Topology and Chiral Symmetry Breaking at Nonzero θ Angle

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FAR BEYOND

Acknowledgments

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Relevant Papers

J.C. Osborn K. Splittorff and J. J. M. Verbaarschot, Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential, Phys. Rev. Lett. 94 (2005) 202001 [arXiv[hep-th/0501210]].

- J.J.M. Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, Phys. Rev. D90 (2014) 070 [arXiv:1410.0883[hep-lat]].

P.H. Damgaard, Topology and the Dirac Operator Spectrum in Finite Volume Gauge Theory, Nucl. Phys. B556 327 (1999).

H. Leutwyler and A. Smilga, Spectrum of Dirac Operator and Role of Winding Number in QCD, Phys. Rev. D 46 (1992) 5607.

M. Creutz, One Flavor QCD, Ann. Phys. 322 (2007) 1518 [arXiv[hep-th/0609187]].

- J.J.M. Verbaarschot and T. Wettig, The Chiral Condensate of One-Flavor QCD and the Dirac Spectrum at $\theta = 0$, PoS LATTICE2014 (2014) 072 [arXiv:1412.5483 [hep-lat]].

- M.K. Kieburg, J.J.M. Verbaarschot and T. Wettig, The Dirac Spectrum and the θ Dependence of the Chiral Condensate (2018), in preparation.

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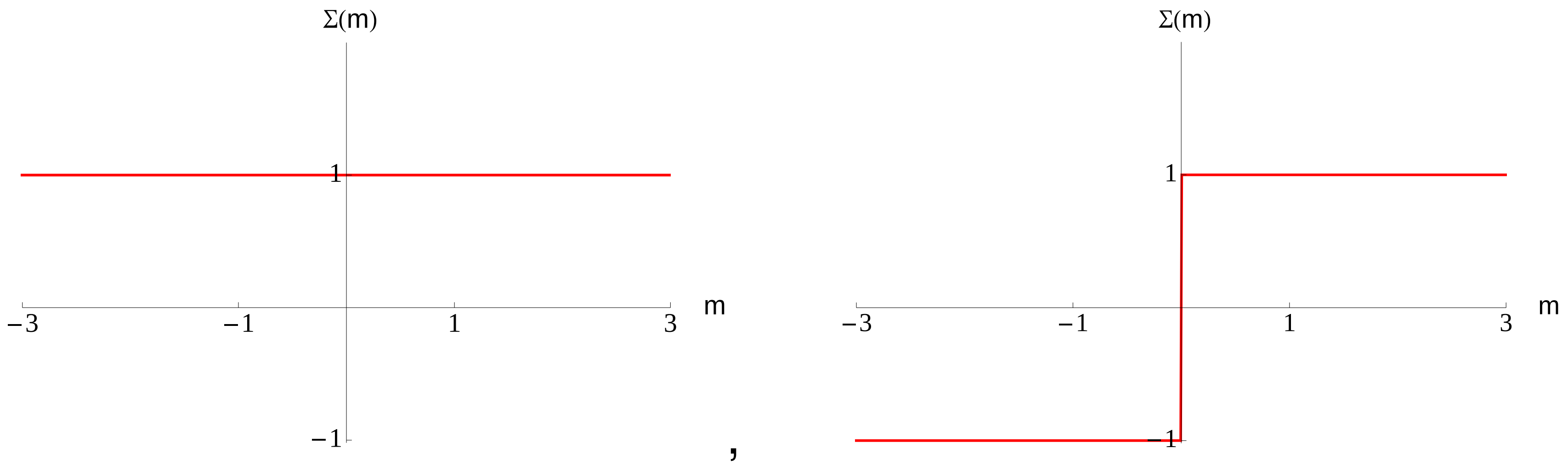
The Problem

Text Book Solutions

The Problem

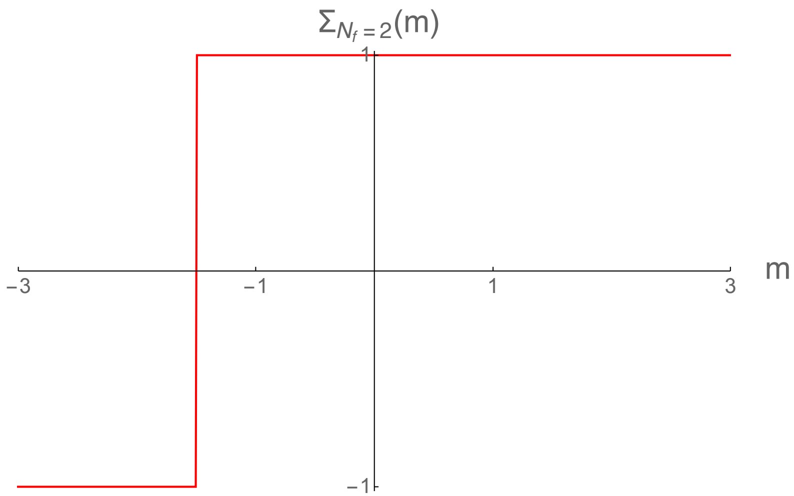


Behavior of the chiral condensate for $N_f = 1$ for QCD at $\theta = 0$ (left) and for QCD at fixed topology (right) .

$$\Sigma(m) \quad = \quad -\langle \bar{q}q \rangle = \frac{1}{V}\frac{d}{dm}\log Z(m) = \left\langle \frac{1}{V}\sum_k \frac{m}{\lambda_k^2 + m^2} \right\rangle .$$

This suggests that the condensate changes sign when the quark mass changes sign.

Two Flavor QCD



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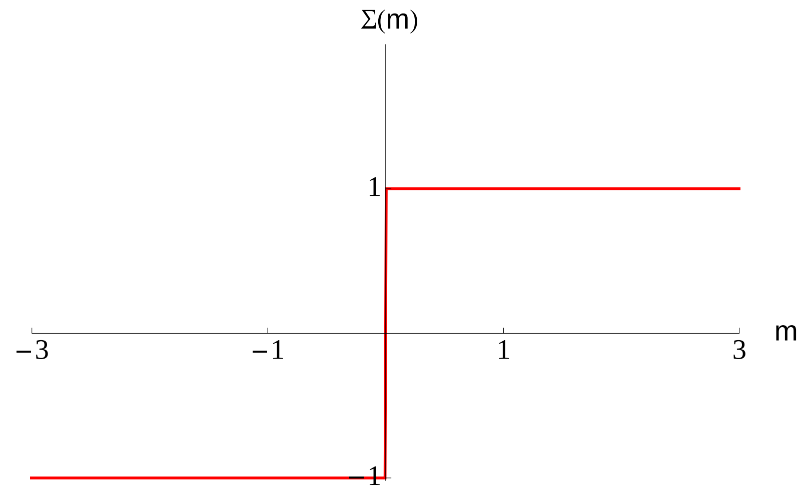
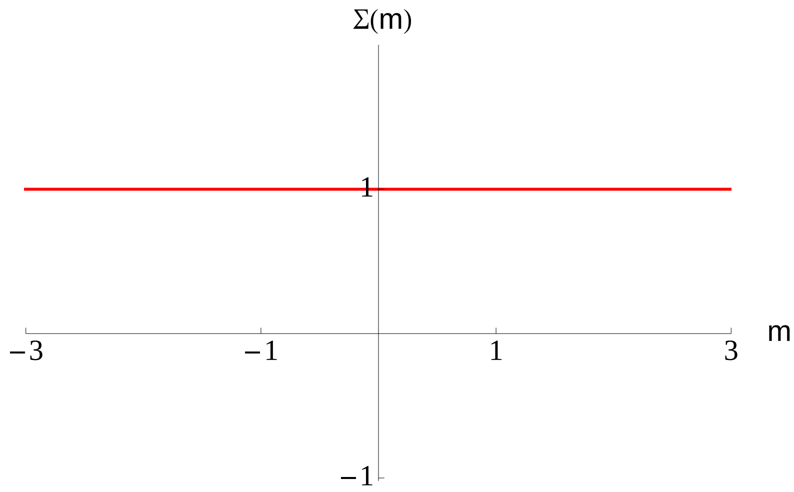


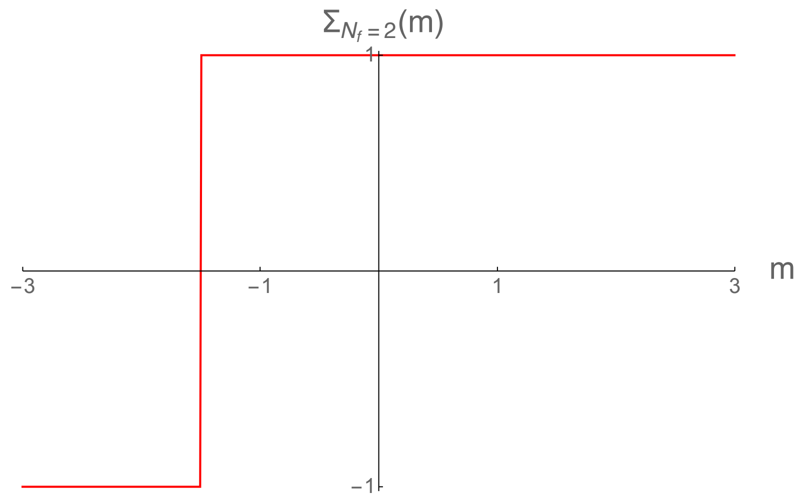
Behavior of the chiral condensate for $N_f = 2$ QCD with $m_1 = 1.5$ and $m_2 = m$ at $\theta = 0$ (left) and the same for fixed topology (right) .

Chiral condensate

$$\Sigma(m) = -\langle \bar{q} q \rangle = \frac{1}{V} \frac{d}{dm} \log Z(m, m_2).$$

Motivation





Behavior of the chiral condensate for $N_f = 1$ (left) and $N_f = 2$ (right).

- What is the role of the zero modes?
- What is the differences between one and two flavors?
- Can we understand this behavior in terms of the Dirac spectrum, also because of statements in the literature that are not satisfactory.

Creutz-2005

One Flavor QCD

- Chiral symmetry is broken by the anomaly.
- There is no spontaneous symmetry breaking and there are no Goldstone bosons.
- The mass dependence of the one flavor QCD partition function is given by

$$Z = e^{mV\Sigma\cos\theta + O(m^2V)}.$$

- Among others, the chiral condensate vanishes for $\theta = \pi/2$.

Text Book Solution for Two-Flavor QCD

Chiral symmetry is broken spontaneously, and in the low energy limit, the partition function is given by the chiral Lagrangian,

$$Z(m_1, m_2) = \int_{U \in \mathsf{SU}(2)} dU \exp[V\Sigma \mathrm{Tr}\, \mathrm{diag}(m_1, m_2)(Ue^{i\theta/2} + U^{-1}e^{-i\theta/2})].$$

In the thermodynamical limit U aligns with the mass term resulting in

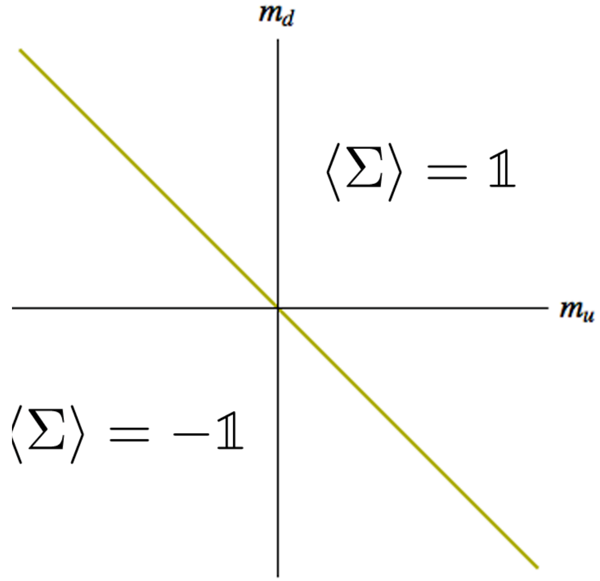
$$Z(m_1, m_2) = \frac{I_1(\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta})}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta}} \approx e^{mV\Sigma|m_1+m_2|}$$

Leutwyler-Smilga-1992

This results in the chiral condensate

$$\langle \bar{\psi}\psi \rangle = \frac{1}{V} \partial_m \log Z(m, m_2) \sim \mathrm{sign}(m + m_2).$$

Phase Diagram of Two-Flavor QCD



Horkel-Sharpe-2015

Phase diagram of two-flavor QCD.

III. Role of the Zero Modes

One Flavor QCD

Two Flavor QCD

Universal Formula

Text Book Solution for $N_f = 1$

$$|\langle\bar{\psi}\psi\rangle| = \left\langle \frac{1}{V}\sum_\nu\sum_k \frac{1}{m+\lambda_k}\frac{m^{|\nu|}\prod(\lambda_n^2+m^2)}{\langle\sum_\nu m^{|\nu|}\prod(\lambda_n^2+m^2)\rangle}\right\rangle.$$

When $m \ll 1/V$ the sum over the eigenvalues is dominated by the zero modes, but only the $\nu = \pm 1$ contributions remain for $m \to 0$.

$$|\langle\bar{\psi}\psi\rangle| = \frac{2}{V}\frac{\left\langle \prod_{\lambda_n\neq 0}(\lambda_n^{\nu=1\,2}+m^2)\right\rangle}{\left\langle \prod_{\lambda_n\neq 0}(\lambda_n^{\nu=0\,2}+m^2)\right\rangle}.$$

Issues:

- The limits are taken in the wrong order. What happens if we take the $V \to \infty$ limit before the chiral limit?
- When $m < 0$ there are large cancellations in the denominator which can make the partition function exponentially small.

Zero Modes and Two Flavor QCD

The arguments of one flavor QCD can also be applied to two-flavor QCD

$$|\langle\bar{\psi}\psi\rangle| = \frac{1}{V}\sum_\nu\sum_k \frac{1}{m+\lambda_k}\frac{(mm_2)^{|\nu|}\prod(\lambda_n^2+m^2)(\lambda_n^2+m_2^2)}{\sum_\nu (mm_2)^{|\nu|}\prod(\lambda_n^2+m^2)(\lambda_n^2+m_2^2)}.$$

When $m \ll 1/V$ at fixed m_2 the sum over the eigenvalues is dominated by the zero modes, but only the $\nu = \pm 1$ contributions remain for $m \to 0$.

$$|\langle\bar{\psi}\psi\rangle| = \frac{2}{V}\frac{\prod_{\lambda_n\neq 0}(\lambda_n^{\nu=1\ 2}+m^2)(\lambda_n^{\nu=1\ 2}+m_2^2)}{\prod_{\lambda_n\neq 0}(\lambda_n^{\nu=0\ 2}+m^2)(\lambda_n^{\nu=0\ 2}+m_2^2)}.$$

Zero Mode Contribution to Chiral Condensate

Issues:

- What happens to these contributions if we take the $V \to \infty$ limit before the chiral limit?
- When $mm_2 < 0$ there are large cancellations in the denominator, and as we will see next, this makes the partition function exponentially small and the condensate exponentially large.

Zero Mode Contribution to Chiral Condensate

$$\Sigma^{\rm ZM}(\vec{m},\theta) = \frac{1}{V}\sum_\nu e^{i\nu\theta}\frac{|\nu|}{m}\frac{Z_\nu(\vec{m})}{Z(\vec{m},\theta)}.$$

Inserting a the $\delta_{\nu\nu'}$ we obtain

$$\Sigma(\vec{m},\theta) = \frac{1}{V}\int\frac{d\phi}{2\pi}\sum_{\nu\nu'} e^{i\nu\phi}\frac{|\nu|}{m}\frac{e^{i\nu'(\theta-\phi)}Z_{\nu'}(\vec{m})}{Z(\vec{m},\theta)}.$$

The sum over ν can be evaluated

$$\sum_\nu |\nu| e^{i\nu\phi} = -\frac{1}{2\sin^2\phi/2}.$$

We then find that the zero mode contribution to the chiral condensate is given by

$$\Sigma^{\rm zm}(\vec{m},\theta) = \frac{1}{mV}\int\frac{d\phi}{2\pi}\frac{-1}{2\sin^2\phi/2}\frac{Z(\vec{m},\theta-\phi)-Z(\vec{m},\theta)}{Z(\vec{m},\theta)}.$$

Zero Mode Contribution

$$\Sigma^{\rm zm}(\vec{m},\theta) = \frac{1}{mV}\int \frac{d\phi}{2\pi}\frac{-1}{2\sin^2\phi/2}\frac{Z(\vec{m},\theta-\phi)-Z(\vec{m},\theta)}{Z(\vec{m},\theta)}.$$

This is a completely general formula for the zero mode contribution to the chiral condensate.

The integral is dominated by the region around $\phi=\theta$, and for $mV \gg 1$

,

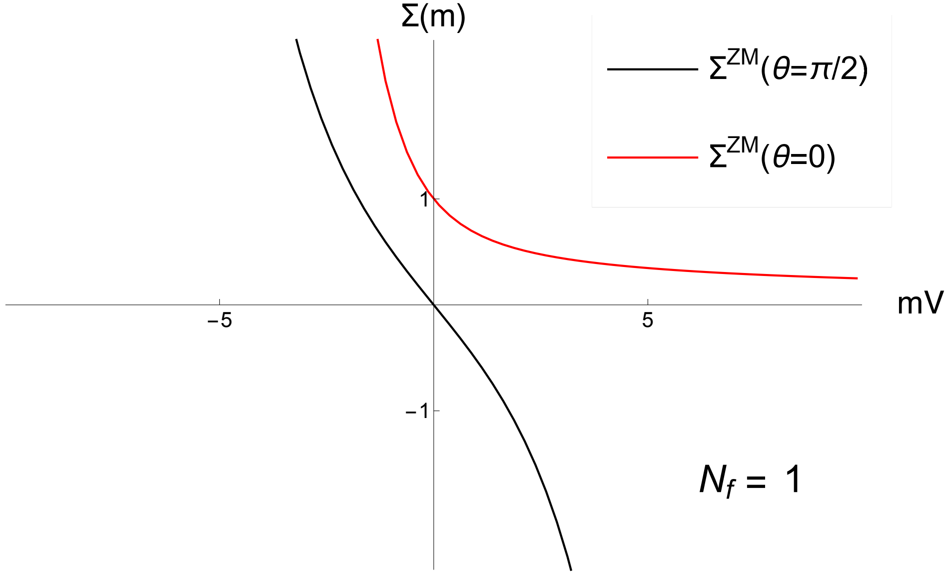
$$\Sigma^{\rm zm}(\vec{m},\theta) \sim \frac{1}{mV}\frac{-1}{2\sin^2\theta/2}\frac{Z(\vec{m},\theta=0)}{Z(\vec{m},\theta)}.$$

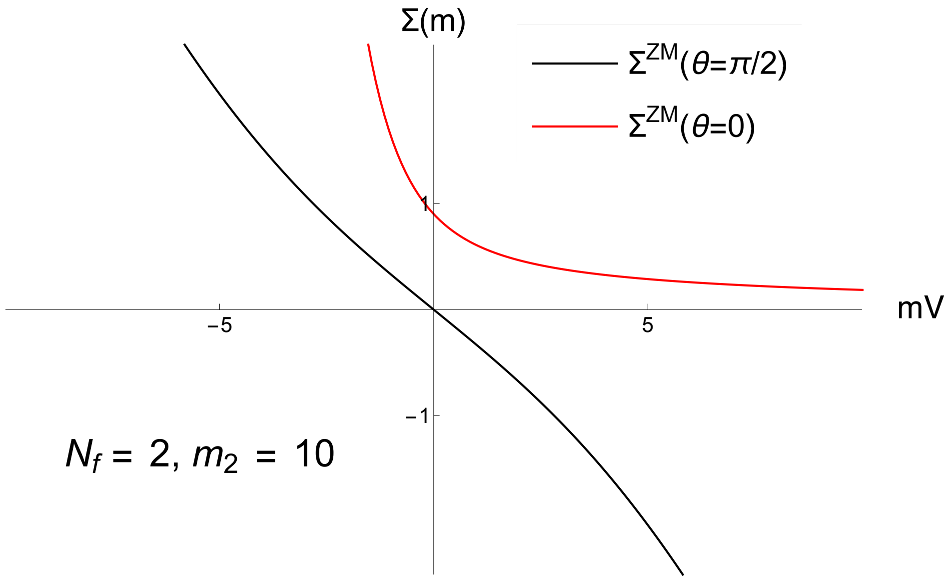
For one and two flavor QCD we obtain for $mV \gg 1$

$$\begin{aligned}\Sigma^{N_f=1}(m,\theta) &\sim \frac{1}{mV}e^{(|m|-m\cos\theta)\Sigma V}\\ \Sigma^{N_f=2}(m,m_2,\theta) &\sim \frac{1}{mV}e^{(|m|+|m_2|-\sqrt{m^2+m_2^2+2mm_2\cos\theta})\Sigma V}.\end{aligned}$$

Diverges exponentially in V except at $\theta=0$.

Zero Modes and the Chiral Condensate





How to Get a Finite Chiral Condensate

- We see that the statement that for $N_f = 1$ the chiral condensate comes from the zero modes is exponentially wrong.
- For two-flavor QCD the contribution from the zero modes is exponentially large as well
- We know that the chiral condensate is finite, so that this contribution should be canceled by another exponentially large contribution.
- Let us looked at the quenched part of the chiral condensate.

III. Decomposition of the Chiral Condensate

Decomposition of the Chiral Condensate

Universal Result for the Quenched Contribution

Cancellation of exponential large Terms

Decomposition of the Spectral Density and Chiral Condensate

$$\rho_\nu(\vec{m}, x) = \rho_\nu^{\rm ZM}(x) + \rho_\nu^{\rm q}(x) + \rho_\nu^{\rm d}(\vec{m}, x).$$

The spectral density at fixed θ can also be decomposed into a zero mode part, a quenched part and a dynamical part,

$$\begin{aligned} \rho(\vec{m}, \theta) &= \sum_\nu e^{i\nu\theta} \frac{Z_\nu(\vec{m})}{Z(\vec{m}, \theta)} \\ &= \rho_\nu^{\rm ZM}(\vec{m}, \theta) + \rho_\nu^{\rm q}(\vec{m}, \theta) + \rho_\nu^{\rm d}(\vec{m}, \theta). \end{aligned}$$

Note that the zero mode part and the quenched part now depend on $\vec{m}$.

The chiral condensate can also be decomposed in the same way, both at fixed ν and at fixed θ. The quenched condensate at fixed ν only depends on one valence mass, but at fixed θ it depends on all masses.

The Microscopic Domain of QCD

We will do calculations in the microscopic domain of QCD.

In this domain, also know as the ϵ -domain, the quark mass and the Dirac eigenvalues scale in the thermodynamic limit as

$$m \sim \frac{1}{V}, \qquad \lambda \sim \frac{1}{V}.$$

Correction terms will enter when $m, \lambda \approx 1/\Lambda_{\rm QCD}\sqrt{V}$.

In this domain, the spectral density can be evaluated analytically.

Spectral Density at Fixed ν for $N_f = 1$

The one-flavor spectral density in the ϵ-domain is given by

$$
\begin{aligned}
\rho_\nu(\lambda, m) &= \rho^{\mathrm{ZM}}(x) + \rho^{\mathrm{q}}(x) + \rho^{\mathrm{d}}(m, x) \\
&= |\nu|\delta(\hat{x}) + \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x})) \\
&\quad - \frac{\hat{x}}{\hat{m}^2 + \hat{x}^2}\left[\hat{x}J_\nu(\hat{x})J_{\nu+1}(\hat{x}) - \hat{m}\frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})}J_\nu^2(\hat{x})\right].
\end{aligned}
$$

Damgaard-Osborn-Toublan-JV-1999

$$
\hat{x} \equiv \lambda\Sigma V, \qquad \hat{m} \equiv m\Sigma V
$$

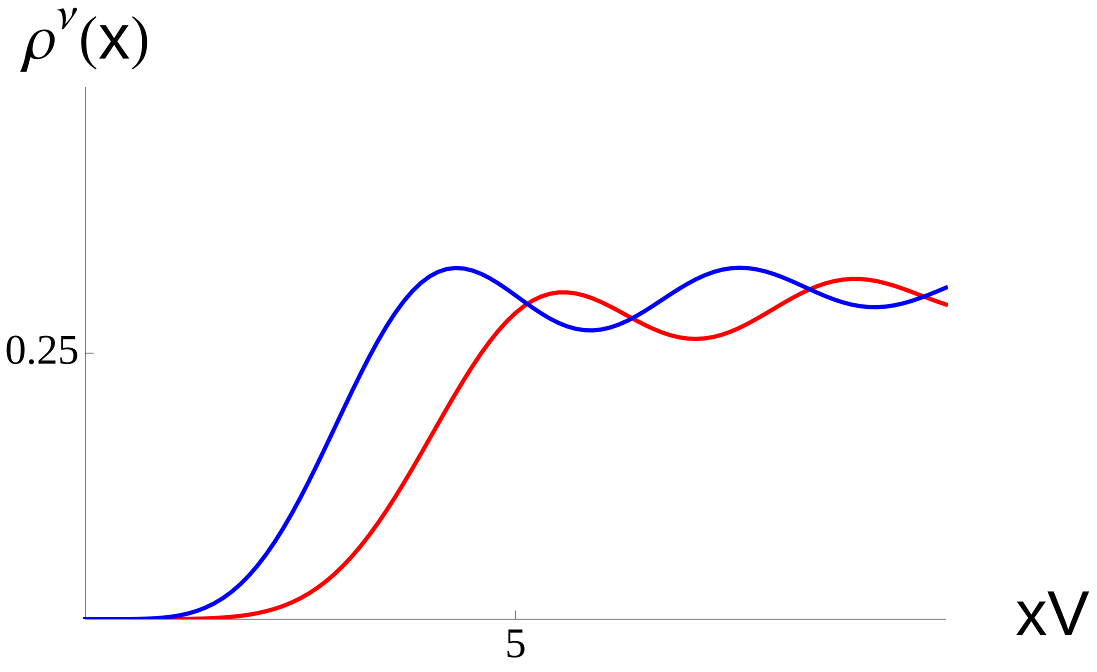
$$
\Sigma_\nu^{\mathrm{q}}(m) = \int dx \frac{2m}{m^2 + x^2}\rho_\nu^{\mathrm{q}}(x).
$$

Quenched Chiral Condensate

Condensate corresponding to $\rho_\nu(x) = \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x}))$ at fixed θ-angle is given by

$$\Sigma^{\mathrm{N_f}=0}(m,\theta) = \frac{1}{2mV\sin^2\theta/2} - \frac{K_1(2V|m\sin\phi/2|)}{\mathrm{sign}(m)|\sin\theta/2|}.$$

Microscopic Spectral Density at fixed ν



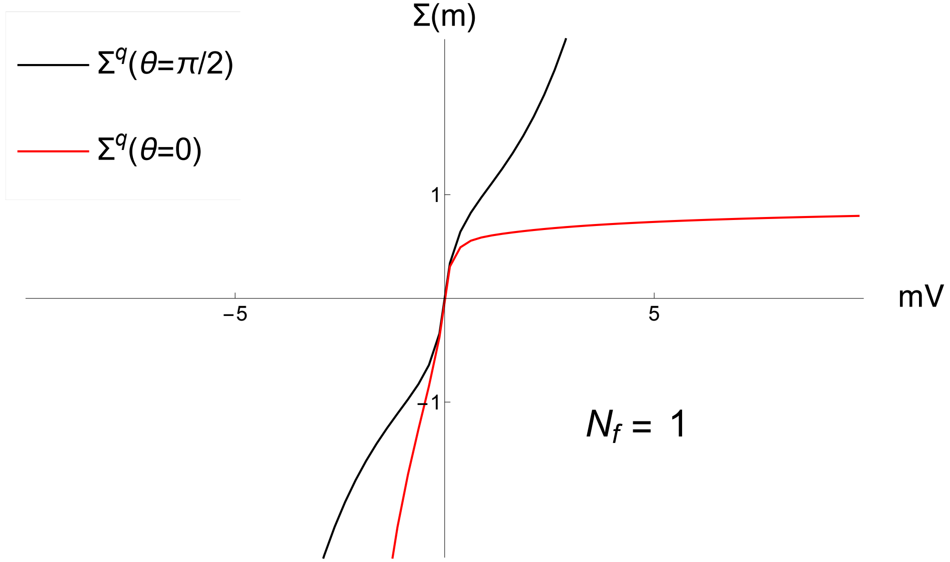
The one microscopic spectral density for $\nu = 2$ and $mV = 1$ (red) compared to the quenched result for $\nu = 2$ (blue).

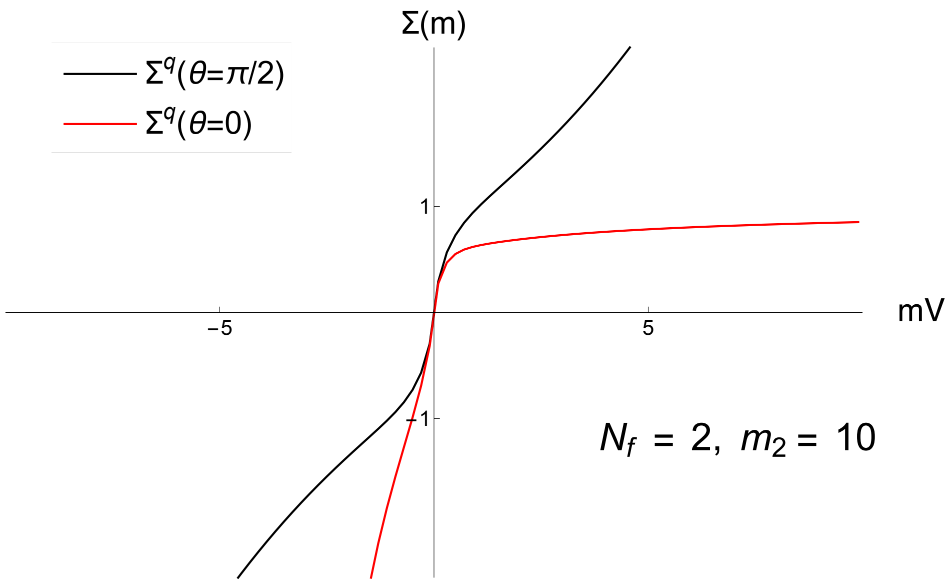
Quenched Part of the Chiral Condensate

$$
\begin{aligned}
\Sigma^{\rm q}(\vec{m},\theta) &= \sum_\nu e^{i\nu\theta}\Sigma^{\rm q}_\nu(m)\frac{Z_\nu(\vec{m})}{Z(\vec{m},\theta)} \\
&= \int\frac{d\phi}{2\pi}\sum_{\nu\nu'} e^{i\nu'\theta}\Sigma^{\rm q}_\nu(m)e^{i(\nu-\nu')\phi}\frac{Z_{\nu'}(\vec{m})}{Z(\vec{m},\theta)} \\
&= \int\frac{d\phi}{2\pi}\Sigma^{N_f=0}(m,\phi)\frac{Z(\vec{m},\theta-\phi)}{Z(\vec{m},\theta)} \\
&= \int\frac{d\phi}{2\pi}\left[\frac{1}{2mV\sin^2\phi/2}-\frac{K_1(2mV|\sin\phi/2|)}{{\rm sign}(m)|\sin\phi/2|}\right]\frac{Z(\vec{m},\theta-\phi)}{Z(\vec{m},\theta)}
\end{aligned}
$$

The first term cancels exactly against the zero mode contribution. Because $K_1(x)\sim\exp(-x)$ the second term is exponentially smaller. It turns out the total exponent in the numerator is the same as in the denominator and the total integral becomes $O(1)$.

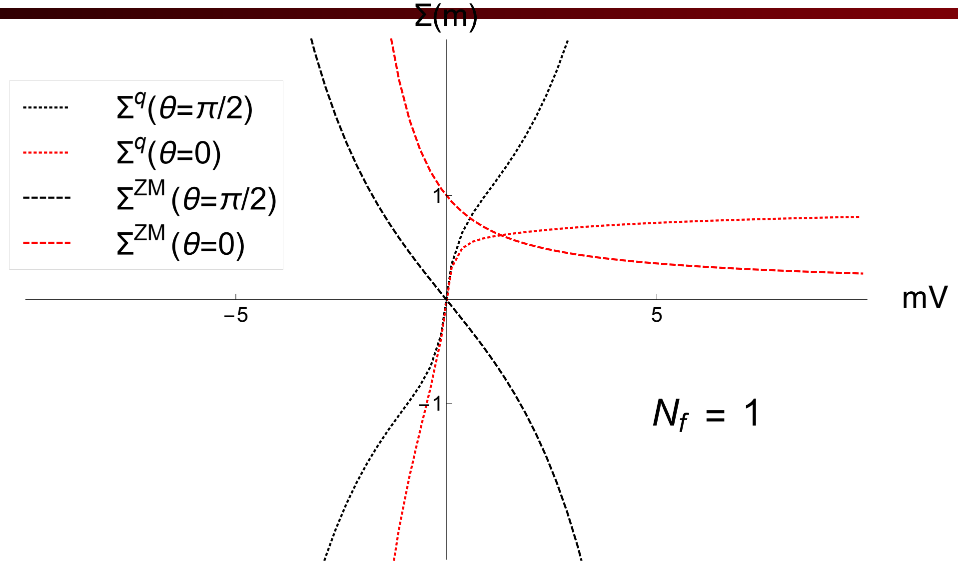
Quenched Part of the Chiral Condensate

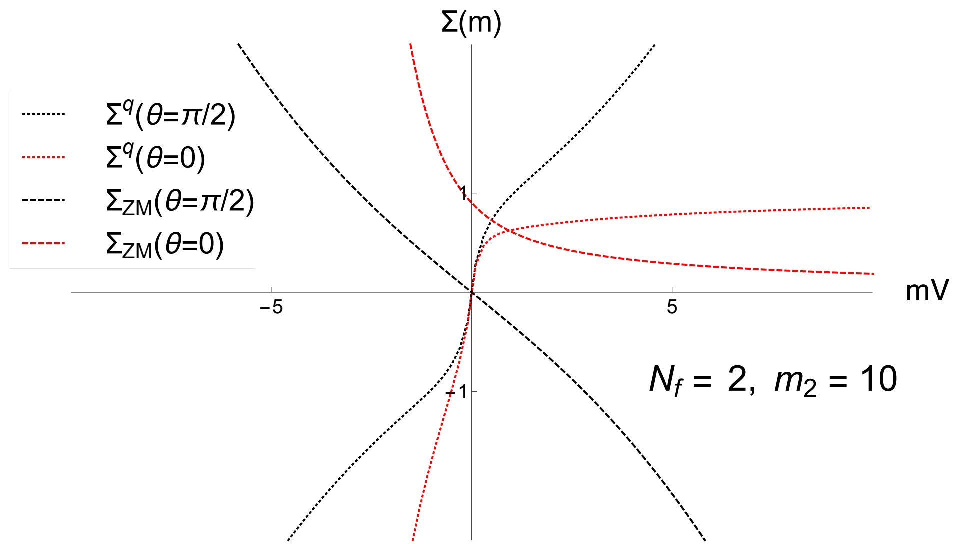




Quenched part of the chiral condensate.

Quenched and Zero Mode Part of the Chiral Condensate



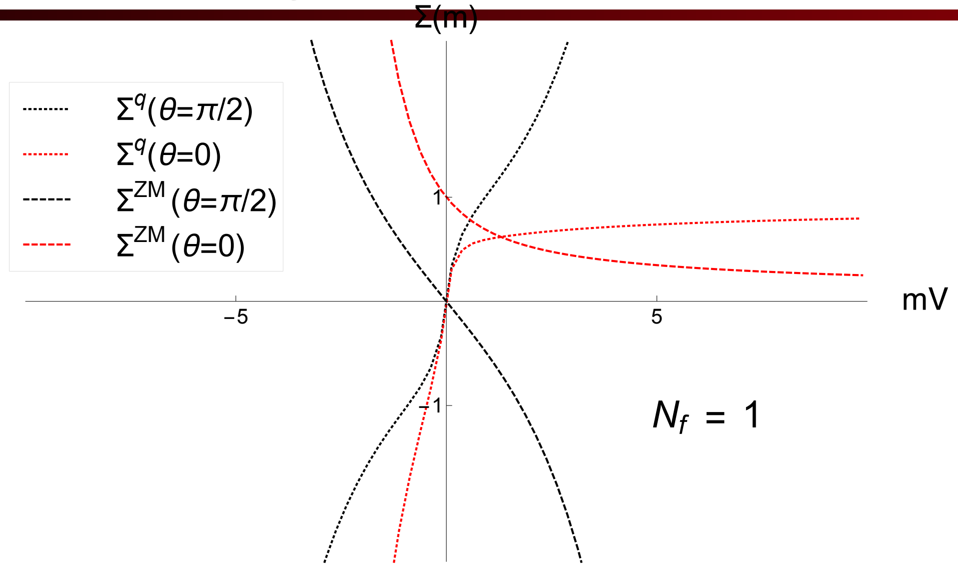


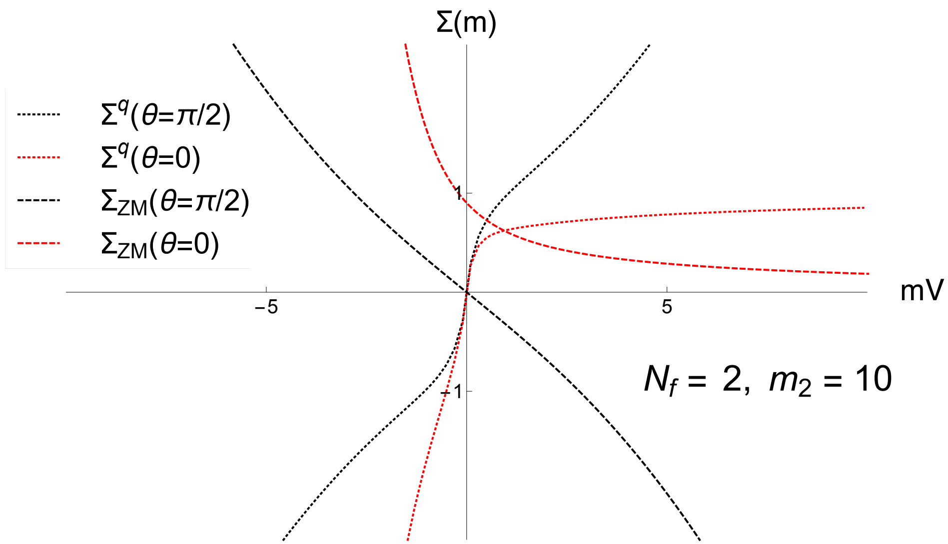
Sum of the Zero Mode part and the Quenched part of the chiral condensate

IV. The Bottom Line

Addition of the Contributions

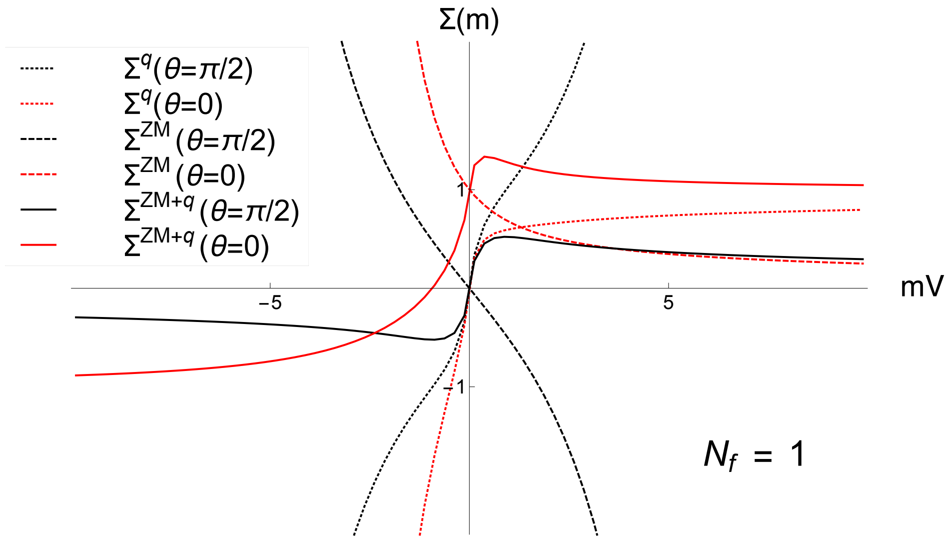
Quenched and Zero Mode Part of the Chiral Condensate

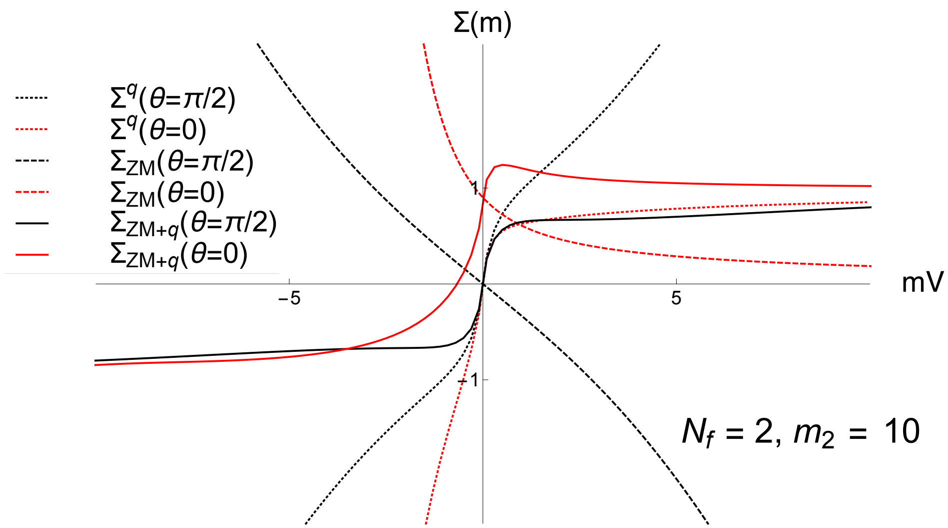




Sum of the Zero Mode part and the Quenched part of the chiral condensate

Quenched and Zero Mode Part of the Chiral Condensate

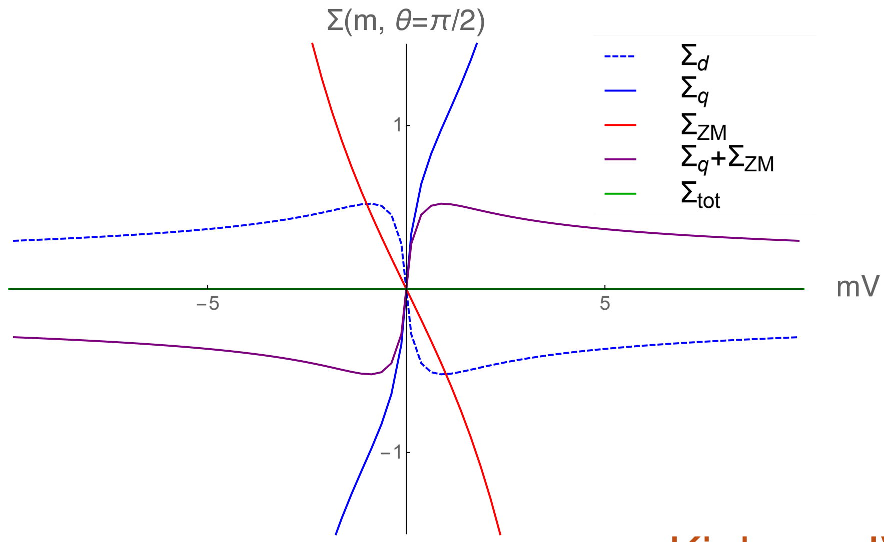




Sum of the Zero Mode part and the Quenched part of the chiral condensate

Kieburg-JV-Wettig-2018

Chiral Condensate for $\theta = \pi/2$

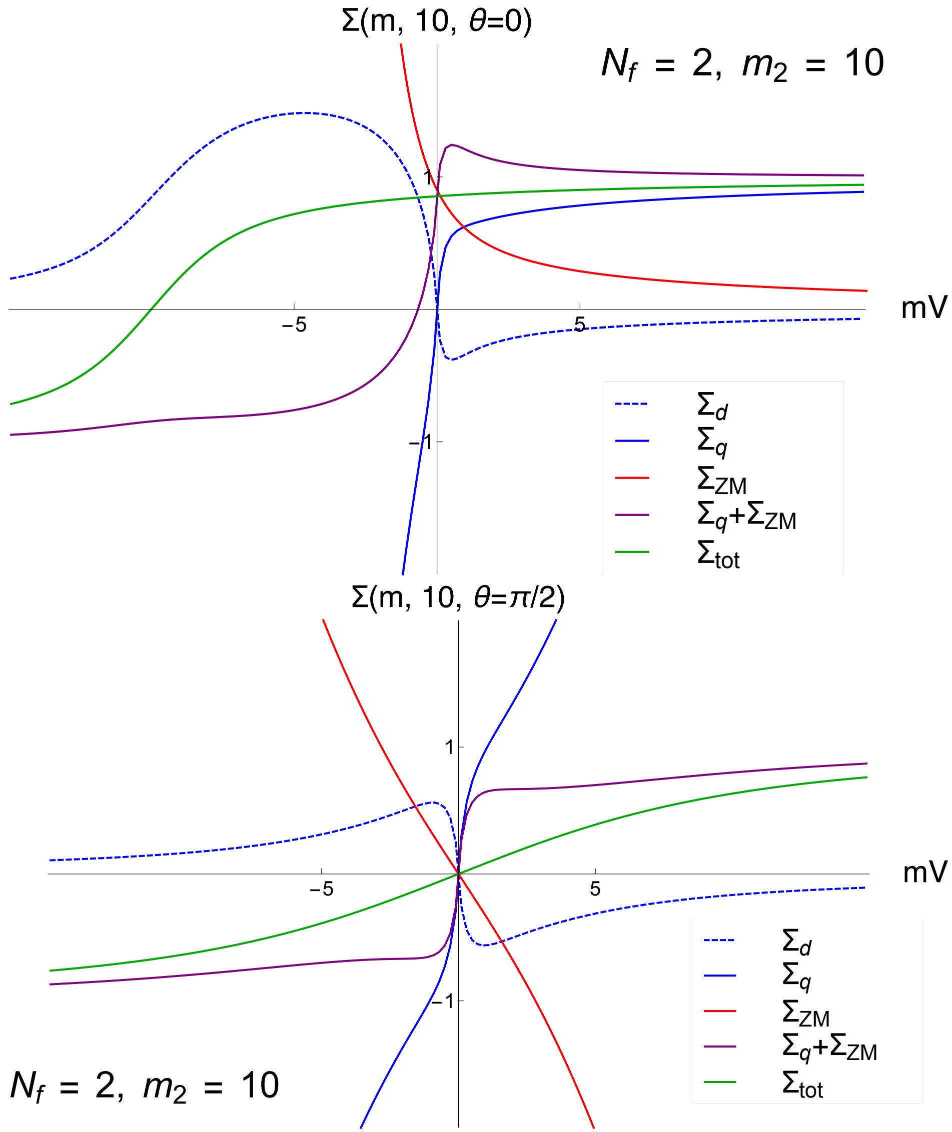


Kieburg-JV-Wettig-2017

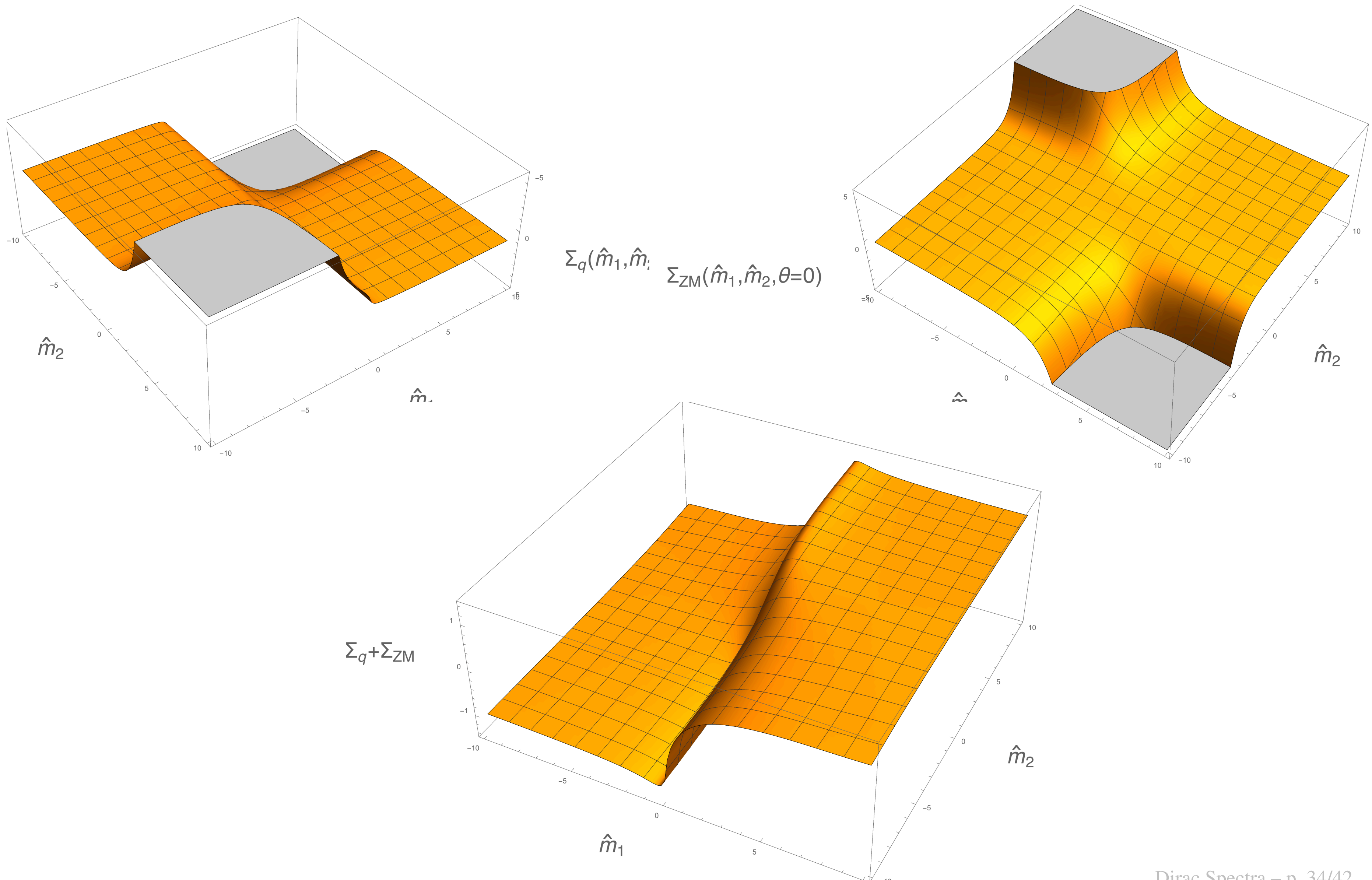
$$\Sigma_{\rm q} \sim \frac{1}{\sqrt{2\pi}|m|^{3/2}} e^{|m|V\Sigma}[1 + O(1/mV\Sigma)]$$

$$\Sigma_{\rm ZM} \sim -\frac{1}{\sqrt{2\pi}|m|^{3/2}} e^{|m|V\Sigma}[1 + O(1/mV\Sigma)]$$

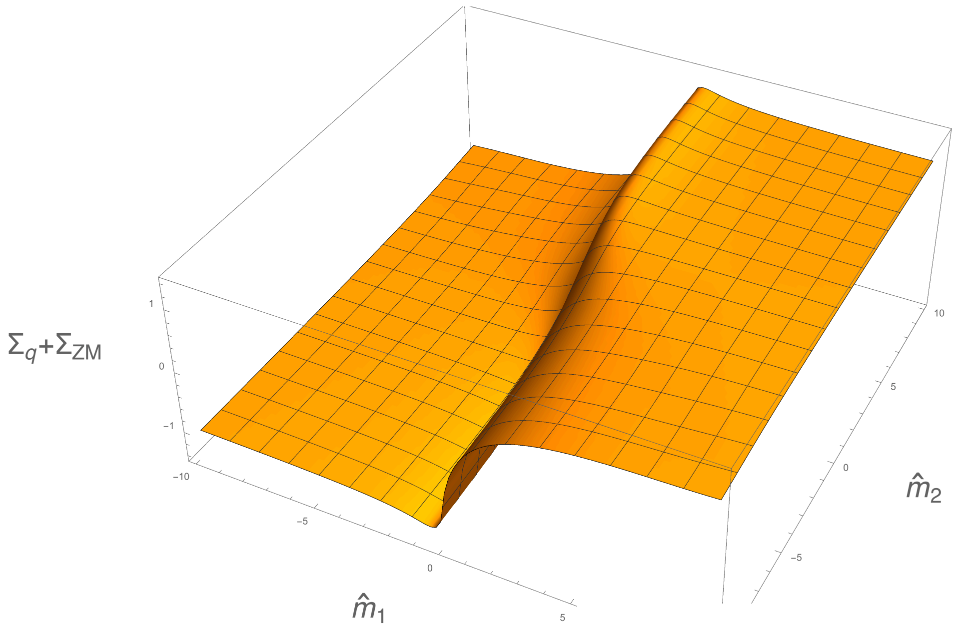
Contributions to the $N_f = 2$ Chiral Condensate

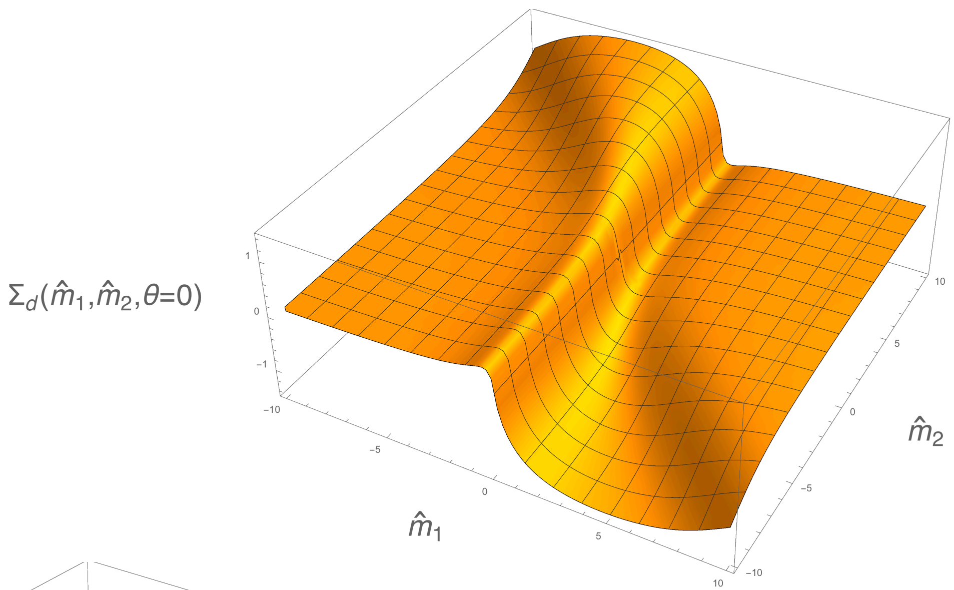


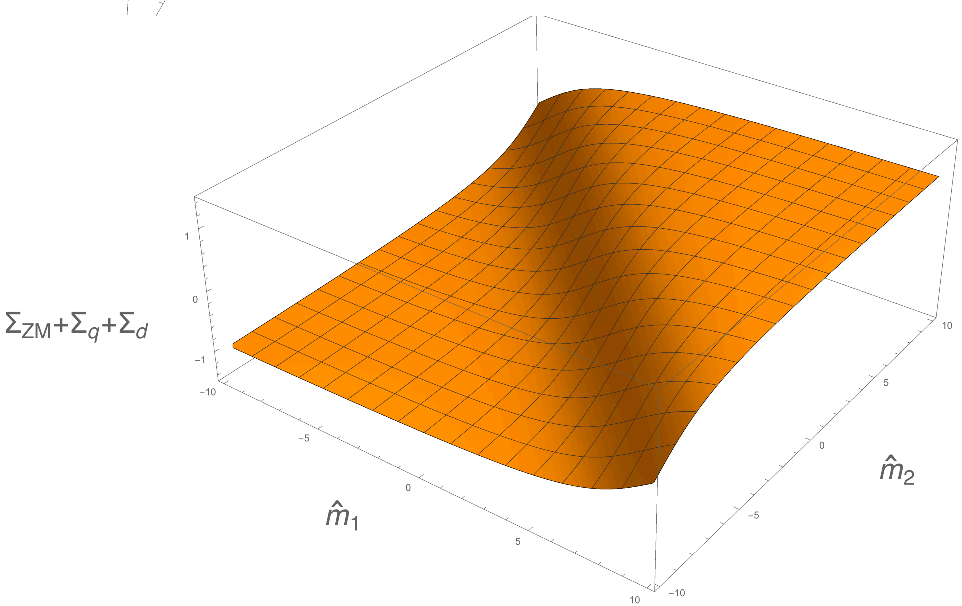
Contributions to the $N_f = 2$ Chiral Condensate



Adding the Contributions







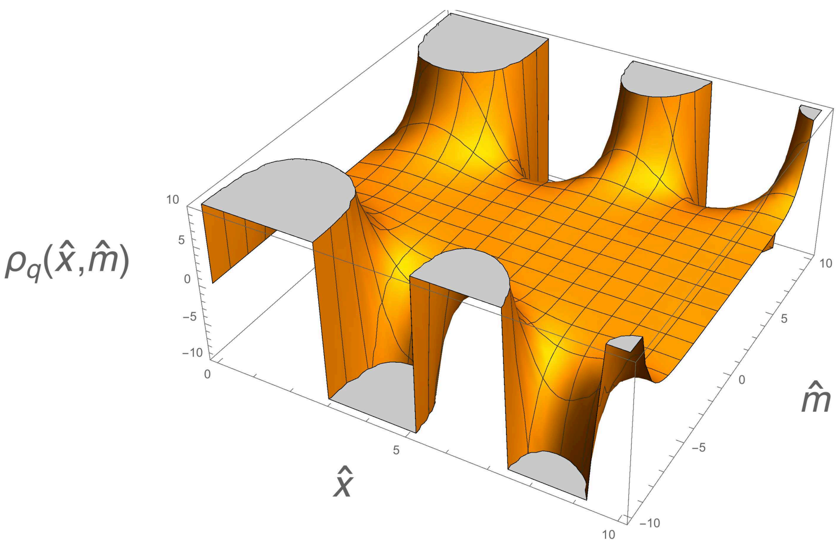
V. Banks-Casher and Dirac Spectrum

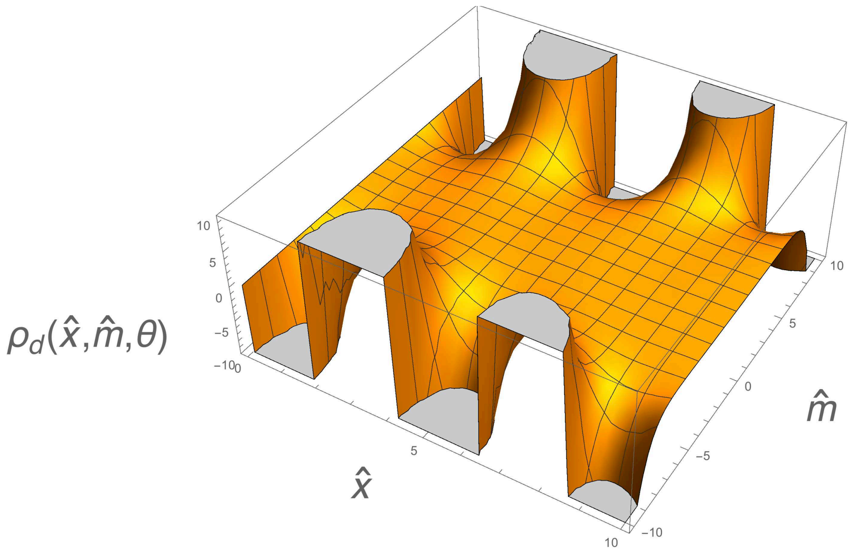
Dirac spectrum

Is the Banks-Casher Relation Still Valid

- Also for $N_f = 1$ the Dirac operator is anti-Hermitian with a Dirac spectrum that is dense on the the imaginary axis.
- The discontinuity of the chiral condensate has to be canceled. This can do done if the spectral oscillates with a period of order $1/V$ and an amplitude that increases exponentially with V . This mechanism was first discovered for QCD at nonzero chemical potential Osborn-Splittorff-JV-1996.
- The conclusion that the spectral density vanishes because the chiral condensate is continuous is incorrect.
- At fixed topology The Dirac spectrum is dense on the imaginary axis and gives rise to a discontinuity of the chiral condensate according to the Banks-Casher formula.

Spectral Density for $\theta = \pi/2$

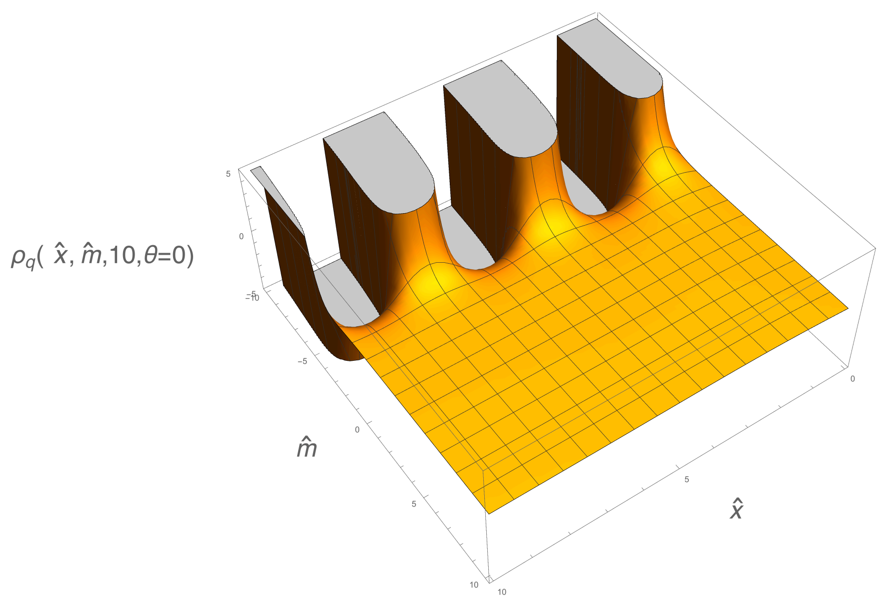


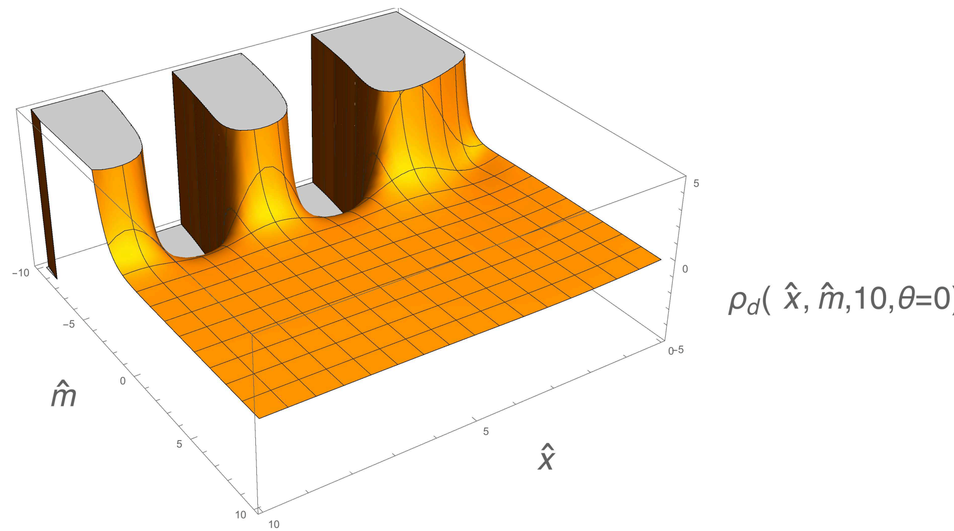


The spectral density of the eigenvalues of the Dirac operator for one-flavor QCD. The quenched part is shown in the left figure and the dynamical part in the right figure. Kieburg-JV-Wettig-2017

The chiral condensate corresponding to this spectral density should vanish!

Spectral Density for Two Flavors





Spectral density of the QCD Dirac operator for $\theta = 0$ and $\hat{m}_2 = 10$. Left we show the quenched contribution and right the correction induced by the fermion determinant.

Analytical Expression for One and Two Flavors

Spectral density in the microscopic domain for one flavor QCD,

$$\rho(x, m, \theta) = \frac{|z|}{z^2 + m^2} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{Z(m, \theta - \phi)}{Z(m, \theta)} [(m^2 + z^2 \cos\phi) \frac{J_1(2z \sin\phi/2)}{2z \sin\phi/2}$$
$$-m \cos(\theta - \phi) J_0(2z \sin\phi/2)]$$

Spectral density in the microscopic domain for two flavor QCD,

$$\rho(z, m_1, m_2, \theta) = \frac{|z|}{2Z(m_1, m_2, \theta)} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left(\left(\frac{-2J_1(2z\cos\phi/2)}{2z\cos\phi/2} \right. \right.$$
$$\left. - \frac{2ze^{-i(\phi/2+\theta)}(2m_1 m_2 - (m_2^2 + m_1^2)e^{i(\theta+\phi)})}{(z^2 + m_1^2)(z^2 + m_2^2)} J_1(2z\cos\phi/2) \right) Z(m_1, m_2, \theta + \phi)$$
$$\left. - \frac{2J_0(2z\cos\phi/2)}{(z^2 + m_1^2)(z^2 + m_2^2)} I_0(\sqrt{m_1^2 + m_2^2 - 2m_1 m_2 \cos(\theta + \phi)}) \left(m_1 m_2 e^{-i(\theta+\phi)} - z^2 e^{i\phi} \right) \right)$$

Kieburg-JV-Wettig-2017

V. Conclusions

- One and two-flavor QCD show a Silver Blaze phenomenon when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.

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- We have obtained a compact formula for the contribution of the zero modes the chiral condensate. The contribution grows exponentially with the volume when $\theta \neq 0$.
- This exponentially increasing part is canceled by the quenched part of the chiral condensate.
- From QCD at nonzero chemical potential we have learnt that the solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume. This also happens for QCD at nonzero θ angle, but in addition, the zero modes are essential for canceling an exponential increasing part of the chiral condensate.

V. Conclusions

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- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta \neq 0$ both for one and two flavors.
- Contributions of zero models and nonzero modes have to be perfectly balanced. Lattice simulations at nonzero θ angle can only be trusted if this balance is preserved by the algorithm.