Entropy production and its time evolution in High Energy QCD

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With Michael Lublinsky and Alex Kovner

Entropy and entanglement entropy

Entanglement and entanglement entropy

Take a general quantum system described by a Hilbert space which is factorisable into different sets of d.o.f. A and B, to which a given density matrix is associated.

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \Leftrightarrow \hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$$

von Neumann entropy: $\hat{\sigma} \equiv -Tr\left(\hat{\rho}_{AB} \log \hat{\rho}_{AB}\right)$

$\hat{\rho}_{AB}$ **can well be the matrix of a pure state, for which** $-Tr\, \hat{\rho}_{AB} \log \hat{\rho}_{AB} = 0$

However, after tracing out some of the d.o.f. the entropy will not vanish any longer, reflecting the now incomplete knowledge that one has about the system.

$$\hat{\rho}'_A = Tr_B\, \hat{\rho}_{AB} \Rightarrow -Tr\left(\hat{\rho}'_A \log \hat{\rho}'_A > 0\right)$$

Textbook example: 2 qbit system

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

$$\hat{\rho}'_A = Tr_B\left[\frac{1}{2}\Big(|0_A 0_B\rangle + |1_A 1_B\rangle\Big)\Big(\langle 0_A 0_B| + \langle 1_A 1_B|\Big)\right] = \frac{1}{2}\Big(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\Big)$$

$$\sigma_A = -Tr\left(\hat{\rho}'_A \log \hat{\rho}'_A > 0\right) = \log 2$$

Some motivation beside pure theoretical interest

Definitely a lot of recent interest in entanglement entropy in high-energy collisions:

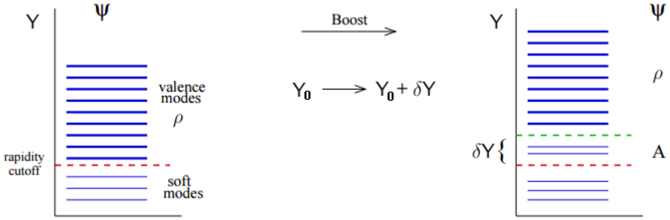
- Early attempt to place an upper bound on the entropy produced in high energy collisions based on expected saturation properties of the gluon distribution - Kutak, Phys.Lett. B705 (2011) 217-221
- Wehrl entropy for parton distribution functions, particularly in the low-x limit - Hagiwara, Hatta, Xiao and Yuan, (2017), arXiv:1801.00087.
- Maybe thermal spectra in final state hadron-hadron collisions can be traced back to a primary entanglement entropy in the projectile wave function, **coordinate space entanglement entropy**- Baker and Kharzeev, arXiv:1712.04558; Kharzeev and Levin, Phys. Rev. D95, 114008
- Thermal spectra in $e^+ e^-$ collisions studied with a toy model for an expanding quantum string , **coordinate space entanglement entropy**- Berges, Florchinger and Venugopalan, Phys. Lett. B778, 442 (2018), JHEP 1804 (2018) 145
- Entanglement entropy in the high energy nucleon and nucleus wave function in the CGC; **momentum space entanglement entropy**- Kovner and Lublinsky 2015 Precursor in general QFT setup: Balasubramanian, McDermott, van Raamsdonk, *Momentum space entanglement and renormalization in QFT*, Phys.Rev. D86 (2012) 045014

CGC 101: splitting hard and soft degrees of freedom

Most of the literature on entanglement entropy is about entanglement between spatial regions.

So first of all: what is entangled with what in momentum space ?

The Born-Oppenheimer valence-soft approximation



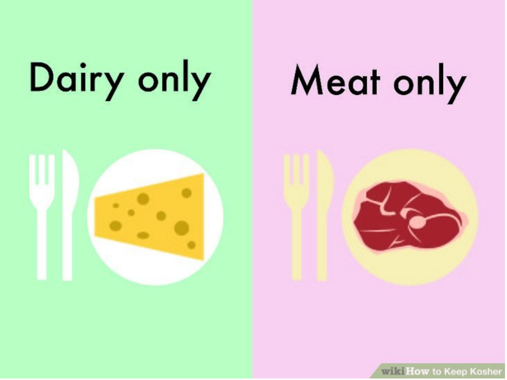
$$\rho^a(\boldsymbol{p}) = -i\, f^{abc} \int_{e^{\delta Y}\Lambda}^{\infty} \frac{dk^+}{2\pi} \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\, a_j^{\dagger b}(k^+, \boldsymbol{k})\, a_j^c(k^+, \boldsymbol{k} - \boldsymbol{p})$$

$$H_g = \int_{\Lambda}^{e^{\delta Y}\Lambda} \frac{dk^+}{2\pi} \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \frac{g\,\boldsymbol{k}^i}{\sqrt{2}|k^+|^{3/2}} \left[a_i^{a\dagger}(k^+, \boldsymbol{k})\rho^a(-\boldsymbol{k}) + a_i^a(k^+, \boldsymbol{k})\rho^a(\boldsymbol{k})\right]$$

Judaism 101 for low-x physicists: what would be "kosher" in the CGC ?

"Thou shall not cook a goat's kid in its mother's milk"
(Exodus 23:19 & 34:26 & Deuteronomy 14:21)

Meet the milk...........and the goat it came from



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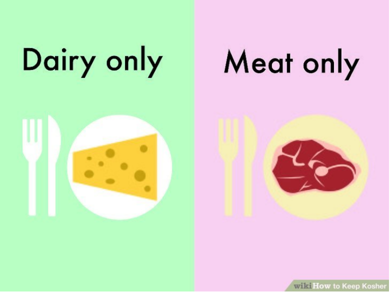


$k^+ \ll \Lambda$

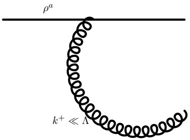
$k^+ \gg \Lambda$

Bottom line: the kosher (read *naïve*) hadron wave function would be:

$$|\psi\rangle = |v\rangle \otimes |s\rangle$$

Judaism for low-x physicists 101: entanglement is NOT Kosher

Cross-talk between soft and valence modes via the WW field which dress up the vacuum state: the goat is entangled with its own milk: only *quasi-factorisation* of the wave function !



In the real world, you've got to gobble up goat and milk at the same time !

If you trace out either part of the wave function, you lose information about the other, i.e. generate entropy

"True" hadron wave function:
$|\psi\rangle = |v\rangle \otimes \Omega|s\rangle$

The CARTOON KRONICLES

$$\Omega = \exp\left\{ i \int_{k^+<\Lambda} \frac{dk^+}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \tilde{b}_i^a(k) \left[a_i^a(k) + a_i^{\dagger a}(k) \right] \right\}, \quad \tilde{b}_i^a(k) = \sqrt{\frac{2}{k^+}} \frac{ig\, \rho^a(\mathbf{k})\, \mathbf{k}_i}{\mathbf{k}^2}$$

Soft-valence entanglement and entropy production in the CGC

Entanglement in the projectile wave function: tracing out the valence modes

MV model for the valence part of the gluon wave function allows to generate a *statistical ensemble* which makes it possible to define a von Neumann entropy.

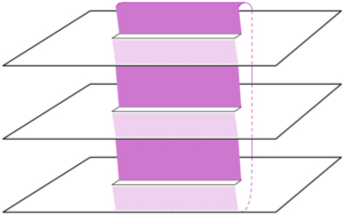
$$\langle \rho^a(\boldsymbol{x})|v\rangle\langle v|\rho^a(\boldsymbol{x})\rangle = N \exp\left\{-\frac{1}{2}\int \rho^a(\boldsymbol{x})\mu^{-2}(\boldsymbol{x}-\boldsymbol{y})\rho^a(\boldsymbol{y})\right\}$$

Tracing out the density matrix of the proton pure state wave function w.r.t. the valence charges, we obtain the *ensemble* entropy

$$\begin{aligned}
\hat{\rho} &= \mathcal{N}\int D[\rho]\; e^{-\int_k \frac{1}{2\mu^2(k)}\rho_a(k)\rho_a(-k)}\, e^{i\int_q b^i_b(q)\phi^i_b(-q)}|0\rangle\langle 0|\; e^{-i\int_p b^j_c(p)\phi^j_c(-p)} \\
&= e^{\frac{\Delta\phi\cdot M\cdot\Delta\phi}{2}}|0\rangle\langle 0|\,, \quad \Delta\phi(\boldsymbol{k}) \equiv \phi^{2a}_i(\boldsymbol{k}) - \phi^{1a}_i(\boldsymbol{k}) \quad \text{acting on different H. spaces} \\
\phi^a_i(\boldsymbol{k}) &= a^a_i(\boldsymbol{k}) + a^{\dagger a}_i(-\boldsymbol{k})\,, \quad M^{ab}_{ij}(\boldsymbol{k}) \equiv g^2\,\mu^2(\boldsymbol{k}^2)\,\frac{\boldsymbol{k}_i\boldsymbol{k}_j}{\boldsymbol{k}^4}\,\delta^{ab}
\end{aligned}$$

Kovner, Lublinsky, Phys.Rev. D92 (2015) no.3, 034016

Entanglement in the projectile wave function: meet the *replica*



Artificial replica space which allows to compute $Tr\,\hat\rho^N$: every plane is one $\hat\rho$, then last and first are connected.

Catch: you need to pray that the result is analytical in N in order to set $N = 1+\epsilon$ and be able to compute

$$\sigma^E = -Tr(\hat\rho\log\hat\rho) = -\lim_{\epsilon\to 0}\frac{Tr\,\hat\rho^{1+\epsilon} - Tr\,\hat\rho}{\epsilon}$$

$$tr[\hat\rho^N] = \left(\frac{\det[\pi]}{2\pi}\right)^{N/2}\int\prod_{\alpha=1}^{N}[D\phi^\alpha]$$
$$\exp\left\{-\frac{\pi}{2}\sum_{\alpha=1}^{N}\phi_i^\alpha\phi_i^\alpha - \frac{1}{2}\sum_{\alpha=1}^{N}(\phi_i^\alpha - \phi_i^{\alpha+1})\right.$$
$$\left.M_{ij}\,(\phi_j^\alpha - \phi_j^{\alpha+1})\right\}$$
$$= \exp\left\{-\frac{1}{2}\ln 2 - \frac{N}{2}tr\left[\ln\frac{M}{\pi}\right]\right.$$
$$\left. - \frac{1}{2}tr\left[\ln\left(\cosh(N\,\mathrm{arcCosh}[1+\frac{\pi}{2M}]) - 1\right)\right]\right\}$$

$$\sigma^E = \frac{1}{2}tr\left[\ln\frac{M}{\pi} + \sqrt{1+\frac{4M}{\pi}}\,\mathrm{arcCosh}\left[1+\frac{\pi}{2M}\right]\right]$$

Entropy production in high energy collisions

Density matrix for a gluon system after scattering:
need to remove the WW cloud in order to consider only *produced* gluons

$$\hat{\rho}_P = \Omega^\dagger U(t)\,\hat{S}\,\Omega|0\rangle \otimes |v\rangle\langle v| \otimes \langle 0|\Omega^\dagger\,\hat{S}^\dagger\,U^\dagger(t)\,\Omega \qquad \left[U(t) = e^{iH_0 t}\right]$$

$$= e^{-iH_0 t} e^{i\int_{q^+}\int d^2x\,\Delta\tilde{b}^a_i(q^+,x)\,\phi^a_i(q^+,x)}\,|0\rangle \otimes |v\rangle\,\langle v| \otimes \langle 0| e^{-i\int_{q^+}\int d^2x\,\Delta\tilde{b}^a_i(q^+,x)\,\phi^a_i(q^+,x)} e^{iH_0 t}$$

$$\Delta b^a_i(\mathbf{x}) \equiv \frac{g}{2\pi}\int d^2\mathbf{z}\,\frac{(\mathbf{x}-\mathbf{z})_i}{(\mathbf{x}-\mathbf{z})^2}\left(S^{ab}(\mathbf{x}) - S^{ab}(\mathbf{z})\right)\rho^b(\mathbf{z})\,,$$

$$\Delta b^a_i(\mathbf{q}) = i\,g\int\frac{d^2\mathbf{l}}{(2\pi)^2}\left[\frac{\mathbf{q}_i}{\mathbf{q}^2} - \frac{\mathbf{l}_i}{\mathbf{l}^2}\right] S^{ab}(\mathbf{q}-\mathbf{l})\,\rho^b(\mathbf{l})\,,$$

Formally, the entropy for the ensemble of collisions is the same;
the different physics is encoded in M^P

$$\sigma^P = \frac{1}{2}\,tr\left[\ln\frac{M^P}{\pi} + \sqrt{1+\frac{4M^P}{\pi}}\,\mathrm{arcCosh}\left[1+\frac{\pi}{2M^P}\right]\right]$$

$$M^{P\,ab}_{ij}(\mathbf{q},\mathbf{p}) = g^2\int_{\mathbf{l}}\frac{\mu^2(\mathbf{l})}{2}\left(\frac{\mathbf{l}_i}{\mathbf{l}^2}+\frac{\mathbf{q}_i}{\mathbf{q}^2}\right)\left(\frac{\mathbf{l}_j}{\mathbf{l}^2}-\frac{\mathbf{p}_j}{\mathbf{p}^2}\right) S^{ca}(-\mathbf{l}-\mathbf{q})\,S^{cb}(\mathbf{l}-\mathbf{p})$$

Kovner, Lublinsky, Phys.Rev. D92 (2015) no.3, 034016

Time evolution of entanglement entropy in the CGC

Re-framing time decoherence as entanglement entropy

We would like to ask the question: can we define and calculate the entropy for a single scattering event, independently of the ensemble of valence charges ?

Tentative answer: we should be able to, because the phases of the produced particles oscillate at different frequencies (energies), so that they de-cohere in a finite amount of time.

How does the density matrix evolve with time, if we start with the pure state below ?

$$|\psi(t)\rangle = \sum_n e^{-iE_n t} c_n |\psi_n\rangle$$

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \, e^{i(E_1 - E_2)t} & \dots \\ c_2 c_1^* \, e^{i(E_2 - E_1)t} & |c_2|^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Re-framing time decoherence as entanglement entropy

A measurement taking a finite amount of time $T \gg |E_1 - E_2|^{-1}$ is actually sensitive only to the averaged density matrix over T, so the information about relative phases is gone for good.

$$\hat{\rho} \sim \begin{pmatrix} |c_1|^2 & 0 & \dots \\ 0 & |c_2|^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

But time is not a dynamical variable, so this implies the problem: how do we actually re-frame this de-coherence in terms of entanglement entropy in some Hilbert space ?

Idea: introduce an auxiliary variable (*white noise*) which couples to energy and lives in an auxiliary Hilbert space
$\Rightarrow$ create tripartite Hilbert space: valence $\otimes$ soft $\otimes$ white noise.

Produced entropy in the weak field limit: leading eigenvalue approximation

Kovner, Lublinsky, MS, in preparation

$$\hat{\rho}_\xi = e^{-iH\xi} U(t)\, \hat{S}\, \Omega|G\rangle \otimes |0\rangle \otimes |v\rangle\langle v| \otimes \langle 0| \otimes \langle G|\Omega^\dagger\, \hat{S}^\dagger\, U^\dagger(t) e^{iH\xi}$$

$$\langle \xi|G\rangle = e^{-\frac{\xi^2}{2T^2}}$$

$$\langle \xi_1|\hat{\rho}|\xi_2\rangle = \frac{1}{\sqrt{\pi}\,T}\, e^{-\frac{\xi_1^2+\xi_2^2}{2T^2}}\, e^{-iH_0\xi_1}\, e^{i\int_q \Delta\tilde{b}_i^a(q)\,\phi_i^a(q^+,\mathbf{q})}|0\rangle\langle 0|e^{-i\int_{\mathbf{p}} \Delta\tilde{b}_j^b(p)\,\phi_j^b(p^+,\mathbf{p})}\, e^{iH_0\xi_2}$$

$$= \frac{1}{\sqrt{\pi}\,T} e^{-\frac{\xi_1^2+\xi_2^2}{2T^2}}\, e^{-\int_q \Delta\tilde{b}^2(q)\,(1-e^{iE_q\Delta\xi_{12}})}$$

Extend the Hilbert space by introducing a calorimeter-like new sector G spanned by Gaussian wave functions...then trace over it !!

The *white noise* variable allows to de-cohere states with parametrically large energy differences; the finite experimental knowledge is interpreted in terms of entanglement with the apparatus, which will allow us to associate an entropy with it !

Produced entropy in the weak field limit: largest eigenvalue approximation

Naïvely expanding the density matrix around $\Delta\tilde{b}^2 = 0$ makes the entropy blows up in our faces...why ?

Because the density matrix is close to a pure state density matrix.

$$\hat{\rho} \sim \begin{pmatrix} 1-\delta_1 & 0 & \dots \\ 0 & \delta_2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

$$\delta_{i\geq 2} \ll 1-\delta_1\,, \quad \delta_{i\geq 2} = \beta_i\,\delta_1\,, \quad \beta_i \geq 0\,, \quad \sum_{i\geq 2}\beta_i = 1\,, \quad \text{because} \;\; Tr\,\hat{\rho} = 1$$

$$\Delta\tilde{b}^2(q) \equiv \sum_i\sum_a \Delta\tilde{b}^a_i(q^+,\boldsymbol{q})\,\Delta\tilde{b}^a_i(q^+,-\boldsymbol{q})$$

$$\sigma^E \approx -\Delta\tilde{b}^2\log\Delta\tilde{b}^2\,, \quad \text{cannot be expanded around } \Delta\tilde{b}^2 = 0 \;\dots\; \text{BUT}$$

$$\sigma^E = -\left[(1-\delta_1)\log(1-\delta_1) + \sum_{i\geq 2}\delta_i\log\delta_i\right]$$

$$= -\delta_1\log\delta_1 - \delta_1\sum_{i\geq 2}\beta_i\ln\beta_i - (1-\delta_1)\log(1-\delta_1) \approx -\delta_1\log\delta_1\,,$$

Produced entropy in the weak field limit: event by event case

$$Tr[\hat{\rho}^N] = (1-\delta_1)^N + O((\Delta\tilde{b}^2)^N) = 1 - N\delta_1 + \frac{N(N-1)}{2}\delta_1^2 + \ldots$$

Let us just put the system on replica space and find the coefficient of N...

$$Tr[\hat{\rho}^N] = \left[\frac{1}{\sqrt{\pi}\,T}\right]^N \int \prod_{\alpha=1}^{N} [d\xi_\alpha]\; e^{-\left\{\int_q \frac{\xi_\alpha^2}{T^2}\right\}}$$
$$\times \quad e^{-\int_q \left[\Delta\tilde{b}_i^a(q)\Delta\tilde{b}_i^a(-q) - \Delta\tilde{b}_i^a(q)\Delta\tilde{b}_i^a(q)\exp\left(iE_q(\xi_{\alpha-1}-\xi_\alpha)\right)\right]}$$

$$\sigma^E_{\Delta b^2 \ll 1} \simeq -\delta_1 \log\delta_1 = -\int_q \Delta\tilde{b}^2(q)\left(1 - e^{-\frac{E_q^2 T^2}{2}}\right)\log\int_q \Delta\tilde{b}^2(q)\left(1 - e^{-\frac{E_q^2 T^2}{2}}\right)$$

What does this entropy mean ? How does it relate, for instance, to $-n\log n$?

$$n = \int_q \Delta\tilde{b}^2(q) \Rightarrow n(T) \equiv \int_q \Delta\tilde{b}^2(q)\left(1 - e^{-\frac{E_q^2 T^2}{2}}\right)$$

$$\sigma^E_{\Delta\tilde{b}^2 \ll 1} = -n(T)\log N(T)$$

Clearly, at $T=0$ no disorder has kicked in yet,
but it steadily increases and asymptotically reaches $-n\log n$.

Produced entropy in the weak field limit: ensemble of events

Now, what happens if we restore the average over an ensemble of events keeping tracing over the white noise component ?

$$\hat{\rho}_{P\xi} = \frac{\mathcal{N}}{\sqrt{\pi}\,T}\int [\mathcal{D}\rho]\, d\xi\, e^{-\int_{\boldsymbol{q}} \frac{\rho^a(\boldsymbol{q})\rho^a(-\boldsymbol{q})}{2\mu^2(\boldsymbol{q})}}\, e^{-\frac{\xi^2}{T^2}}$$
$$\times \quad e^{-iH_0\xi}\, e^{i\int_q \Delta\tilde{b}^a_i(q)\,\phi^a_i(q^+,\boldsymbol{q})}|0\rangle\langle 0|e^{-i\int_p \Delta\tilde{b}^b_j(p)\,\phi^b_j(p^+,\boldsymbol{p})}\, e^{iH_0\xi}\,,$$

Again, put it all on replica space and trace over both sets of variables: valence charges (MV model) and white noise (gaussian detector wave function)

$$Tr[\hat{\rho}^N_{P\xi}] = \left[\frac{\mathcal{N}}{\sqrt{\pi}\,T}\right]^N \int \prod_{\alpha=1}^{N} [\mathcal{D}\rho_\alpha d\xi_\alpha]\; e^{-\left\{\int_{\boldsymbol{q}} \frac{\rho^a_\alpha(\boldsymbol{q})\rho^a_\alpha(-\boldsymbol{q})}{2\mu^2(\boldsymbol{q})} + \frac{\xi^2_\alpha}{T^2}\right\}}$$
$$\times \quad e^{-\int_q \left[\Delta\tilde{b}^a_{i\alpha}(q)\Delta\tilde{b}^a_{i\alpha}(-q) - \Delta\tilde{b}^a_{i(\alpha-1)}(q)\Delta\tilde{b}^a_{i\alpha}(q)\exp\left(iE_q(\xi_{\alpha-1}-\xi_\alpha)\right)\right]}$$

Again, impossible to diagonalise the matrix; must approximate, but small coupling expansion blows up
$\Rightarrow$ resort again to the largest eigenvalue approximation

$$\sigma^E_{P\xi} = -\delta_1 \log \delta_1$$

Produced entropy in the weak field limit: result for an ensemble of events

$$\sigma^{E}_{1\,P\xi} = -\delta_1 \log \delta_1$$

Computed up to second order in the coupling constant
Again, a monotonic expression which increases with time.
The entanglement of soft and valence modes grants a non vanishing result also at $T = 0$

$$\delta_1 = \langle \int_q \Delta\tilde{b}^a_i(q)\Delta\tilde{b}^a_i(-q)\rangle_\rho$$
$$-\frac{1}{2}\left[\langle\left[\int_q \Delta\tilde{b}^a_i(q)\Delta\tilde{b}^a_i(-q)\right]^2\rangle_\rho + \int_{q,p}\langle\Delta\tilde{b}^a_i(q)\Delta\tilde{b}^b_j(p)\rangle_\rho\langle\Delta\tilde{b}^a_i(-q)\Delta\tilde{b}^b_j(-p)\rangle_\rho e^{-\frac{(E_q+E_p)^2T^2}{2}}\right]$$

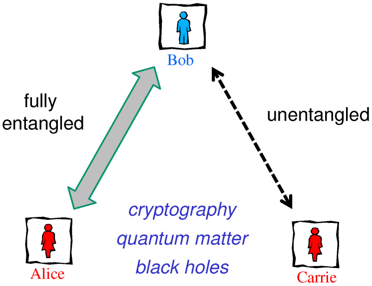
Let me stress the most interesting feature:
averaging over the ensemble, the time dependence is pushed one order higher up in the coupling constant.
Is it expected on the grounds of general properties of Quantum Mechanics ?

Weaker time dependence of the ensemble entropy: intepretation

Koashi, Wiener, *Monogamy of entanglement and other correlations, Phys. Rev. A 69(2):022309, 2004*

Monogamy of entanglement:
no *menage à trois* allowed for maximal entanglement !!!
More general constraint:
the more A is entangled with B, the less it can be entangled with C

Monogamy is *frustrating*!



Conclusions and perspectives

- Entanglement entropy between soft and valence gluons is known at LO in α_s
- Entropy production in QCD for dilute-dense high energy collisions was computed as well
- The time evolution of the produced entropy was investigated al leading logarithmic order. Central idea: extension of the Hilbert space via an auxiliary "white noise variable".
- For a single event, this entropy has a time dependence at LO in the Weitsäcker-Williams field; average over the ensemble of events pushes such a dependence one order down.

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- Perspectives: we'll see....stay tuned !

Thank you for your attention