

Gluon propagator and vertex from lattice QCD: a pathway to the instanton liquid parameters?

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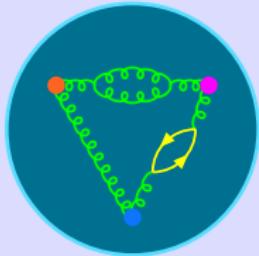
Motivation

QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_{q=u,d,\dots} \underbrace{\bar{q}(i\gamma^\mu D_\mu - m_q)q}_{\text{quarks}} - \underbrace{\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}}_{\text{gluons}}$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_0 f_{bc}^a A_\mu^b A_\nu^c$$

$$D_\mu = \partial_\mu + i g_0 A_\mu^a \frac{\lambda_a}{2}$$



Perturbative methods not useful for low-energy QCD: confinement and chiral symmetry breaking.

Non-perturbative techniques and phenomenologic approaches:

- Lattice QCD
- Schwinger-Dyson Equations
- Operator Product Expansion
- SVZ sum rules
- Semiclassical approximations

Motivation

Classical solutions of \mathcal{L}_{QCD}

$$S = \frac{1}{2} \int d^4x \ G_{\mu\nu}^2 \geqslant 0 ; \quad \frac{\delta S}{\delta G_\nu} = D_\mu G_{\mu\nu} = 0 \quad \rightarrow \quad A_I$$

- U(1) axial anomaly

$$\partial_\mu J_{5,\mu}(x) = \frac{g^2 N_F}{8\pi^2} \text{Tr}(G_{\mu\nu} \tilde{G}_{\mu\nu})$$

- Dirac operator zero modes

$$n_R(\psi_0); n_L(\gamma_5 \psi_0)$$

- Topological charge

$$Q = n_L - n_R$$

Motivation

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- Genuinely non-perturbative
- QCD vacuum
- Condensates
- SB χ S?
- Deep inelastic scattering?
- Mass generation?
- Confinement?
- Gluon Green functions?

[Rev. Mod. Phys. 70, (1998) 323]

Instanton gauge field

Only exact solution [PLB59 (1975) 85]:

$$A_I \equiv A_\mu^a(x) = 2\bar{\eta}_{\mu\nu}^a \frac{x_\nu}{x^2} \phi\left(\frac{x}{\rho}\right)$$

with

$$\phi_{BPST}(z) = \frac{1}{1+z^2}$$

- Multi-instanton solutions are constructed by the sum of those solutions.

$$A = \sum A_I$$

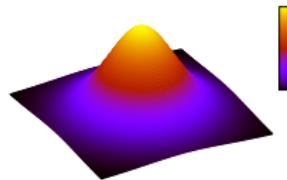
- The profile ϕ is modified at large distances due to instanton *interactions* [NPB245 (1984) 259].

Finite action

$$S_I = \int d^4x S(x) = \frac{8\pi^2}{g^2}$$

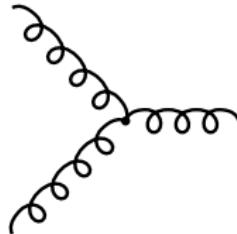
topological

$$Q = \int \partial_\mu K_\mu = \pm 1$$



Computing α_{MOM} from the lattice

I Landau gauge fixing $\partial_\mu A_\mu = 0$



II Gluon propagator

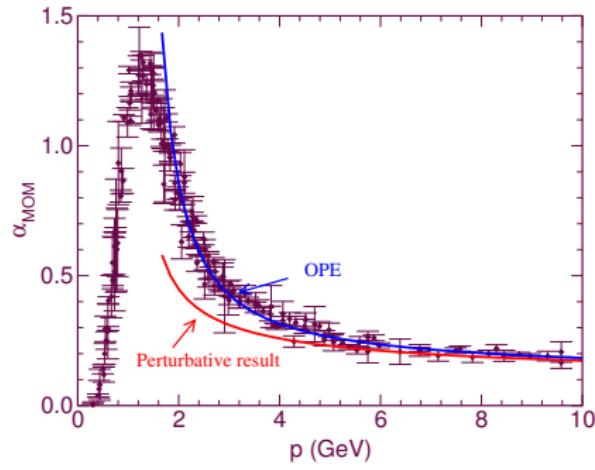
$$\langle \tilde{A}_\mu^a(k) \tilde{A}_\mu^a(-k) \rangle \rightarrow G^{(2)}(k^2)$$

III Three gluon vertex

$$\langle \tilde{A}_\mu^a(k_1) \tilde{A}_\nu^b(k_2) \tilde{A}_\rho^b(k_3) \rangle_{k_1^2 = k_2^2 = k_3^2} \rightarrow G^{(3)}(k^2)$$

IV The coupling is defined by the ratio:

$$\alpha_{\text{MOM}}(k^2) = \frac{1}{4\pi} \frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3}$$

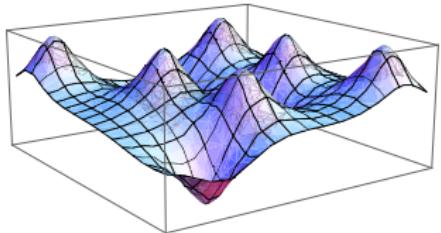


Gluon Green functions in an instanton background

| Independent pseudo-particle sum ansatz:

$$A_\mu^a(x) = \sum_i 2R_i^{a\alpha} \bar{\eta}_{\mu\nu}^a \frac{x_\nu - z_\nu^i}{(x - z^i)^2} \phi\left(\frac{x - z^i}{\rho_i}\right)$$

in $\langle A^2 \rangle_{min}$ Landau gauge.

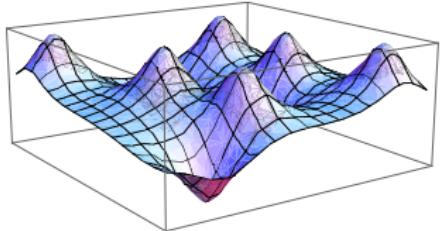


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II Fourier transform

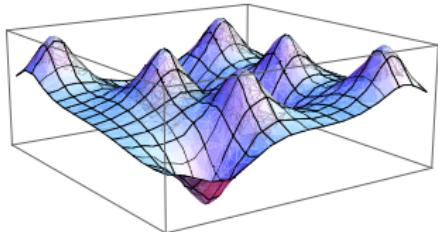
$$\mathbf{A}_\mu^a(x)[\phi(z)] \rightarrow \tilde{\mathbf{A}}_\mu^a(k)[I(k\rho)] ; \quad I(s) = \frac{8\pi^2}{s} \int_0^\infty dz z J_2(sz) \phi(z)$$

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III Neglecting instanton correlations [PRD70, (2004) 114503]

$$\mathbf{G}^{(m)}(k^2) \sim n k^{2-m} \langle \rho^{3m} I^m(k\rho) \rangle$$

with $\langle \dots \rangle$ the average over instanton sizes.

α_{MOM} in an instanton background

The ratio of Green functions defining $\alpha_{\text{MOM}}(k)$ is then:

$$\alpha_{\text{MOM}}(k^2) = \frac{1}{4\pi} \frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3}$$

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Instanton detector

In an instanton background $\alpha_{\text{MOM}}(k)$ should exhibit a $\sim k^4$ behavior sensitive to the instanton density n [JHEP 04 (2003) 005].

- This result is independent of the instanton profile function ϕ / I
- For small k depends on the width of the instanton distribution, $\delta\rho/\rho$

Results in SU(3) pure gauge theory

Lattice setup's for $N_f = 0$:

- Tree-level Symanzik

β	a	V	confs.	$\alpha_{\text{MOM}}(k)$
3.8	0.285fm	$(13.7\text{fm})^4$	1050	
3.9	0.243fm	$(15.6\text{fm})^4$	2000	
4.2	0.141fm	$(4.5\text{fm})^4$	420	

- Wilson

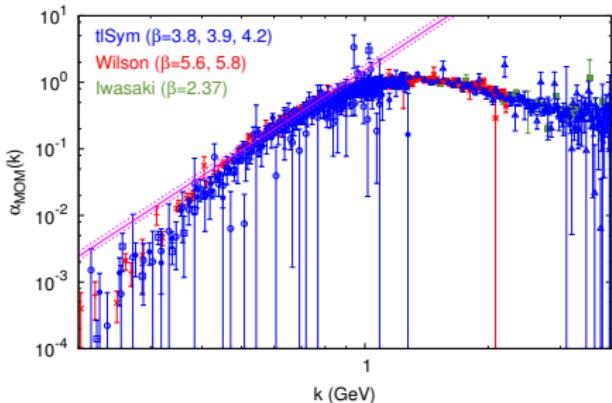
5.8	0.140fm	$(6.72\text{fm})^4$	960
5.6	0.235fm	$(11.3\text{fm})^4$	1920
5.6	0.235fm	$(12.3\text{fm})^4$	1790

- Iwasaki

$$2.37 \quad 0.140\text{fm} \quad 2.8^3 \times 5.6\text{fm}^4$$

Details for the lattice setup in:

[PLB 760 (2016) 354]



Instanton prediction $\alpha_{\text{MOM}}(k) \approx ck^4 / 18\pi n$
works for $k \in (0.3, 0.9)\text{GeV}^{-1}$

$$n \approx 12(1)\text{fm}^{-4}$$

(Assuming $c \approx 1.6$, from $\frac{\delta\rho^2}{(\rho)^2} \approx 0.014$
[PRD58 (1998) 014505])

Achievement

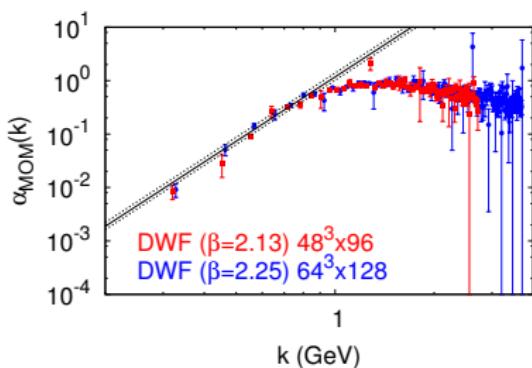
This method can be applied to compute instanton density without applying any smoothing technique, and also for $N_F \neq 0$ (JHEP 02 (2018) 140)

Unquenched lattice-QCD results

Achievement

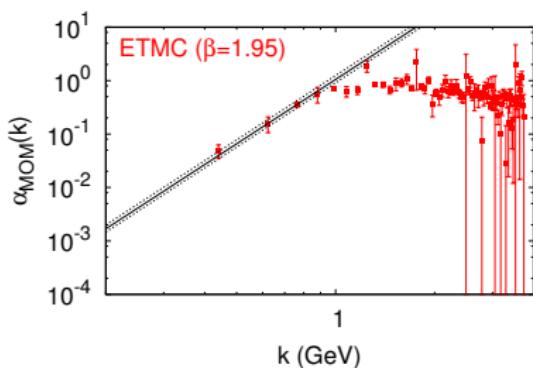
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DWF ($N_F = 2 + 1$, RBC & UKQCD)



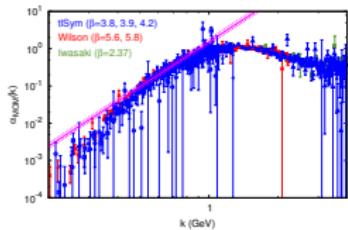
$$\frac{n_{N_F=2+1}}{n_{N_F=0}} = 1.3(1)$$

tmQCD ($N_F = 2 + 1 + 1$, ETMC)

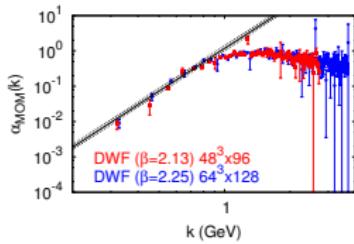


$$\frac{n_{N_F=2+1+1}}{n_{N_F=0}} = 1.5(1)$$

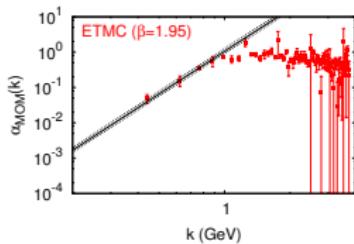
Instanton density from IR running: *resumé*



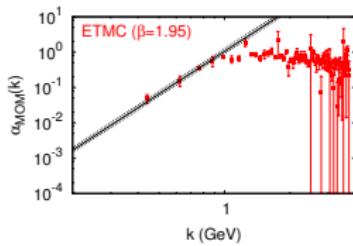
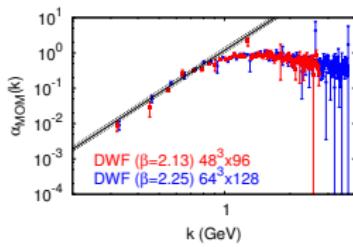
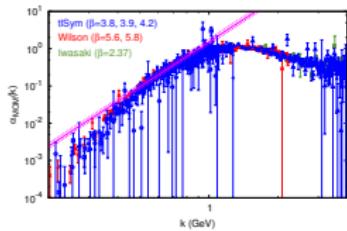
- Instanton k^4 law works for $0.3 \text{ GeV} \lesssim k \lesssim 1 \text{ GeV}$



- Rather large instanton density $\sim 10 \text{ fm}^{-4}$
- $\sim 40\%$ density increase due to dynamical quarks



Instanton density from IR running: *resumé*



- Instanton k^4 law works for $0.3\text{GeV} \lesssim k \lesssim 1\text{GeV}$
- Rather large instanton density $\sim 10\text{fm}^{-4}$
- $\sim 40\%$ density increase due to dynamical quarks
- What if we eliminated quantum fluctuations?

Smoothing the lattice

Topological charge density (2D cut)

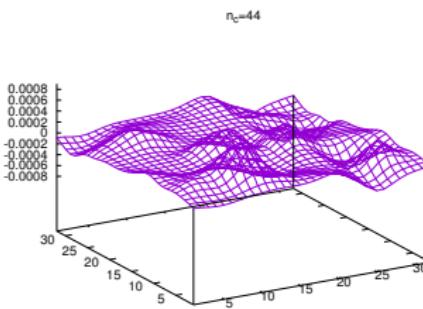
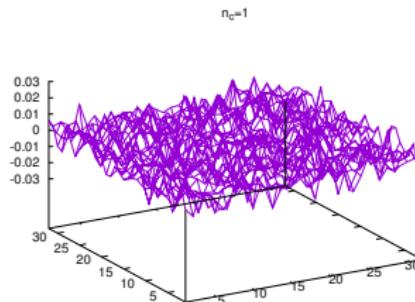
Instantons have been traditionally studied from lattice QCD via the removal of short-range (UV) fluctuations...

UV removal

Different methods have been used:

- Cooling
- Gradient flow
- APE spearing
- HYP smearing
- STOUT smearing

(c.f. Athenodorou's talk)



Cooling, Wilson Flow and smearing techniques

Wilson Flow

- Flow Equation:

$$\partial_\tau B_\mu(\tau, x) = D_\nu G_{\mu\nu}(\tau, x) ,$$

with $B_\mu(0, x) = A_\mu(x)$

- The flowed fields are given by:

$$B_\mu(\tau, x) = \int d^4y \frac{e^{-\frac{(x-y)^2}{4\tau}}}{(4\pi\tau)^2} A_\mu(y)$$

- Has a smoothing effect of radius $\sqrt{8\tau}$
[JHEP 08 (2010) 071]
- Flow time usually expressed in terms of t_0 defined by $\sqrt{8t_0} = 0.3fm$

Cooling

- Updates gauge fields U by U^{new} minimizing the action.

Smearing

- The different types of smearing create *fat links* from the original ones and surrounding staples.

They provide similar topological properties [C. Alexandrou *et al.* PRD92, 125014 (2015) & arXiv:1708.00696]

Equivalence of smoothing techniques...

It has been shown that the smoothing techniques are equivalent for some observables [C. Alexandrou, arxiv:1708.00696].

$$\tau_{\text{WF}} = \frac{n_c}{3} = \frac{\alpha_{\text{APE}}}{6} n_{\text{APE}} = \rho_{\text{ST}} n_{\text{ST}}$$

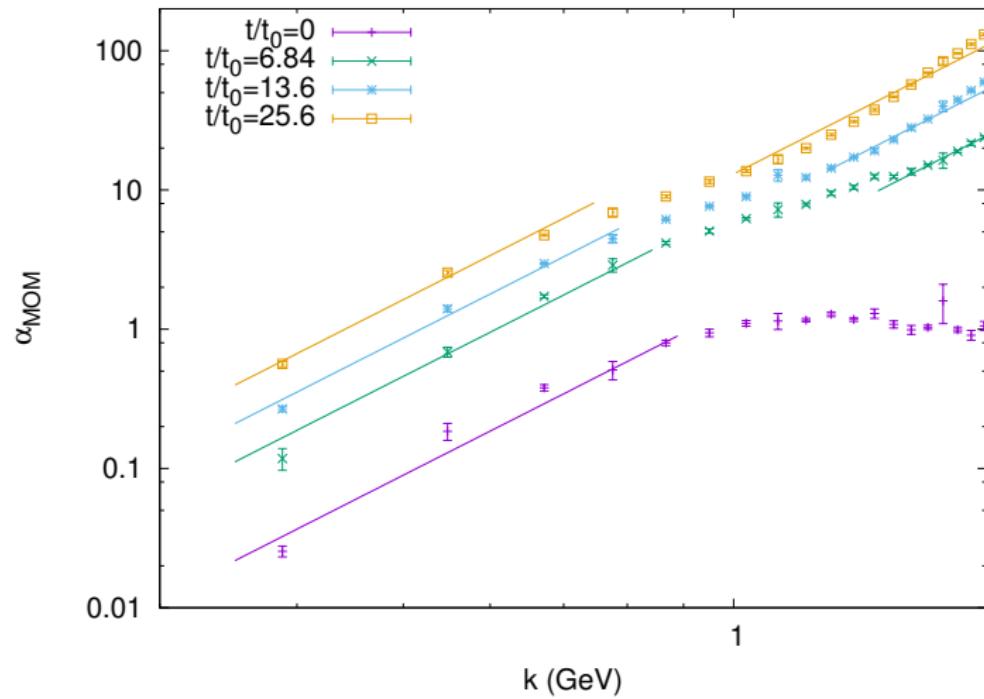
We have chosen WF for $\tau = 4, 8$ and 15 and the equivalent number of steps for each smearing technique:

WF	Cooling	APE ($\alpha = 0.5$)	STOUT ($\rho = 0.05$)
$\tau = 4$ ($t/t_0 = 6.8$)	$n_c = 12$	$n_c = 48$	$n_{st} = 80$
$\tau = 8$ ($t/t_0 = 13.6$)	$n_c = 24$	$n_c = 96$	$n_{st} = 160$
$\tau = 15$ ($t/t_0 = 25.6$)	$n_c = 45$	$n_c = 180$	$n_{st} = 300$

(c.f. A. Athenodorou @ this conference)

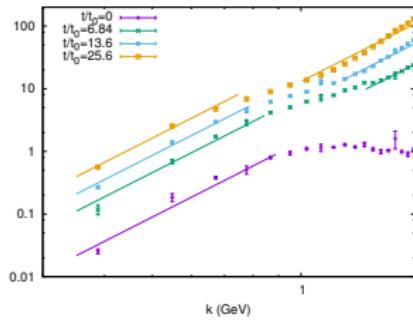
$\alpha_{\text{MOM}}(k)$ after Wilson Flow

Asymptotic freedom disappears and k^4 -law emerges [PLB760 (2016) 354]



$\alpha_{\text{MOM}}(k)$ after Wilson Flow

α_{MOM}



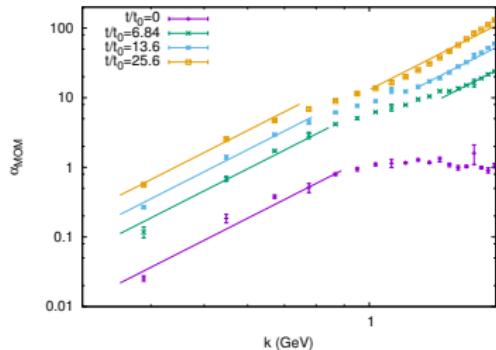
- Different slopes at small and large k are associated to the instanton distribution.

$$\alpha_{\text{MOM}}(k^2) = \frac{k^4}{18\pi n} \times \begin{cases} 1 + 48 \frac{\delta\rho^2}{\langle\rho\rangle^2} & k \ll \rho^{-1} \\ 1 & k \gg \rho^{-1} \end{cases}$$

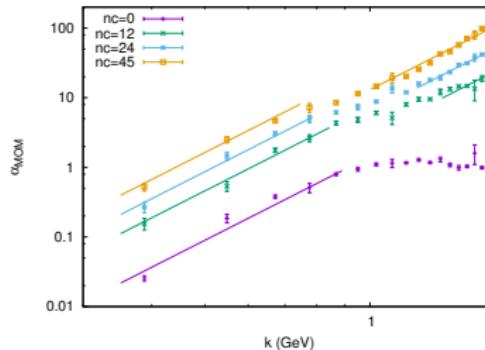
- At large momenta we can obtain instanton density n .
- At small momenta the slope is a factor $c = 1 + 48 \frac{\delta\rho^2}{\langle\rho\rangle^2}$ times larger.
- If $\frac{\delta\rho^2}{\langle\rho\rangle^2} \approx 0.014$ [PRD58 (1998) 014505], $c \approx 1.6$ and $n = 12(1)\text{fm}^{-4}$

Comparison with cooling and smearing techniques

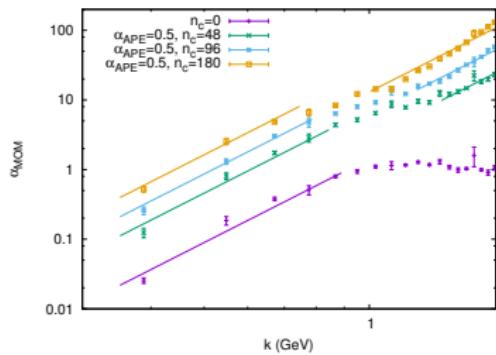
Wilson Flow



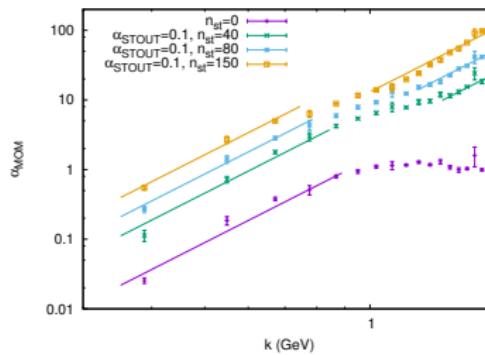
Cooling



APE smearing

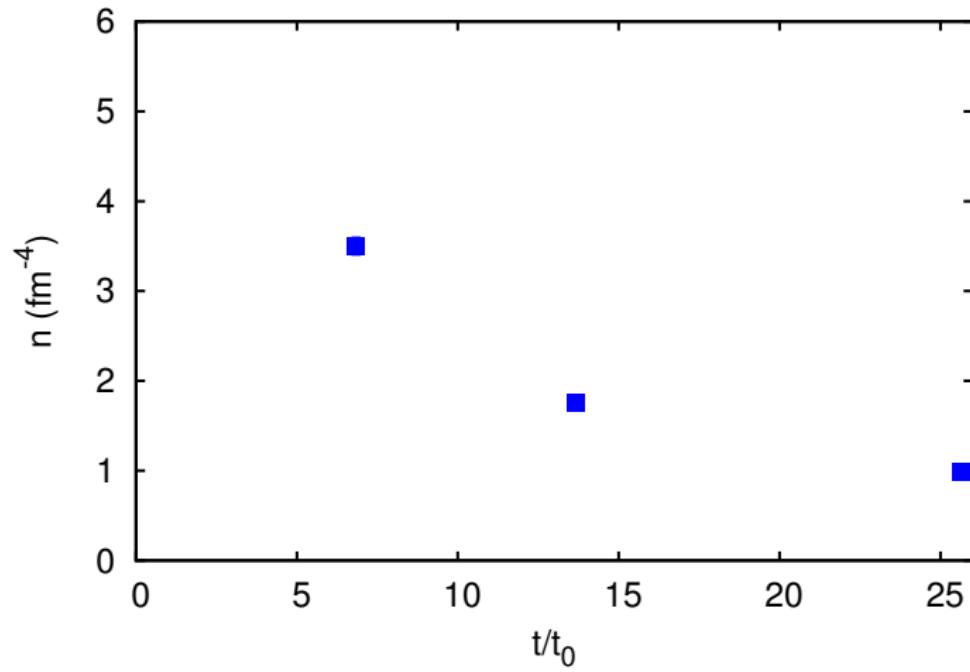


STOUT smearing



Instanton density strongly depends on the smoothing

The different slopes correspond to different instanton densities, i.e., the density decreases with the smoothing...



The same is obtained with equivalent cooling and APE & STOUT smearing

α_{MOM} and individual Green functions...

If the profile $\phi(x)$ is a BPST one (or when $k \gg \rho^{-1}$):

$$\mathbf{G}^{(m)}(\mathbf{k}^2) \sim \mathbf{n} \ k^{2-m} \langle \rho^{3m} \mathbf{I}^m(\mathbf{k}\rho) \rangle$$

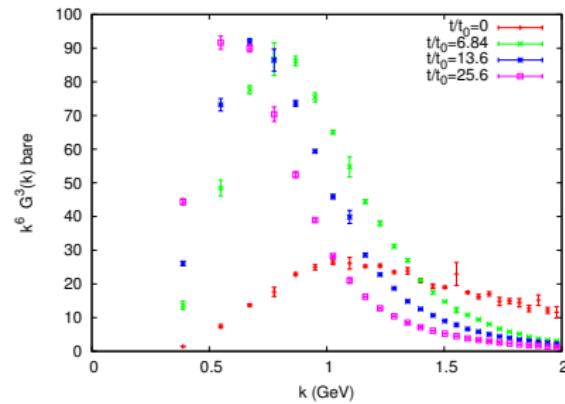
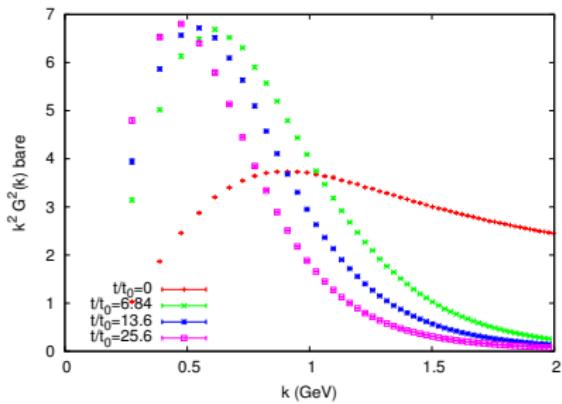
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$$\mathbf{G}^{(m)}(k^2) \sim \textcolor{red}{n} k^{2-m} \langle \rho^{3m} I^m(k\rho) \rangle \quad \sim \quad \begin{cases} k^2 G^{(2)}(k^2) \sim nk^{-4} \\ k^6 G^{(3)}(k^2) \sim nk^{-4} \end{cases}$$

[Phys. Rev. D70 (2004) 114503]

...



Locating instantons after smoothing

Wilson flow / cooling / smearing eliminate short-distance fluctuations

Topological charge density

For BPST instantons,

$$Q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma}$$

behave as:

$$Q(x) = \frac{6}{\pi^2 \rho^4} \left(\frac{\rho^2}{x^2 + \rho^2} \right)^4$$

For modified profiles this shape

remains unchanged for small x

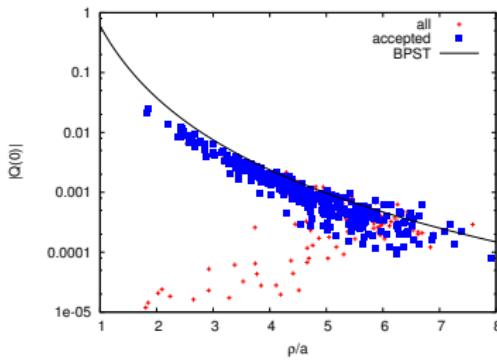
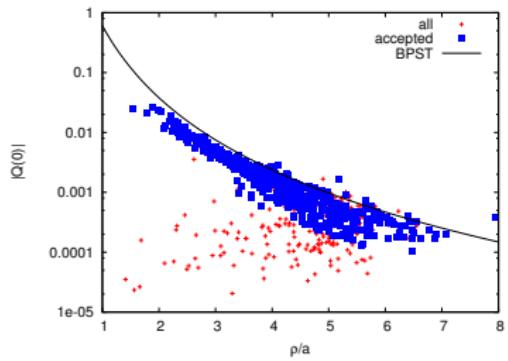
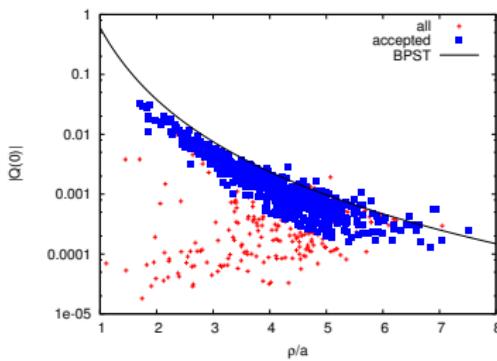
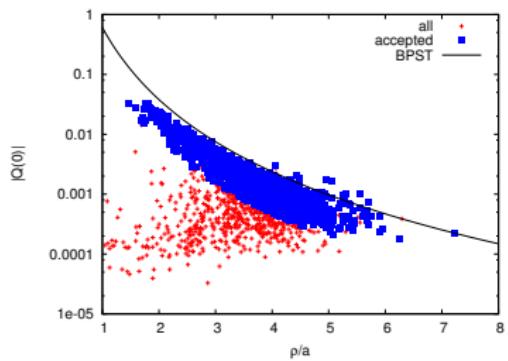
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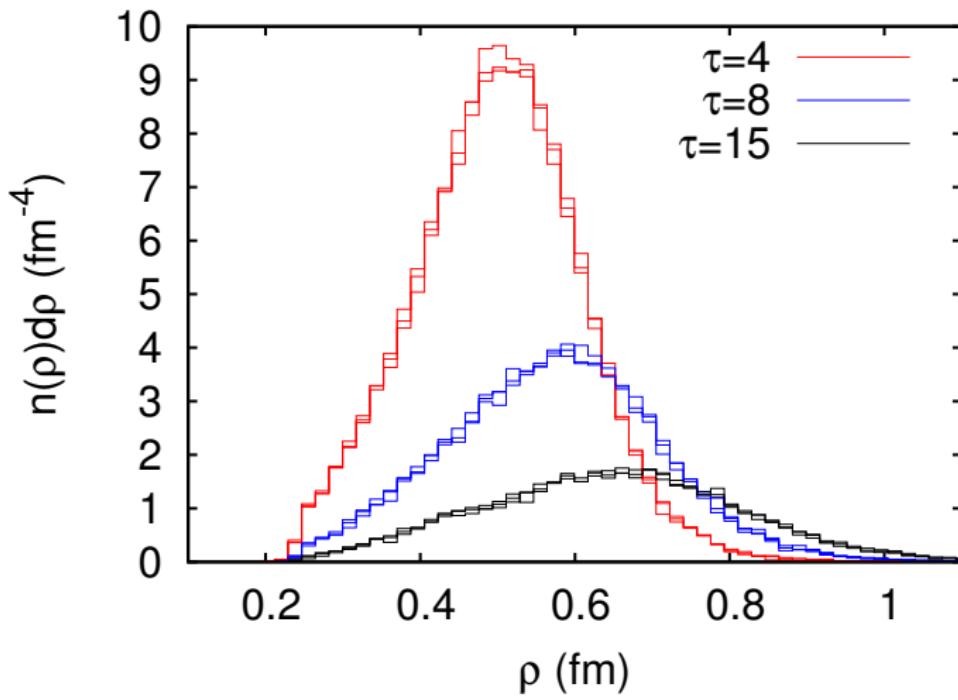
- Search for local extremes
 $x/Q(x) > Q(x \pm \hat{\mu})$
- Fit $Q(x \pm \hat{\mu})$ to BPST (like PRD 88 (2013) 034501)
- Check self-duality
- Discard close pairs when
 $r_{ij}^2 < \rho_i \rho_j$

Results (instanton localization)

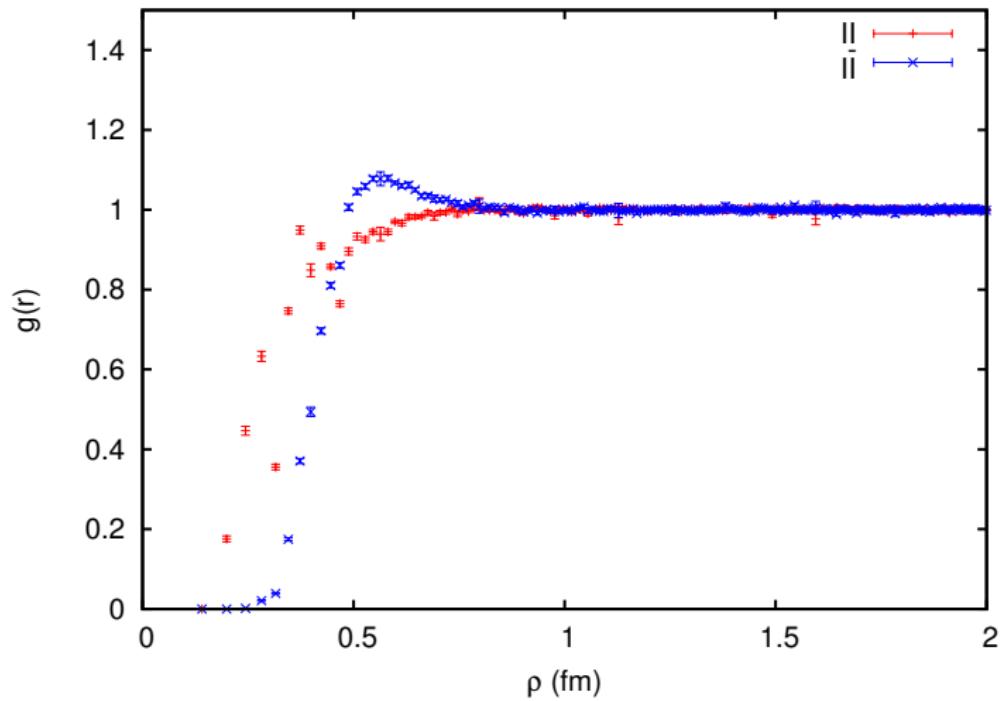
tISym 32^4 , $\beta = 4.2$ at $\tau = 4, 8, 10$ and 15



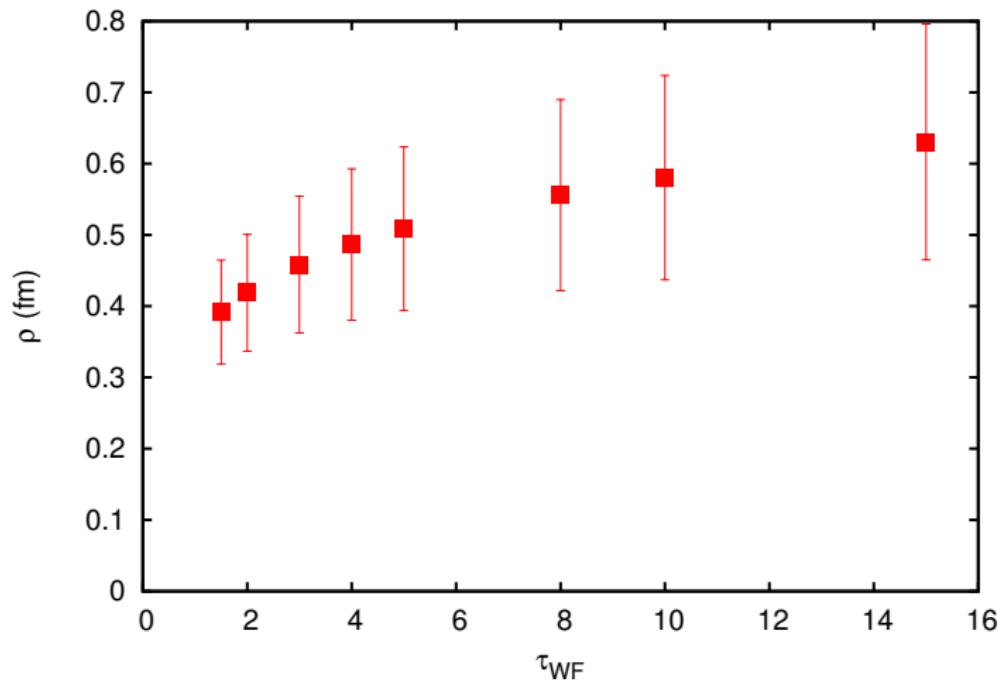
Instanton sizes (tlSym 32^4 , $\beta = 4.2$)



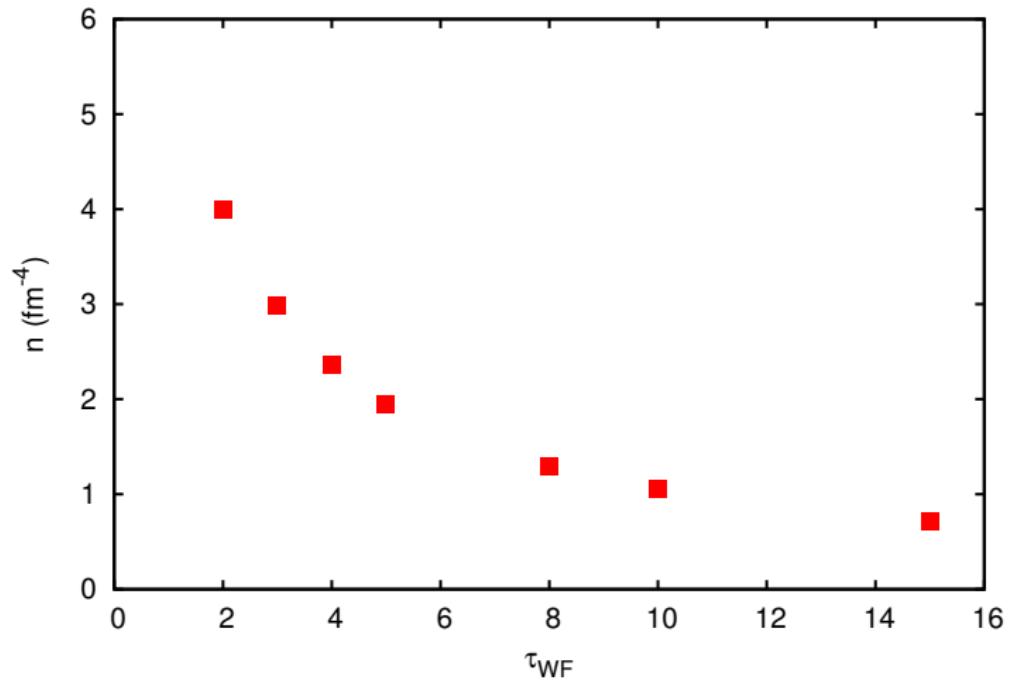
Radial distribution function ($t\text{ISym } 32^4$, $\beta = 4.2$)



Instanton sizes (tlSym 32^4 , $\beta = 4.2$)

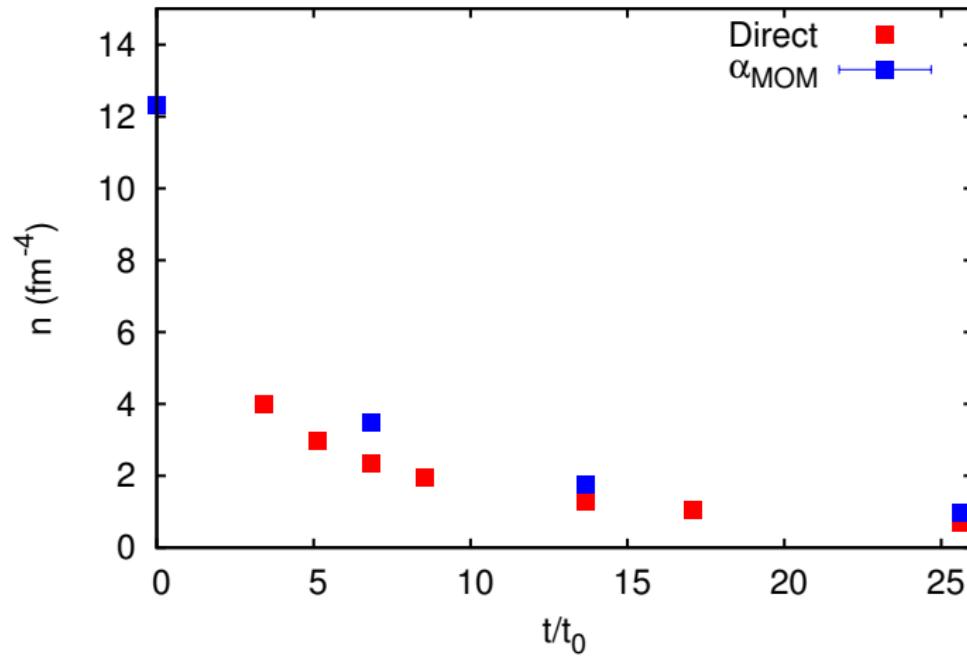


Instanton density (tlSym 32^4 , $\beta = 4.2$)

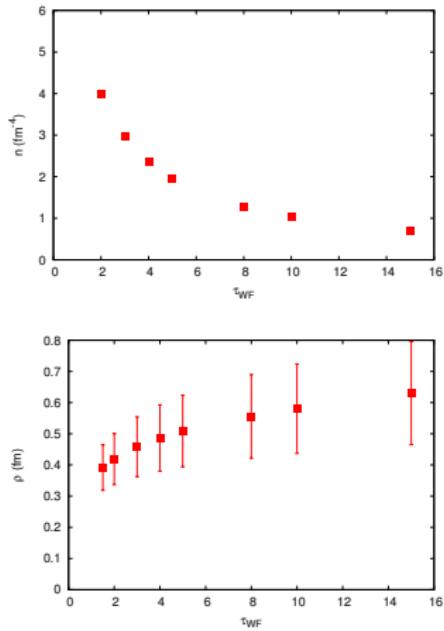


Direct counting vs $\alpha_{\text{MOM}}(k^2)$ (tlSym 32^4 , $\beta = 4.2$)

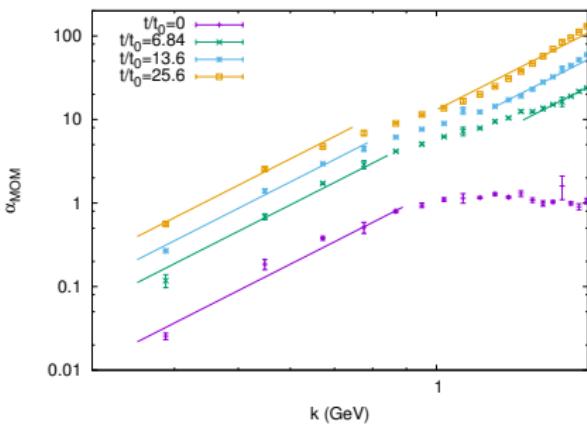
We can compare the densities extracted from direct detection and α_{MOM} :



Direct counting vs $\alpha_{\text{MOM}}(k^2)$ (tlSym 32^4 , $\beta = 4.2$)



The evolution of the Instanton Liquid parameters can be observed both from $\alpha_{\text{MOM}}(k)$ and the direct localisation.



Annihilation model

If instanton annihilation is a first order process:

$$\frac{dn_I}{d\tau} = \frac{dn_A}{d\tau} = -\lambda n_I n_A$$

and

$$n = n_I + n_A \gg q = n_I - n_A,$$

$$\frac{dn}{d\tau} \approx -\frac{\lambda}{2} n^2$$

whose solution for $\lambda = cte$ is

$$\frac{1}{n(\tau)} = \frac{1}{n(0)} + \frac{1}{2} \lambda \tau$$

Instanton evolution with smoothing

Annihilation model

If instanton annihilation is a first order process:

$$\frac{dn_I}{d\tau} = \frac{dn_A}{d\tau} = -\lambda n_I n_A$$

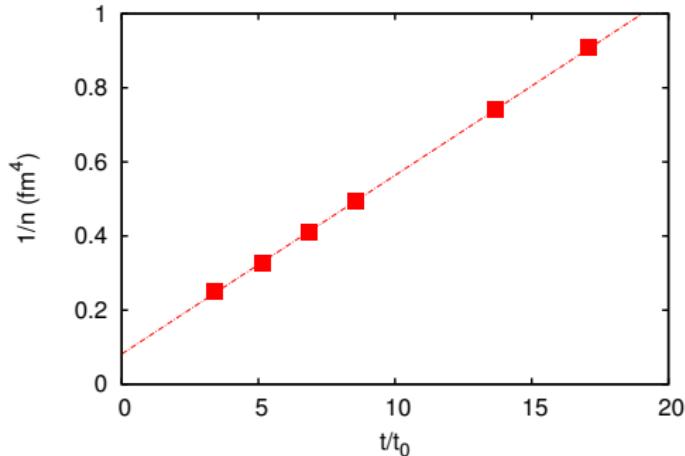
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$$n = n_I + n_A \gg q = n_I - n_A,$$

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whose solution for $\lambda = cte$ is

$$\frac{1}{n(\tau)} = \frac{1}{n(0)} + \frac{1}{2} \lambda \tau$$



$$n \sim 12 \text{ fm}^{-4}$$

Conclusions

- Instantons explain the running of α_{MOM} in the IR
- Instanton density increases with N_F
- After smoothing, instantons dominate also the running at large momenta (Λ_{QCD} disappeared!)
- Instanton density n and size ρ evolve with the smoothing
- A simple model for IA annihilation allows to understand the evolution.