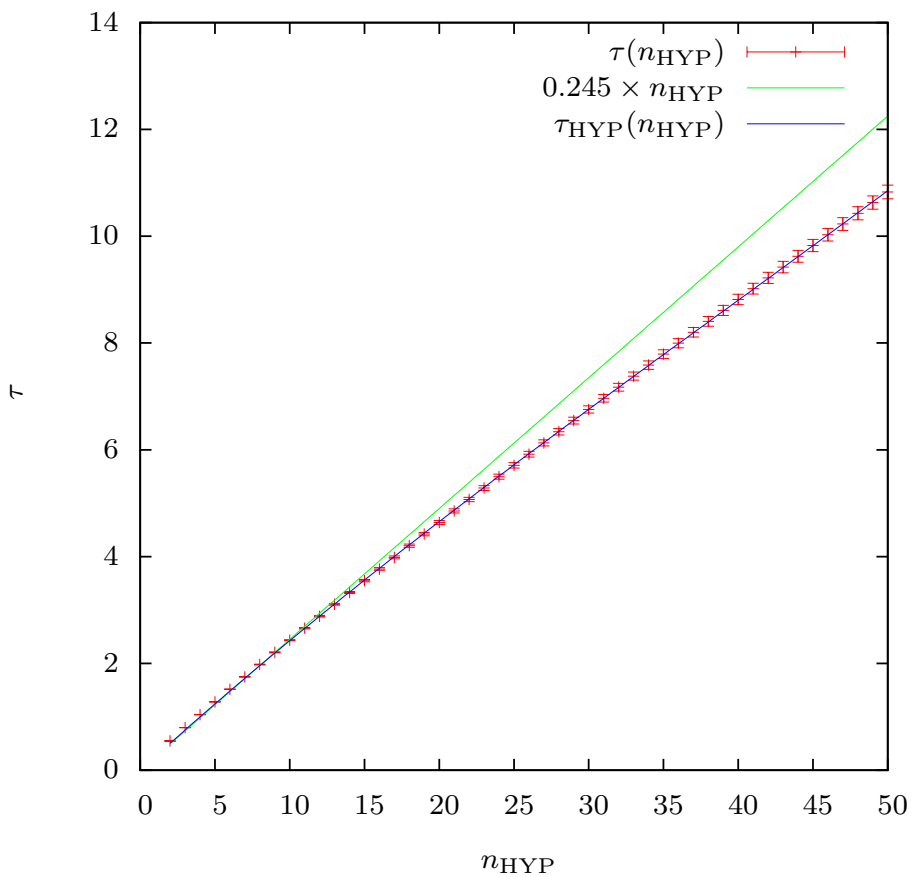
- Hence, HYP smearing has the same effect as the Wilson flow if

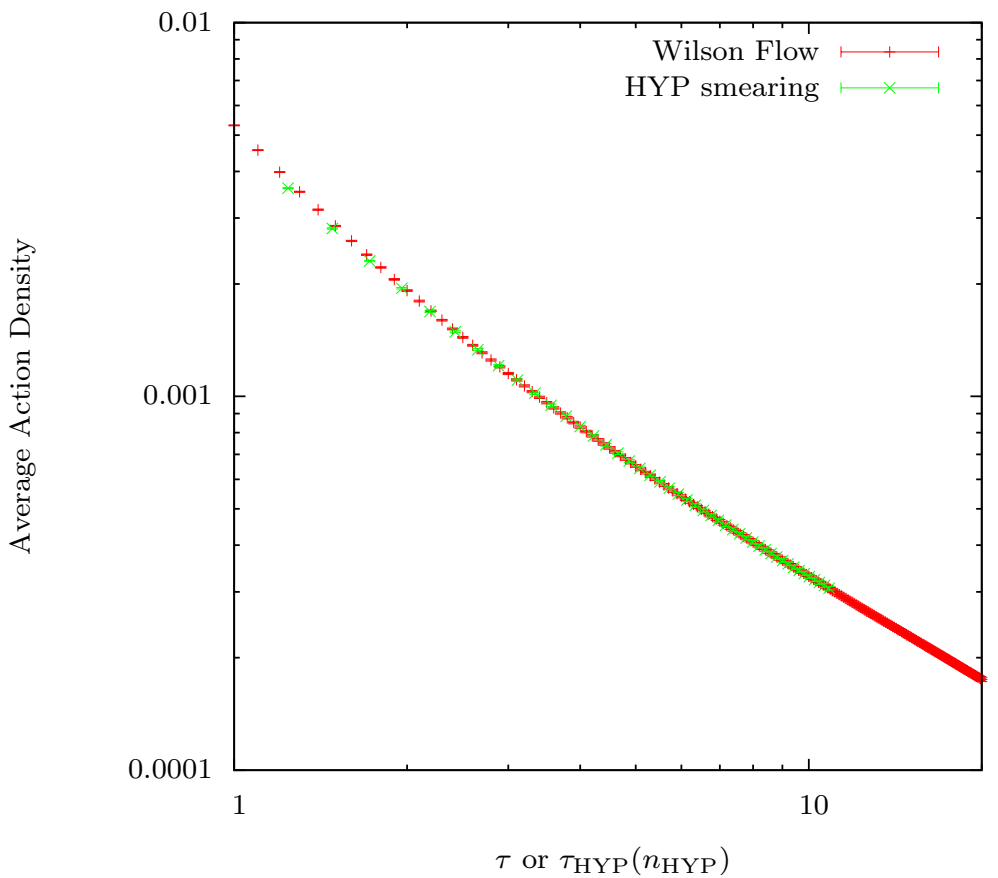
$$\tau \simeq \tau_{\mathrm{HYP}}(n_{\mathrm{HYP}})\,.$$

Wilson flow Vs. HYP

Numerical matching condition: $\tau_{\rm HYP}(n_{\rm HYP}) = A\ n_{\rm HYP} + B\ n_{\rm HYP}^2 + C\ n_{\rm HYP}^3\,.$

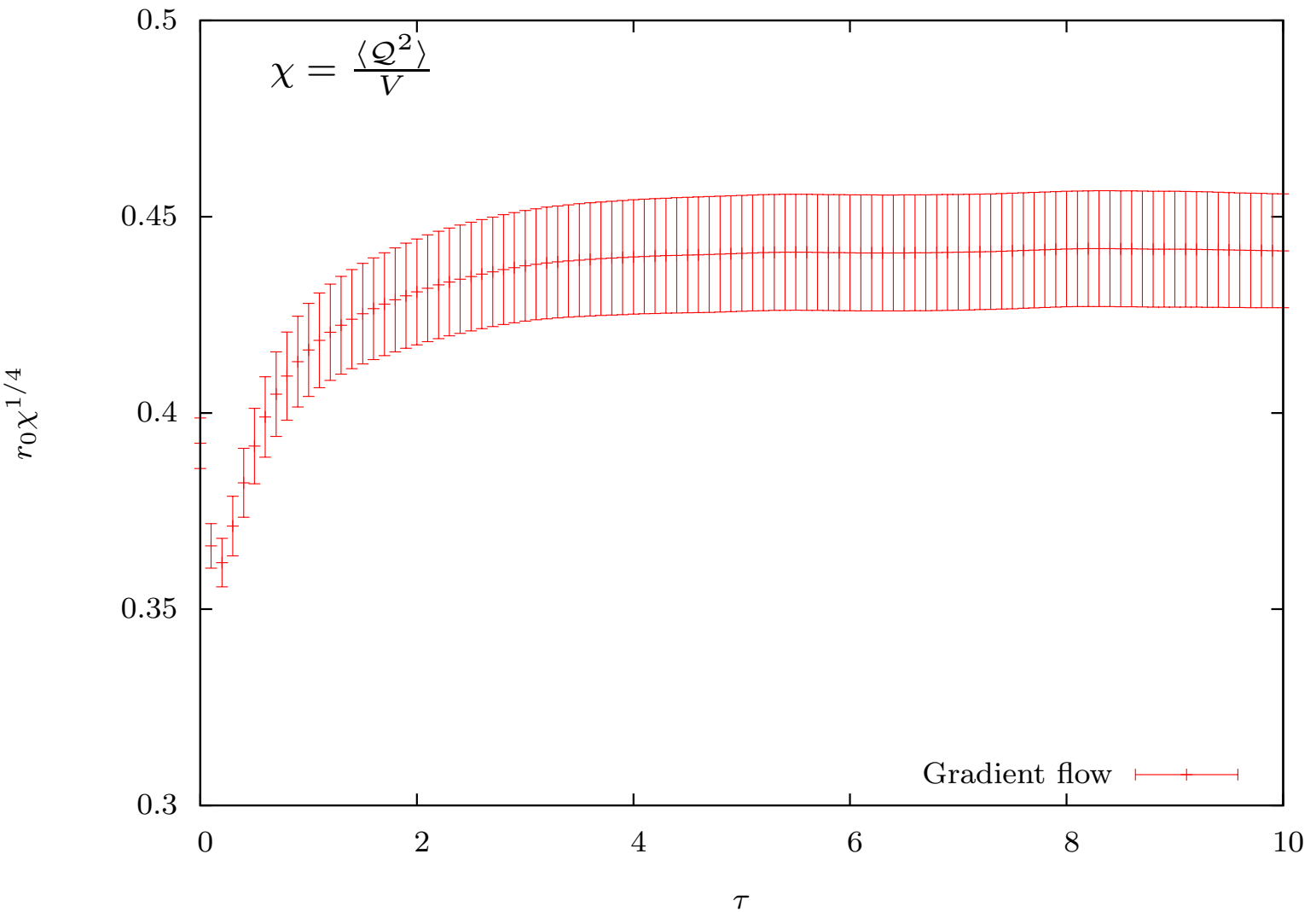
Define function $\tau(n_{\rm HYP})$ such as τ and $n_{\rm HYP}$ changes action by the same amount





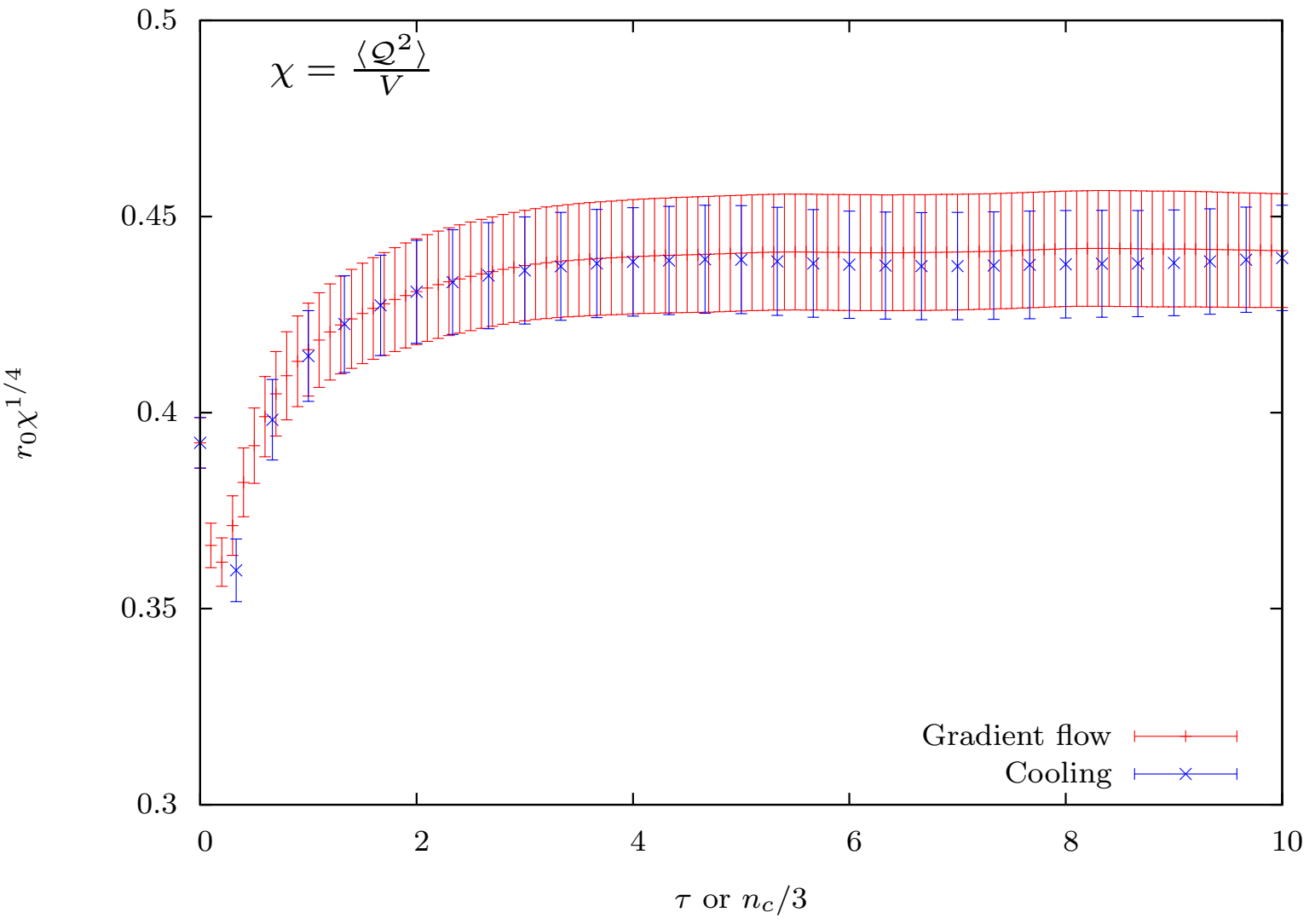
Topological Susceptibility - The Wilson flow

The Wilson flow time $t_0 \simeq 2.5a^2$



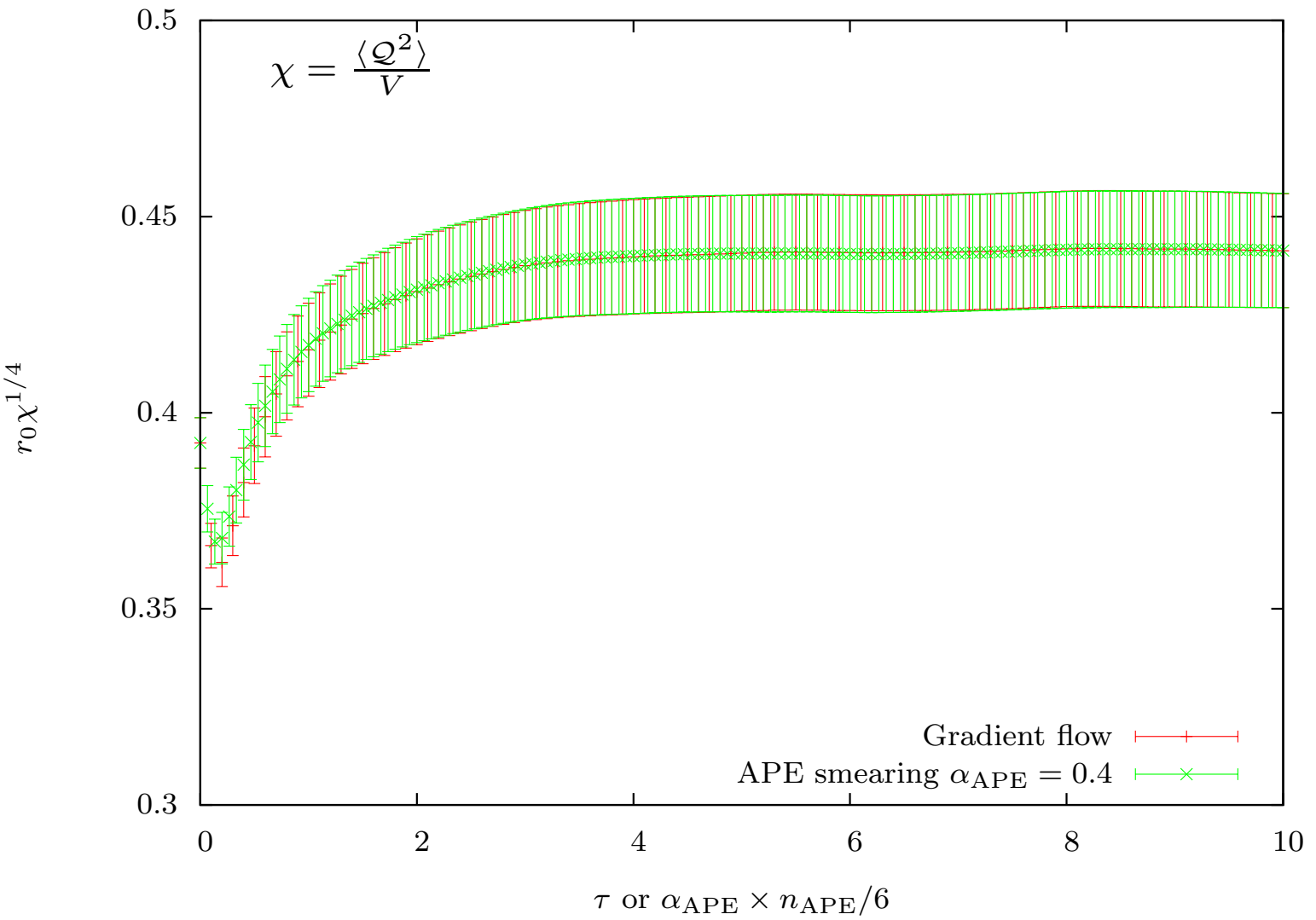
Topological Susceptibility - Cooling

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_c = 7.5$ cooling steps



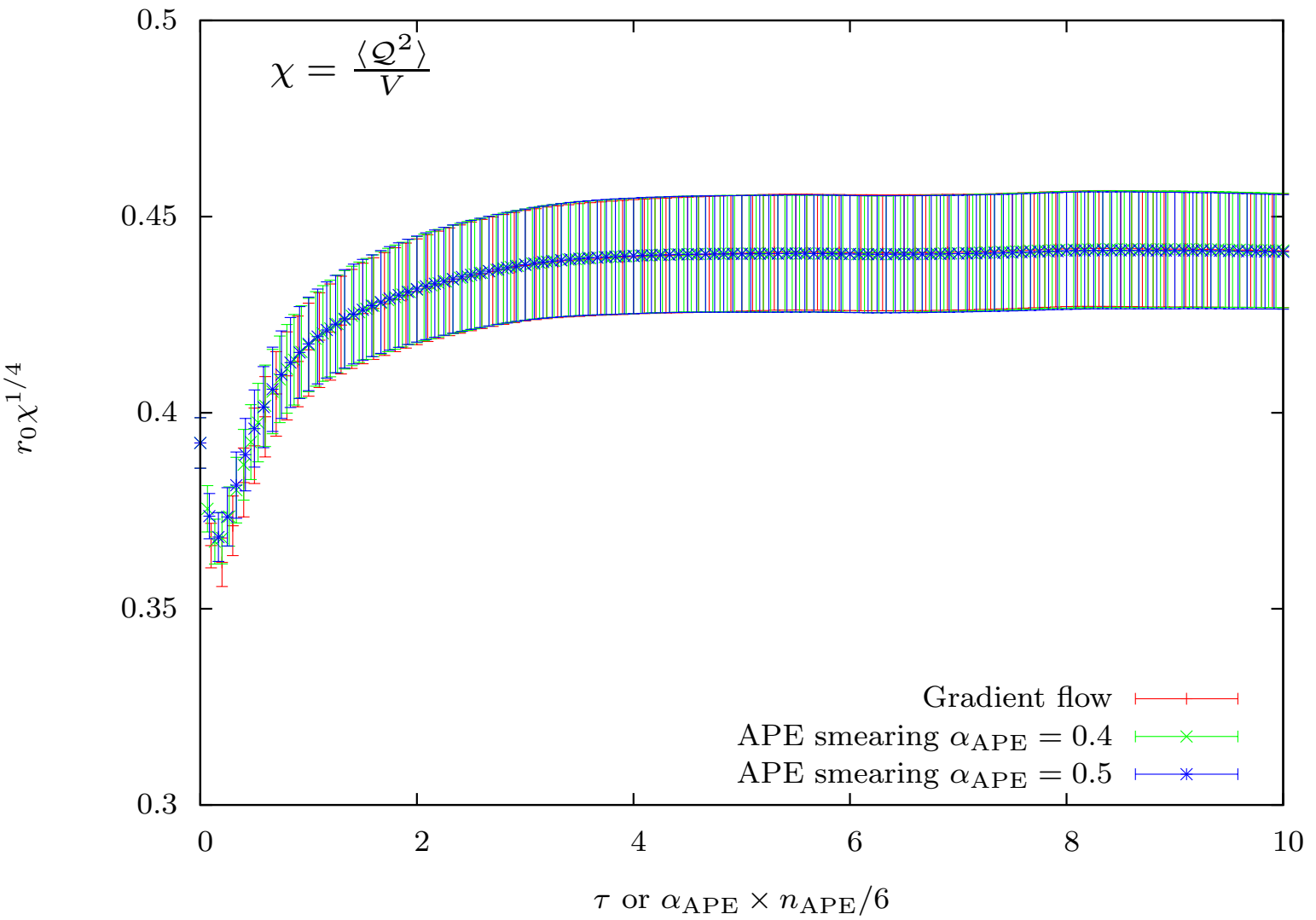
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 37.5$ APE smearing steps for $\alpha_{\rm APE} = 0.4$



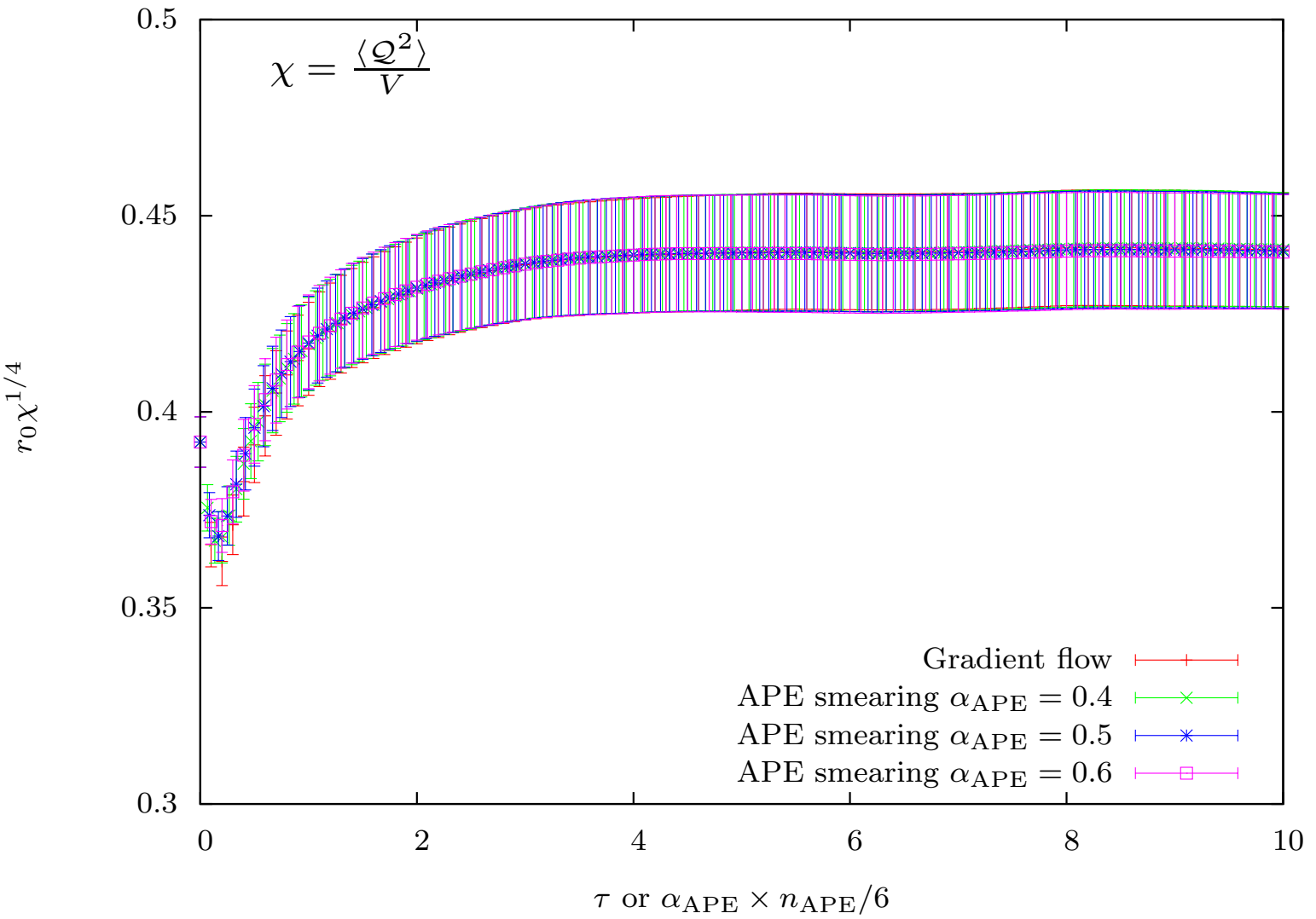
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 30$ APE smearing steps for $\alpha_{\rm APE} = 0.5$



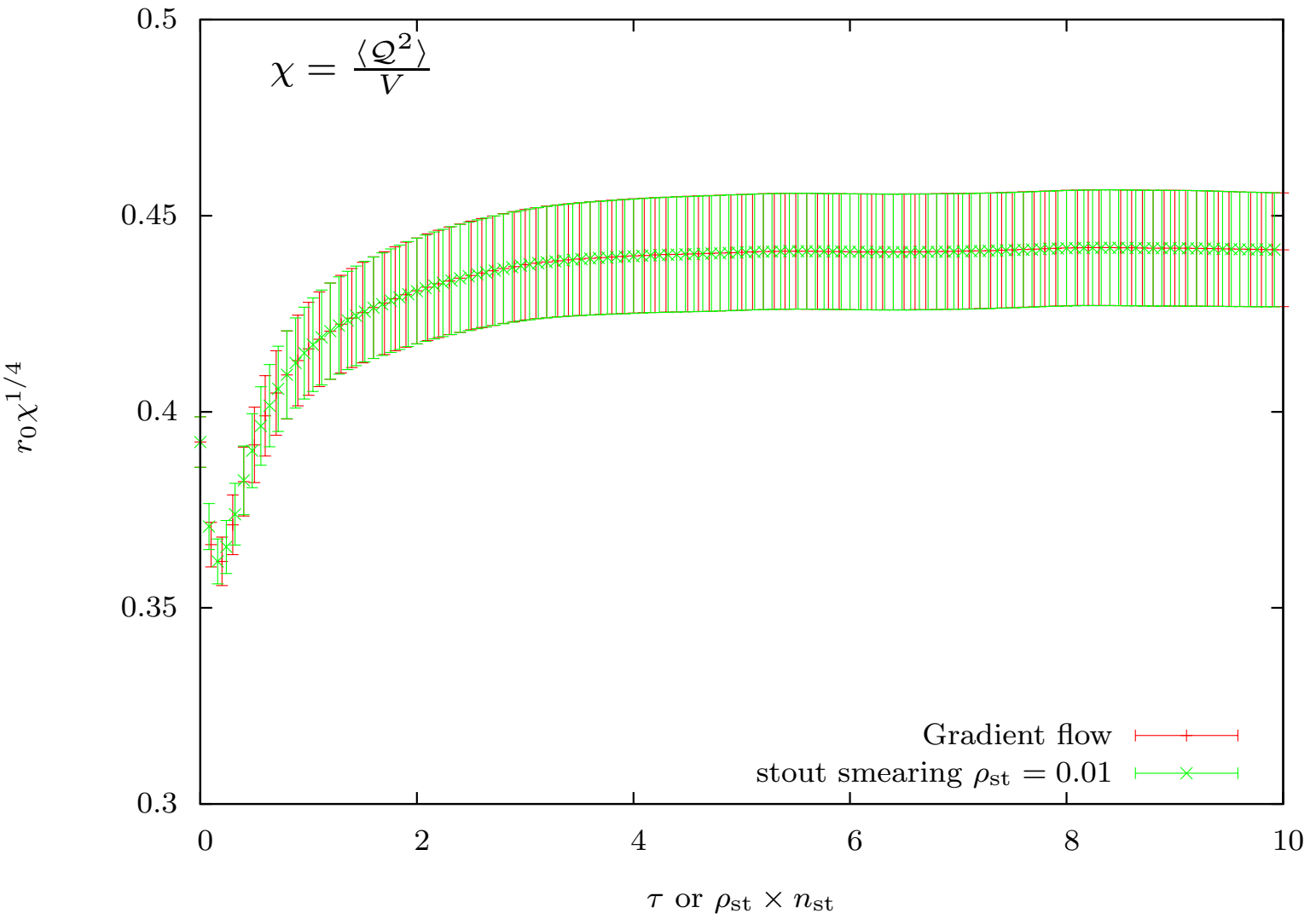
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 25$ APE smearing steps for $\alpha_{\rm APE} = 0.6$



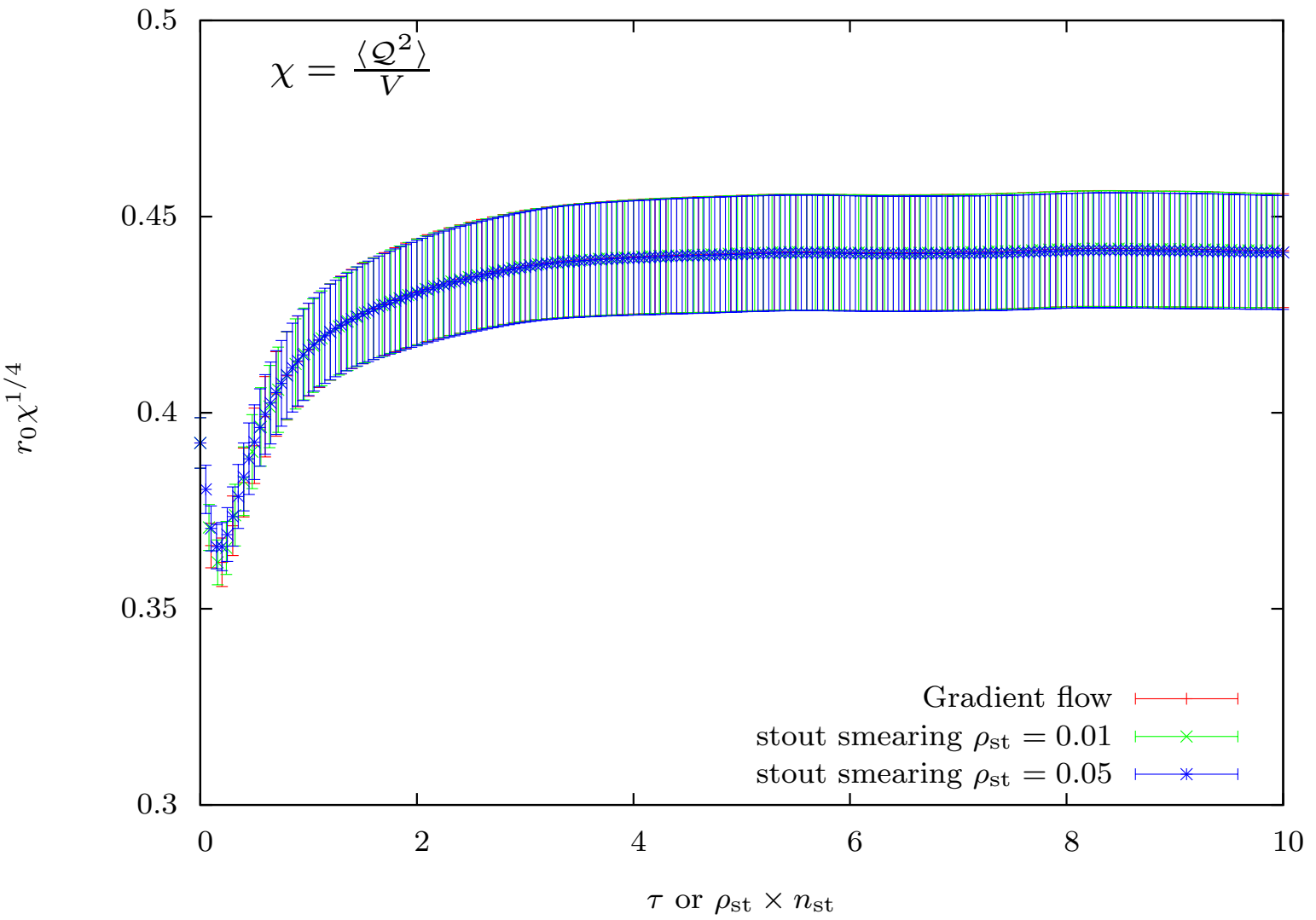
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 250$ stout smearing steps for $\rho_{\rm st} = 0.01$



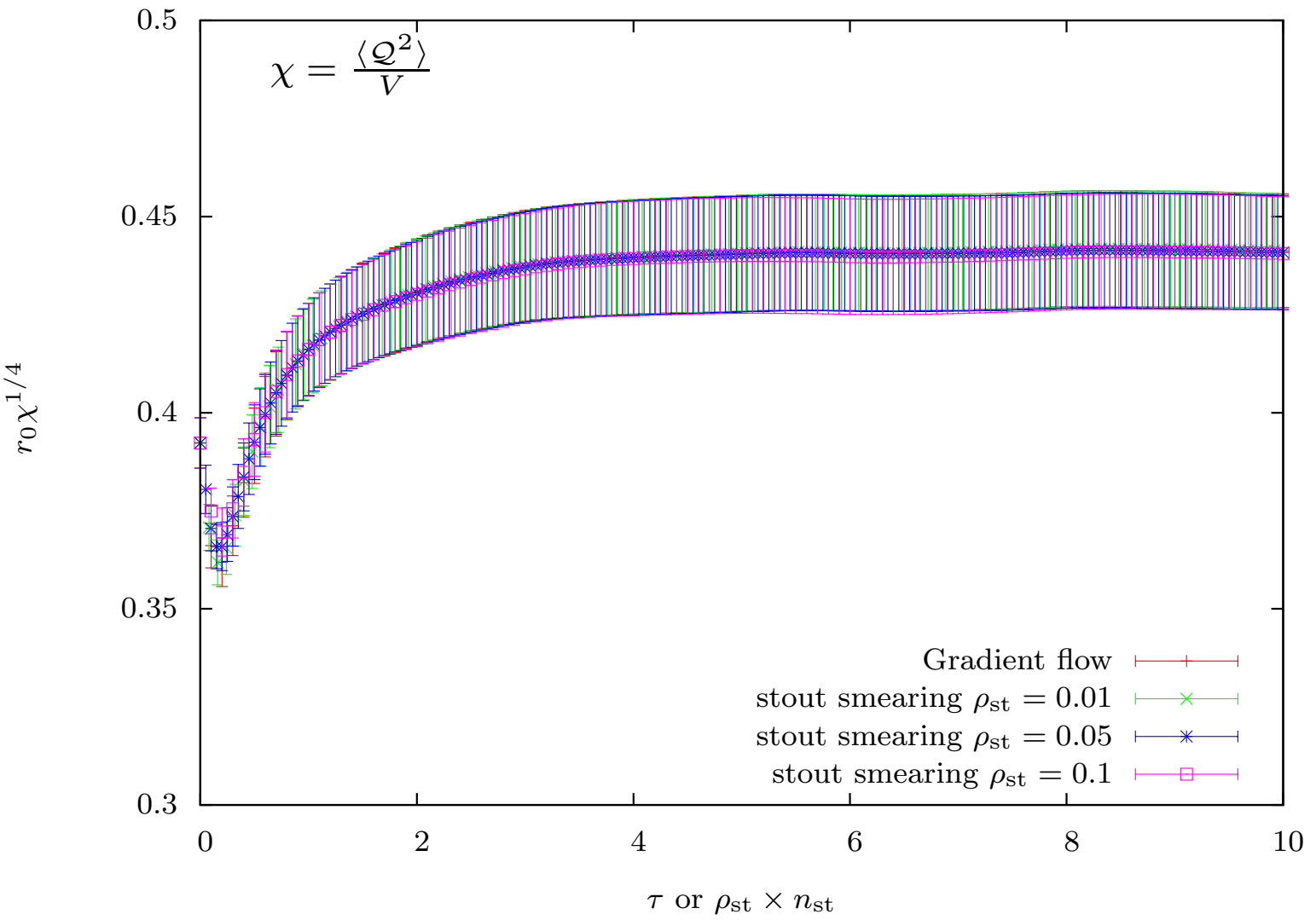
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 50$ stout smearing steps for $\rho_{\rm st} = 0.05$



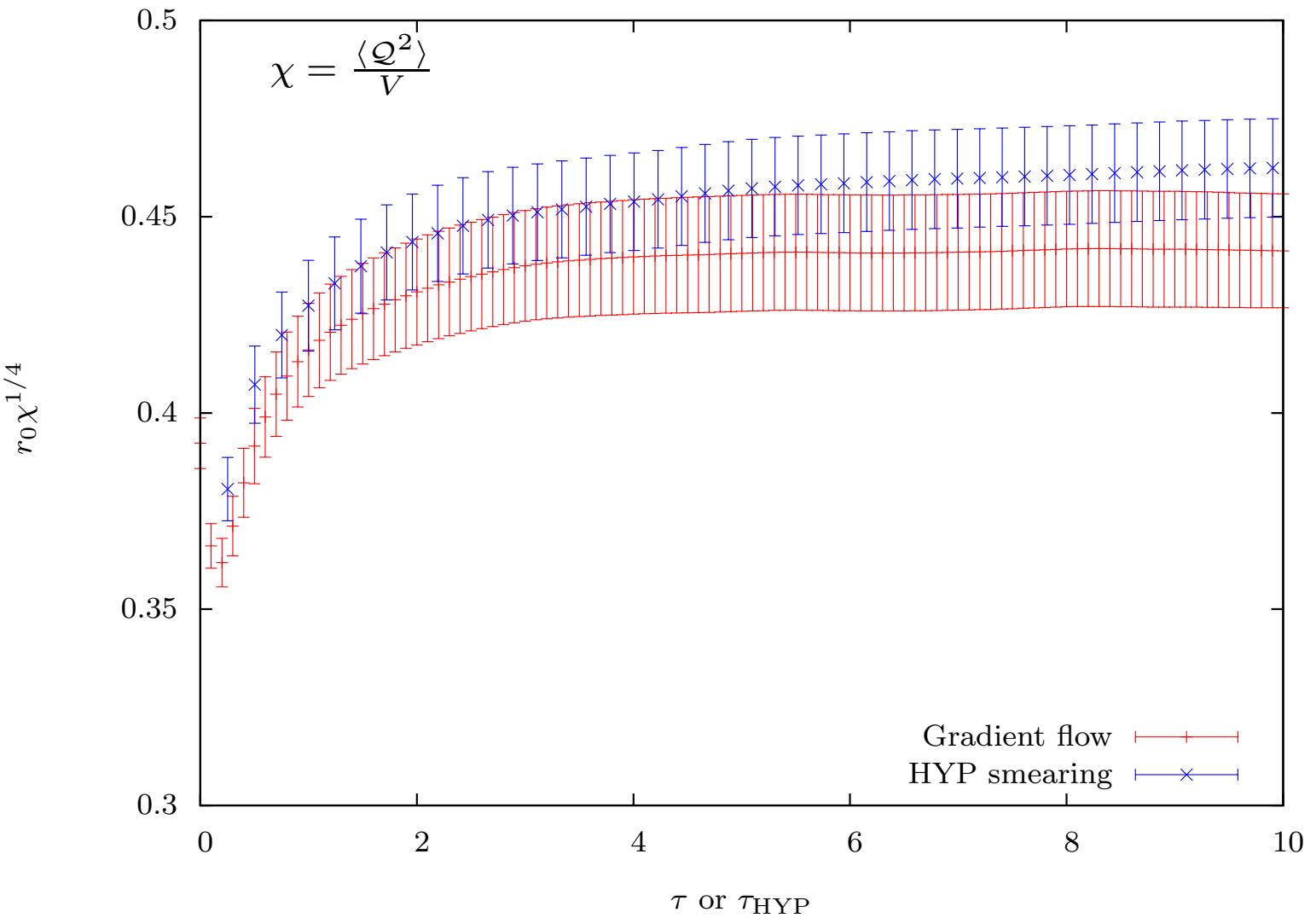
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 25$ stout smearing steps for $\rho_{\rm st} = 0.1$



Topological Susceptibility - HYP

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 10$ HYP smearing steps



Correlation between different smoothers

Let us have a look at the correlation coefficient

$$c_{\mathcal{Q}_1,\mathcal{Q}_2} = \frac{\left\langle \left(\mathcal{Q}_1 - \overline{\mathcal{Q}}_1\right)\left(\mathcal{Q}_2 - \overline{\mathcal{Q}}_2\right)\right\rangle}{\sqrt{\left\langle \left(\mathcal{Q}_1 - \overline{\mathcal{Q}}_1\right)^2\right\rangle \left\langle \left(\mathcal{Q}_2 - \overline{\mathcal{Q}}_2\right)^2\right\rangle}} \,.$$

	WF, t_0	cool, t_0	APE, t_0	stout, t_0	HYP, t_0
WF, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
cool, t_0	0.97(0)	1.00(0)	0.97(0)	0.97(0)	0.94(0)
APE, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
stout, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
HYP, t_0	0.97(0)	0.94(0)	0.97(0)	0.97(0)	1.00(0)

Topological charges are highly correlated!

In the continuum all numbers become 1.00

Fixing the smoothing scale

- One can fix a physical flow time:

$$\lambda_S \simeq \sqrt{8t}\,.$$

- Similarly for cooling $\sim 120 \times$ faster than Gradient Flow:

$$\lambda_S \simeq a\sqrt{\frac{8n_c}{3}}\,.$$

- For the APE smearing $\sim 20 \times$ faster than Gradient Flow:

$$\lambda_S \simeq a\sqrt{\frac{4\alpha_{\rm APE} n_{\rm APE}}{3}}\,.$$

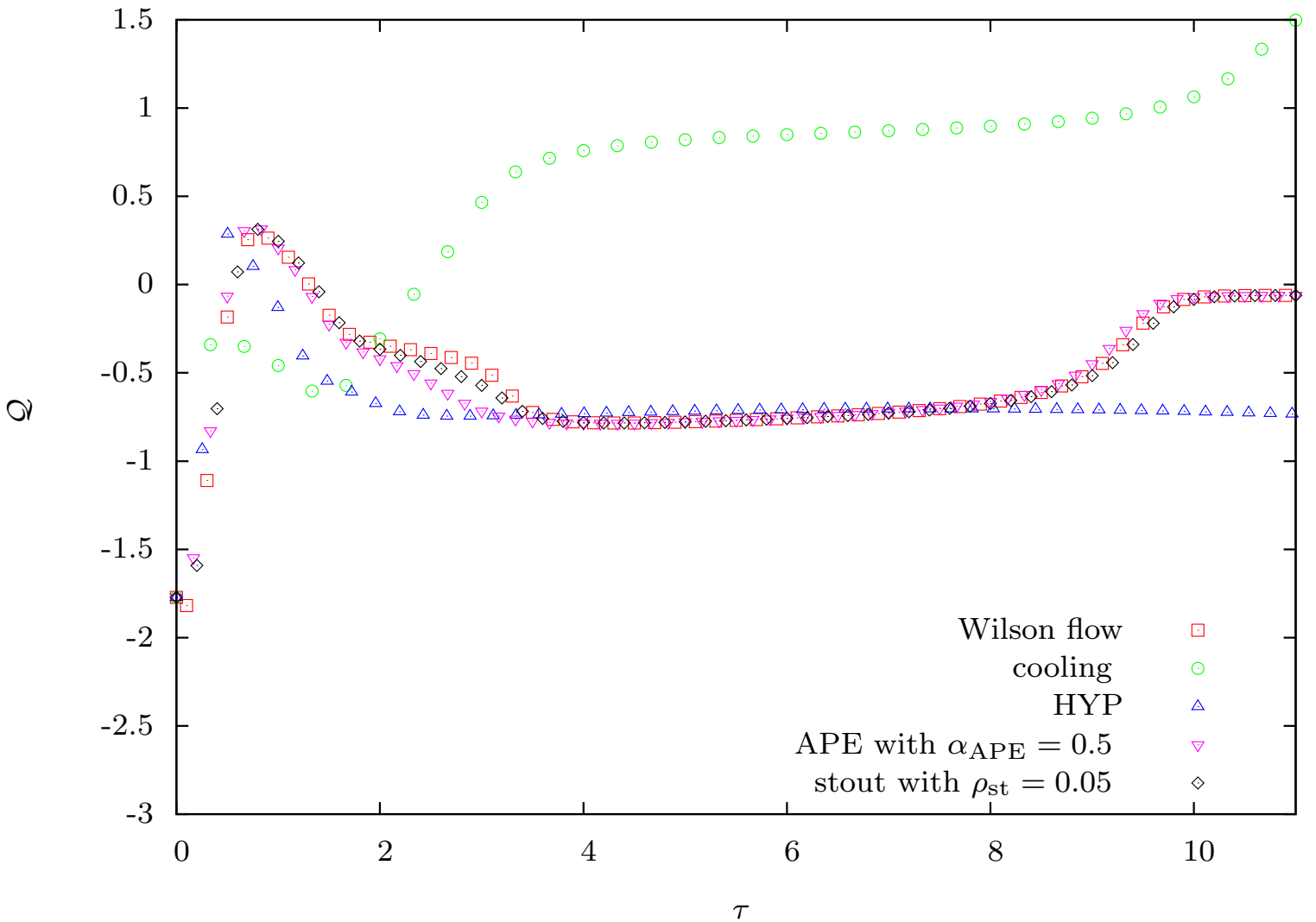
- For the stout smearing $\sim 30 \times$ faster than Gradient Flow

$$\lambda_S \simeq a\sqrt{8\rho_{\rm st} n_{\rm st}}\,.$$

- Similar procedure can be applied to t_0.

Level of agreement on the topological charge

- Why topological susceptibility has such a high level of agreement?



Fermionic Definitions of the Topological Charge

- (f) Index definition with different steps of HYP smearing.

  M. F. Atiyah and I. M. Singer, *Annals Math.* 93 (1971) 139149

- (f) Spectral-flow with different steps of HYP smearing.

  S. Itoh, Y. Iwasaki and T. Yoshie, *Phys. Rev.* D 36 (1987) 527

- (f) Spectral projectors with different cutoffs M^2.

  L. Giusti and M. Lüscher, *JHEP* 0903 (2009) 013 and M. Lüscher and F. Palombi, *JHEP* 1009 (2010) 110

Index definition

- Chiral Symmetry can be realized on the Lattice!
- Modified definition of chiral symmetry on the lattice
  - Dirac operator satisfies the Ginsparg-Wilson relation is chirally symmetric

$$\gamma_5 D + D\gamma_5 = a\gamma_5 D\gamma_5$$

  - Overlap Dirac operator, introduced by Neuberger

    H. Neuberger, Phys. Lett. B427 (1998) 353355, [hep-lat/9801031].
  - Atiyah-Singer index theorem: relates the number of Zero modes to topological charge

$$\mathcal{Q} = n_- - n_+$$

  - It gives integer values of $\mathcal{Q}$.
  - Massless Overlap Operator

$$D = \frac{1}{a}\left(1 - \frac{A}{\sqrt{A^\dagger A}}\right) , \qquad A = 1 + s - aD_W ,$$

  - s can be tuned to optimize the locality of the Overlap operator
  - In the continuum limit the dependence on s vanishes
- **Several orders of magnitude slower than Gradient Flow**

Spectral Projectors

- L. Giusti and M. Lüscher, [arXiv:0812.3638] & M. Lüscher and F. Palombi, [arXiv:1008.0732].
- Introduce the projector $\mathbb{P}_M$ to the subspace of eigenmodes of the Hermitian Dirac operator $D^\dagger D$ with eigenvalues below M^2.
- $\mathbb{P}_M$ can be calculated stochastically vs. explicit computation of eigenmodes

pros. For explicit computation of eigenmodes, comp. cost drops from $\mathcal{O}(V^2)$ to $\mathcal{O}(V)$

cons. Introducing stochastic noise

- To avoid the stochastic noise we opted for an explicit computation of eigenmode.

$\longrightarrow$ The bare topological charge is given by

$$\mathcal{Q}_0 = \sum_i^{\lambda_i < M_0^2} R_i, \quad R_i = u_i^\dagger \gamma_5 u_i$$

$\longrightarrow$ The topological susceptibility

$$\chi_0 = \frac{\langle \mathcal{Q}_0^2 \rangle}{V}.$$

$\longrightarrow$ With renormalized quantities

$$\mathcal{Q} = \left(\frac{Z_S}{Z_P}\right) \mathcal{Q}_0, \quad \chi = \left(\frac{Z_S}{Z_P}\right)^2 \chi_0, \quad M = Z_P^{-1} M_0.$$

Spectral Projectors

- For chirally symmetric fermions $\mathcal{Q} = \mathrm{Tr}\,\{\gamma_5 \mathbb{P}_M\}$.

$\longrightarrow$ This definition is then equivalent to the index definition

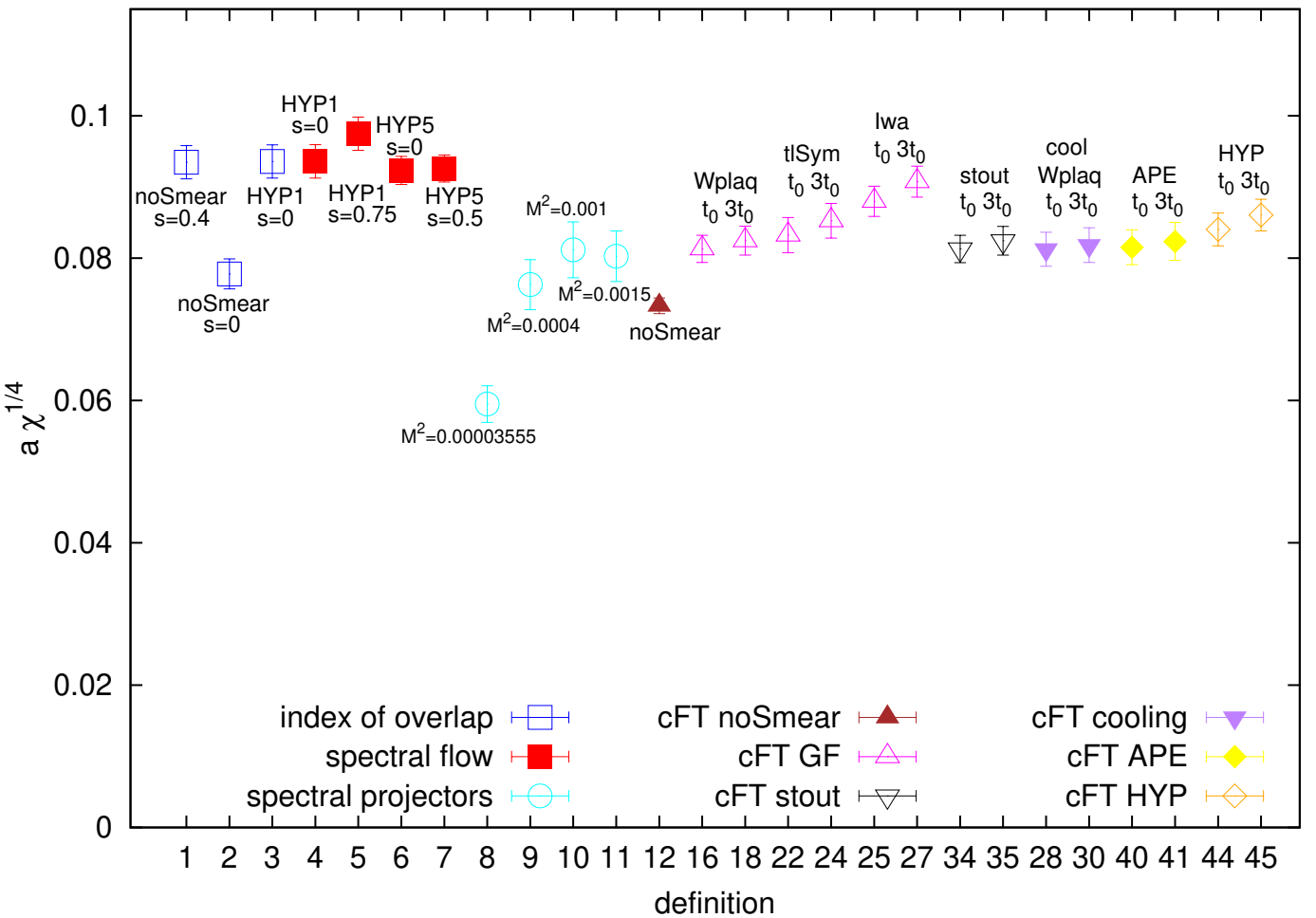
- Wilson twisted mass fermions lead to a shift of $\mathcal{O}(a^2)$
- Results depend on the renormalized spectral threashold M^2
- The 400 lowest eigenmodes allow M up to 160 MeV for all the ensembles
- Investigate the dependence in M

$\rightarrow$ M should not be too small because of large cutoff effects

$\rightarrow$ M should not be too large because of enhanced noise

Comparison of topological Susceptibility

- Comparison of results for the topological susceptibility.

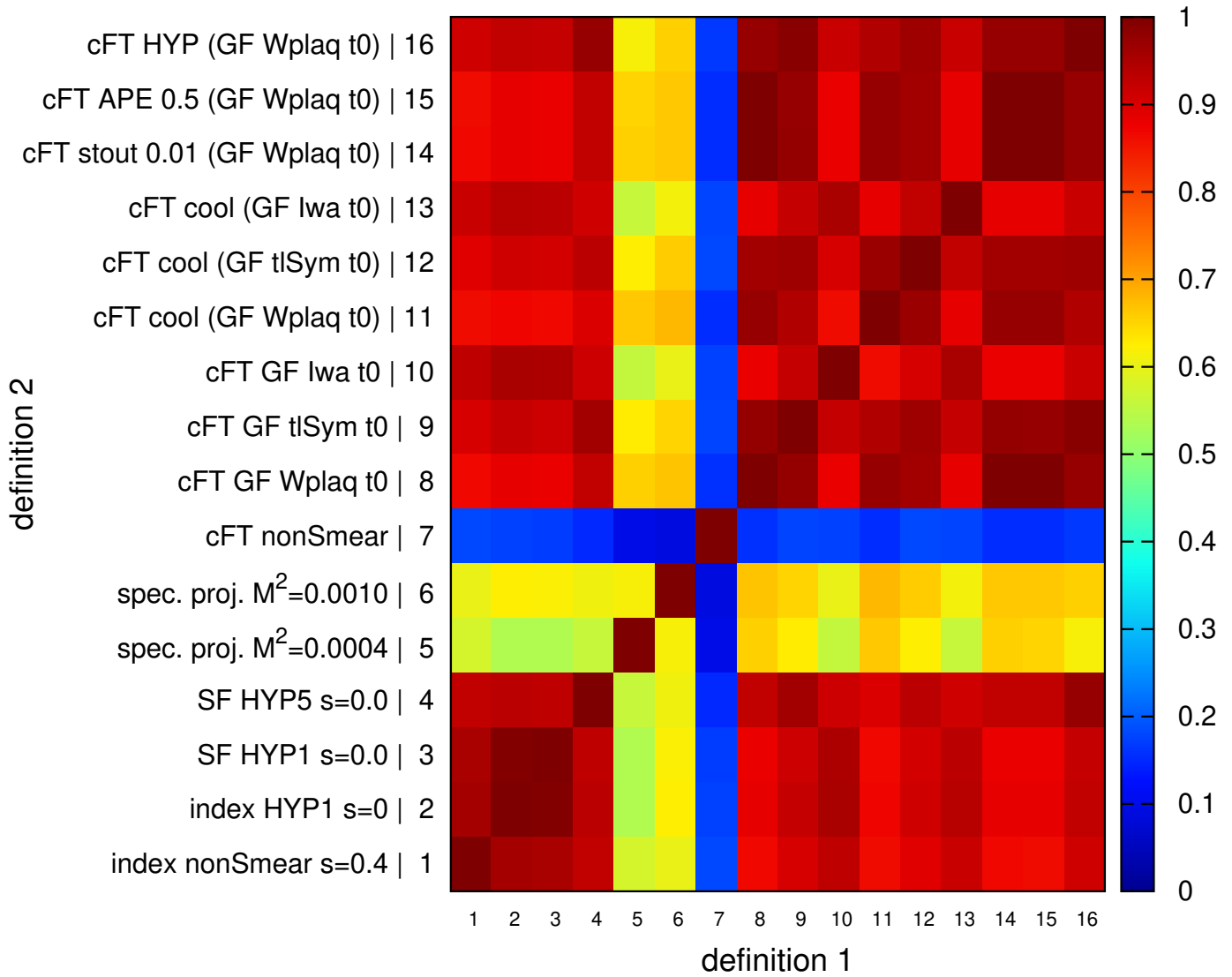


- Using $N_f = 2$ twisted mass configuration with:

$\beta = 3.90$, $a \simeq 0.085$fm, $r_0/a = 5.35(4)$, $m_\pi \simeq 340$ MeV, $m_\pi L = 2.5$, $L/a = 16$

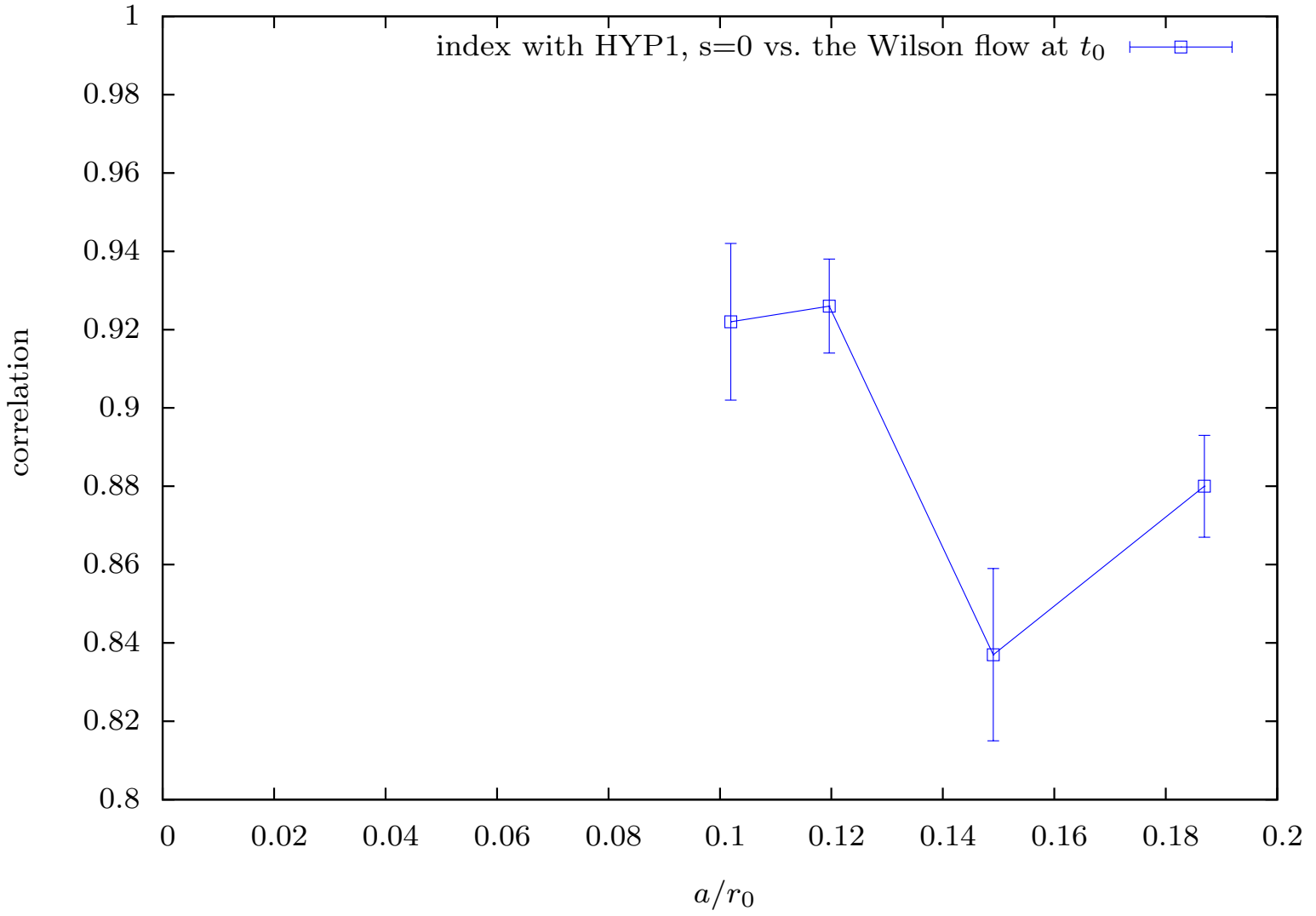
Correlation Coefficient

- Comparison of the correlation coefficient between fermionic and gluonic definitions.



Fermionic Vs. Gluonic - Continuum limit

- Correlation for a fermionic and gluonic definitions as we approach the continuum limit.



Comparison in Neutron Electric Dipole Moment

- The Electric Dipole Moment is given by

$$|\vec{d}_N| = \lim_{q\to 0} \theta \frac{F_3(q^2)}{2m_N} \,.$$

- From Nucleon-Nucleon matrix element of the electromagnetic current $J_\mu^{\rm em}$ we obtain

$\rightarrow$ CP-odd electromagnetic Form-Factor $F_3(Q^2)$:

$$\langle N(\vec{p}_f, s_f)|J_\mu^{\rm em}|N(\vec{p}_i, s_i)\rangle \quad = \quad \bar{u}(\vec{p}_f, s_f)\left[\cdots + \theta \frac{F_3(Q^2)}{2m_N} Q_\nu \sigma_{\mu\nu}\gamma_5 + \cdots\right] u(\vec{p}_i, s_i)\,.$$

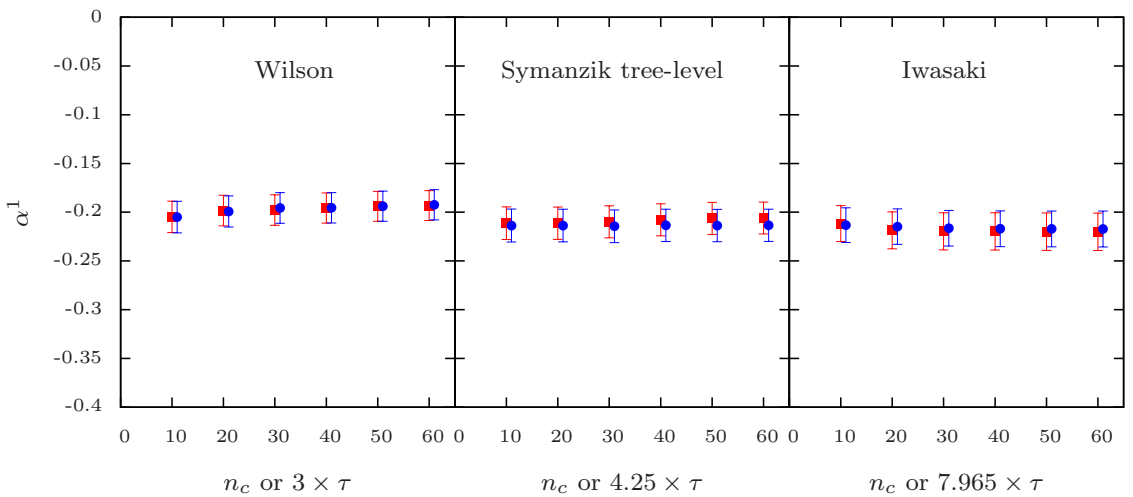
- To extract the CP-odd Form Factor we need to calculate

$$\langle J_N(\vec{p}_f, t_f) J_\mu^{\rm em}(\vec{q}, t) \overline{J}_N(\vec{p}_i, t_i)\mathcal{Q}\rangle. = f(F_3, \alpha^1)$$

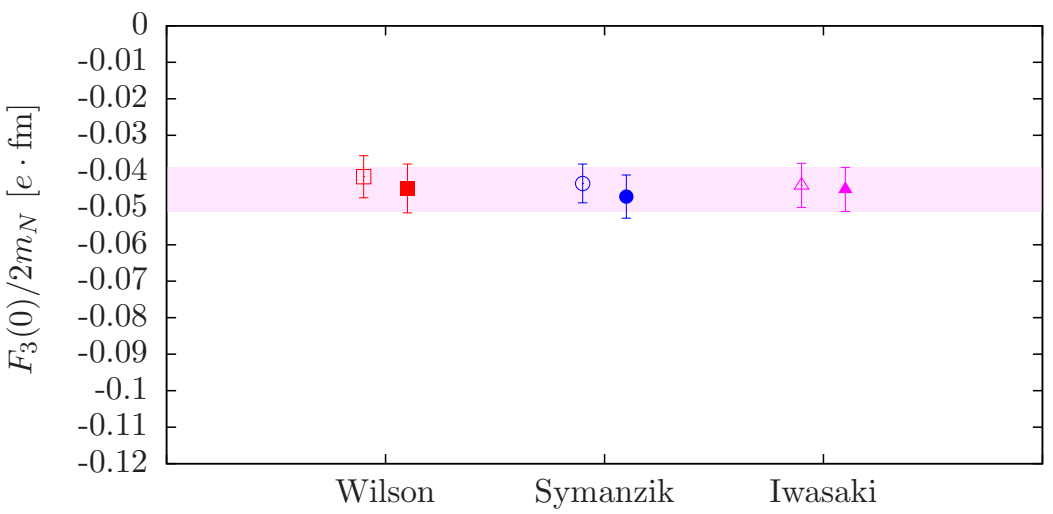
- The phase α^1 appears due to the CP-breaking in the θ-vacuum.

Comparison in Neutron Electric Dipole Moment

The phase α^1 (blue: cooling, red: Gradient Flow):

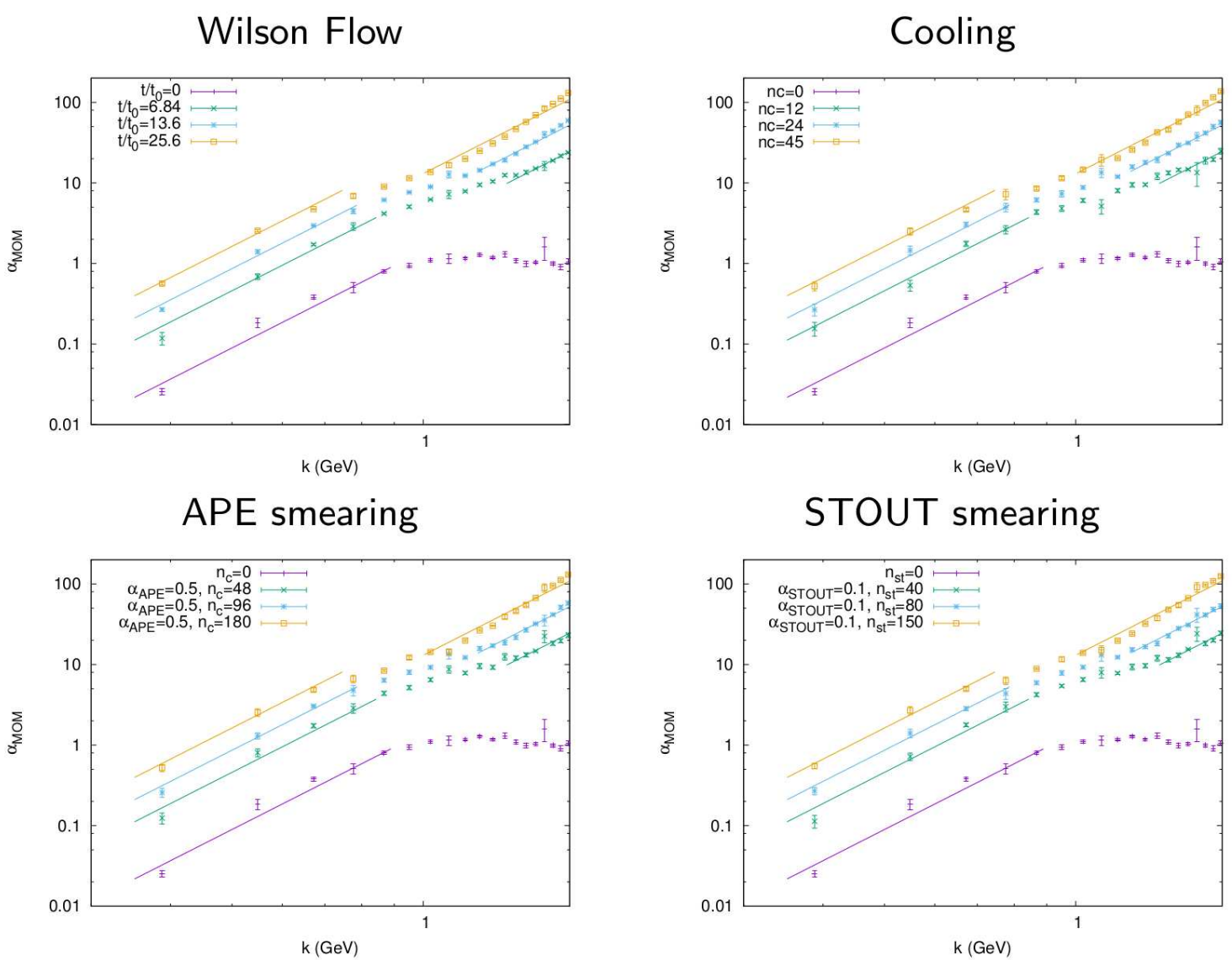


The CP-odd form factor:



Comparison in $\alpha_{\mathrm{mom}}(k)$ after smoothing

From Feliciano De Soto's presentation:



What we learned?

- Topological susceptibilities are in the same ballpark: $a\chi^{1/4} \in [0.08, 0.09]$.
- Correlation coefficient appears to increase towards to 1 as $a \to 0$.
- Different definitions influenced by different lattice artifacts.
- Most correlation coefficients are above 80 %.
- Cooling, APE smearing, stout smearing are numerically equivalent if matched:

$$\tau \simeq \frac{n_c}{3}, \qquad \tau \simeq \alpha_{\mathrm{APE}} \frac{n_{\mathrm{APE}}}{6} \qquad \tau \simeq \rho_{\mathrm{st}} n_{\mathrm{st}} \, .$$

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$\sim 120 \times$ faster, $\sim 20 \times$ faster $\sim 30 \times$ faster .

Topological Susceptibility - What we want to learn

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  $\longrightarrow$ Different definitions influenced by different lattice artifacts.

- Comparison between
  - Fermionic Definition
  - Gradient flow

- Do both approaches lead to compatible results?

  $\longrightarrow$ Pion mass dependence

  $\longrightarrow$ Universality of Topological Susceptibility

  $\longrightarrow$ Universality in Pure $SU(3)$ M. Lüscher and F. Palombi, *JHEP* 1009 (2010) 110

  $\longrightarrow$ This has **never been demonstrated before for Full QCD.**

  $\longrightarrow$ Work with $N_f = 2 + 1 + 1$ twisted mass fermions with

β	a [fm]	Z_P	Z_P/Z_S	r_0/a
1.90	0.0885(36)	0.529(9)	0.699(13)	5.231(38)
1.95	0.0815(30)	0.504(5)	0.697(7)	5.710(41)
2.10	0.0619(18)	0.514(3)	0.740(5)	7.538(58)

  $\longrightarrow$ This leads to the extraction of the Chiral Condensate

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Topological Susceptibility in Chiral Perturbation Theory

- We consider the partition function of QCD with non-zero θ (Mao and Chiu, 0903.2146):

$$Z(\theta) = \sum_{\mathcal{Q}} \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\psi^\dagger e^{iS\{\mathrm{QCD},\theta=0\} - i\theta\mathcal{Q}[U]}.$$

- Topological Susceptibility is the second derivative of the energy density at $\theta = 0$

$$\chi = \left.\frac{\partial^2 \varepsilon(\theta)}{\partial\theta^2}\right|_{\theta=0} = \frac{\langle \mathcal{Q}^2 \rangle}{V}, \qquad \varepsilon(\theta) = -\frac{\log(Z)}{V} \ .$$

- The lowest order, two flavor χPT Lagrangian can be written as:

$$\mathcal{L}^{(2)}_{\chi PT} = \frac{f_\pi^2}{4} \mathrm{Tr}\left[\partial_\mu U(x) \partial^\mu U(x)^\dagger\right] + \Sigma\, \mathrm{Re}\left(\mathrm{Tr}\left[\mathcal{M} U(x)^\dagger\ \right]\right),$$

- The partition function can be written in terms of $\mathcal{L}^{(2)}_{\chi PT}$:

$$Z(\theta) = \int \mathcal{D}U e^{i\int d^4x \mathcal{L}^{(2)}(U(x),\theta)}.$$

- Topological susceptibility yields to:

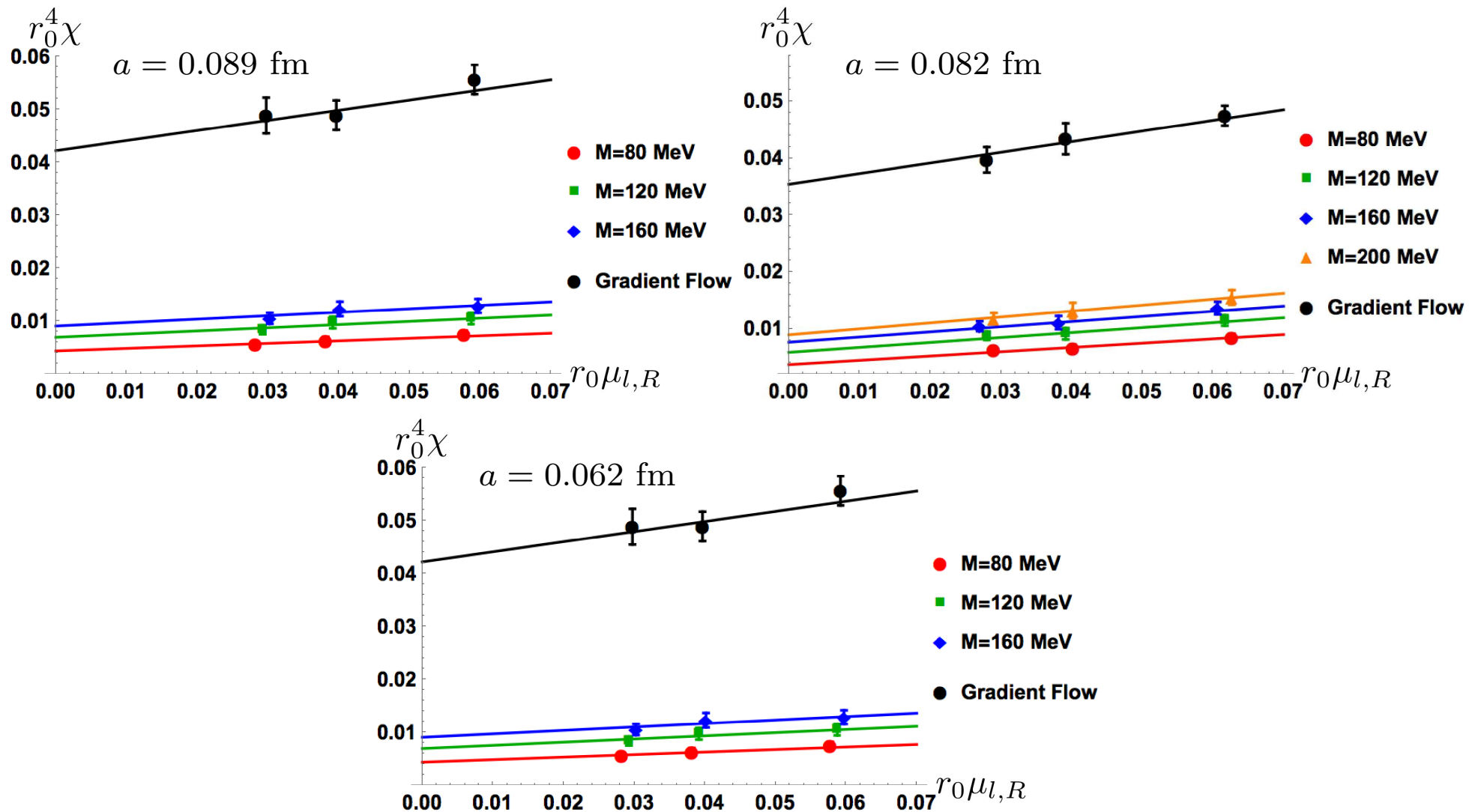
$$\chi = \left.\frac{\partial^2 \varepsilon(\theta)}{\partial\theta^2}\right|_{\theta=0} = \Sigma \frac{\mu_l}{2},$$

- 3-parameter fit (plus $\mathcal{O}(a^2)$)

$$r_0^4 \chi = [r_0^3 \Sigma] \frac{r_0 \mu_{l,R}}{2} + [c] \left(\frac{a}{r_0}\right)^2 + \left[\frac{\alpha}{r_0}\right] r_0 \mu_{l,R} \left(\frac{a}{r_0}\right)^2,$$

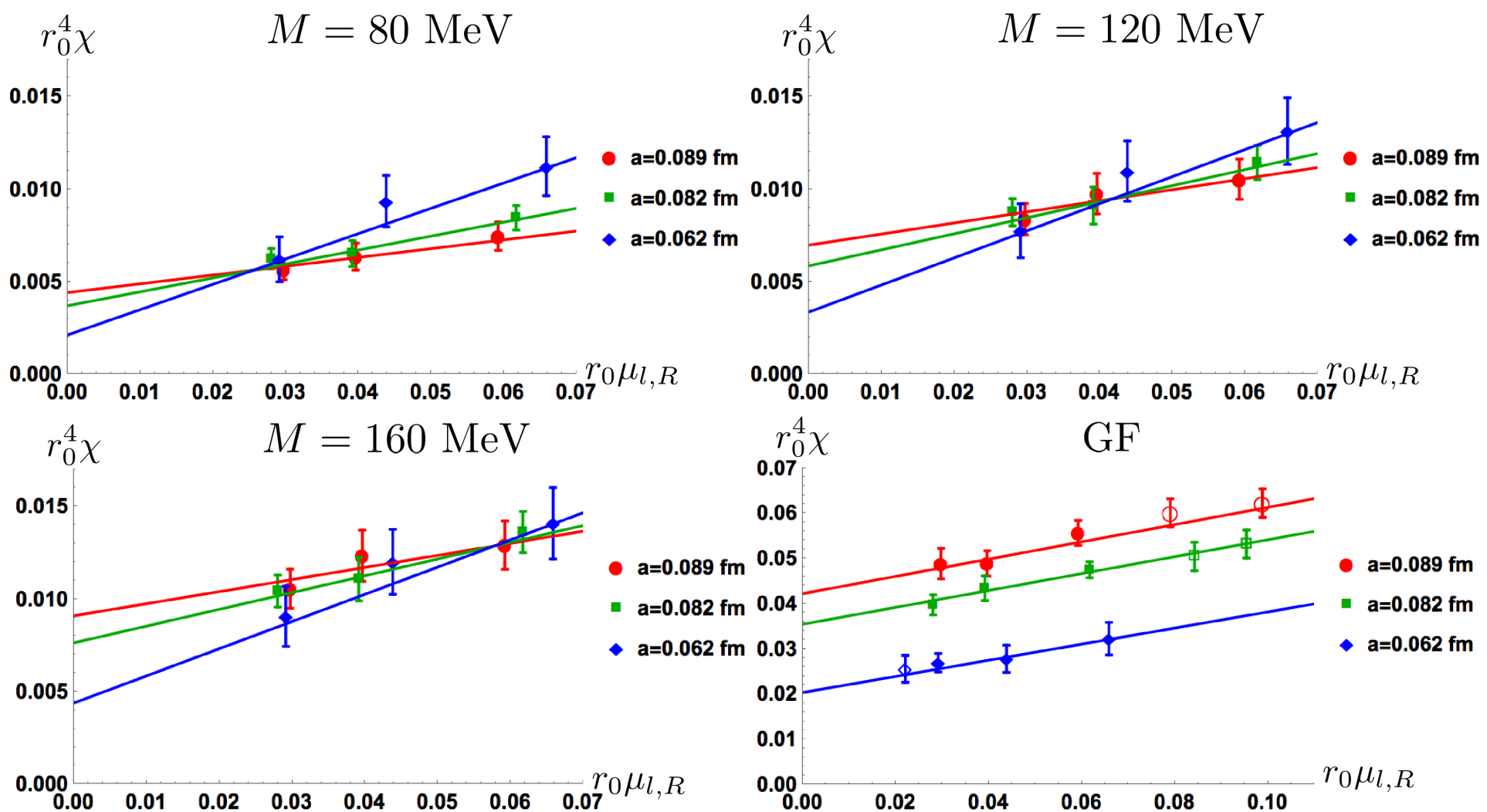
$$r_0^4\chi = f(r_0\mu_{l,R})$$

- Cross sections for the three lattice spacings:



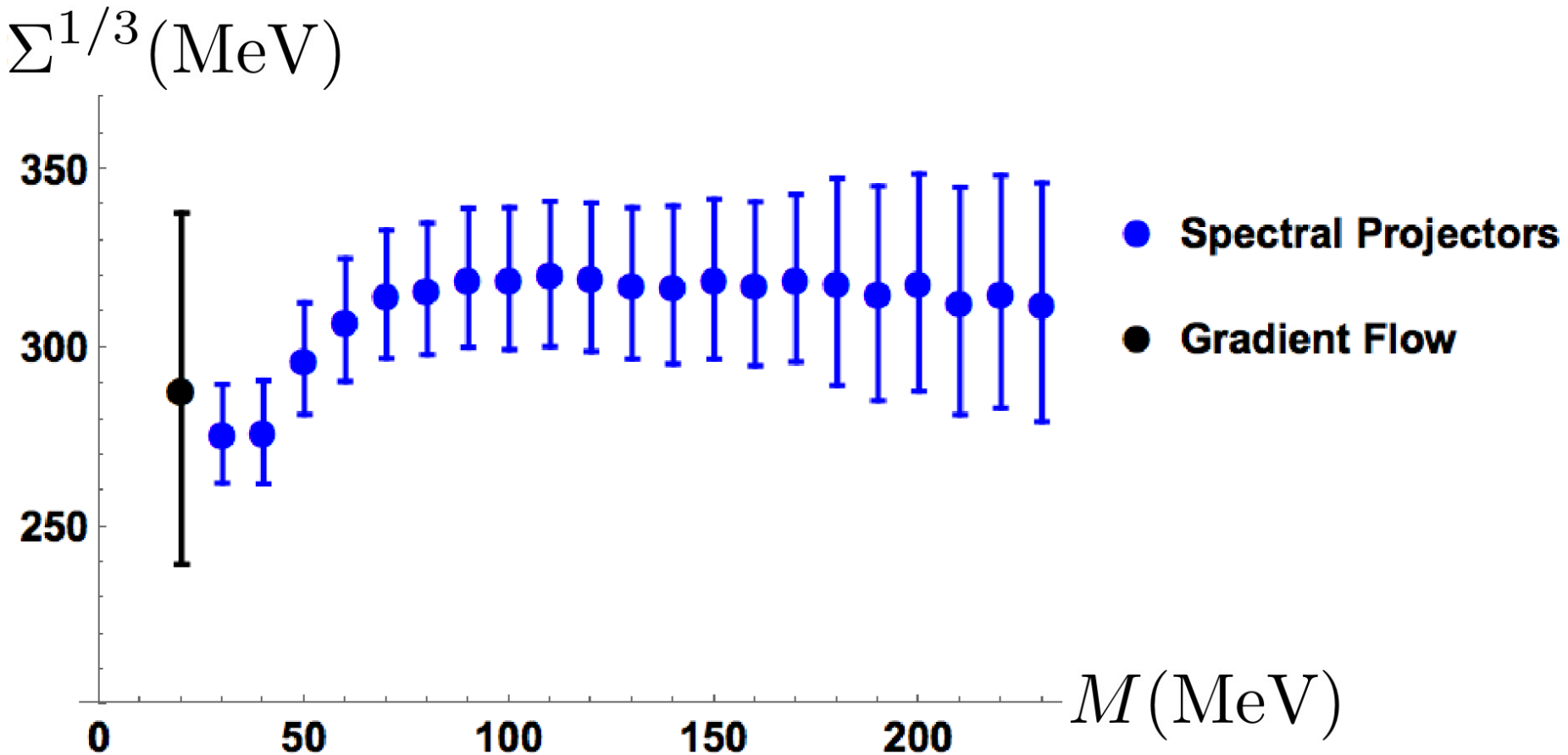
$$r_0^4\chi = f(r_0\mu_{l,R})$$

- Global fits



Chiral Condensate: $\Sigma^{1/3} = f(M)$

- Comparison of $\Sigma^{1/3}$ for Gradient Flow and Spectral Projectors
- $\Sigma^{1/3}$ as a function of M



Total discretization error

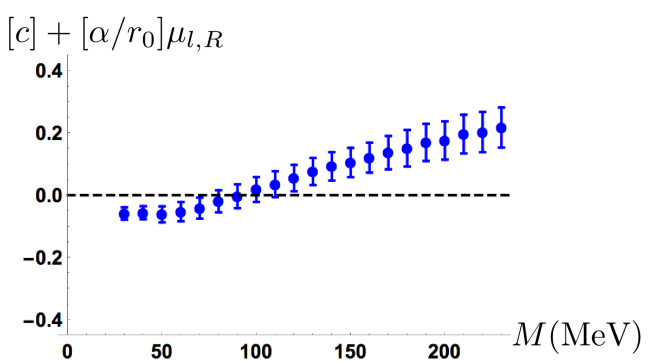
- From the 3-parameters fit:

$$r_0^4\chi = [r_0^3\Sigma]\,\frac{r_0\mu_{l,R}}{2} + [c]\left(\frac{a}{r_0}\right)^2 + \left[\frac{\alpha}{r_0}\right] r_0\mu_{l,R}\left(\frac{a}{r_0}\right)^2 ,$$

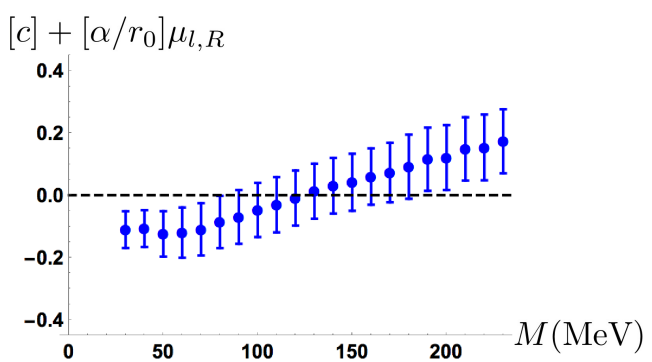
- Total discretisation error:

$$[c] + \left[\frac{\alpha}{r_0}\right] r_0\mu_{l,R} ,$$

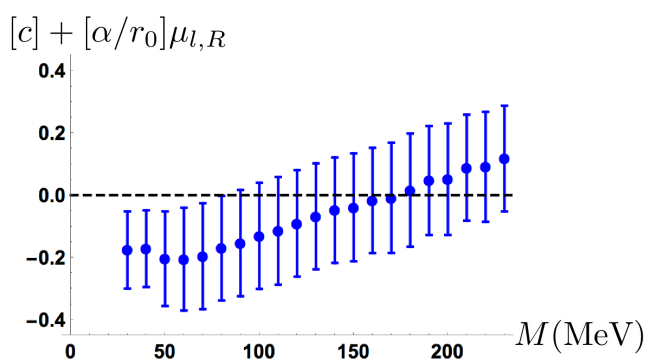
- Global fits for gradient Flow



$\mu_{l,R} = 12.6$ MeV



$\mu_{l,R} = 18.5$ MeV



$\mu_{l,R} = 26.0$ MeV

Fit parameters

M [MeV]	$\Sigma^{1/3}$ [MeV]	$r_0\Sigma^{1/3}$	c	α/r_0	N_{ens}	$\chi^2/d.o.f.$
80	316(18)	0.760(44)	0.121(32)	-4.7(1.4)	9	0.281
120	318(21)	0.764(50)	0.191(46)	-4.6(1.8)	9	0.233
160	318(23)	0.764(55)	0.249(54)	-4.3(2.0)	9	0.198
200	318(30)	0.764(73)	0.291(60)	-3.9(2.5)	6	0.066
GF	289(49)	0.69(12)	1.15(12)	0.7(3.8)	9	0.261

- Spectral projector method has larger mass dependent discretization effects
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- Value of Chiral Condensate agrees between Gradient Flow and Spectral Projectors

$\longrightarrow$ Universality is confirmed

$\chi(a \to 0)$ LO Fit

- Alternative fit
- For each ensemble we fix the quark mass and extract χ
- We perform a linear fit of $r_0^4\chi$ as a function of $(a/r_0)^2$ using

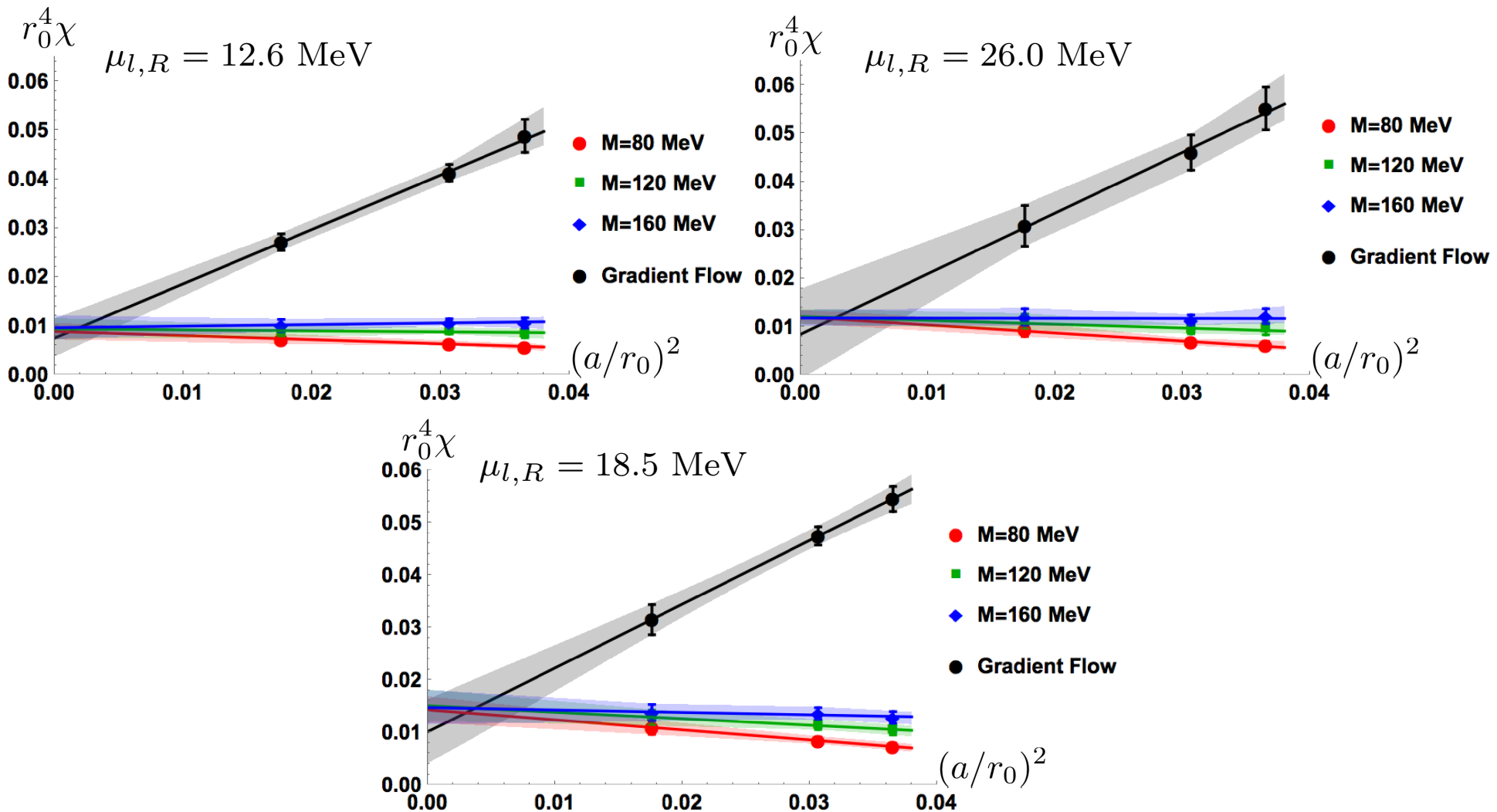
$$r_0^4\chi(a) = [K](a/r_0)^2 + [r_0^4\chi(a=0)],$$

- To calculate the chiral condensate, we fit the continuum values of $r_0^4\chi(a=0)$ as a function of $r_0\mu_{l,R}$ using

$$r_0^4\chi(a=0) = [r_0^3\Sigma]\frac{r_0\mu_l}{2}.$$

$$r_0^4\chi = f(a^2/r_0^2)$$

- Continuum extrapolations:



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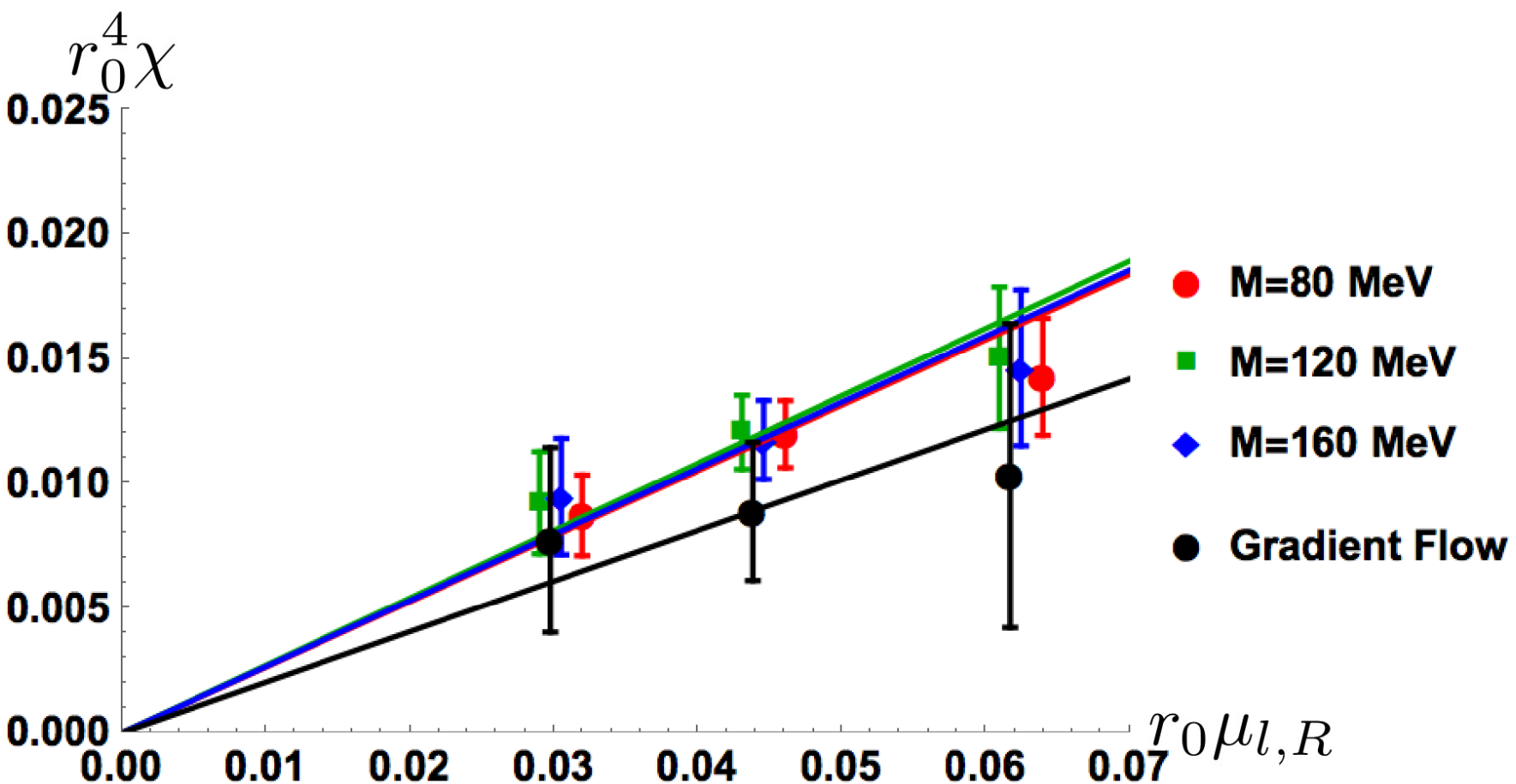
$$r_0^4\chi = f(\mu_{l,R})$$

- Continuum fit to the lowest order χPT expression for topological susceptibility.

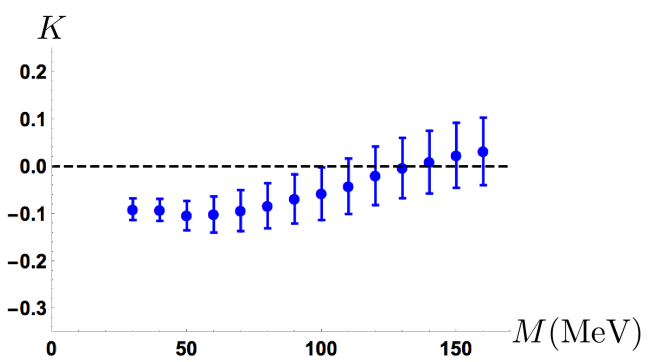


$\chi(a \to 0)$ LO Fit

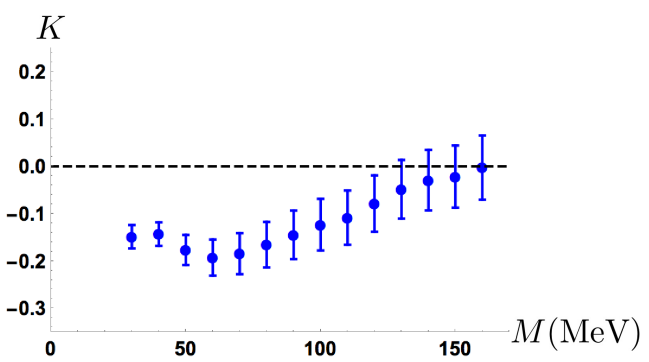
- Values of the chiral condensate

M [MeV]	$\Sigma^{1/3}$ [MeV]	$r_0\Sigma^{1/3}$	$\chi^2/d.o.f.$
80	336(10)	0.807(23)	0.555
120	339(11)	0.815(26)	0.344
160	337(12)	0.809(29)	0.387
GF	308(26)	0.740(62)	0.171

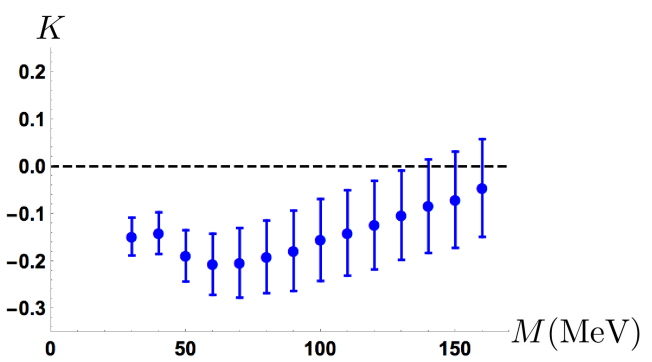
- Agreement between the Spectral Projectors and Gradient-Flow
- Total discretization error



$\mu_{l,R} = 12.6$ MeV



$\mu_{l,R} = 18.5$ MeV



$\mu_{l,R} = 26.0$ MeV

What we learned by investigating the topological susceptibility?

- The topological susceptibility computed using spectral projectors has much smaller discretization effects than Gradient-Flow.
- Universality has been confirmed
- Spectral cutoff as small as $M \sim 30$ MeV is sufficient for extracting fit quantities.
- We can minimize the cutoff effects by choosing the right M (depending on the quark mass)
- We use $M = 120$ MeV to quote

$$r_0 \Sigma^{1/3} = 0.754(50)_{\rm stat}(26)_{\rm sys}, \quad \Sigma^{1/3} = 318(21)_{\rm stat}(11)_{\rm sys} \text{ MeV}.$$

- This result is in agreement with other investigations (Banks-Casher relation)
- This is highly relevant for
  - calculating topological charge dependent quantities

  when there is only
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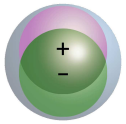
THANK YOU!

The nEDM: General Considerations

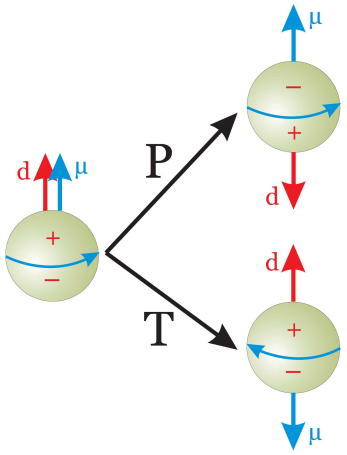
The structure of the Standard Model

→ Based on the role of the discrete symmetries C, P, T

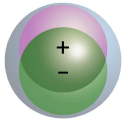


Possible experimental observation of nEDM

→ Flags violation of P and T



→ Due to CPT symmetry → violation of CP



Best experimental bound:

$$|\vec{d}_N| < 2.9 \times 10^{-13} e \cdot \text{fm}$$

[C. A. Baker *et al*, Phys. Rev. Lett. **97**, 131801 (2006) [arXiv:hep-ex/0602020]]

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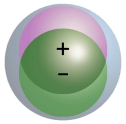

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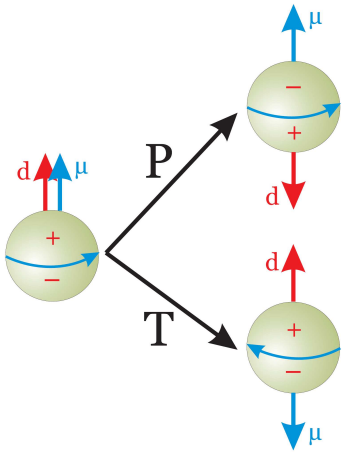

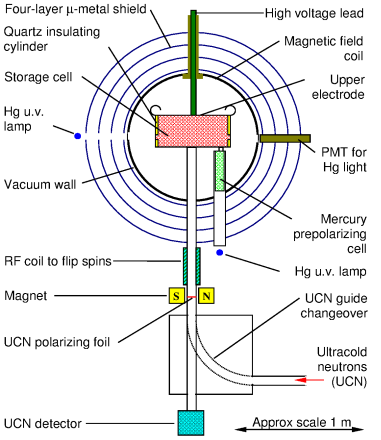

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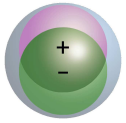

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- Supersymmetric predictions:

$$|\vec{d}_N| \sim (10^{-12} \sim 10^{-15}) e \cdot \text{fm}$$

[M. Pospelov and A. Ritz, [hep-ph/0504231]]

- Weak Interactions prediction:

$$|\vec{d}_N| \sim 10^{-19} e \cdot \text{fm}$$

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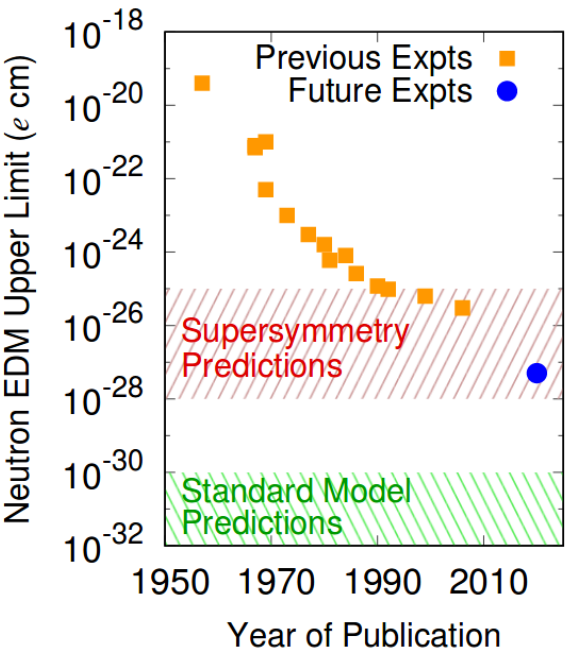
- Missing the neutron electric dipole moment resulting from QCD.



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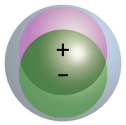

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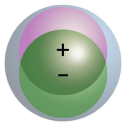

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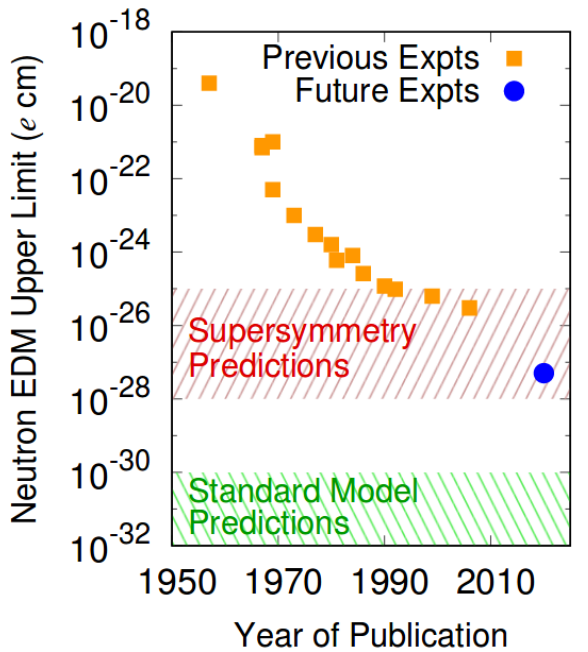
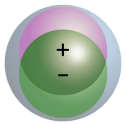
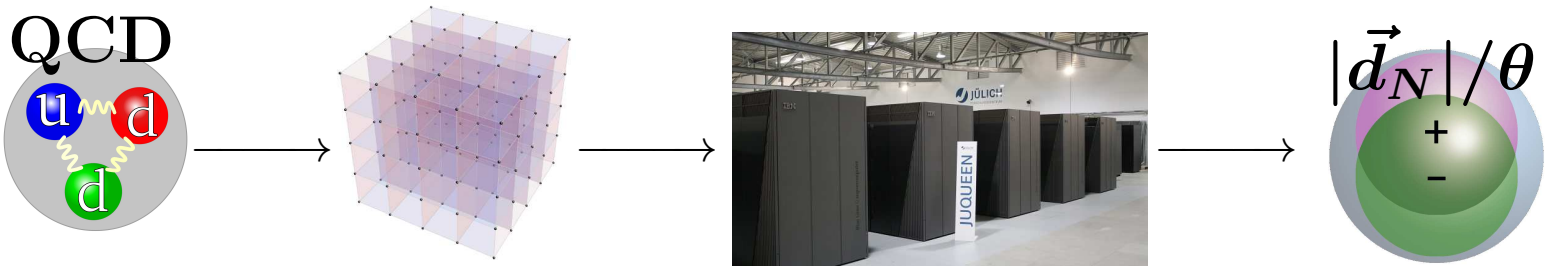




Missing the neutron electric dipole moment resulting from QCD.

→ Only from *ab initio* Lattice Calculations



The nEDM: General Considerations

- Consider violation of CP in strong interactions...
- QCD Lagrangian density with no additional insertions:

$$\mathcal{L}_{\mathrm{QCD}}(x) = \frac{1}{2g^2}\mathrm{Tr}\left[G_{\mu\nu}(x)\,G^{\mu\nu}(x)\right] + \sum_f \overline{\psi}_f(x)\left(i\gamma_\mu D^\mu + m_f\right)\psi_f(x)\ ,$$

  is invariant under C, P and T transformations.
- Hence, it cannot induce a non-vanishing nEDM.
- We need to insert the CP-violating Cherns-Simons (CS) term:

$$\mathcal{L}_{\mathrm{CS}}(x) \equiv -i\theta\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G_{\mu\nu}(x)\,G_{\rho\sigma}(x)\right]\ .$$

- We, now, consider QFT with Lagrangian density:

$$\mathcal{L}(x) = \mathcal{L}_{\mathrm{QCD}}(x) + \mathcal{L}_{\mathrm{CS}}(x)\ .$$

- Model dependend studies as well as ChPT predictions:

$$|d_N| \sim \theta\cdot\mathcal{O}\left(10^{-2}\sim 10^{-3}\right)e\cdot\mathrm{fm}\,.$$

$\rightarrow\ \theta \lesssim \mathcal{O}\left(10^{-10}\sim 10^{-11}\right)$

The nEDM: General Considerations

- Interaction between nucleon fields $u_N(x)$ and the electromagnetic field tensor $F_{\mu\nu}$:

$$-\frac{1}{2}\theta\frac{F_3(q^2)}{2m_N}\overline{u}_N(p_f)\sigma_{\mu\nu}\gamma_5 u_N(p_i)F^{\mu\nu}\,.$$

- In the static limit ($q^2 = (p_f - p_i)^2 \to 0$) the above gives the Electric Dipole Moment of the nucleon

$$|\vec{d}_N| = \lim_{q\to 0}\theta\frac{F_3(q^2)}{2m_N}\,.$$

- Hence, what needs to be done is to **Extract CP-odd Form-Factor $F_3(0)$ for $\theta \neq 0$** from expectation values such as:

$$\langle\mathcal{O}(x_1,...,x_n)\rangle_\theta = \frac{1}{Z_\theta}\int d[U]d[\psi_f]d[\bar{\psi}_f]\;\mathcal{O}(x_1,...,x_n)\;e^{-S_{\rm QCD}+i\theta\int q(x)d^4x}\,.$$

- How?
  - ? External Electric Field Method
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The nEDM: General Considerations

- This is a first test on the applicability of the implemented methods!
- Our final goal is to Extract the nEDM for
  - **physical pion mass**
  - **continuum limit**
- Extract the nEDM on configurations produced with $N_f = 2 + 1 + 1$ twisted mass fermions,
- Produced with **Iwasaki Gauge Action**.
- Lattice spacing $a \simeq 0.082$ **fm**.
- $L/a = 32$.
- B55.32.
  - $\beta = 1.95$, $a = 0.0823(10)$ fm
  - $a\mu = 0.0055$, $M_\pi = 0.372$ GeV
  - $r_0/a = 5.710(41)$
  - $32^3 \times 64$, $L = 2.6$ fm
  - No. of confs = 4695

The nEDM: Lattice Calculation

- From Nucleon-Nucleon matrix element of the electromagnetic current $J_\mu^{\rm em}$ we obtain

$\rightarrow$ CP-odd electromagnetic Form-Factor $F_3(Q^2)$:

$$\langle N(\vec{p}_f, s_f)|J_\mu^{\rm em}|N(\vec{p_i}, s_i)\rangle \quad = \quad \bar{u}(\vec{p}_f, s_f)\left[\cdots + \theta\frac{F_3(Q^2)}{2m_N}Q_\nu\sigma_{\mu\nu}\gamma_5 + \cdots\right]u(\vec{p_i}, s_i)\,.$$

- **Assuming a theory where CP symmetry is violated ...**
- Problem: CP-odd Form-Factor $\propto Q^k\ F_3(Q^2)$
  - Parametrize $F_3(Q^2)$ in Q^2 and peform a Dipole fit.
  - Use the "Application of the derivative to the ratio technique".
  - Use the "The elimination of the momentum in the plateau region technique".
- Now let's extract $F_3(Q^2)$!!!
  - What are the desired quantities without treating the CS term **perturbatively**?
  - How 3-pt functions modify if we treat the CS term **perturbatively**.
  - Equate the above two: **perturbatively** $\equiv$ **non-perturbatively** (in θ).

[E. Shintani *et al*, arXiv:hep-lat/0505022]

The nEDM: non-perturbative treatment (in θ)

- Consider the 3-pt function:

$$G_{3\mathrm{pt}}^{\mu,(\theta)}(\vec{q},t_f,t_i,t) \quad \equiv \quad \langle J_N(\vec{p}_f,t_f)J_\mu^{\mathrm{em}}(\vec{q},t)\overline{J}_N(\vec{p}_i,t_i)\rangle_\theta\,.$$

- By inserting two complete sets of states etc...

$$\begin{aligned} G_{3\mathrm{pt}}^{\mu,(\theta)}(\vec{q},t_f,t_i,t) \quad &\simeq \quad e^{-E^f_{N^\theta}(t_f-t)}e^{-E^i_{N^\theta}(t-t_i)} \\ &\times \quad \sum_{s_f,s_i}\langle J_N|N(\vec{p}_f,s_f)\rangle_\theta\langle N(\vec{p}_f,s_f)|J_\mu^{\mathrm{em}}|N(\vec{p}_i,s_i)\rangle_\theta\langle N(\vec{p}_i,s_i)|\overline{J}_N\rangle\,. \end{aligned}$$

- with $E^f_{N^\theta}=\sqrt{\vec{p}_f^{\,2}+m^2_{N^\theta}}$ and $E^i_{N^\theta}=\sqrt{\vec{p}_i^{\,2}+m^2_{N^\theta}}$.

- with $\langle J_N^\theta|N^\theta(\vec{p},s)\rangle_\theta = Z_N^\theta u_N^\theta(\vec{p},s)$, and $\langle N^\theta(\vec{p},s)|\overline{J}_N^\theta\rangle_\theta = (Z_N^\theta)^* \bar{u}_N^\theta(\vec{p},s)\,.$

- and $\left(i\gamma^\mu p_\mu + m_{N^\theta}e^{-i2\alpha(\theta)\gamma_5}\right)u_N^\theta(\vec{p},s) = \bar{u}_N^\theta(\vec{p},s)\left(i\gamma^\mu p_\mu + m_{N^\theta}e^{-i2\alpha(\theta)\gamma_5}\right) = 0\,.$

- The phase $e^{i2\alpha(\theta)\gamma_5}$ appears due to the CP-breaking in the θ-vacuum.

The nEDM: non-perturbative treatment (in θ)

- The phase $e^{i2\alpha(\theta)\gamma_5}$ appears due to the CP-breaking in the θ-vacuum:
  $\rightarrow$ CP-even: $m_{N^\theta} = m_N + O(\theta^2), \quad Z_N^\theta = Z_N + O(\theta^2)\,.$
  $\rightarrow$ CP-odd: $\alpha(\theta) = \alpha^1\theta + O(\theta^3)$.
- Form-Factors:

$$\langle N^\theta(\vec{p}_f\,,s_f)|J_\mu^{\rm em}|N^\theta(\vec{p}_i\,,s_i)\rangle_\theta = \bar{u}_N^\theta(\vec{p}_f\,,s_f)\,W_\mu^\theta(Q)\,u_N^\theta(\vec{p}_i\,,s_i)\,,$$

with

$$W_\mu^\theta(Q) \quad = \quad g(\theta^2)\;W_\mu^{\rm even}(Q) + i\theta\;h(\theta^2)\;W_\mu^{\rm odd}(Q)\,.$$

and $g(\theta^2) = 1 + O(\theta^2)$ and $h(\theta^2) = 1 + O(\theta^2)$

$$W_\mu^{\rm even}(Q) \quad = \quad \gamma_\mu F_1(Q^2) - i\,\frac{F_2(Q^2)}{2m_N}\,Q_\nu\,\sigma_{\nu\mu}\,,$$

$$W_\mu^{\rm odd}(Q) \quad = \quad -i\,\frac{F_3(Q^2)}{2m_N}\,Q_\nu\,\sigma_{\nu\mu}\gamma_5 + F_A(Q^2)\left(Q_\mu \not{Q} - \gamma_\mu\,Q^2\right)\gamma_5\,.$$

$\Rightarrow$

$$G_{\rm 3pt}^{\mu,(\theta)}(\vec{q},t_f,t_i,t) \quad = \quad |Z_N|^2 e^{-E_N^f(t_f-t)}e^{-E_N^i(t-t_i)}\,\frac{-i\not{p}_f + m_N(1+2i\alpha^1\theta\gamma_5)}{2E_N^f}$$

$$\times \quad \left[W_\mu^{\rm even}(Q) + i\theta W_\mu^{\rm odd}(Q)\right]\frac{-i\not{p}_i + m_N(1+2i\alpha^1\theta\gamma_5)}{2E_N^i} + O(\theta^2)\,.$$

The nEDM: non-perturbative treatment (in θ)

- The phase $e^{i2\alpha(\theta)\gamma_5}$ appears due to the CP-breaking in the θ-vacuum:
  $\rightarrow$ CP-even: $m_{N^\theta} = m_N + O(\theta^2), \quad Z_N^\theta = Z_N + O(\theta^2)\,.$
  $\rightarrow$ CP-odd: $\alpha(\theta) = \alpha^1\theta + O(\theta^3)$.
- Form-Factors:

$$\langle N^\theta(\vec{p}_f\,,s_f)|J_\mu^{\rm em}|N^\theta(\vec{p}_i\,,s_i)\rangle_\theta = \bar{u}_N^\theta(\vec{p}_f\,,s_f)\,W_\mu^\theta(Q)\,u_N^\theta(\vec{p}_i\,,s_i)\,,$$

with

$$W_\mu^\theta(Q) \quad = \quad g(\theta^2)\;W_\mu^{\rm even}(Q) + i\theta\;h(\theta^2)\;W_\mu^{\rm odd}(Q)\,.$$

and $g(\theta^2) = 1 + O(\theta^2)$ and $h(\theta^2) = 1 + O(\theta^2)$

$$W_\mu^{\rm even}(Q) \quad = \quad \gamma_\mu F_1(Q^2) - i\,\frac{F_2(Q^2)}{2m_N}\,Q_\nu\,\sigma_{\nu\mu}\,,$$

$$W_\mu^{\rm odd}(Q) \quad = \quad -i\,\frac{F_3(Q^2)}{2m_N}\,Q_\nu\,\sigma_{\nu\mu}\gamma_5 + F_A(Q^2)\left(Q_\mu \not{Q} - \gamma_\mu\,Q^2\right)\gamma_5\,.$$

$\Rightarrow$

$$G_{\rm 3pt}^{\mu,(\theta)}(\vec{q},t_f,t_i,t) \quad = \quad |Z_N|^2 e^{-E_N^f(t_f-t)}e^{-E_N^i(t-t_i)}\,\frac{-i\not{p}_f + m_N(1+2i\alpha^1\theta\gamma_5)}{2E_N^f}$$

$$\times \quad \left[W_\mu^{\rm even}(Q) + i\theta W_\mu^{\rm odd}(Q)\right]\frac{-i\not{p}_i + m_N(1+2i\alpha^1\theta\gamma_5)}{2E_N^i} + O(\theta^2)\,.$$

The nEDM: non-perturbative treatment (in θ)

Correction came from Abramczyk, et al., arXiv:1701.07792

→ CP-even

- Dirac Equation: $(i\gamma^\mu p_\mu + m_N)\, u_N(\vec{p})$
- Parity Operator: γ_4
- Under Parity $u_N(\vec{p}) \to \gamma_4 u_N(-\vec{p})$

→ CP-odd

- Dirac Equation: $\left(i\gamma^\mu p_\mu + m_{N^\theta} e^{-i2\alpha(\theta)\gamma_5}\right) u_N^\theta(\vec{p})$
- Parity Operator: $e^{i2\alpha(\theta)\gamma_5}\gamma_4$
- Under Parity $u_N^\theta(\vec{p}) \to e^{i2\alpha(\theta)\gamma_5}\gamma_4 u_N^\theta(-\vec{p})$

- γ_4 is no longer parity operator of neutron state.
- This accounts to rotating:

$$\begin{aligned} \tilde{F}_2 &= \cos(2\alpha)F_2 - \sin(2\alpha)F_3 \\ \tilde{F}_3 &= \sin(2\alpha)F_2 + \cos(2\alpha)F_3 \end{aligned}$$

The nEDM: perturbative treatment (in θ)

- Perturbative treatment of the CS term according to:

$$e^{i\theta \int d^4x q(x)} \equiv e^{i\theta \mathcal{Q}} = 1 + i\theta\mathcal{Q} + O(\theta^2)$$

- Let us apply the **perturbative expansion in 3-pt** functions:

$$\begin{aligned} G^{\mu,(\theta)}_{3\mathrm{pt}}(\vec{q}, t_f, t_i, t) &= \langle J_N(\vec{p}_f, t_f) J^{\mathrm{em}}_\mu(\vec{q}, t) \overline{J}_N(\vec{p}_i, t_i) \rangle_\theta \\ &= G^{\mu,(0)}_{3\mathrm{pt}}(\vec{q}, t_f, t_i, t) + i\,\theta\, G^{\mu,(0)}_{3\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i, t) + O\left(\theta^2\right), \end{aligned}$$

where:

$$\begin{aligned} G^{\mu,(0)}_{3\mathrm{pt}}(\vec{q}, t_f, t_i, t) &= \langle J_N(\vec{p}_f, t_f) J^{\mathrm{em}}_\mu(\vec{q}, t) \overline{J}_N(\vec{p}_i, t_i) \rangle, \\ G^{\mu,(0)}_{3\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i, t) &= \langle J_N(\vec{p}_f, t_f) J^{\mathrm{em}}_\mu(\vec{q}, t) \overline{J}_N(\vec{p}_i, t_i) \mathcal{Q} \rangle. \end{aligned}$$

- **What is new here is the topological charge $\mathcal{Q}$**

The nEDM: equate non-perturbative with perturbative in θ

- Equating the results from the two treatments we obtain:

$$
\begin{aligned}
G^{(0)}_{3\mathrm{pt}}(\vec{q}, t_f, t_i, t) &= |Z_N|^2 e^{-E^f_N (t_f - t)} e^{-E^i_N (t - t_i)} \frac{-i\not{p}_f + m_N}{2E^f_N} W^{\mathrm{even}}_\mu(Q) \frac{-i\not{p}_i + m_N}{2E^i_N}, \\
G^{(0)}_{3\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i, t) &= |Z_N|^2 e^{-E^f_N (t_f - t)} e^{-E^i_N (t - t_i)} \Big[\frac{-i\not{p}_f + m_N}{2E^f_N} W^{\mathrm{odd}}_\mu(Q) \frac{-i\not{p}_i + m_N}{2E^i_N} \\
&+ \frac{2\alpha^1 m_N}{2E^f_N} \gamma_5 W^{\mathrm{even}}_\mu(Q) \frac{-i\not{p}_i + m_N}{2E^i_N} + \frac{-i\not{p}_f + m_N}{2E^f_N} W^{\mathrm{even}}_\mu(Q) \frac{2\alpha^1 m_N}{2E^i_N} \gamma_5 \Big].
\end{aligned}
$$

- Similarly for 2-pt functions we obtain:

$$
G^{(\theta)}_{2\mathrm{pt}}(\vec{q}, t_f, t_i) = G^{(0)}_{2\mathrm{pt}}(\vec{q}, t_f, t_i) + i\,\theta\, G^{(0)}_{2\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i) + O\left(\theta^2\right),
$$

where

$$
\begin{aligned}
G^{(0)}_{2\mathrm{pt}}(\vec{q}, t_f, t_i) &= \langle J_N(\vec{q}, t_f) \overline{J}_N(\vec{q}, t_i) \rangle &&= |Z_N|^2 e^{-E_N t} \frac{-i\not{Q} + m_N}{2E_N}, \\
G^{(0)}_{2\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i) &= \langle J_N(\vec{q}, t_f) \overline{J}_N(\vec{q}, t_i) \mathcal{Q} \rangle &&= |Z_N|^2 e^{-E_N t} \frac{2\alpha^1 m_N}{2E_N} \gamma_5 .
\end{aligned}
$$

- **We need to measure the topological charge $\mathcal{Q}$ via:**

$\Rightarrow$ **Cooling**

$\Rightarrow$ **Gradient-Flow**

The nEDM: Topological charge

- We calculate the topological charge:

$$\mathcal{Q} = \int d^4x q(x) \,,$$

- with topological charge density:

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left\{G_{\mu\nu} G_{\rho\sigma}\right\} .$$

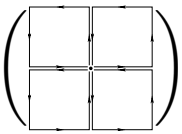
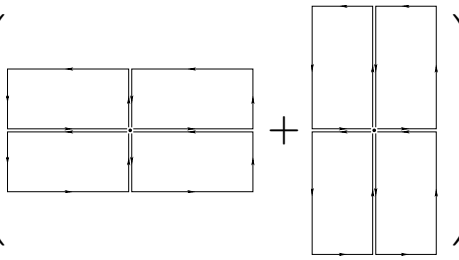
- We use the improved definition with leading lattice artifacts (clasically) of $O(a^4)$:

$$q(x) = c_0 q_L^{\mathrm{clov}}(x) + c_1 q_L^{\mathrm{rect}}(x) \,,$$

with $c_0 = 5/3$ and $c_1 = -1/12$ as well as

$$q_L^{\mathrm{clov}}(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{clov}} C_{\rho\sigma}^{\mathrm{clov}}\right) \quad \text{and} \quad q_L^{\mathrm{rect}}(x) = \frac{2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{rect}} C_{\rho\sigma}^{\mathrm{rect}}\right) \,,$$

where

$C_{\mu\nu}^{\mathrm{clov}}(x) = \frac{1}{4} \mathrm{Im}\Big($  $\Big)$ and $C_{\mu\nu}^{\mathrm{rect}}(x) = \frac{1}{8} \mathrm{Im}\Big($  $\Big)$.

- We need to smooth out the ultraviolet fluctuations...

The nEDM: Topological charge

Cooling

- Cooling $U_\mu(x) \in SU(N)$: $U_\mu^{\rm old}(x) \to U_\mu^{\rm new}(x)$ with

$$P(U) \propto e^{(\lim_{\beta\to\infty} \beta \frac{1}{N} {\rm ReTr} X_\mu{}^\dagger U_\mu)}.$$

- Choose a $U_\mu^{\rm new}(x)$ that maximizes:

$${\rm ReTr}\{\, U_\mu^{\rm new}(x) X_\mu^\dagger(x)\}.$$

- One full cooling iteration is noted as $n_c = 1$.

Gradient Flow

- Solution of the evolution equations:

$$\begin{aligned} \dot{V}_\mu\,(x,\tau) &= -g_0^2\,[\partial_{x,\mu} S_G(V(\tau))]\, V_\mu\,(x,\tau) \\ V_\mu\,(x,0) &= U_\mu\,(x)\ , \end{aligned}$$

- With link derivative defined as:

$$\begin{aligned} \partial_{x,\mu} S_G(U) &= i \sum_a T^a \frac{\rm d}{{\rm d}s} S_G\left(e^{isY^a} U\right)\Big|_{s=0} \\ &\equiv i \sum_a T^a \partial_{x,\mu}^{(a)} S_G(U)\,, \end{aligned}$$

- Total gradient flow time is expressed as $\tau = n_{\rm int}\epsilon$.

The nEDM: Topological charge - Equivalence between smoothers

- Recently shown that cooling exhibits equivalence with gradient flow:
  - Smoothing with **Wilson Action**
  - Perturbatively $\tau \simeq n_c/3$

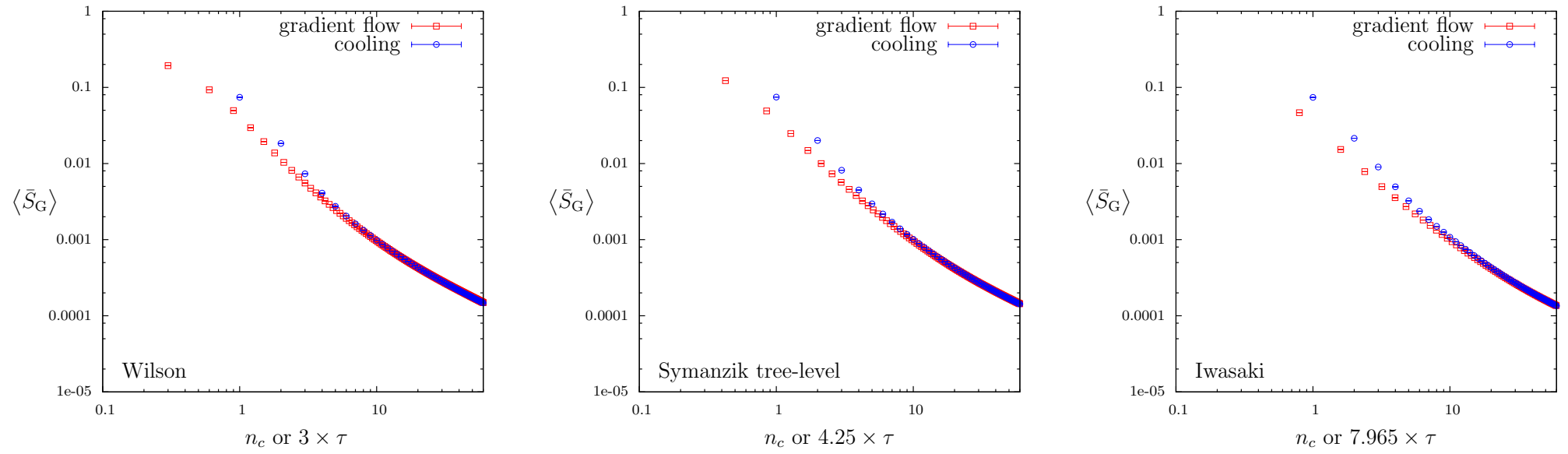
    [C. Bonati and M. D'Elia, Phys. Rev. D 89, 105005 (2014) [arXiv:1401.2441]]
- Configurations **produced** with **Iwasaki** Gauge Action
- Generalize the equivalence for Symanzik improved actions with rectangles
  - Smoothing with:

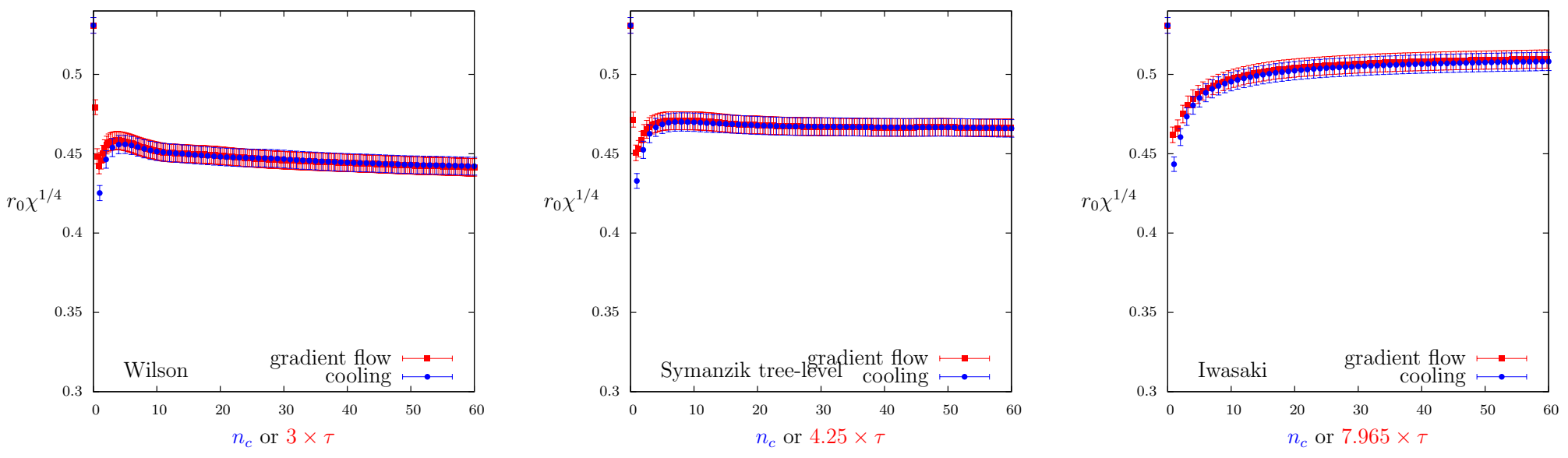
$$S_g = \frac{\beta}{N} \sum_x \left(c_0 \sum_{\substack{\mu,\nu=1 \\ 1\leq\mu<\nu}}^{4} \left\{ 1 - \mathrm{ReTr}(U^{1\times 1}_{x,\mu,\nu}) \right\} + c_1 \sum_{\substack{\mu,\nu=1 \\ \mu\neq\nu}}^{4} \left\{ 1 - \mathrm{ReTr}(U^{1\times 2}_{x,\mu,\nu}) \right\} \right),$$

    with $c_0 + 8c_1 = 1$
  - Shown that perturbatively $\tau \simeq n_c/(3 - 15c_1)$
  - Consider
    * Wilson: $c_1 = 0$, $\tau \simeq n_c/3$
    * Symanzik tree-level improved: $c_1 = -1/12$, $\tau \simeq n_c/4.25$
    * Iwasaki: $c_1 = -0.331$, $\tau \simeq n_c/7.965$

    [C. Alexandrou, A. A and K. Jansen, Phys. Rev. D 92, 125014 (2015) [arXiv:1509.04259]].

The nEDM: Topological charge - Equivalence between smoothers





The nEDM: Topological charge - Equivalence between smoothers

- We read an observable at a fixed value of $a\sqrt{8\tau} = O(0.3\text{fm})$.

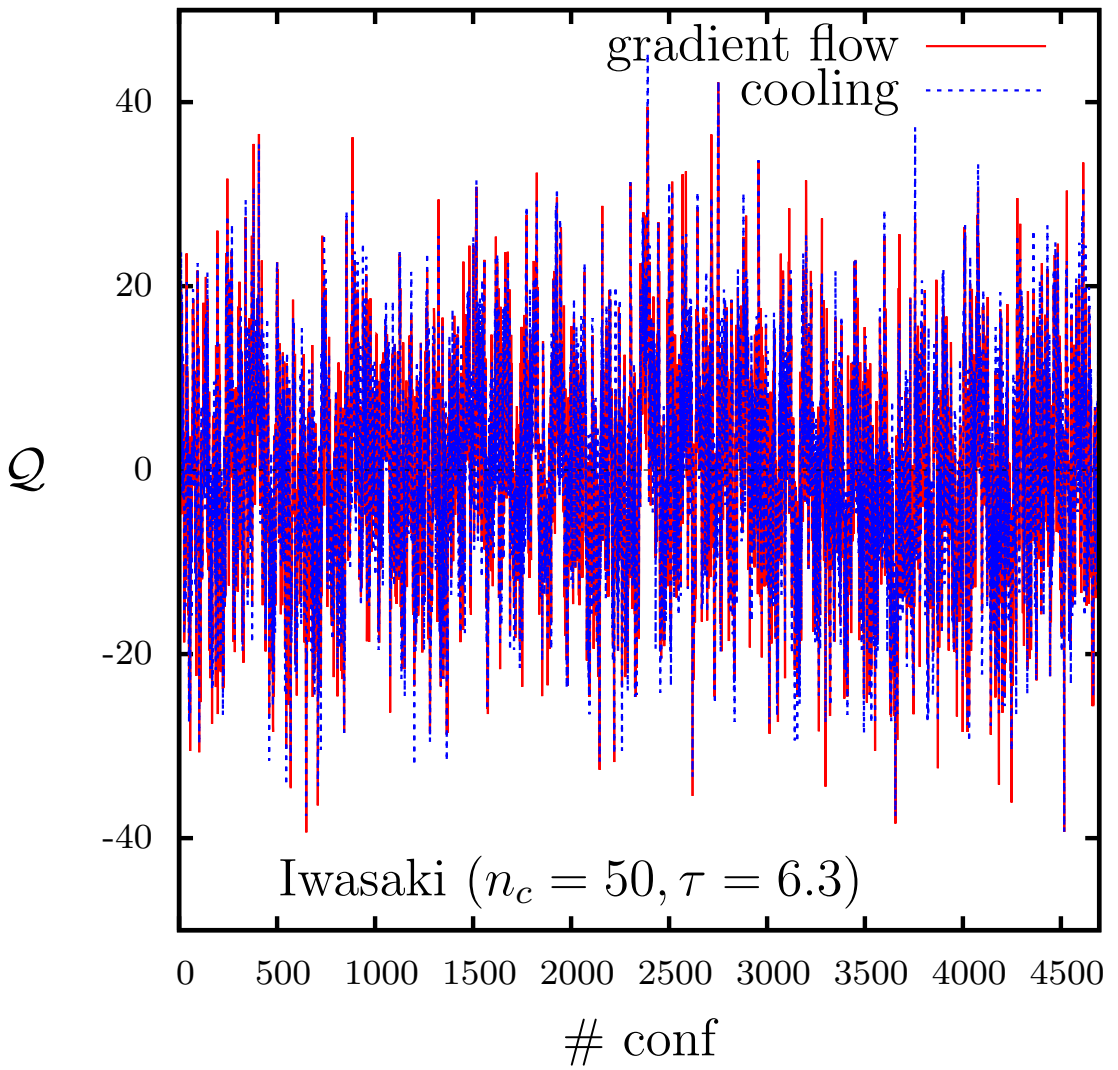
  [M. Lüscher, JHEP 1008:071 2010 [arXiv:1006.4518v3]].

- For practical reasons we choose a value of $a\sqrt{8\tau} \approx 0.6\text{fm}$.

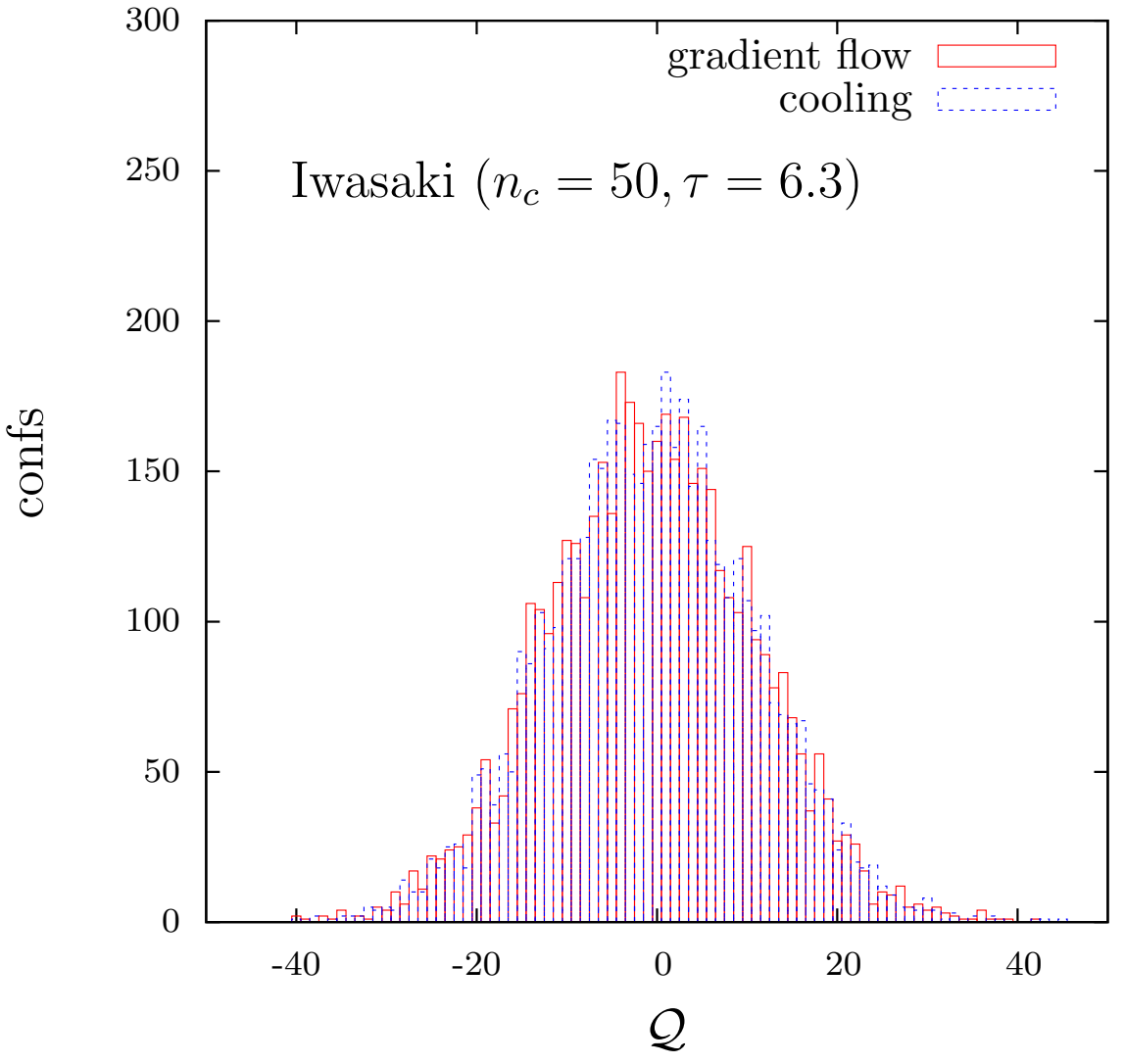
- This corresponds to:

Smoothing action	c_0	c_1	n_c/τ	$n_c\,(a\sqrt{8\tau} \approx 0.6\text{fm})$	$\tau\,(n_c)$
Wilson	1	0	3	20	6.7
Symanzik tree-level improved	$\frac{5}{3}$	$-\frac{1}{12}$	4.25	30	7.1
Iwasaki	3.648	-0.331	7.965	50	6.3

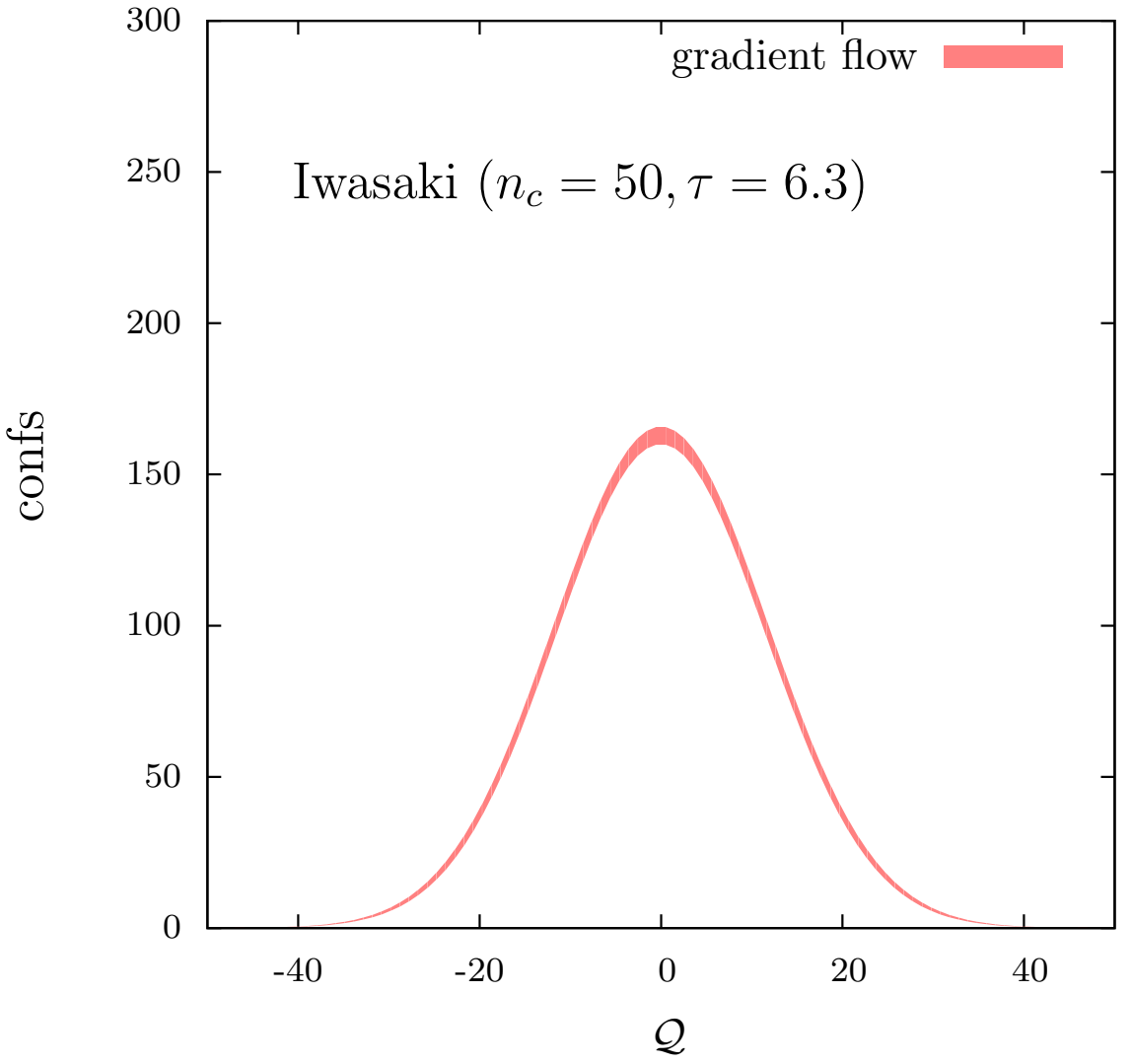
The nEDM: Topological charge - History



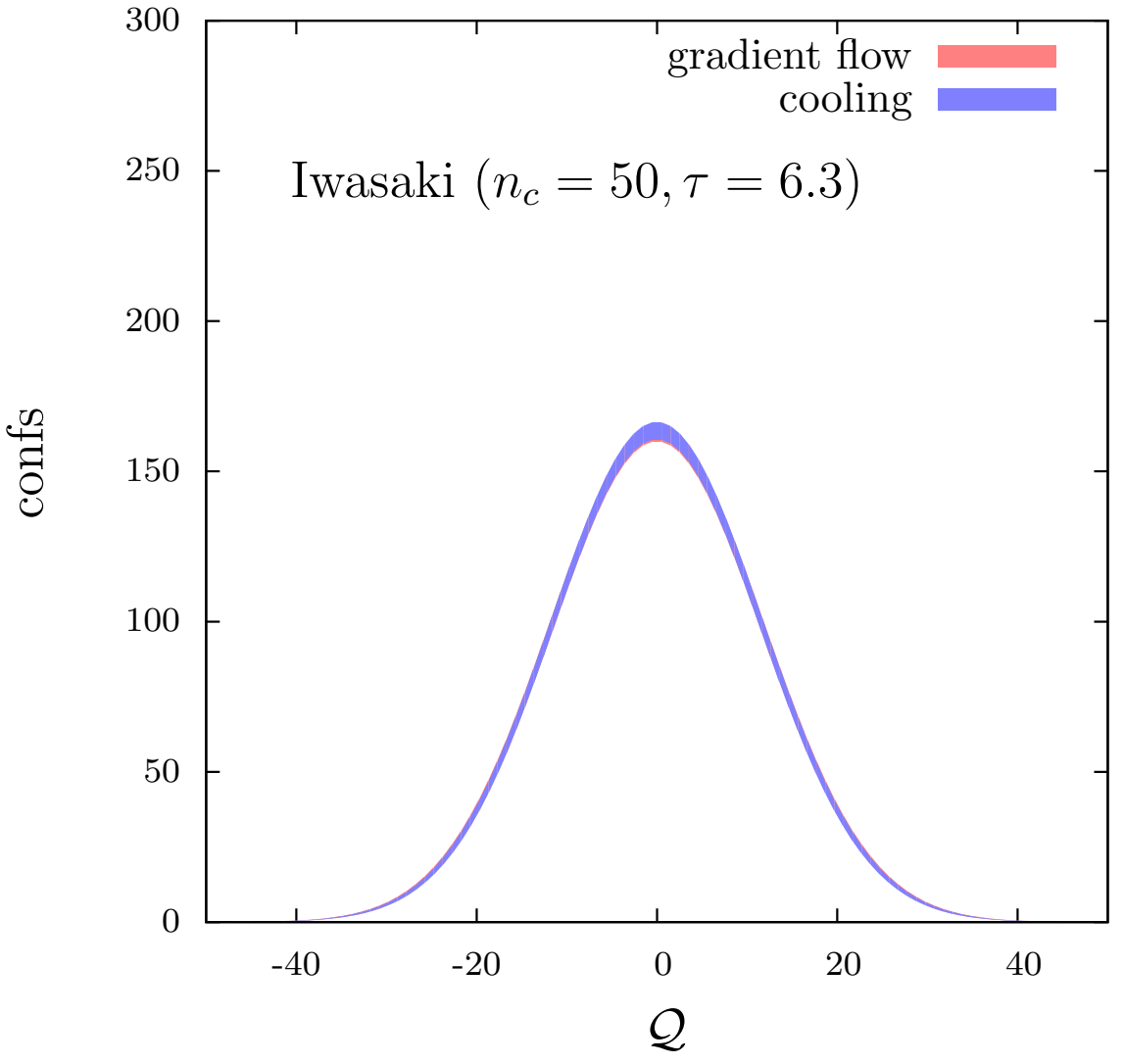
The nEDM: Topological charge - Distribution



The nEDM: Topological charge - Distribution

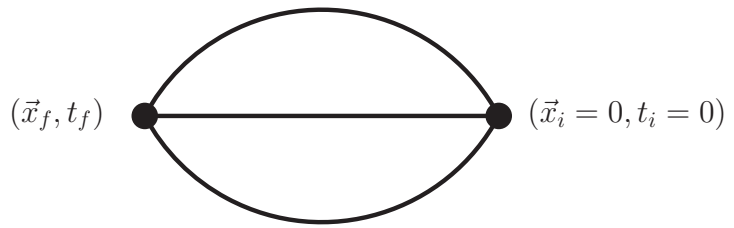


The nEDM: Topological charge - Distribution



The nEDM: 2-pt functions

- We calculate the 2-pt function:



- In practice we evaluate the 2-pt functions:

$$G_{2\mathrm{pt}}(\vec{q}, t_f, t_i, \Gamma_0) \equiv |Z_N|^2 e^{-E_N(t_f - t_i)} {\Gamma_0}^{\alpha\beta} \left[\Lambda_{1/2}(\vec{q})\right]_{\alpha\beta} ,$$

$$G_{2\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i, \Gamma_5) \equiv |Z_N|^2 e^{-E_N(t_f - t_i)} {\Gamma_5}^{\alpha\beta} \left[\frac{\alpha^1 m_N}{E_N}\gamma_5\right]_{\alpha\beta} .$$

- With projectors

$$\Gamma_0 = \frac{1}{4}\left(\mathbb{1} + \gamma_0\right) , \quad \Gamma_5 = \frac{\gamma_5}{4} ,$$

- We determine the value of α^1 by:

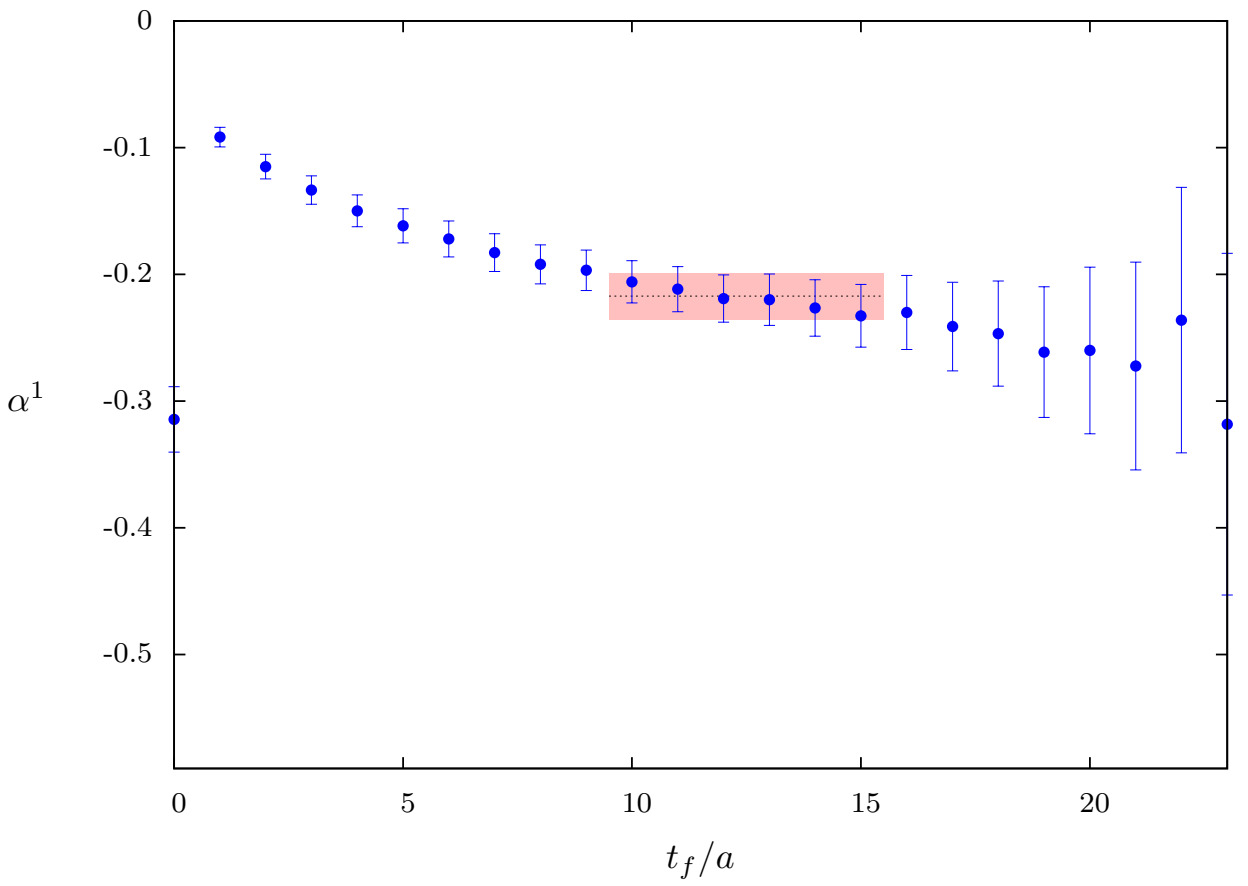
$$\mathrm{R}_{2\mathrm{pt}}(\alpha^1, t_f, t_i) = \frac{G_{2\mathrm{pt},\mathcal{Q}}\left(0, t_f, t_i, \Gamma_5\right)}{G_{2\mathrm{pt}}\left(0, t_f, t_i, \Gamma_0\right)} .$$

The nEDM: 2-pt Plateaus

- We calculate the plateau:

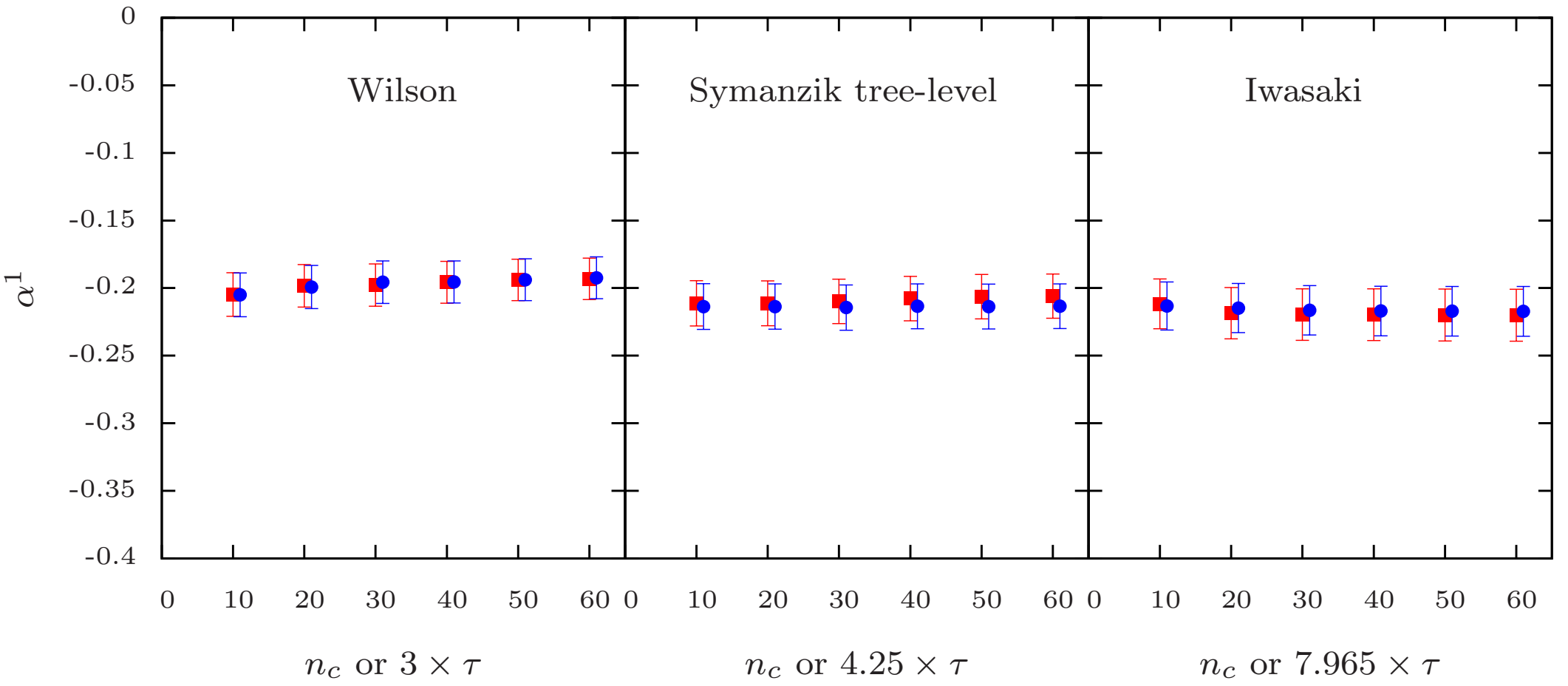
$$\Pi_{2\mathrm{pt}}(\alpha^1) = \lim_{t_f - t_i \to \infty} \mathrm{R}_{2\mathrm{pt}}(\alpha^1, t_f, t_i) = \alpha^1 \, .$$

- Example of a plateau for Iwasaki action, gradient flow at $\tau = 6.3$



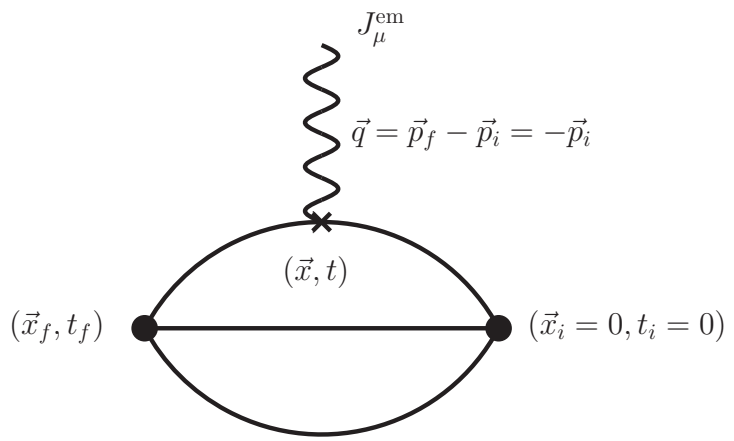
The nEDM: α^1 - Gradient Flow VS Cooling

α^1 via cooling (n_c) and gradient flow (τ) for Wilson smoothing action



The nEDM: 3-pt functions

- Calculate:



- We extract the CP-odd Form-Factors $F_3(q^2)$ from:

$$\begin{aligned} \mathrm{R}^{\mu}_{3\mathrm{pt},\mathcal{Q}}\left(\vec{q}, t_f, t_i, t, \Gamma_k\right) &= \frac{G^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{q}, t_f, t_i, t, \Gamma_k)}{G_{2\mathrm{pt}}(\vec{q}, t_f - t_i, \Gamma_0)} \\ &\times \sqrt{\frac{G_{2\mathrm{pt}}(\vec{q}, t_f - t, \Gamma_0) G_{2\mathrm{pt}}(\vec{0}, t - t_i, \Gamma_0) G_{2\mathrm{pt}}(\vec{0}, t_f - t_i, \Gamma_0)}{G_{2\mathrm{pt}}(\vec{0}, t_f - t, \Gamma_0) G_{2\mathrm{pt}}(\vec{q}, t - t_i, \Gamma_0) G_{2\mathrm{pt}}(\vec{q}, t_f - t_i, \Gamma_0)}} \,. \end{aligned}$$

- We calculate the plateau:

$$\Pi^{\mu}_{3\mathrm{pt}}\left(\Gamma_k\right) = \lim_{t_f - t \to \infty} \lim_{t - t_i \to \infty} \mathrm{R}^{\mu}_{3\mathrm{pt}}\left(\vec{q}, t_f, t_i, t, \Gamma_k\right) \,.$$

- Projectors:

$$\Gamma_k = \frac{i}{4}(\mathbb{1} + \gamma_0)\,\gamma_5\,\gamma_k \quad (k = 1, 2, 3)\,.$$

The nEDM: 3-pt functions

- The above plateau gives:

$$\Pi^0_{3\mathrm{pt},\mathcal{Q}}(\Gamma_k) = i\sqrt{\frac{2m_N^2}{E_N(E_N+m_N)}}\,Q_k\left[\frac{\alpha^1 F_1(Q^2)}{2m_N} + \frac{(E_N+3m_N)\alpha^1 F_2(Q^2)}{4m_N^2} + \frac{(E_N+m_N)F_3(Q^2)}{4m_N^2}\right].$$

- We elliminate F_1 and F_2 through the CP-even plateaus:

$$\begin{aligned}\Pi^k_{F_3} &= -i\Pi^0_{3\mathrm{pt},\mathcal{Q}}(\Gamma_k) + i\,\alpha^1\,\Pi^k_{3\mathrm{pt}}(\Gamma_0) + \alpha^1\,\frac{1}{2}\sum_{i,j=1}^{3}\epsilon_{jki}\Pi^j_{3\mathrm{pt}}(\Gamma_i)\,,\\ &= \sqrt{\frac{E_N+m_N}{8E_N m_N^2}}\,Q_k\,F_3(Q^2)\,.\end{aligned}$$

- Momentum Q_k hinders a direct evaluation of $F_3(Q^2=0)$
  - Usual parametrization in Q^2 and then fit with Dipole ansatz
  - "Application of the derivative to the ratio technique".

    [D. Guadagnoli *et al*, JHEP 0304 (2003) 019, [arXiv:hep-lat/0210044]]
  - "The elimination of the momentum in the plateau region technique"

    [C. Alexandrou *et al*, PoS LATTICE2014 (2014) 075, [arXiv:1410.8818]]

The nEDM: Extraction of $F_3(0)$ via Dipole Fit

- We extract the plateau $\Pi^k_{F_3}\left(Q^2, \Gamma_k\right)$ for momentum transfer Q^2:

  (Averaging over all momentum directions and index k...)

  for $t_f - t_i = 10a$, $t_f - t_i = 12a$ and $t_f - t_i = 14a$ (we chose results for $t_f - t_i = 12a$)

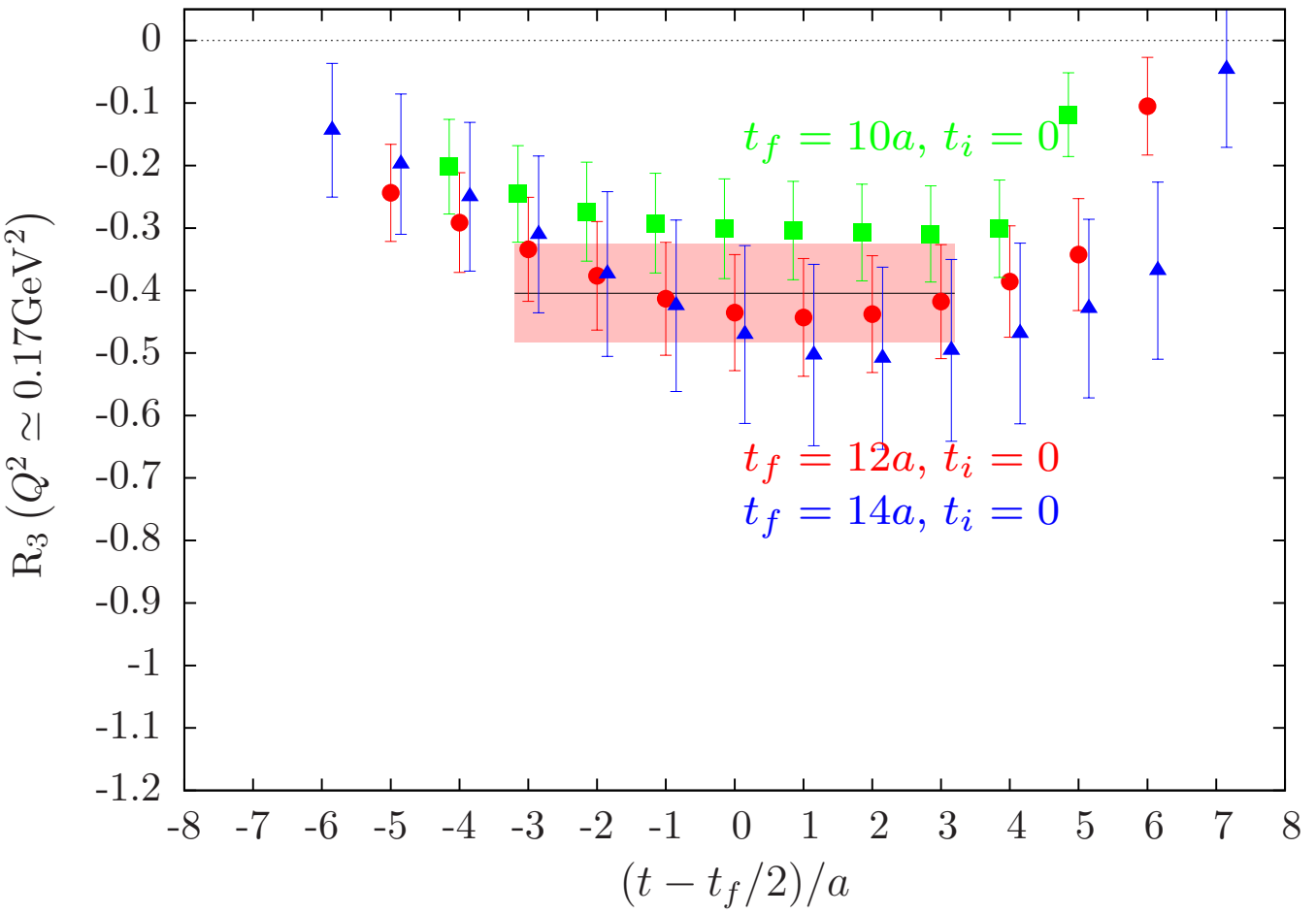
- We use the value of α^1 to build the ratio which leads to $F_3(Q^2)$ via $\Pi^k_{F_3}$.
- We parametrize $F_3(Q^2)$ in momentum transfer Q^2.

- We finally perform a dipole fit:

$$F_3\left(Q^2\right) = \frac{F_3\left(0\right)}{\left(1 + \frac{Q^2}{m^2_{F_3}}\right)^2}.$$

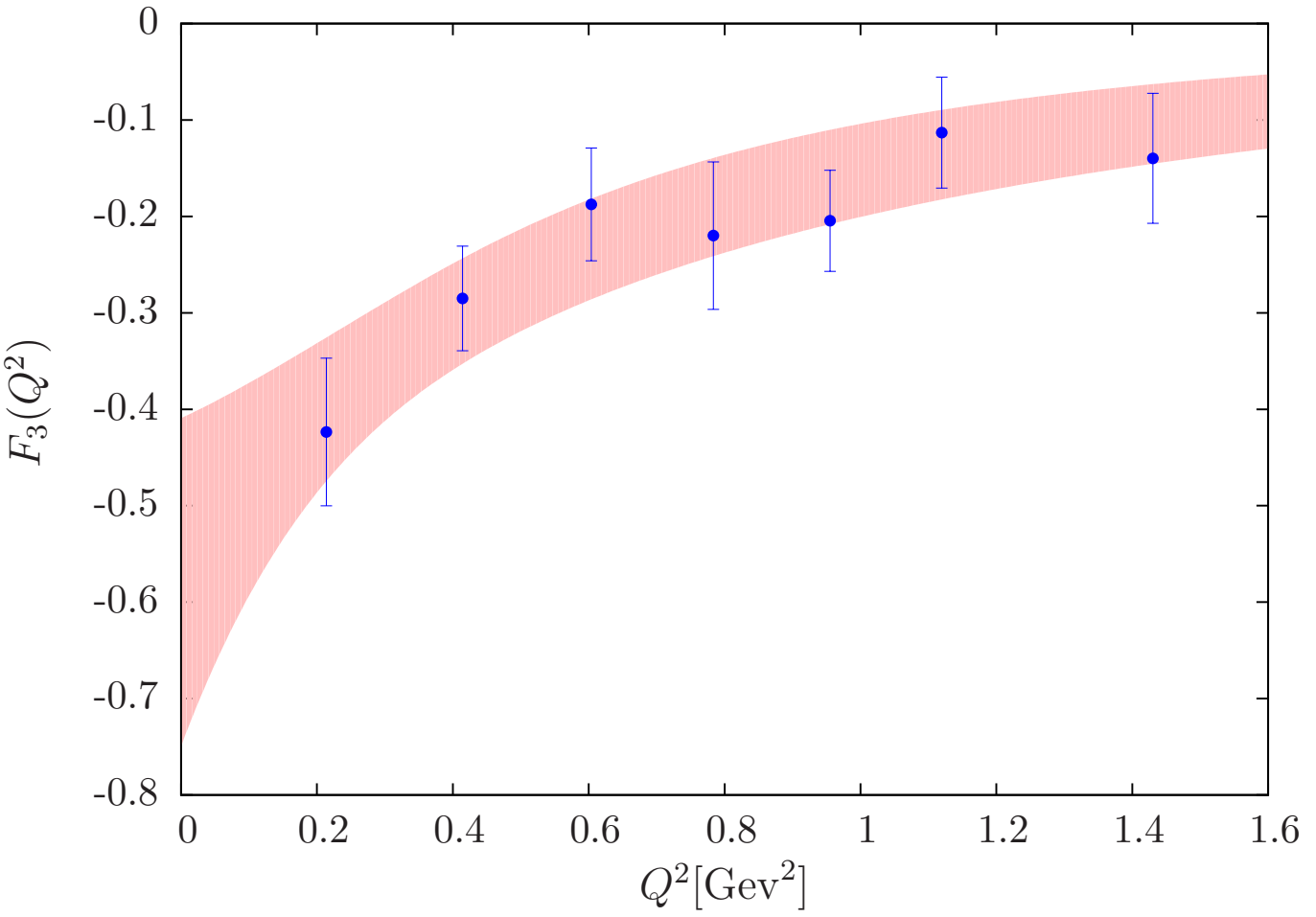
The nEDM: Extraction of $F_3(0)$ via Dipole fit

Example of a plateau for Iwasaki action, gradient flow at $\tau = 6.3$ and $Q^2 \simeq 0.17\text{GeV}^2$:



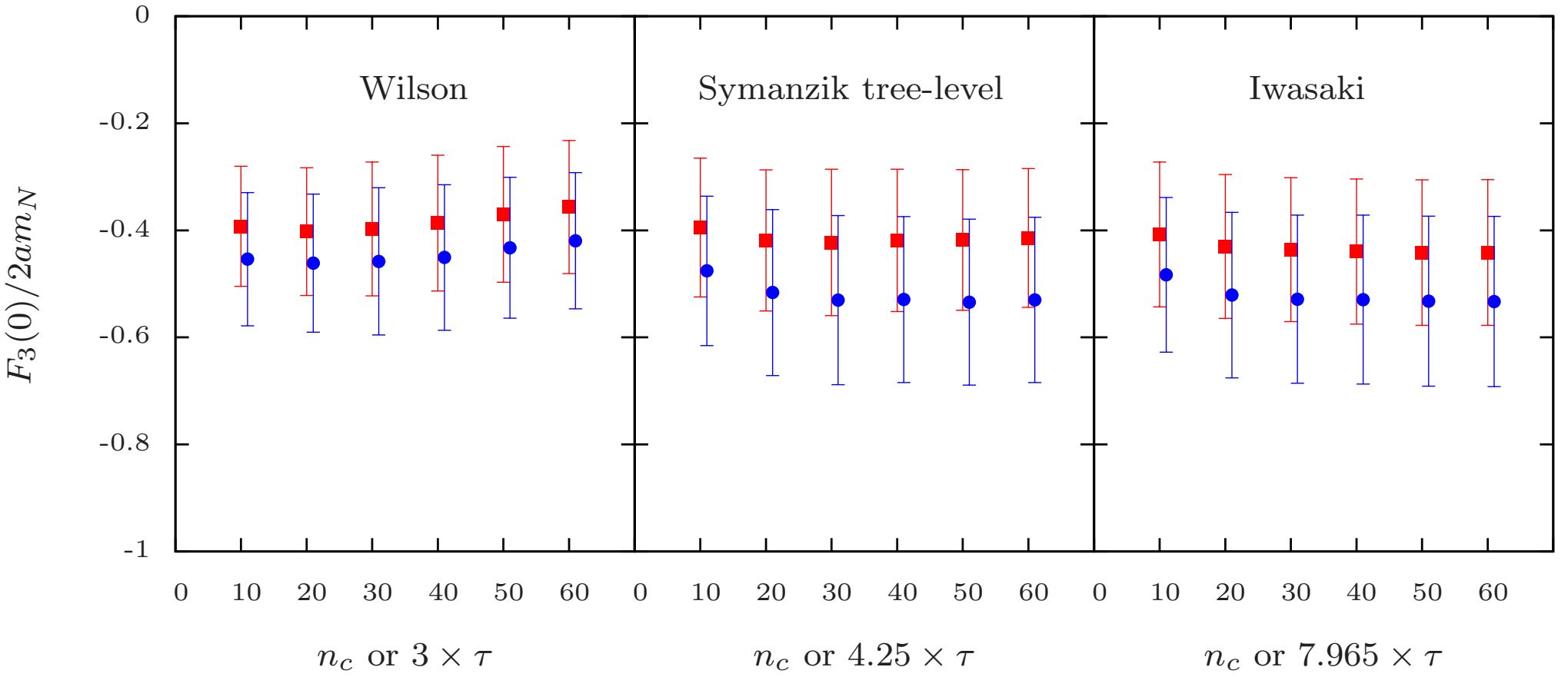
The nEDM: Extraction of $F_3(0)$ via Dipole fit

Example of a dipole fit for Iwasaki action, gradient flow at $\tau = 6.3$ and $Q^2 \leq 1\text{GeV}^2$:



The nEDM: Extraction of $F_3(0)$ via Dipole fit

$F_3(0)/2am_N$ via cooling (n_c) and gradient flow (τ):



The three smoothing actions give consistent results!

The nEDM: Extraction of $F_3(0)$ via Derivative on the ratio

- Assuming continuous momenta one can remove the Q_k dependence in front of $F_3(Q^2)$:

$$\lim_{Q^2\to 0}\frac{\partial}{\partial Q_j}\Pi^k_{F_3}(\vec{Q}) = \sqrt{\frac{E_N+m_N}{8E_N m_N^2}}\,\delta_{kj}F_3(0)\,.$$

- We build ratios such as:

$$\lim_{Q^2\to 0}\frac{\partial}{\partial Q_j}\mathrm{R}^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{q},t_f,t_i,t,\Gamma_k) = \frac{1}{G_{2\mathrm{pt}}(\vec{0},t_f,t_i,\Gamma_0)}\cdot\sum_{x_j=-L/2+a}^{L/2-a}\left(\sum_{\substack{x_i=0\\ i\neq j, i=1,2,3}}^{L-a} i x_j G^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{x},t_f,t_i,t,\Gamma_k)\right).$$

- Requires position space correlators $G^{(0)}_{\mu J_N J^{\mathrm{em}}_\mu \mathcal{Q} J_N}(\vec{x},t_f,t_i,t,\Gamma_k)$
- In finite volume this approximates the derivative of δ-function:

$$a^3\sum_{x_j=-L/2+a}^{L/2-a}\left(\sum_{\substack{x_i=0\\ i\neq j}}^{L-a} i x_j G^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{x},t_f,t_i,t,\Gamma_k)\right) = \frac{1}{V}\sum_{\vec{k}}\left(a^3\sum_{x_j=-L/2+a}^{L/2-a}\left(\sum_{\substack{x_i=0\\ i\neq j}}^{L-a} i x_j \exp\left(i\vec{k}\vec{x}\right)\right)\right) G^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{k},t_f,t_i,t,\Gamma_k)$$

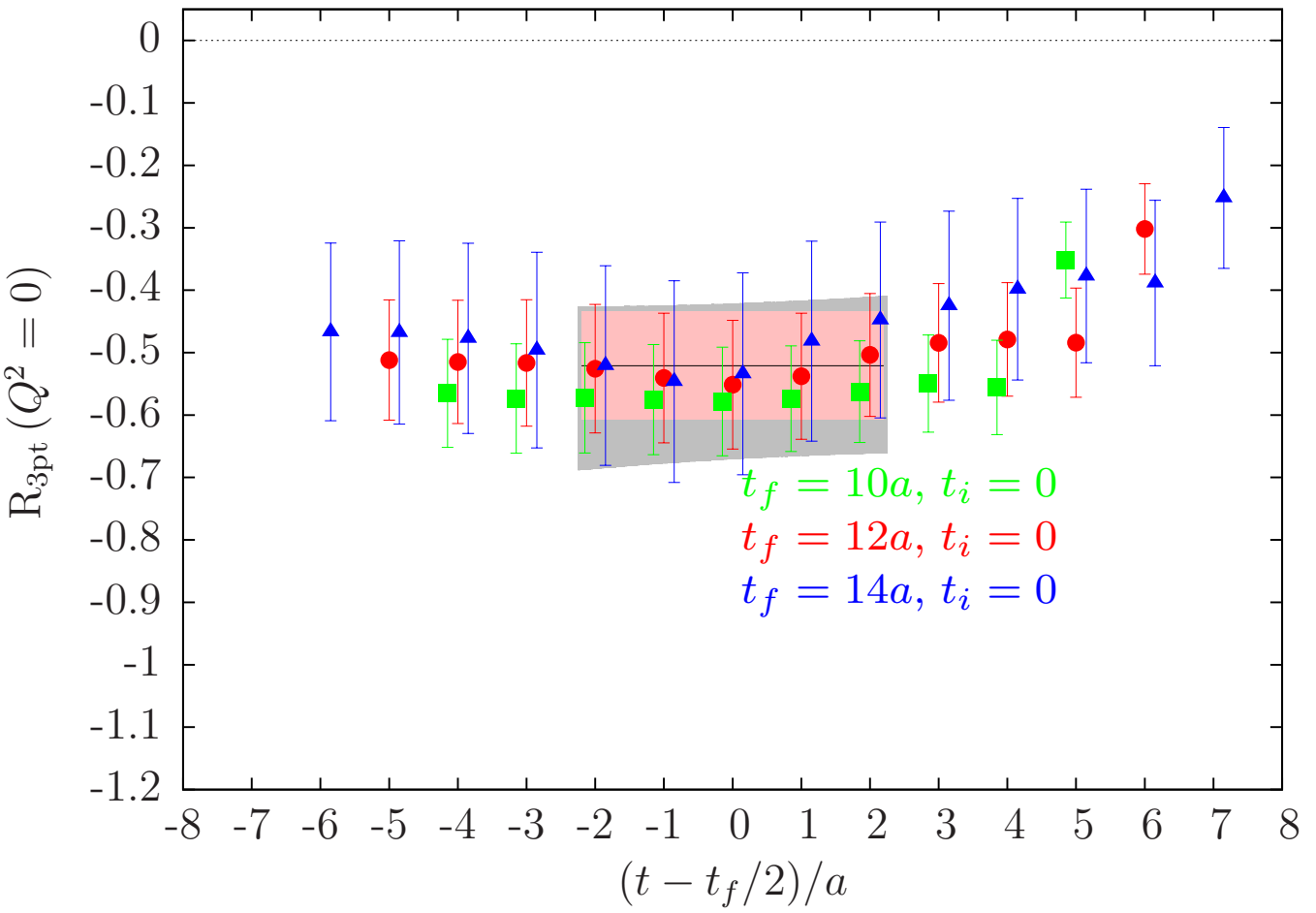
$$\xrightarrow{L\to\infty}\frac{1}{(2\pi)^3}\int d^3\vec{k}\,\frac{\partial}{\partial k_j}\delta^{(3)}(\vec{k})\,G^{\mu}_{3\mathrm{pt},\mathcal{Q}}(\vec{k},t_f,t_i,t,\Gamma_k).$$

- Residual t-dependence $G^{\mu,(0)}_{3\mathrm{pt},\mathcal{Q}}(\vec{q},t_f,t_i,t,\Gamma_k)\sim\exp\left(-\Delta E_N t\right)$, with $\Delta E_N\to 0$ for $L\to\infty$

[W. Wilcox, Phys.Rev. D66 (2002) 017502, [arXiv:hep-lat/0204024]]

The nEDM: Extraction of $F_3(0)$ via Derivative on the ratio

Example of a plateau for Iwasaki action and gradient flow at $\tau = 6.3$:



The nEDM: Extraction of $F_3(0)$ via Momentum Elimination

- Define a suitable ratio such that:

$$\Pi^k_{F_3} = -i\Pi^0_{3\mathrm{pt},\mathcal{Q}}(\Gamma_k) + i\,\alpha^1\,\Pi^k_{3\mathrm{pt}}(\Gamma_0) + \alpha^1\,\frac{1}{2}\sum_{i,j=1}^{3}\epsilon_{jki}\Pi^j_{3\mathrm{pt}}(\Gamma_i) \quad = \quad \sqrt{\frac{E_N+m_N}{8E_N m_N^2}}\,Q_k\,F_3(Q^2)\,.$$

- On-axis momenta e.g. $\vec{q} = (\pm\tilde{Q}, 0, 0)^T$
- Fourier transform on $\Pi(\tilde{Q}) \to \Pi(y)$ in position space; with $(n = y/a)$

$$\Pi(y) = \begin{cases} +\Pi(n), & n = 0, \ldots, N/2 \\ -\Pi(N-n), & n = N/2+1, \ldots, N-1,\ N = L/a \end{cases},$$

- Average over pos. and neg. y we get $\to \overline{\Pi}(n)$
- Finally transform back and introduce continuous momenta k:

$$\Pi(k) = \Big[\exp(ikn)\overline{\Pi}(n)\Big]_{n=0,\,N/2} + 2i\sum_{n=1}^{N/2-1}\overline{\Pi}(n)\sin\left(\frac{k}{2}\cdot(2n)\right)$$

- We define $\hat{k} \equiv 2\sin\left(\frac{k}{2}\right)$ and $P_n\left(\hat{k}^2\right) = P_n\left(\left(2\sin\left(\frac{k}{2}\right)\right)^2\right) = \sin(nk)/\sin\left(\frac{k}{2}\right)$ and obtain:

$$\Pi(\hat{k}) - \Pi(0) = i\sum_{n=1}^{N/2-1}\hat{k}\,P_n\left(\hat{k}^2\right)\overline{\Pi}(n)\,.$$

- $P_n\left(\hat{k}^2\right)$ is related to Chebyshev polynomials of the 2nd kind

The nEDM: Extraction of $F_3(0)$ via Momentum Elimination

- By applying the derivative in respect to $\hat{k}$:

$$\frac{F_3(\hat{k}^2)}{2m_N} = i \sum_{n=1}^{N/2-1} P_n(\hat{k}^2)\,\overline{\Pi}(n)\,,$$

- Generalize to off-axis momentum classes
$M(\tilde{Q}, Q^2_{\rm off}) = \left\{ \vec{q} \mid \vec{q} = \{\pm\tilde{Q}, Q_1, Q_2\}\,,\ Q_1^2 + Q_2^2 = q^2_{\rm off} \right\}$

  where $\{\pm\tilde{Q}, Q_1, Q_2\}$ denotes all possible permutations of $\pm\tilde{Q}$, Q_1 and Q_2

- To combine results for $F_3(Q^2)/(2m_N)$ for different momentum classes $Q_{\rm off}$ and arrive at (Euclidean) $Q^2(\hat{k}, Q^2_{\rm off}) = 0$: Analytic Continuation:

$$\begin{aligned} k &\rightarrow i\kappa \\ \hat{k} &\rightarrow i\hat{\kappa} = -2\sinh\left(\frac{\kappa}{2}\right) \\ P_n(\hat{k}^2) \rightarrow P_n\left(\hat{\kappa}^2\right) &= \sinh(n\kappa)/\sinh\left(\frac{\kappa}{2}\right) \end{aligned}$$

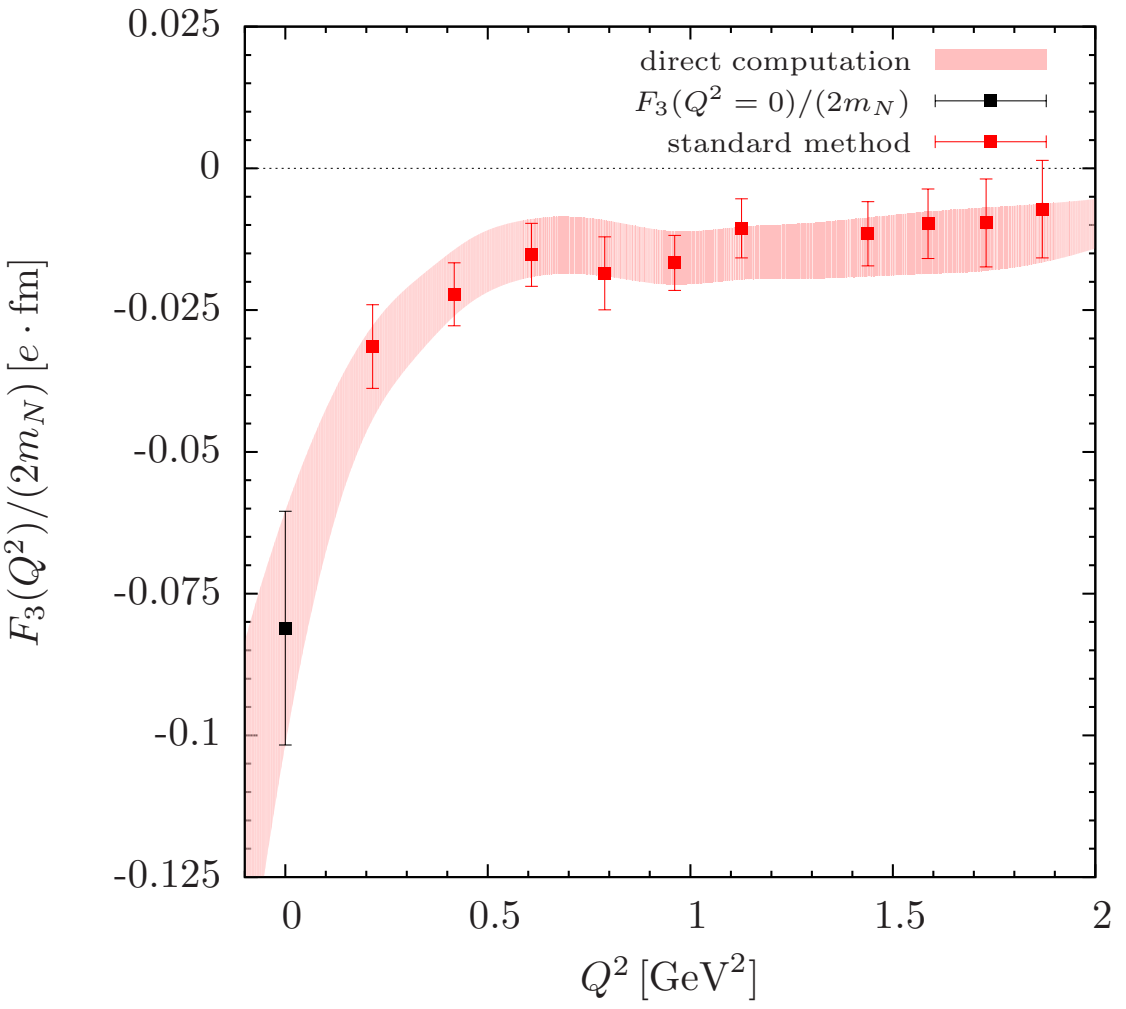
  Final formular takes similar form:

$$\frac{F_3(\hat{\kappa}^2)}{2m_N} = i \sum_{n=1}^{N/2-1} P_n(\hat{\kappa}^2)\,\overline{\Pi}(n)\,,$$

  Combine results from different sets of $M(\tilde{Q}, Q^2_{\rm off})$ by taking the error weighted average.

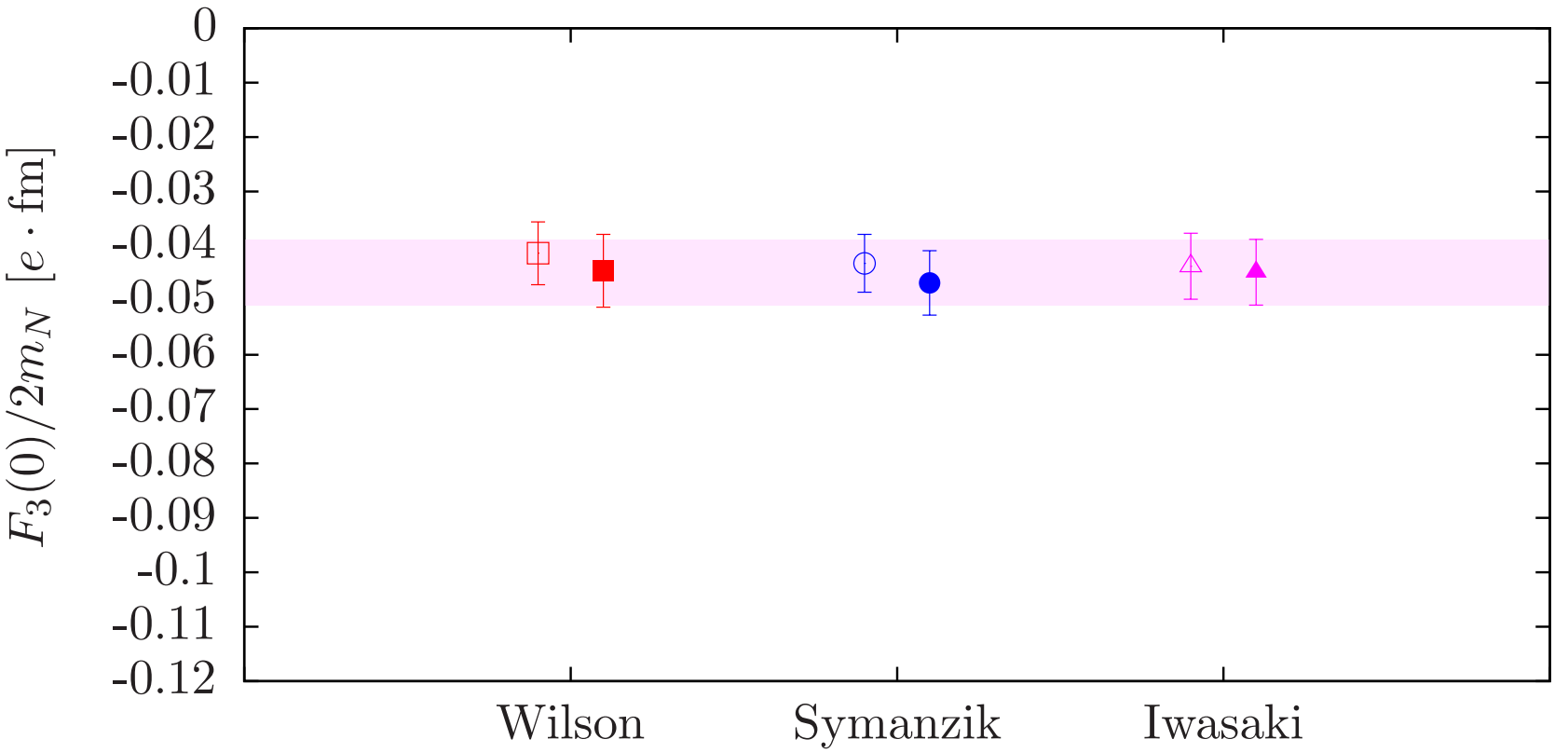
The nEDM: Extraction of $F_3(0)$ via Elimination of momentum

$F_3(Q^2)/2m_N$ via gradient flow ($\tau = 6.3$) with the Iwasaki smoothing action extracted for $Q^2_{\text{offmax}} = 5(2\pi/L)^2$:



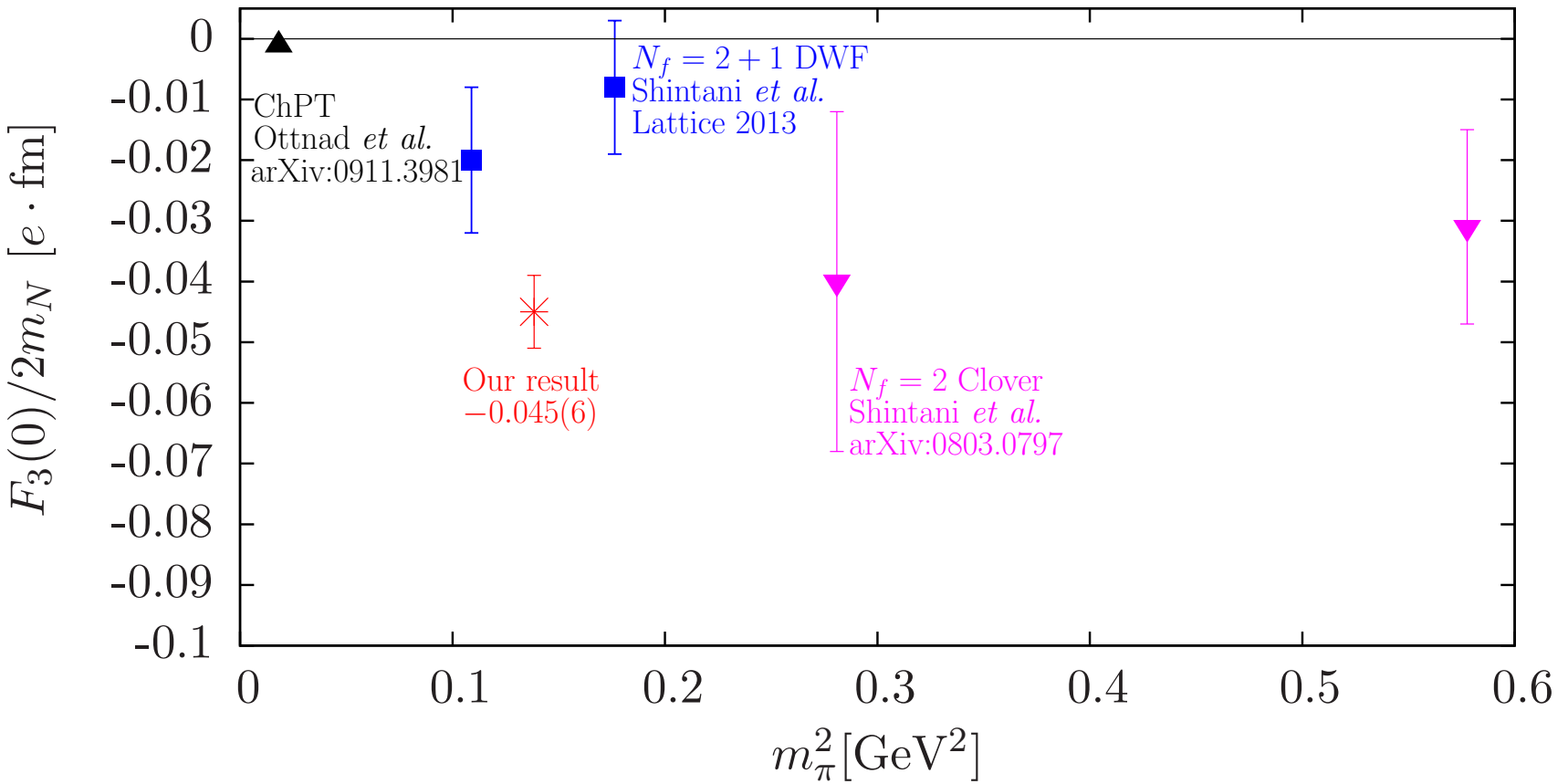
The nEDM: Comparison

Weighted results for the **three different smoothing actions** with $\mathcal{Q}$ extracted via cooling (open symbols) and the gradient flow (closed symbols):



The nEDM: Comparison

Our result (∗) compared to other works:

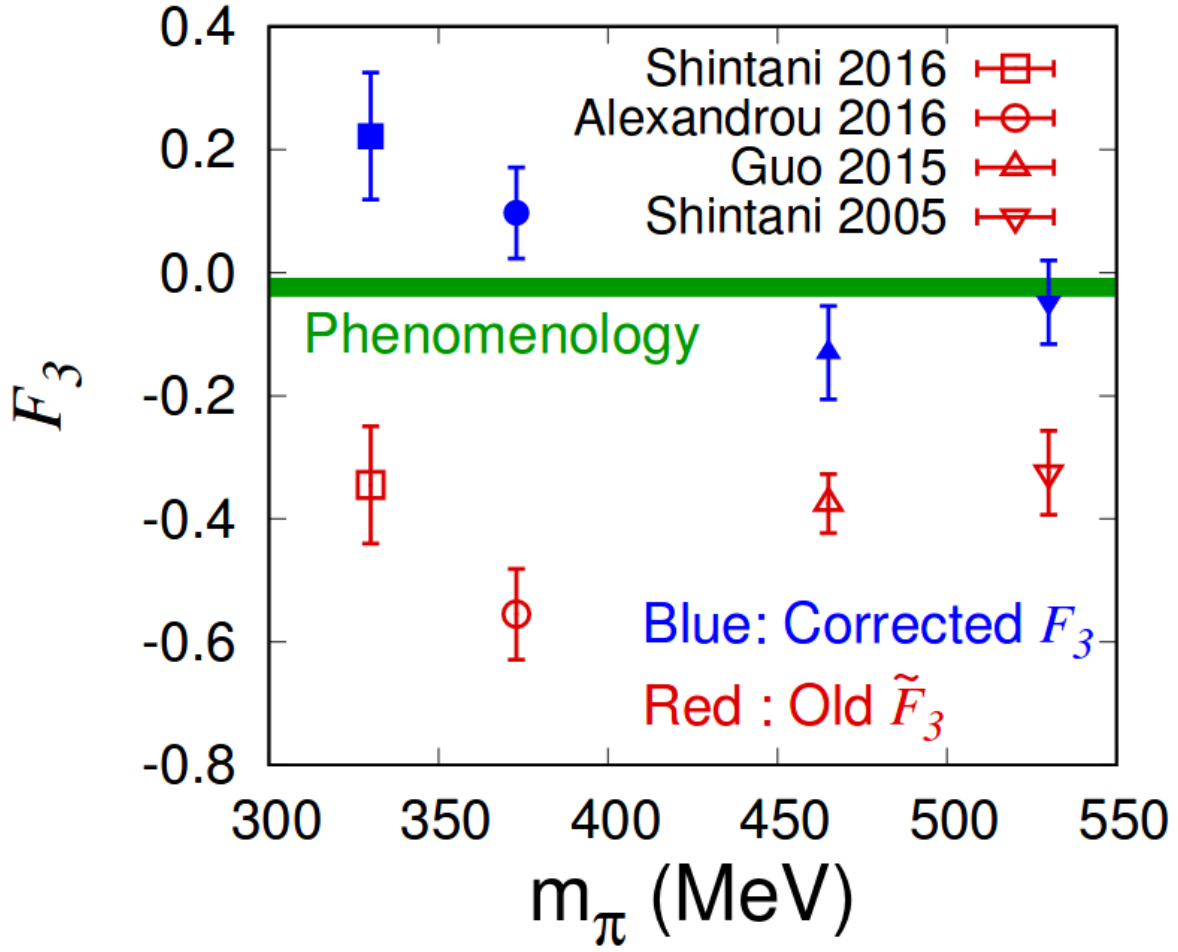


(▲) Ottnad *et. al.* ChPT, [arXiv:0911.3981 [hep-ph]], (■) Shintani *et al*, (▼) Shintani *et al* [arXiv:0803.0797].

The nEDM: Corrected Values

- Without rotation Phase $e^{i\alpha\gamma_5}$ there is mixing of F_2 and F_3. [Abramczyk, *et al*, arXiv:1701.07792]

$$\tilde{F}_2 = \cos(2\alpha)F_2 - \sin(2\alpha)F_3$$
$$\tilde{F}_3 = \sin(2\alpha)F_2 + \cos(2\alpha)F_3$$



The nEDM: What we learn?

- Calculate the nEDM for B55.32 at $M_\pi \simeq 370$ MeV.
- Implemented three momentum dependence treating techniques:
  - Dipole fit.
  - Application of the derivative to the ratio.
  - Elimination of the momentum in the plateau region.
- Implemented two smoothers and demonstrated equivalence:
  - Cooling.
  - Gradient Flow.
- Used three smoothing actions:
  - Wilson.
  - Symanzik tree-level improved.
  - Iwasaki.
- All combinations give similar results!
- Our result agrees with older estimations!
- We are looking forward to the **physical point − continuum limit**.