Forward particle production in proton-nucleus collisions at NLO: solving the running-coupling puzzle

Bertrand Ducloué
(IPhT Saclay)

Probing QCD at the high energy frontier
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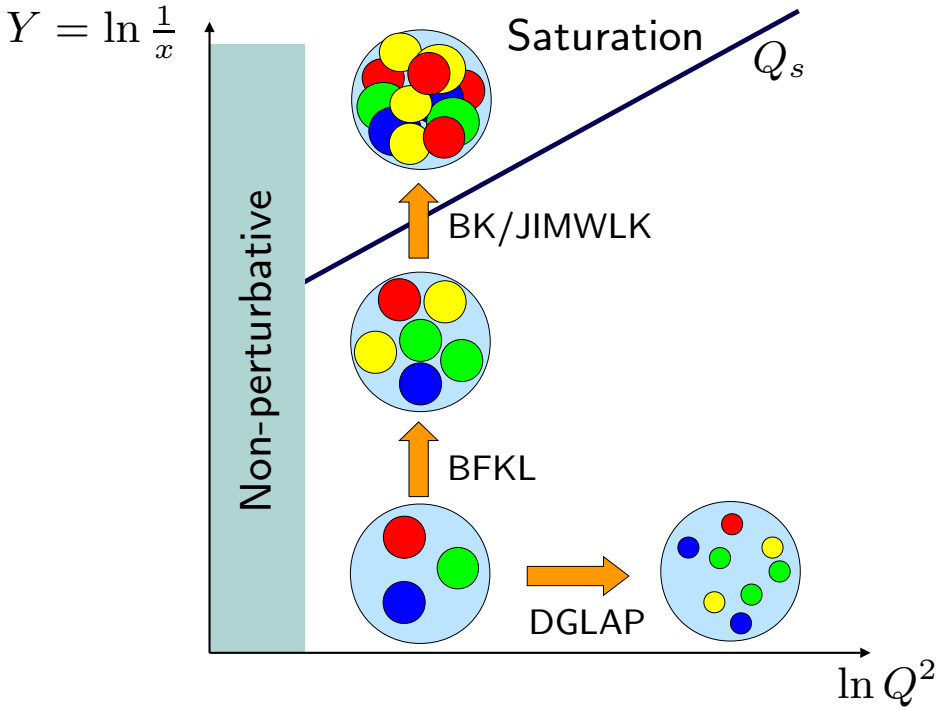
B. D., T. Lappi, Y. Zhu, PRD 93 (2016) 114016 [arXiv:1604.00225]

B. D., T. Lappi, Y. Zhu, PRD 95 (2017) 114007 [arXiv:1703.04962]

B. D., E. Iancu, T. Lappi, A.H. Mueller, G. Soyez, D.N. Triantafyllopoulos, Y. Zhu, PRD 97 (2018) 054020 [arXiv:1712.07480]

Our goal is to study QCD in the saturation regime



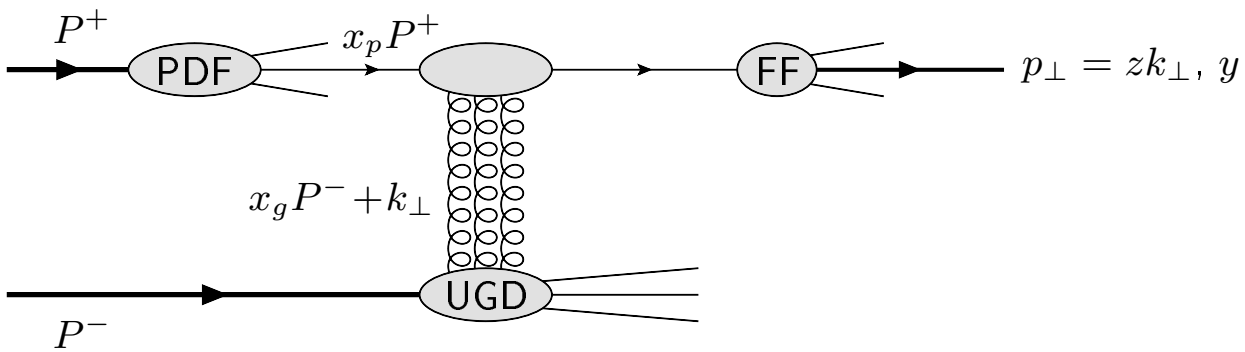
The production of forward particles is a crucial tool to probe small x values

Saturation effects stronger in pA collisions ($Q_s^2 \sim A^{1/3}$)

Here we study the inclusive production of forward hadrons in proton-nucleus collisions: $pA \to hX$

Forward hadron production at LO

Single inclusive forward hadron production at LO in the $q \to q$ channel:



Dilute projectile: $x_p = \dfrac{k_\perp}{\sqrt{s}} e^{y}$, described by collinear PDFs

Dense target: $x_g = \dfrac{k_\perp}{\sqrt{s}} e^{-y} \ll 1$, described by unintegrated gluon distribution $\mathcal{S}$

LO quark multiplicity: $\dfrac{\mathrm{d}N}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y} \propto \mathrm{PDF} \otimes \mathcal{S} \otimes \mathrm{FF}$ $\qquad \left(\dfrac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y} = \int \mathrm{d}^2\mathbf{b} \dfrac{\mathrm{d}N}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y} \right)$

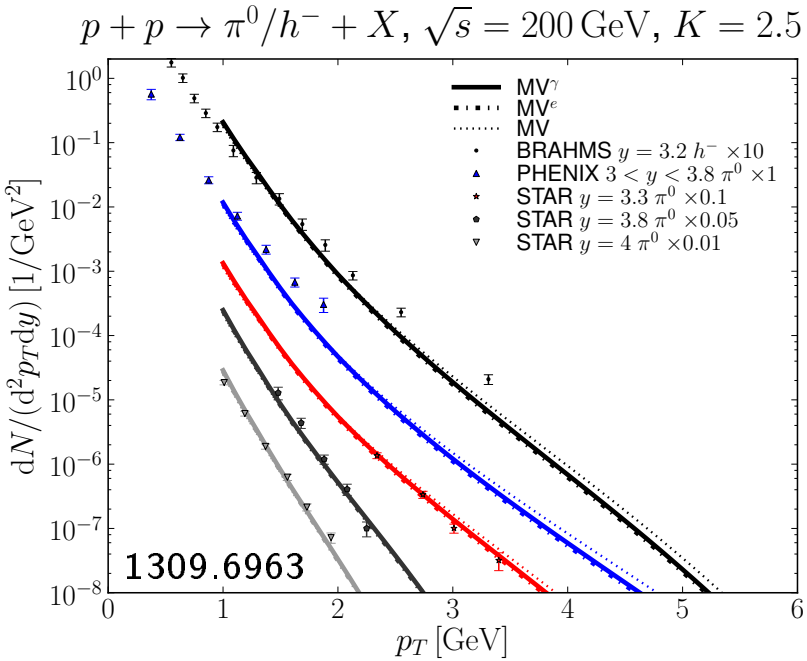
$\mathcal{S}$ is the Fourier transform of the dipole correlator $S(\mathbf{r})$:

$$\mathcal{S}(k_\perp) = \int \mathrm{d}^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} S(\mathbf{r}) \ , \quad S(\mathbf{r} = \mathbf{x} - \mathbf{y}) = \left\langle \frac{1}{N_c} \operatorname{Tr} V(\mathbf{x}) V^\dagger(\mathbf{y}) \right\rangle$$

Rapidity (or x) dependence of S : governed by the Balitsky-Kovchegov equation

Forward hadron production at LO

Using these LO expressions together with dipole correlators constrained by HERA DIS data (Lappi, Mäntysaari):

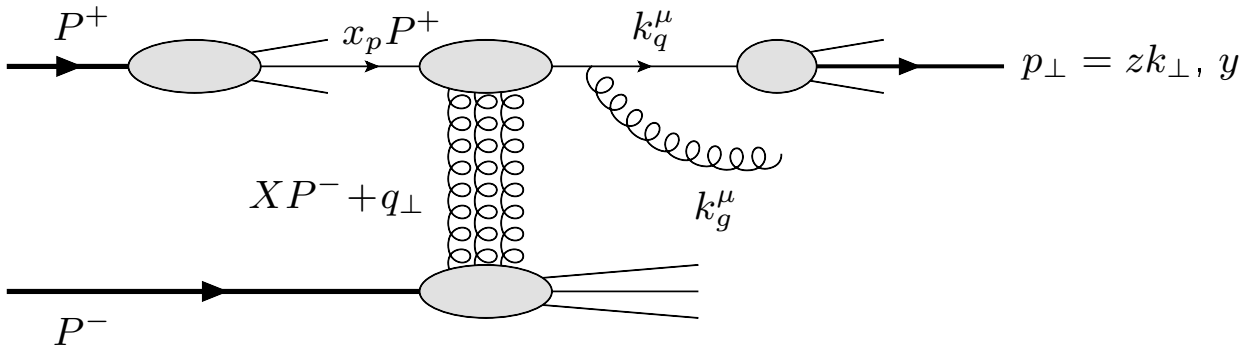


Reasonable description of the trend of the data (but large K factor needed)

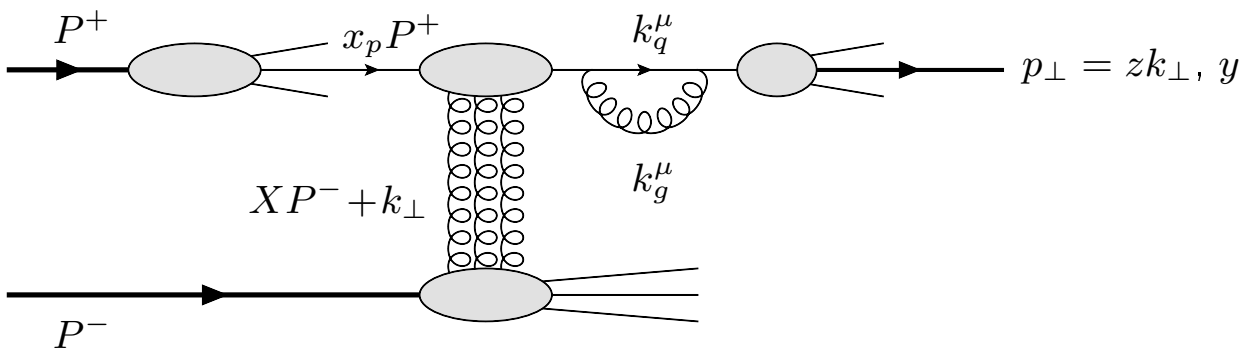
This is only leading order. What about NLO corrections?

NLO corrections to the impact factor for this process: Chirilli, Xiao, Yuan

Example of real $q \to q$ contribution:



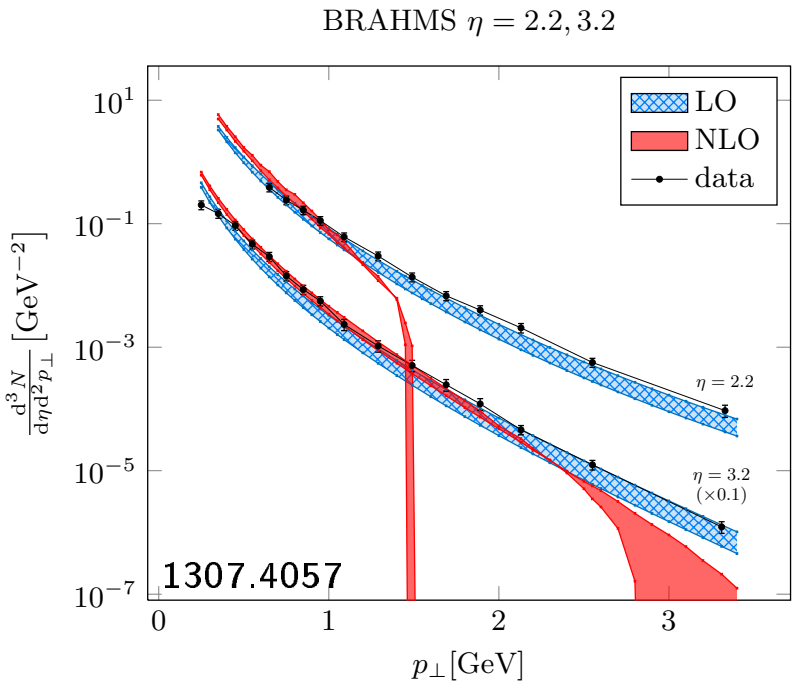
Example of virtual $q \to q$ contribution:



$1 - \xi = \frac{k_g^+}{x_p P^+}$ is the momentum fraction of the incoming quark carried by the gluon

Forward hadron production at NLO

First numerical implementation of these expressions (Staśto, Xiao, Zaslavsky):



Negative cross section above some $p_\perp \sim Q_s$

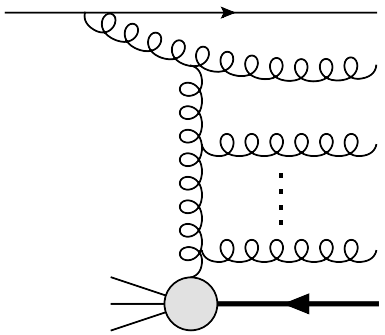
Many works devoted to solving this issue, using for example the kinematical constraint / Ioffe time cutoff (Altinoluk, Armesto, Beuf, Kovner, Lublinsky). Numerical implementation: Watanabe, Xiao, Yuan, Zaslavsky. Can extend the positivity range but doesn't solve the problem completely.

It turned out that the cause of the negativity is the subtraction of the LO contribution from the NLO corrections

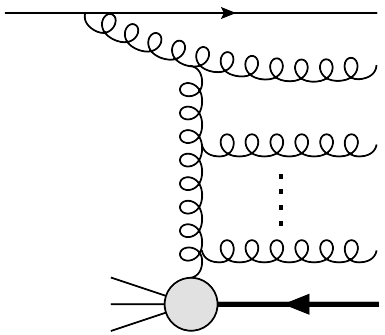
LO vs. NLO

Balitsky-Kovchegov (BK) evolution: resummation of $(\alpha_s \ln 1/x)^n$, corresponding to any number of soft gluons already at LO

LO: all gluons are soft:



NLO impact factor: the first gluon can be hard:

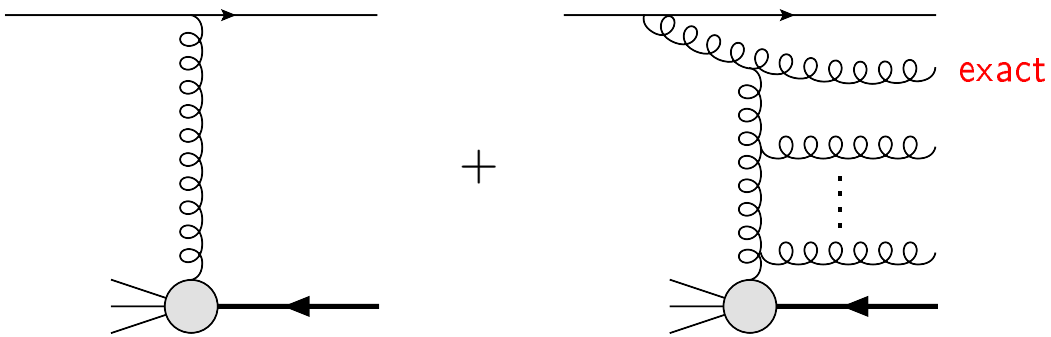


The case where the first gluon is soft is already included in the leading order
$\Rightarrow$ Need to avoid double counting between LO and NLO

Two possible solutions to avoid double counting:

1) Subtract the case where the gluon in the NLO impact factor is soft
Chirilli, Xiao, Yuan ('CXY')

2) Rearrange the terms to avoid doing a subtraction
Iancu, Mueller, Triantafyllopoulos



These two choices should be equivalent

The expression for the (quark production) multiplicity at NLO reads

$$\frac{\mathrm{d}N^{pA\to qX}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp,x_0)}{(2\pi)^2} \quad \leftarrow \text{Lowest order}$$

$$+\frac{\alpha_s}{2\pi^2}\int_{x_p}^{\xi_{\max}}\mathrm{d}\xi\frac{1+\xi^2}{1-\xi}\frac{x_p}{\xi}q\left(\frac{x_p}{\xi}\right)\left\{C_\mathrm{F}\mathcal{I}(k_\perp,\xi,X(\xi))+\frac{N_\mathrm{c}}{2}\mathcal{J}(k_\perp,\xi,X(\xi))\right\} \quad \leftarrow \text{real}$$

$$-\frac{\alpha_s}{2\pi^2}\int_{0}^{\xi_{\max}}\mathrm{d}\xi\frac{1+\xi^2}{1-\xi}x_p q\left(x_p\right)\left\{C_\mathrm{F}\mathcal{I}_v(k_\perp,\xi,X(\xi))+\frac{N_\mathrm{c}}{2}\mathcal{J}_v(k_\perp,\xi,X(\xi))\right\} \quad \leftarrow \text{virt.}$$

with e.g.

$$\mathcal{J}(k_\perp,\xi,X(\xi)) = \int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{2(\mathbf{k}-\xi\mathbf{q})\cdot(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\xi\mathbf{q})^2(\mathbf{k}-\mathbf{q})^2}\mathcal{S}(q_\perp,X(\xi))$$

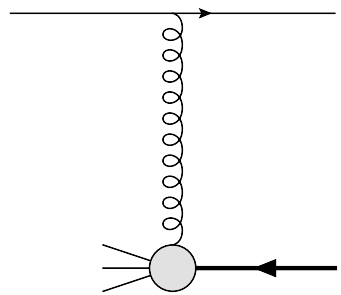
$$-\int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\frac{2(\mathbf{k}-\xi\mathbf{q})\cdot(\mathbf{k}-\mathbf{l})}{(\mathbf{k}-\xi\mathbf{q})^2(\mathbf{k}-\mathbf{l})^2}\mathcal{S}(q_\perp,X(\xi))\mathcal{S}(l_\perp,X(\xi))$$

$$\mathcal{J}_v(k_\perp,\xi,X(\xi)) = \int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{2(\xi\mathbf{k}-\mathbf{q})\cdot(\mathbf{k}-\mathbf{q})}{(\xi\mathbf{k}-\mathbf{q})^2(\mathbf{k}-\mathbf{q})^2}\mathcal{S}(k_\perp,X(\xi))$$

$$-\int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\frac{2(\xi\mathbf{k}-\mathbf{q})\cdot(\mathbf{l}-\mathbf{q})}{(\xi\mathbf{k}-\mathbf{q})^2(\mathbf{l}-\mathbf{q})^2}\mathcal{S}(k_\perp,X(\xi))\mathcal{S}(l_\perp,X(\xi))$$

The NLO cross section

In the previous expressions:



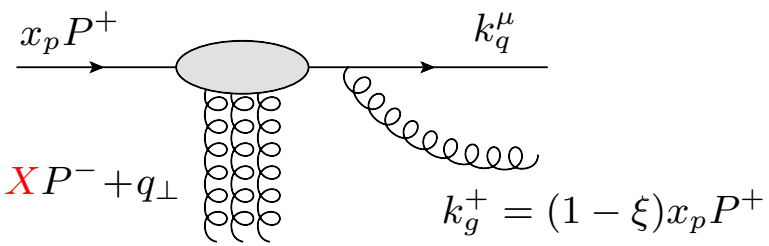
- $x_p q(x_p) \dfrac{\mathcal{S}(k_\perp, x_0)}{(2\pi)^2}$ represents the lowest order contribution

  (no phase space for BK evolution. x_0: initial condition)

- $X(\xi)$ is the rapidity scale at which the dipole correlators are evaluated

  At LO: the P^- fraction needed from the target is $\dfrac{k_\perp}{\sqrt{s}} e^{-y} \equiv x_g$

  At NLO:



$$X = \frac{k_\perp}{\sqrt{s}} e^{-y} \left(1 + \frac{\xi}{1-\xi} \frac{(q_\perp - k_\perp)^2}{k_\perp^2}\right)$$
$$\approx \frac{x_g}{1-\xi} \equiv X(\xi) \text{ when } k_\perp \gtrsim Q_s$$

The limit $\xi < 1 - \frac{x_g}{x_0} \equiv \xi_{\text{max}}$ enforces $X(\xi) < x_0$

The 'CXY' approximation corresponds to making the replacements $X(\xi) \to x_g$ and $\xi_{\text{max}} \to 1$

The terms proportional to C_F in the cross section involve the integrals

$$\mathcal{I}(k_\perp, \xi, X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \left[\frac{\mathbf{k}-\mathbf{q}}{(\mathbf{k}-\mathbf{q})^2} - \frac{\mathbf{k}-\xi\mathbf{q}}{(\mathbf{k}-\xi\mathbf{q})^2} \right]^2 \mathcal{S}(q_\perp, X(\xi))$$

$$\mathcal{I}_v(k_\perp, \xi, X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \left[\frac{\mathbf{k}-\mathbf{q}}{(\mathbf{k}-\mathbf{q})^2} - \frac{\xi\mathbf{k}-\mathbf{q}}{(\xi\mathbf{k}-\mathbf{q})^2} \right]^2 \mathcal{S}(k_\perp, X(\xi))$$

Both $\mathcal{I}$ and $\mathcal{I}_v$ vanish in the limit $\xi \to 1$. Thus the integral over ξ is not logarithmic and the C_F terms are a pure α_s correction

However these terms are divergent when the additional gluon at NLO is collinear to the initial or final state quark

These divergences are absorbed in the DGLAP evolution of the PDFs and fragmentation functions

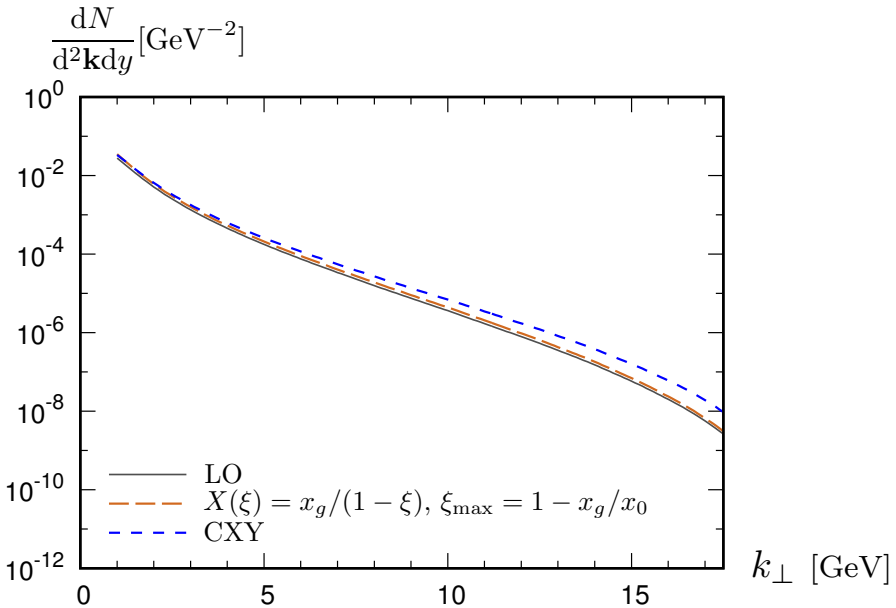
After using dimensional regularization to subtract the corresponding $1/\varepsilon$ poles, $\mathcal{I}$ and $\mathcal{I}_v$ are replaced by

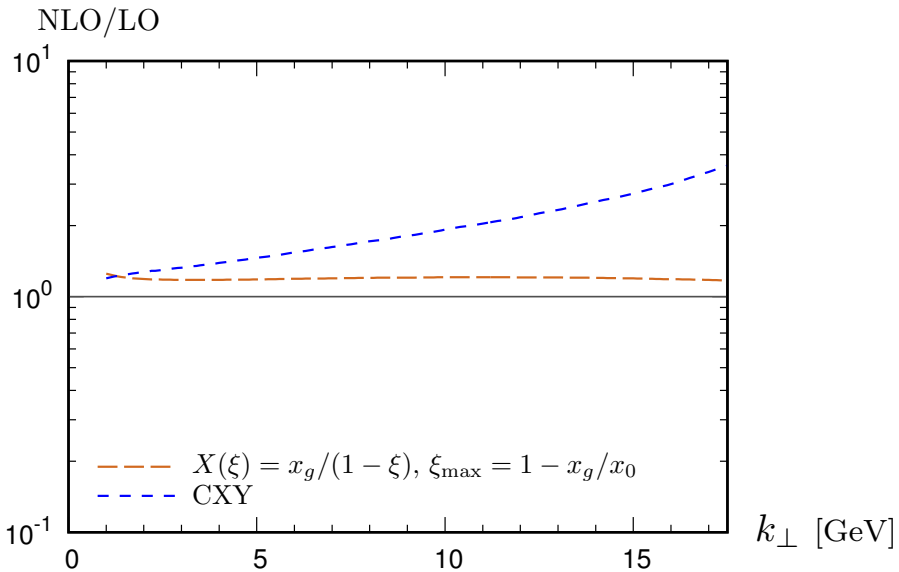
$$\mathcal{I}^{\text{finite}}(k_\perp,\xi,X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{r}}{4\pi} S(\mathbf{r},X(\xi)) \ln\frac{c_0^2}{\mathbf{r}^2\mu^2}\left(e^{-i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\xi^2}e^{-i\frac{\mathbf{k}}{\xi}\cdot\mathbf{r}}\right)$$
$$-2\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{(\mathbf{k}-\xi\mathbf{q})\cdot(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\xi\mathbf{q})^2(\mathbf{k}-\mathbf{q})^2}\mathcal{S}(q_\perp,X(\xi))$$

$$\mathcal{I}_v^{\text{finite}}(k_\perp,\xi,X(\xi)) = \frac{\mathcal{S}(k_\perp,X(\xi))}{2\pi}\left(\ln\frac{k_\perp^2}{\mu^2} + \ln(1-\xi)^2\right)$$

One can show that with these expressions the C_F terms still vanish when $\xi \to 1$

Results for the LO+C_F NLO corrections at fixed coupling ($\alpha_s = 0.2$):





The 'CXY' approximation corresponds to replacing $X(\xi) \to x_g$ and $\xi_{\max} \to 1$

In both cases the NLO corrections proportional to C_F are positive
$\to$ not the cause of the negativity

We can write the sum of the LO and N_c terms as

$$\frac{\mathrm{d}N^{\mathrm{LO}+N_c}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp, x_0)}{(2\pi)^2} + \alpha_s \int_0^{1-x_g/x_0} \frac{\mathrm{d}\xi}{1-\xi}\mathcal{K}(k_\perp, \xi, X(\xi)) \equiv \frac{\mathrm{d}N^{\mathrm{LO}+N_c, unsub}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y},$$

$$\mathcal{K}(k_\perp, \xi, X) = \frac{N_c}{(2\pi)^2}(1+\xi^2)\left[\theta(\xi - x_p)\frac{x_p}{\xi} q\left(\frac{x_p}{\xi}\right)\mathcal{J}(k_\perp, \xi, X) - x_p q\left(x_p\right)\mathcal{J}_v(k_\perp, \xi, X)\right]$$

$\mathcal{J}$ and $\mathcal{J}_v$ do not vanish when $\xi \to 1$
The second term contains both the LL evolution and NLO corrections

Using the integral BK equation,

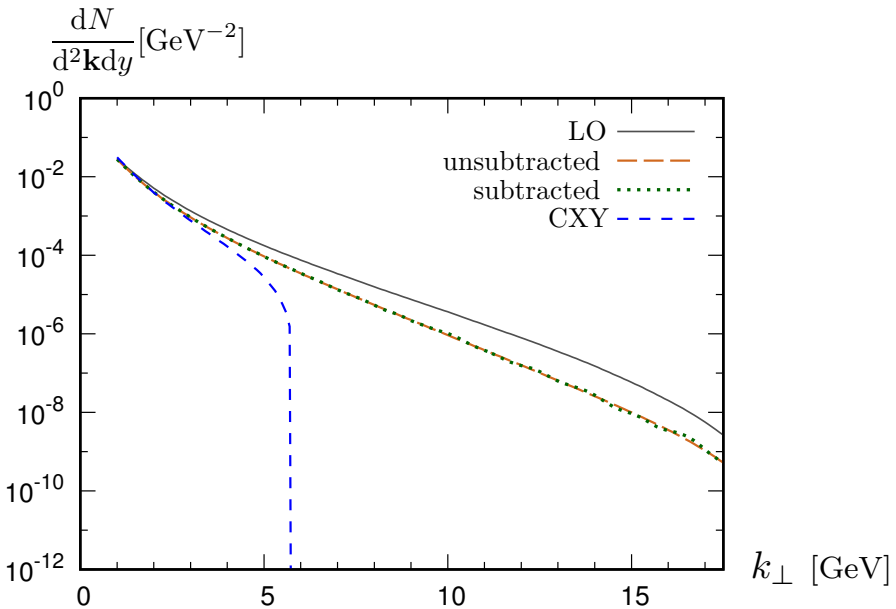
$$\mathcal{S}(k_\perp, x_g) = \mathcal{S}(k_\perp, x_0) + 2\alpha_s N_c \int_0^{1-x_g/x_0} \frac{\mathrm{d}\xi}{1-\xi}\left[\mathcal{J}(k_\perp, 1, X(\xi)) - \mathcal{J}_v(k_\perp, 1, X(\xi))\right]$$

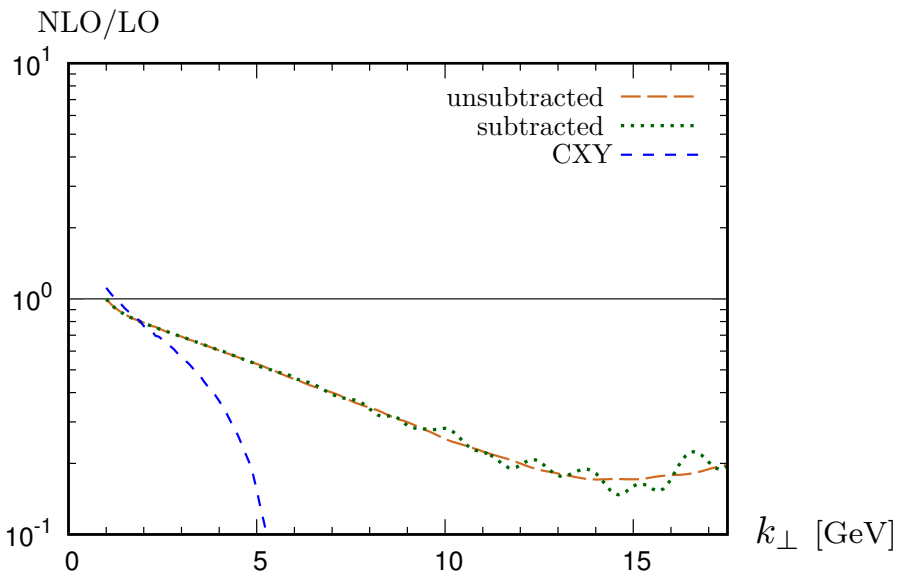
the LO+N_c cross section can be rewritten as

$$\frac{\mathrm{d}N^{\mathrm{LO}+N_c, sub}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp, x_g)}{(2\pi)^2} + \alpha_s \int_0^{1-x_g/x_0} \frac{\mathrm{d}\xi}{1-\xi}\left[\mathcal{K}(k_\perp, \xi, X(\xi)) - \mathcal{K}(k_\perp, 1, X(\xi))\right]$$

The 'CXY' approximation corresponds to making the replacements $X(\xi) \to x_g$ and $\xi_{\mathrm{max}} \to 1$ in this subtracted version

Results for the LO+N_c NLO corrections at fixed coupling ($\alpha_s = 0.2$):



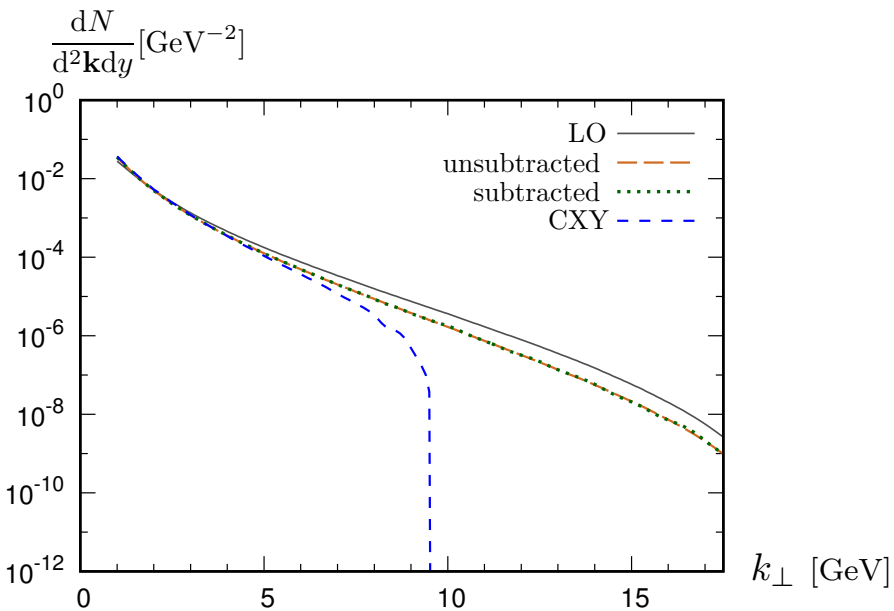


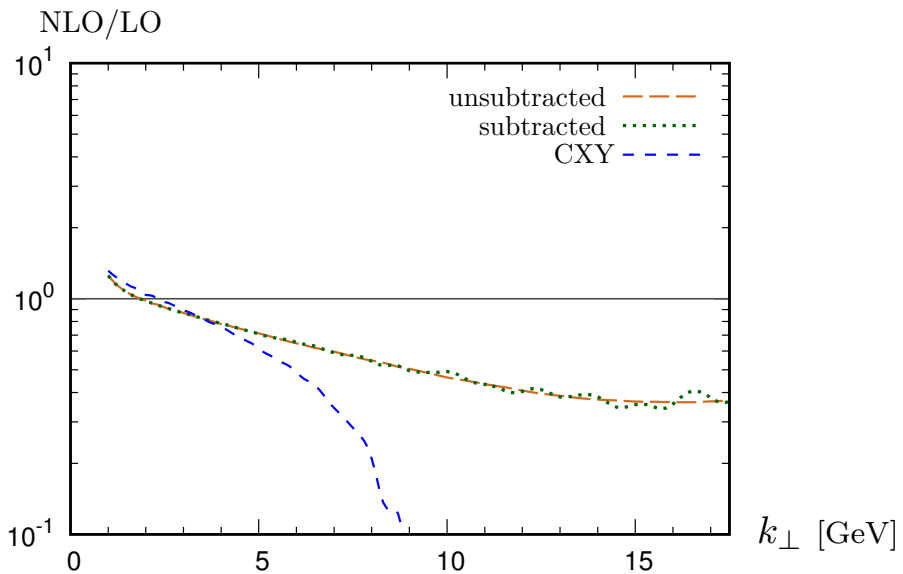
The 'subtracted' and 'unsubtracted' expressions give the same (positive) results

The 'CXY' approximation leads to negative results for $k_\perp \gtrsim 5$ GeV

Total NLO multiplicity

Total (LO+C_F+N_c) multiplicity ($\alpha_s = 0.2$):





Similar conclusions (the C_F terms are positive at large $k_\perp$)

Conclusions at fixed coupling

The negativity issue observed in the first implementation of the NLO impact factor can be attributed to approximations made in the LO subtraction

In the 'subtracted' formulation, we add and subtract a large contribution. If we use the CXY approximation what we add and subtract is no longer the same which can make the final result negative

Without this approximation the cross section has a physical behavior at all $k_\perp$
$\Rightarrow$ Problem solved? No!

So far we discussed only the fixed coupling case. But the running of the coupling is an important effect that has to be taken into account in realistic calculations

The equivalence between the 'subtracted' and 'unsubtracted' formulations holds only if one uses the same coupling α_s when computing the cross section and when solving the BK equation

In practice the BK equation is usually solved in coordinate space, while the cross section is written in momentum space

Fixed coupling BK equation:

$$\frac{\partial S(\mathbf{r},X)}{\partial \ln X} = 2\alpha_s N_c \int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2} \frac{\mathbf{r}^2}{\mathbf{x}^2(\mathbf{r}-\mathbf{x})^2} \left[S(\mathbf{r},X) - S(\mathbf{x},X)S(\mathbf{r}-\mathbf{x},X)\right]$$

In QCD one usually expects the hardest scale to determine the running of the coupling. This leads to the smallest dipole prescription:

$$\frac{\partial S(\mathbf{r},X)}{\partial \ln X} = 2N_c \int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2} \frac{\mathbf{r}^2}{\mathbf{x}^2(\mathbf{r}-\mathbf{x})^2} \alpha_s(r^2_{\mathrm{min}}) \left[S(\mathbf{r},X) - S(\mathbf{x},X)S(\mathbf{r}-\mathbf{x},X)\right]$$

with $r_{\mathrm{min}} \equiv \min\{|\mathbf{r}|, |\mathbf{x}|, |\mathbf{r}-\mathbf{x}|\}$

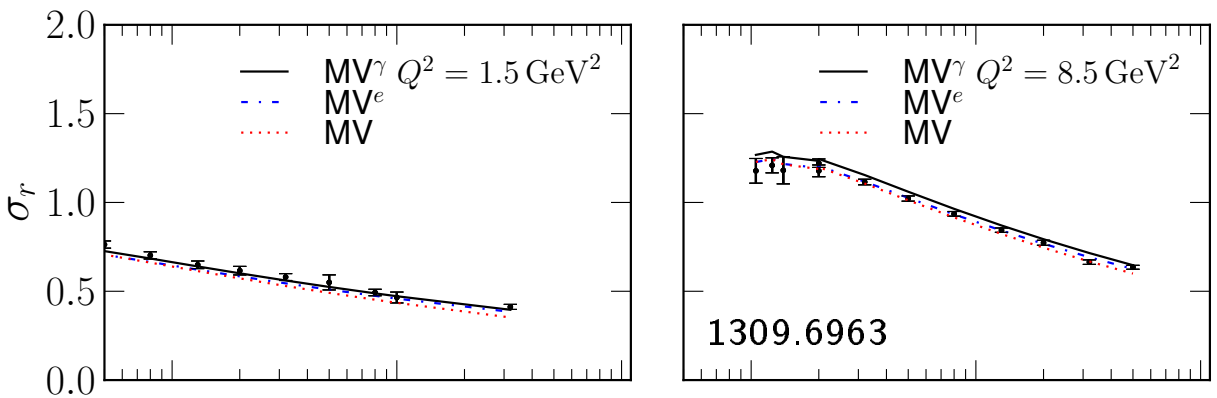
On the other hand the only natural choice in momentum space is $\alpha_s(\mathbf{k}^2)$

Many phenomenological studies use instead Balitsky's prescription:

$$\frac{\partial S(\mathbf{r},X)}{\partial \ln X} = 2\alpha_s(\mathbf{r}^2)N_c \int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2}\left[S(\mathbf{r},X) - S(\mathbf{x},X)S(\mathbf{r}-\mathbf{x},X)\right] \times \left[\frac{\mathbf{r}^2}{\mathbf{x}^2(\mathbf{r}-\mathbf{x})^2} + \frac{1}{\mathbf{x}^2}\left(\frac{\alpha_s(\mathbf{x}^2)}{\alpha_s((\mathbf{r}-\mathbf{x})^2)} - 1\right) + \frac{1}{(\mathbf{r}-\mathbf{x})^2}\left(\frac{\alpha_s((\mathbf{r}-\mathbf{x})^2)}{\alpha_s(\mathbf{x}^2)} - 1\right)\right]$$

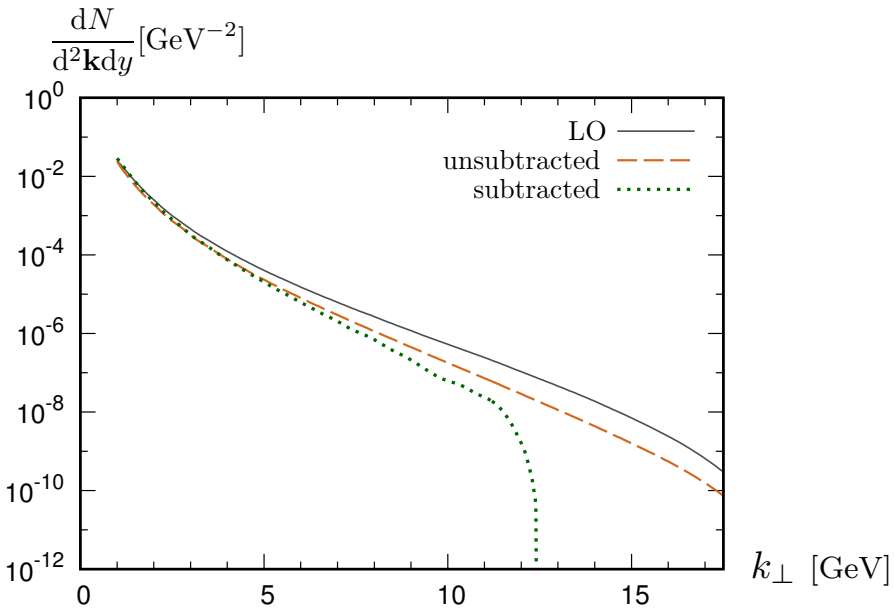
which reduces to the smallest dipole prescription when one of the three dipoles is much smaller than the other two

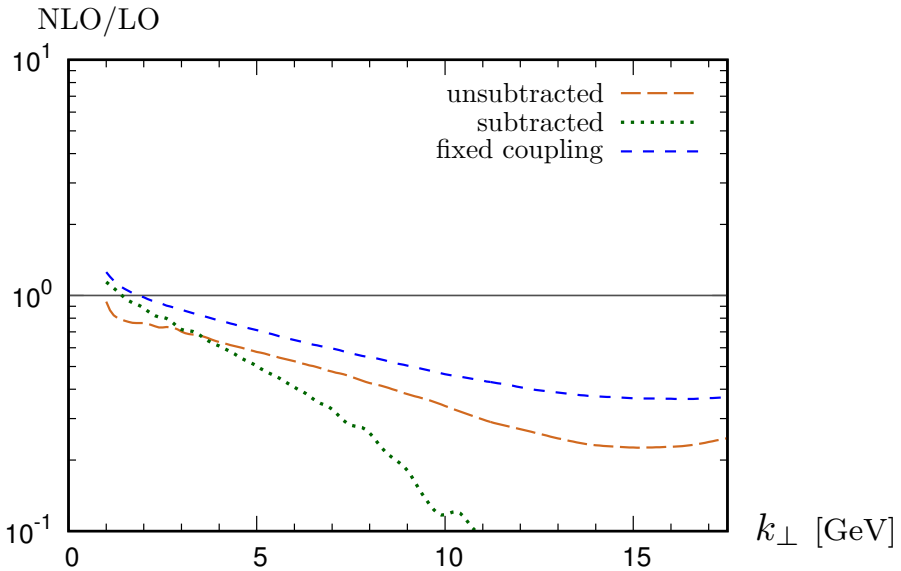
Here we use the dipole correlators obtained by solving the rcBK equation with Balitsky's prescription and constrained by HERA DIS data (Lappi, Mäntysaari)



Running coupling

Using these dipole correlators and $\alpha_s(\mathbf{k}^2)$ in the cross section:





The 'subtracted' and 'unsubtracted' expressions are no longer equivalent since we don't use exactly the same α_s in the cross section and when solving BK

- 'Subtracted' version: very different results compared to fixed coupling, becomes negative again at large $k_\perp$
- 'Unsubtracted' version: physical results but does not have the correct LO limit

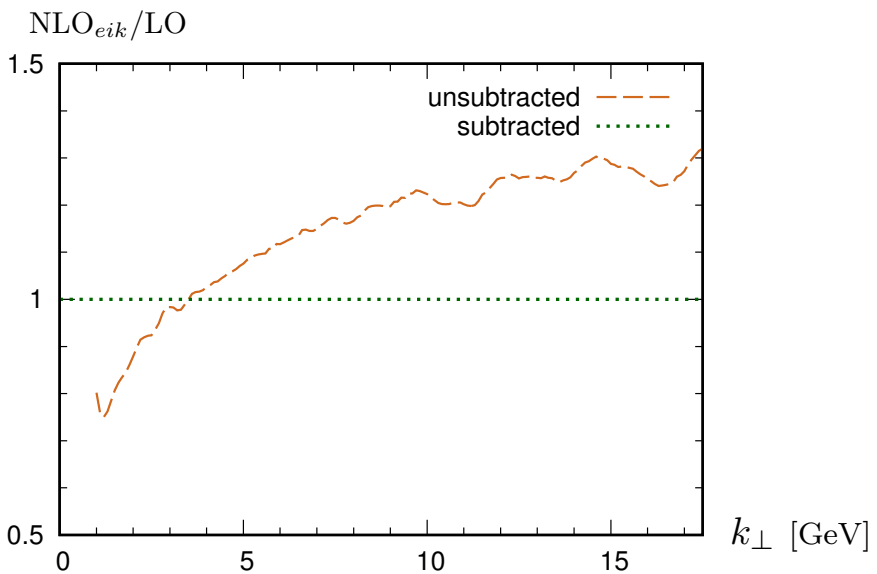
The LO limit of the 'subtracted' cross section is simply obtained by $\alpha_s \to 0$:

$$\frac{\mathrm{d}N^{\mathrm{LO}+N_c,sub}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp, x_g)}{(2\pi)^2} + \alpha_s \int_0^{1-x_g/x_0} \frac{\mathrm{d}\xi}{1-\xi}\left[\mathcal{K}(k_\perp,\xi,X(\xi)) - \mathcal{K}(k_\perp,1,X(\xi))\right]$$

For the 'unsubtracted' formulation it is obtained by setting $\xi \to 1$:

$$\frac{\mathrm{d}N^{\mathrm{LO}+N_c,unsub}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp, x_0)}{(2\pi)^2} + \alpha_s \int_0^{1-x_g/x_0} \frac{\mathrm{d}\xi}{1-\xi}\mathcal{K}(k_\perp,\xi,X(\xi))$$

If we don't use the same α_s when solving BK and in the cross section, the 'eikonal' limit of the unsubtracted formulation is not the correct LO result:



For these reasons it would be preferable to use the same coupling everywhere

Possible way to do that: rewrite the cross section expression in coordinate space

We write $\mathcal{J} = \int \mathrm{d}^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \widetilde{\mathcal{J}}$ and $\mathcal{J}_v = \int \mathrm{d}^2\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \widetilde{\mathcal{J}}_v$, with

$$\widetilde{\mathcal{J}}(\mathbf{r},\xi,X) = 2\int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2} \frac{\mathbf{x}\cdot(\mathbf{x}-\mathbf{r})}{\mathbf{x}^2(\mathbf{r}-\mathbf{x})^2} \left[S(\mathbf{r}-(1-\xi)\mathbf{x},X) - S(\xi\mathbf{x},X)S(\mathbf{r}-\mathbf{x},X)\right]$$

$$\widetilde{\mathcal{J}}_v(\mathbf{r},\xi,X) = 2\int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2} \frac{1}{\mathbf{x}^2} \left[S(\mathbf{r}+(1-\xi)\mathbf{x},X) - S(\mathbf{x},X)S(\mathbf{r}-\xi\mathbf{x},X)\right]$$

(and similarly for the C_F terms)

In these notations the BK equation reads

$$\frac{\partial S(\mathbf{r},X)}{\partial \ln X} = -2\alpha_s N_\mathsf{c} \left[\widetilde{\mathcal{J}}(\mathbf{r},1,X) - \widetilde{\mathcal{J}}_v(\mathbf{r},1,X)\right]$$

Recall rcBK with Balitsky's prescription:

$$\frac{\partial S(\mathbf{r},X)}{\partial \ln X} = 2\alpha_s(\mathbf{r}^2)N_c \int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2}\left[S(\mathbf{r},X) - S(\mathbf{x},X)S(\mathbf{r}-\mathbf{x},X)\right] \times \left[\frac{\mathbf{r}^2}{\mathbf{x}^2(\mathbf{r}-\mathbf{x})^2} + \frac{1}{\mathbf{x}^2}\left(\frac{\alpha_s(\mathbf{x}^2)}{\alpha_s((\mathbf{r}-\mathbf{x})^2)} - 1\right) + \frac{1}{(\mathbf{r}-\mathbf{x})^2}\left(\frac{\alpha_s((\mathbf{r}-\mathbf{x})^2)}{\alpha_s(\mathbf{x}^2)} - 1\right)\right]$$

This can be generalized to $\xi \neq 1$ by replacing $\widetilde{\mathcal{J}}_v$ with

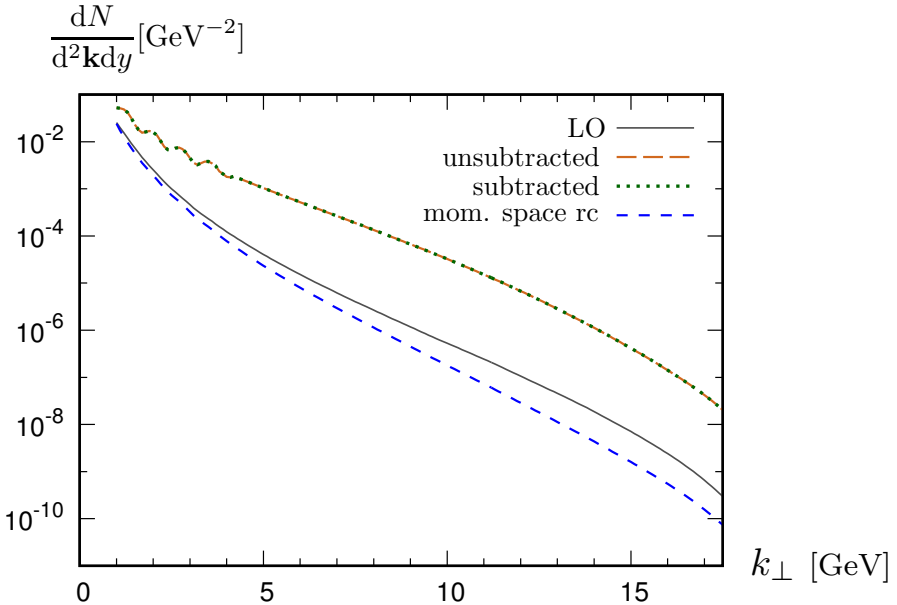
$$\widetilde{\mathcal{J}}_v^{\mathrm{rc}}(\mathbf{r},\xi,X) = 2\int \frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2}\frac{1}{\mathbf{x}^2}\frac{\alpha_s(\mathbf{x}^2)}{\alpha_s((\mathbf{r}-\xi\mathbf{x})^2)}\left[S(\mathbf{r}+(1-\xi)\mathbf{x},X) - S(\mathbf{x},X)S(\mathbf{r}-\xi\mathbf{x},X)\right],$$

and by replacing the global α_s factor by $\alpha_s(\mathbf{r}^2)$

Not a unique choice but:

- $\xi = 1$: recovers Balitsky's prescription
- Fixed coupling results unchanged

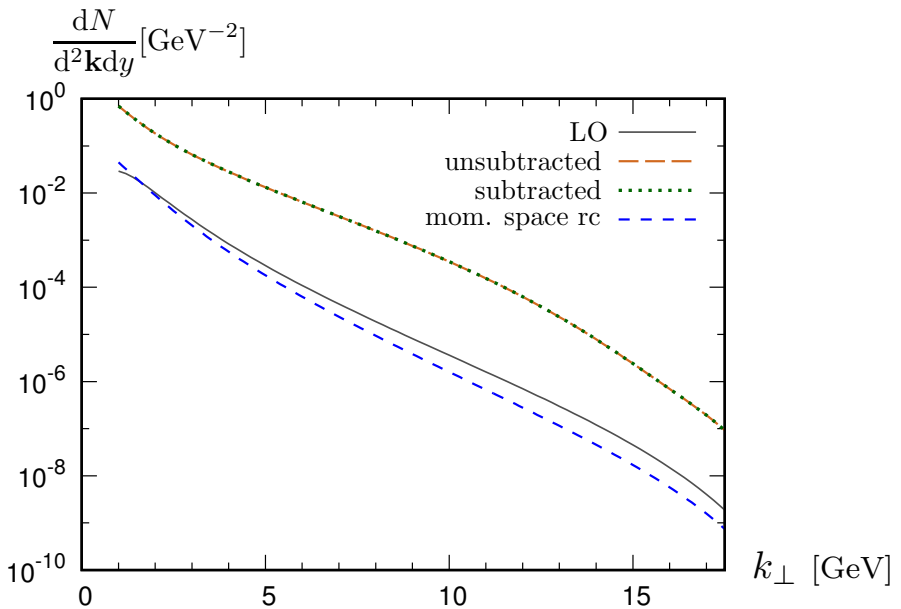
Results with this 'generalized' Balitsky prescription:



The 'subtracted' expression gives the same results as the 'unsubtracted' one

Completely different results compared to fixed coupling or $\alpha_s(k_\perp)$, absurdly large NLO corrections

The issue is not specific to our choice for the running coupling. Indeed it also appears using the simple parent dipole prescription $\alpha_s(\mathbf{r}^2)$:



Coordinate space formulation

To illustrate the problem, let's look at the following simple example:

$$\mathcal{N}_k \equiv \bar{\alpha}_s(k_\perp)\,\mathcal{S}(k_\perp) = \bar{\alpha}_s(k_\perp)\int \mathrm{d}^2\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} S(\mathbf{r})$$

$$\mathcal{N}_r \equiv \int \mathrm{d}^2\mathbf{r}\, \bar{\alpha}_s(r_\perp) e^{-i\mathbf{k}\cdot\mathbf{r}} S(\mathbf{r})$$

These two quantities do not differ by only a small factor. Indeed, using the McLerran-Venugopalan model $S(r_\perp) = \exp\left(-\frac{r_\perp^2 Q_s^2}{4}\,\ln\frac{1}{r_\perp^2\Lambda^2}\right)$, we find at large $k_\perp$

$$\mathcal{N}_k \sim \frac{4\pi\bar{\alpha}_s(k_\perp)Q_s^2}{k_\perp^4} \qquad \text{while} \qquad \mathcal{N}_r \sim -\frac{4\pi}{\bar{b}[\ln(k_\perp^2/\Lambda^2)]^2}\,\frac{1}{k_\perp^2}$$

which are opposite in sign and have different tails

The choice of the running coupling prescription and the Fourier transform do not 'commute'

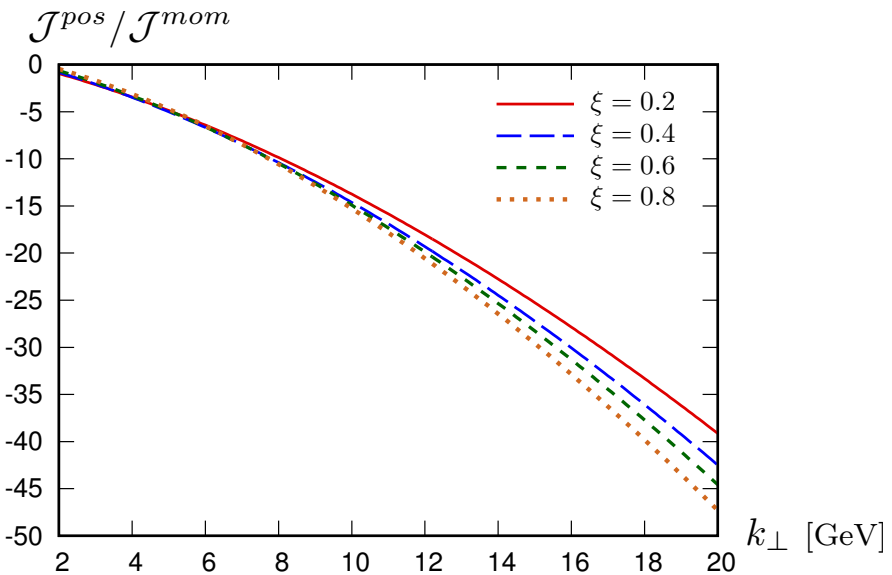
Note that the problem comes from the perturbative region $\mathbf{r} \to 0$, not the IR region

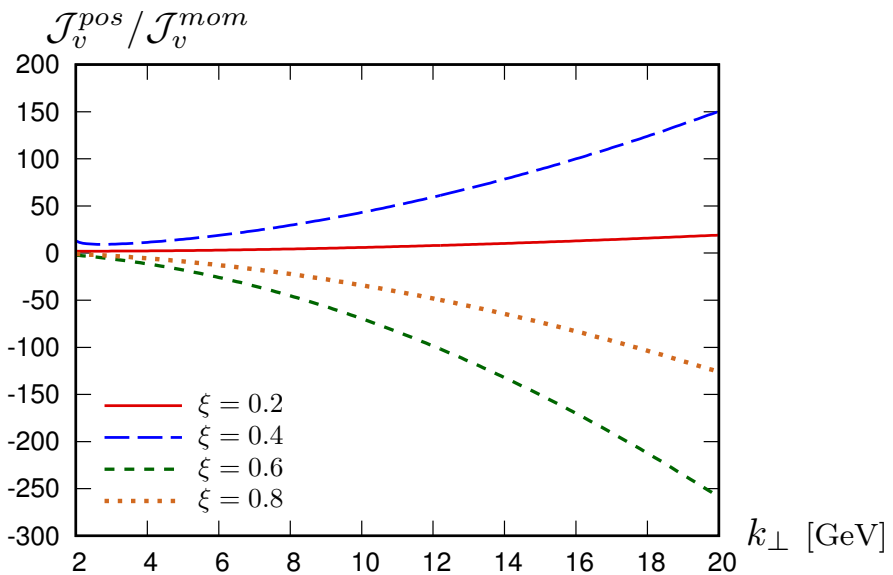
The problem comes from large daughter dipoles contributions $x_\perp \gg r_\perp$. Indeed, in this limit we have for example

$$\mathcal{J}(\mathbf{k},\xi) \sim \frac{\bar{\alpha}_\mathbf{s}}{2\pi^2}\int \mathrm{d}^2\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \underbrace{\int \frac{\mathrm{d}^2\mathbf{x}}{\mathbf{x}^2}\,[S((1-\xi)\mathbf{x}) - S(-\xi\mathbf{x})S(\mathbf{x})]}_{\mathbf{r}-\text{independent}} = 0 \text{ for } k_\perp \neq 0$$

On the contrary, if we move the coupling under the integral and replace $\alpha_s \to \alpha_s(\mathbf{r}^2)$, we get a large contribution from the F.T. of the coupling

Numerically we can study how $\mathcal{J}$ and $\mathcal{J}_v$ are modified when using $\alpha_s(\mathbf{r}^2)$ instead of $\alpha_s(\mathbf{k}^2)$:



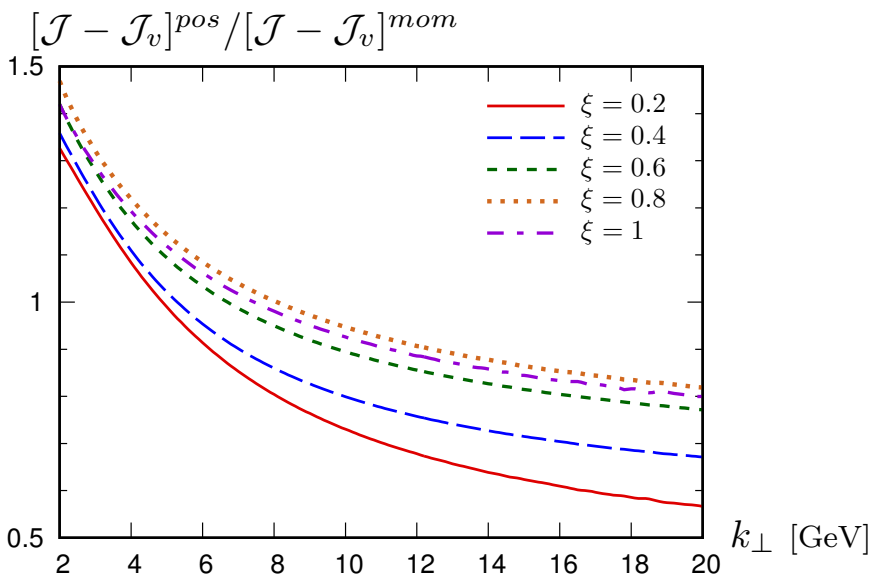


However we know that such severe issues don't appear when solving BK with these prescriptions. Why?

The BK equation involves the difference between $\widetilde{\mathcal{J}}$ and $\widetilde{\mathcal{J}}_v$ at $\xi = 1$:

$$\frac{\partial S(\mathbf{r}, X)}{\partial \ln X} = -2\alpha_s N_c \left[\widetilde{\mathcal{J}}(\mathbf{r}, 1, X) - \widetilde{\mathcal{J}}_v(\mathbf{r}, 1, X)\right]$$

Thus the spurious contributions generated by large daughter dipoles cancel:

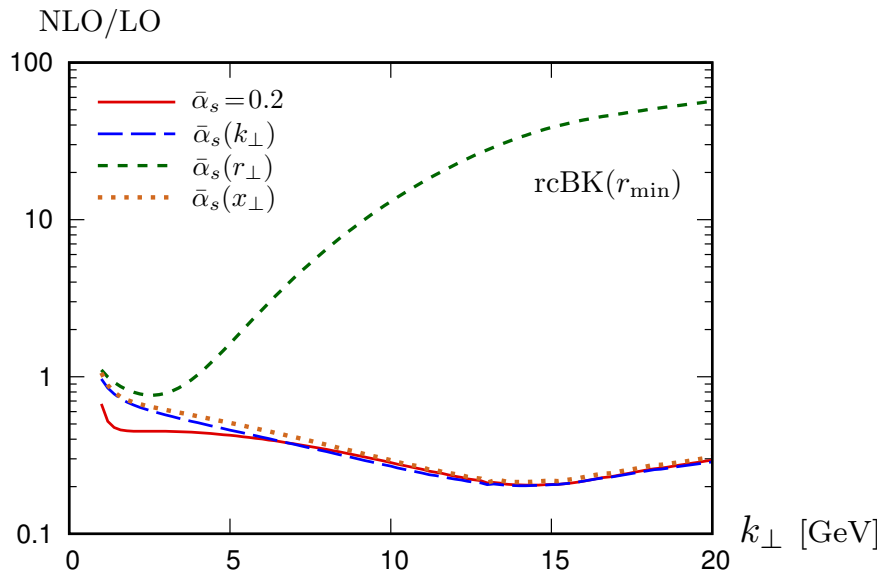


But in the cross section we have $\left[\widetilde{\mathcal{J}} \times q(x_p/\xi) - \widetilde{\mathcal{J}}_v \times q(x_p)\right] \Rightarrow$ No cancellation

Coordinate space formulation

Based on this we expect that the daughter dipole prescription, $\alpha_s(\mathbf{x}^2)$, should lead to physical results: the spurious contributions generated by large daughter dipoles will remain independent of $\mathbf{r}$, and will thus be eliminated by the Fourier transform

With this prescription the cross section indeed has a physical behavior, similar to the results with fixed or momentum space running coupling:



$\Rightarrow$ By using the daughter dipole prescription we can get:

- Physical results at all $k_\perp$
- Correct LO limit

So far we focused mostly on the N_c NLO corrections. Recall the NLO multiplicity:

$$\frac{\mathrm{d}N^{pA\to qX}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = x_p q(x_p)\frac{\mathcal{S}(k_\perp, x_0)}{(2\pi)^2} + \frac{\alpha_s}{2\pi^2}\int_{x_p}^{\xi_{\max}} \mathrm{d}\xi \frac{1+\xi^2}{1-\xi}\frac{x_p}{\xi} q\left(\frac{x_p}{\xi}\right)\left\{C_F\mathcal{I}(k_\perp,\xi,X(\xi)) + \frac{N_c}{2}\mathcal{J}(k_\perp,\xi,X(\xi))\right\} - \frac{\alpha_s}{2\pi^2}\int_0^{\xi_{\max}} \mathrm{d}\xi \frac{1+\xi^2}{1-\xi} x_p q(x_p)\left\{C_F\mathcal{I}_v(k_\perp,\xi,X(\xi)) + \frac{N_c}{2}\mathcal{J}_v(k_\perp,\xi,X(\xi))\right\}$$

The C_F terms involve the integrals

$$\mathcal{I}(k_\perp,\xi,X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\left[\frac{\mathbf{k}-\mathbf{q}}{(\mathbf{k}-\mathbf{q})^2} - \frac{\mathbf{k}-\xi\mathbf{q}}{(\mathbf{k}-\xi\mathbf{q})^2}\right]^2 \mathcal{S}(q_\perp, X(\xi))$$

$$\mathcal{I}_v(k_\perp,\xi,X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\left[\frac{\mathbf{k}-\mathbf{q}}{(\mathbf{k}-\mathbf{q})^2} - \frac{\xi\mathbf{k}-\mathbf{q}}{(\xi\mathbf{k}-\mathbf{q})^2}\right]^2 \mathcal{S}(k_\perp, X(\xi))$$

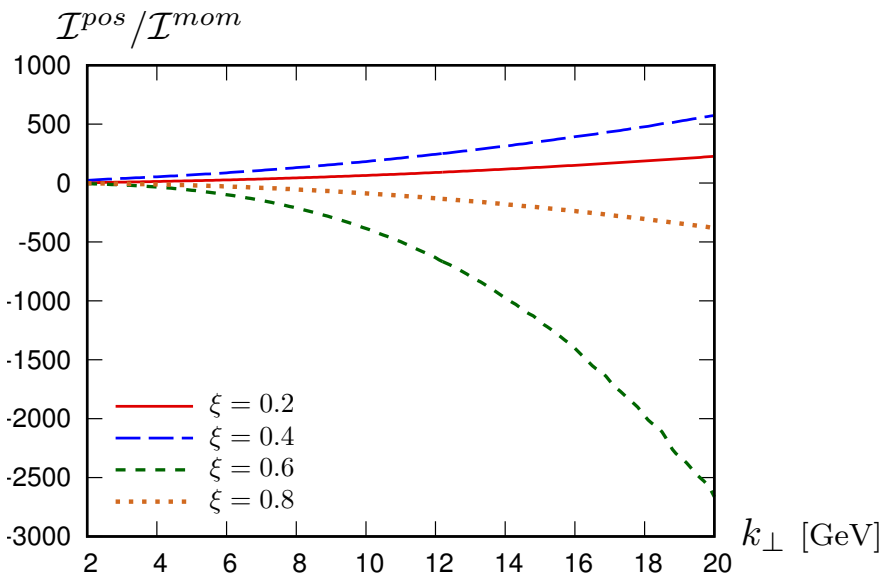
After subtracting the collinear divergence, we should replace $\mathcal{I}$ and $\mathcal{I}_v$ by

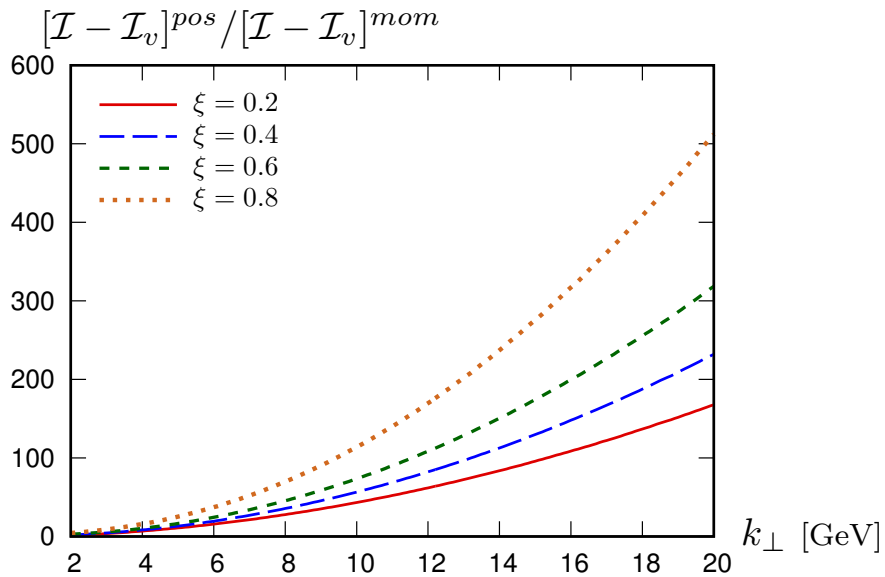
$$\mathcal{I}^{\text{finite}}(k_\perp,\xi,X(\xi)) = \int \frac{\mathrm{d}^2\mathbf{r}}{4\pi} S(\mathbf{r},X(\xi)) \ln\frac{c_0^2}{\mathbf{r}^2\mu^2}\left(e^{-i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\xi^2}e^{-i\frac{\mathbf{k}}{\xi}\cdot\mathbf{r}}\right)$$
$$-2\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{(\mathbf{k}-\xi\mathbf{q})\cdot(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\xi\mathbf{q})^2(\mathbf{k}-\mathbf{q})^2}\mathcal{S}(q_\perp,X(\xi))$$

$$\mathcal{I}_v^{\text{finite}}(k_\perp,\xi,X(\xi)) = \frac{\mathcal{S}(k_\perp,X(\xi))}{2\pi}\left(\ln\frac{k_\perp^2}{\mu^2} + \ln(1-\xi)^2\right)$$

With these expressions the C_F terms still vanish when $\xi \to 1$, i.e. they do not generate a large longitudinal logarithm

From the previous expressions it is clear that $\mathcal{I}_v^{\text{finite}}$ will be affected by the same Fourier transform problem as the N_c terms. This is also the case of $\mathcal{I}^{\text{finite}}$:





But contrary to what happens with $\mathcal{J}$ and $\mathcal{J}_v$ the problem is still there if we consider the difference $\mathcal{I}^{\text{finite}} - \mathcal{I}_v^{\text{finite}}$

In addition to this Fourier-transform artefact, another problem is that if we use for example $\alpha_s(\mathbf{r}^2)$, the C_F terms will generate a spurious large longitudinal logarithm

Anyway we can't solve the F.T. problem in the same way as for the N_c terms: because of the collinear subtraction we have no control on the size of the daughter dipoles anymore, so we cannot use the coupling $\alpha_s(\mathbf{x}^2)$

$$\mathcal{I}^{\text{finite}}(k_\perp,\xi,X(\xi)) = \int \mathrm{d}^2\mathbf{r}\left[\frac{S(\mathbf{r},X(\xi))}{4\pi}\ln\frac{c_0^2}{\mathbf{r}^2\mu^2}\left(e^{-i\mathbf{k}\cdot\mathbf{r}}+\frac{1}{\xi^2}e^{-i\frac{\mathbf{k}}{\xi}\cdot\mathbf{r}}\right)\right.$$
$$\left.-2e^{-i\mathbf{k}\cdot\mathbf{r}}\int\frac{\mathrm{d}^2\mathbf{x}}{(2\pi)^2}\frac{\mathbf{x}\cdot(\mathbf{x}+\mathbf{r})}{\mathbf{x}^2(\mathbf{x}+\mathbf{r})^2}S(\xi\mathbf{r}-(1-\xi)\mathbf{x},X(\xi))\right]$$

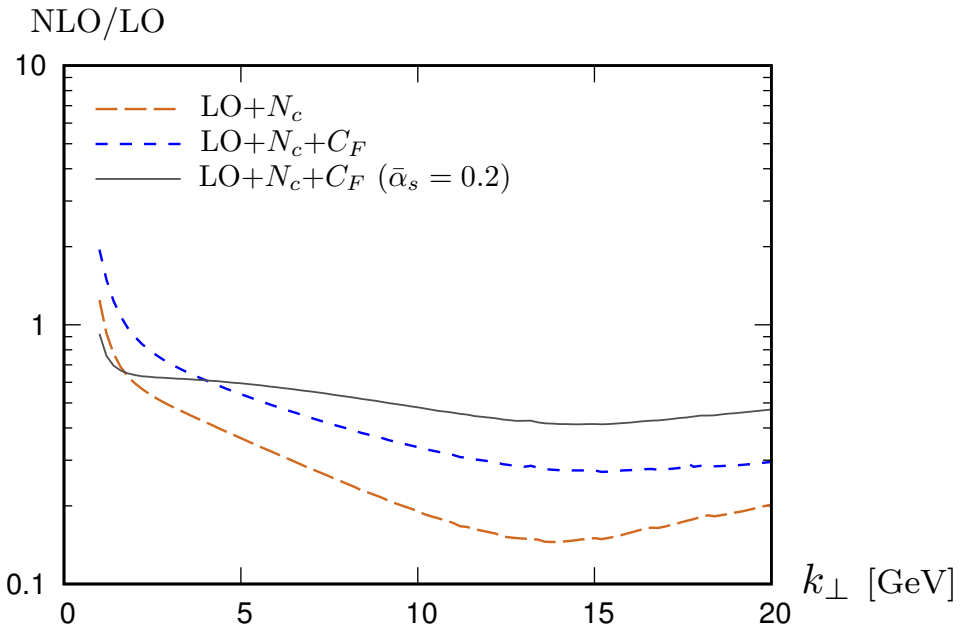
$$\mathcal{I}_v^{\text{finite}}(k_\perp,\xi,X(\xi)) = \int \mathrm{d}^2\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{S(\mathbf{r},X(\xi))}{2\pi}\left(\ln\frac{k_\perp^2}{\mu^2}+\ln(1-\xi)^2\right)$$

Therefore the only reasonable choice for these terms seems to be a coupling running with a transverse scale, such as $\alpha_s(\mathbf{k}^2)$

This is not as problematic as for the N_c terms: the C_F terms are not related to high energy evolution and are a pure α_s correction

If one insists on using the same coupling everywhere, the most practical way could be to perform the whole calculation in momentum space

Results using $\alpha_s(\mathbf{k}^2)$ in the C_F terms and $\alpha_s(\mathbf{x}^2)$ everywhere else:



- Physical results at all $k_\perp$
- Correct LO limit
- Same results for the 'subtracted' and 'unsubtracted' formulations

However it is not natural to use the daughter dipole prescription when solving BK

Conclusions

The negativity problem at NLO originally observed for this process is now understood and solved as long as fixed coupling is considered

Running coupling: additional complications appear

- Mixed coordinate/momentum space calculation: mismatch between 'subtracted' and 'unsubtracted' formulations
- 'Naive' coordinate space formulation: non-physical results due to spurious large daughter dipoles contributions which should cancel in the end
- The problem can be avoided for the N_c terms by using the daughter dipole prescription
- The use of a momentum space running coupling seems mandatory for the C_F terms

Directions towards phenomenology:

- Add the $q \to g$, $g \to q$ and $g \to g$ channels + fragmentation functions
- Use NLO BK for the rapidity evolution of the dipole correlators
- The initial condition for the BK evolution of the target must be obtained by a fit (e.g. to HERA DIS data) also performed at NLO accuracy