Inferring the dense nuclear matter equation of state
with neutron star tides

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in collaboration with
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Neutron stars as multi-messenger laboratories for dense matter
ECT*-EMMI/GSI workshop

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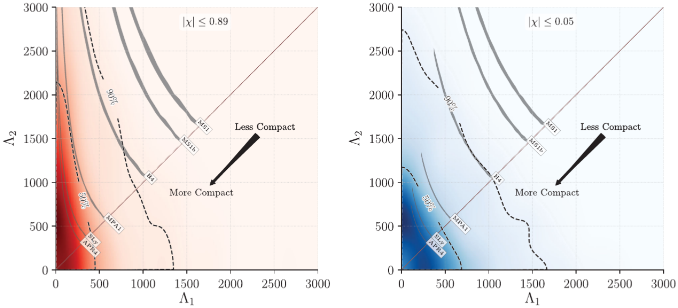
- Tidal deformability is parametrised through the *Love number* k_l

$$l\text{-th multipole moment} = k_l \times (l\text{-th harmonic of tidal potential})$$

- The Love number increases with equation of state "stiffness"
  - Can be extracted from GW data analysis and constrain the equation of state

Stiffer equation of state ⇔ Less compact ("puffier") star ⇔ Easier to deform

GW170817 tidal deformability constraints



Credit: Abbott *et al.* (2017) PRL **119**, 161101

$$\Lambda = \frac{2k_2}{3}\left(\frac{Rc^2}{GM}\right)^5$$

- But: k_l is defined through the *equilibrium tide* approximation
  - Equilibrium tide: tidally-perturbed star is always in hydrostatic equilibrium
  - Dynamical tide: non-instantaneous fluid response, oscillation mode resonances

The tidal problem

- Equation of motion for tidal perturbation, driven by tidal potential

$$\ddot{\boldsymbol{\xi}} + \frac{\nabla \delta p}{\rho} - \frac{\nabla p}{\rho^2}\delta\rho + \nabla\delta\Phi = -\nabla U \quad \Rightarrow \quad \boxed{k_{lm} = \frac{\delta\Phi_{lm}(R)}{2U_{lm}(R)}}$$

- **Full solution:** $\boldsymbol{\xi} = [\xi_r(r)\hat{e}_r + \xi_h(r)\nabla]\, Y_l^m(\theta,\phi)\, e^{im\omega_{\rm orb}t}$

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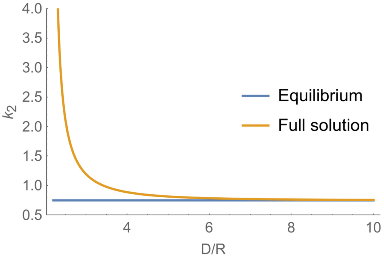
◦ Simplest possible neutron star model: incompressible star ($\rho = \text{const.}$)

– Equilibrium tide: $k_l = \dfrac{3}{4(l-1)}$

– Full solution: $k_{lm}^{\rm eff} = k_l \left[1 - \dfrac{(m\omega_{\rm orb})^2}{\omega_f^2}\right]^{-1}$

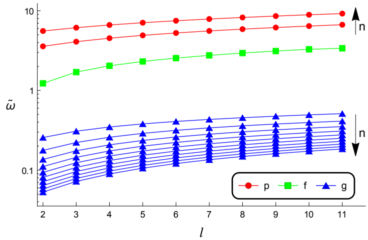
with $\omega_f^2 = \dfrac{2l(l-1)}{2l+1}\dfrac{GM}{R^3}$ (f-mode)



◦ *Effective* Love number contains f-mode resonances at $\omega_{\rm orb} = \pm\omega_f/m$

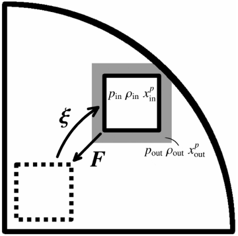
- Tidal perturbation decomposition with respect to *oscillation modes*

$$\boldsymbol{\xi} = \sum_{\alpha} a_\alpha(t)\boldsymbol{\xi}_\alpha(\boldsymbol{r}), \quad \ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = -\int \boldsymbol{\xi}_\alpha^* \cdot \nabla U \rho \mathrm{d}^3\boldsymbol{r}$$



- Fluid (polar) modes:
  - f-modes: fundamental oscillations ($n = 0$)
  - p-modes: acoustic oscillations } $n > 0$
  - g-modes: buoyancy oscillations } $n > 0$

- *Buoyancy* is present if nuclear reaction timescale > perturbation timescale



Credit: Herbrik and Kokkotas (2017) MNRAS **466**, 1330

  - β reactions (Urca reactions):

$$n\,(+\,n) \to p\,(+\,n) + e^- + \bar{\nu}_e, \quad p\,(+\,n) + e^- \to n\,(+\,n) + \nu_e$$

  - Relevant timescales:

$$t_m \sim 2\left(\frac{10^9\,\mathrm{K}}{T}\right)^6 \text{ months} \quad \text{(modified Urca reactions)}$$

$$t_d \sim 20\left(\frac{10^9\,\mathrm{K}}{T}\right)^4 \text{ sec} \quad \text{(direct Urca reactions)}$$

- During last stages of inspiral, composition is "frozen"

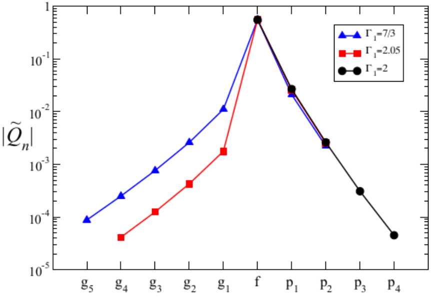
$$k_l = \frac{2\pi G}{(2l+1)R^{2l+1}} \sum_\alpha \frac{Q_\alpha^2}{\omega_\alpha^2}$$

$\Gamma_1 = 2$		$\Gamma_1 = 2.05$		$\Gamma_1 = 7/3$	
mode	k_l	mode	k_l	mode	k_l
f	0.27528	f	0.27055	f	0.24685
$+p_1$	0.25887	$+p_1$	0.25526	$+g_1$	0.26115
$+p_2$	0.26021	$+p_2$	0.25653	$+p_1$	0.25052
$+p_3$	0.26015	$+g_1$	0.25878	$+g_2$	0.25556
		$+g_2$	0.25960	$+p_2$	0.25653
		$+g_3$	0.25993	$+g_3$	0.25856
		$+g_4$	0.26008	$+g_4$	0.25944
				$+g_5$	0.25983
	9×10^{-4}		7×10^{-4}		3×10^{-4}

Results for $\Gamma = 2$

$$Q_\alpha \sim \int \boldsymbol{\xi}_\alpha^* \cdot \nabla U \rho \mathrm{d}^3\boldsymbol{r}$$

Overlap between the mode and the tide

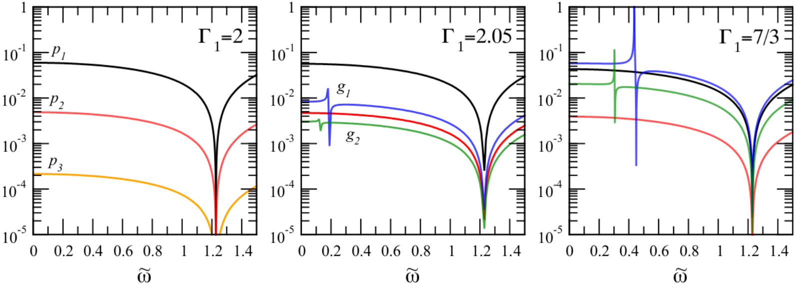


- Equation of state: $p = K\rho^\Gamma$
- Buoyancy arises when $\Gamma_1 - \Gamma \neq 0$, with $\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{\text{composition}}$

- Love number converges to its expected value ($k_2 \approx 0.2599$)
- Equilibrium tide is oblivious to composition gradients

$$k_{lm}^{\mathrm{eff}} = \frac{2\pi G}{(2l+1)R^{2l+1}} \sum_{\alpha} \frac{Q_\alpha^2}{\omega_\alpha^2 - (m\omega_{\mathrm{orb}})^2}$$

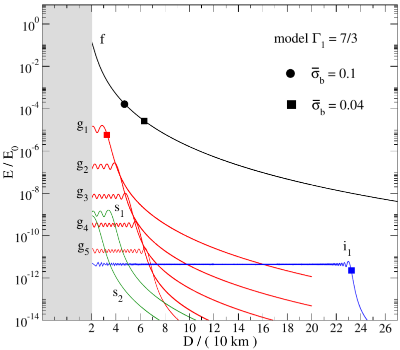
Mode contributions to the Love number (relative to f-mode)



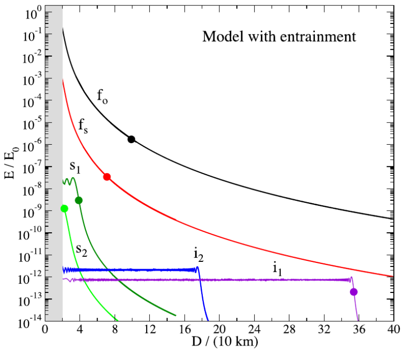
- Contribution of each mode is large near its resonance
- f-mode prevails, as expected
- Composition dependence enters at $\sim 10^{-2}$
  - Possibly within reach of 3G detectors

- Crust surface layer ⇒ elastic stresses
  - *s*-modes: shear oscillations
  - *i*-modes: oscillations at the crust-core interface
- Superfluidity ⇒ additional fluid degree of freedom
  - Superfluid counterparts of ordinary fluid modes
- *f*-mode contributes the most (again) to the Love number

With crust



With crust and superfluidity



- Resonant excitation of modes can *fracture the crust* [Tsang *et al.* (2012) PRL **108**, 011102]
  - Crust breaking strain is exceeded for *i*-mode and *f*-mode during the inspiral

A phenomenological model

- Account only for f-mode contribution (other modes treated as a "systematic error")

$$k_{lm}^{\rm eff} = k_l \left[1 - \frac{(m\omega_{\rm orb})^2}{\omega_f^2}\right]^{-1} + \frac{(m\omega_{\rm orb})^2}{\omega_f^2}\left[\frac{1}{2} - \frac{\omega_f^2}{GM/R^3}\frac{\xi_h(R)}{\xi_r(R)}\left(k_l + \frac{1}{2}\right)\right]\left[1 - \frac{(m\omega_{\rm orb})^2}{\omega_f^2}\right]^{-1}$$

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- Can we meaningfully "extend" to relativity?
  - Assume expression is similar for relativistic stars
  - Use *universal relation* between ω_f and k_l

$$\omega_f = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4, \quad y = f(k_l)$$

[Chan *et al.* (2014) PRD **90**, 124023]

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- Replace $\frac{\xi_h(R)}{\xi_r(R)} \equiv \frac{\epsilon}{l}$
  - Incompressible stars: $\epsilon = 1$
  - Appropriately stiff equations of state: $\epsilon \sim 1$

$\Rightarrow$ One-parameter expression for k^{eff}_{lm}

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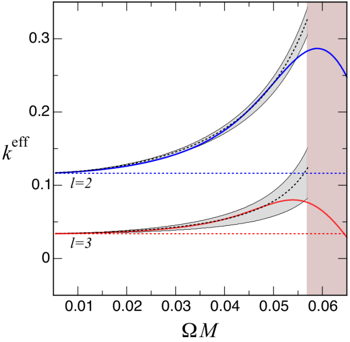


$\epsilon = 0.85 - 0.90$

Precise match: $\epsilon = 0.875$

- Replace $\dfrac{\xi_h(R)}{\xi_r(R)} \equiv \dfrac{\epsilon}{l}$
  - Incompressible stars: $\epsilon = 1$
  - Appropriately stiff equations of state: $\epsilon \sim 1$

$\Rightarrow$ One-parameter expression for $k_{lm}^{\rm eff}$

- Matches results where dynamical effects are incorporated in the effective-one-body framework [Hinderer *et al.* (2016) PRL **116**, 181101; Steinhoff *et al.* (2016) PRD **94**, 104028]

Summary

- Tidal effects on GW signal can constrain dense matter equation of state
- Dynamical effects are missed from equilibrium tide description
  - Oscillation mode contributions become important near *resonances*
  - Near merger, f-mode dominates
- *Effective tidal deformability* affected by neutron star composition (at % level)
  - Evidence about neutron star internal structure
- Phenomenological prescription for the dynamical tide works surprisingly well
  - Useful for inexpensive implementation of tidal effects in GW data analyses
- *Crust* and *superfluid* properties could be probed via crustal fracture precursor signals

◦ Future tasks
  - Relativistic prescription

N. Andersson and PP (2020) PRD **101**, 083001; (2021) MNRAS **503**, 533
A. Passamonti, N. Andersson and PP (2021) MNRAS **504**, 1273; (2022) **514**, 1494
PP, F. Gittins, A. Nanda, N. Andersson and D. I. Jones (2022) 2205.07577

Additional material: The tidal problem

- Perturbed hydrodynamic equations in the rotating frame:

$$\text{Continuity:}\quad \delta\rho + \nabla\cdot(\rho\boldsymbol{\xi}) = 0,$$

$$\text{Euler:}\quad \ddot{\boldsymbol{\xi}} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{\xi}} + \frac{\nabla\delta p}{\rho} - \frac{\nabla p}{\rho^2}\delta\rho + \nabla\delta\Phi = -\nabla U,$$

$$\text{Poisson:}\quad \nabla^2\delta\Phi = 4\pi G\delta\rho,$$

$$\text{Laplace:}\quad \nabla^2 U = 0,$$

$$\text{Equation of state:}\quad \frac{\delta\rho}{\rho} = \frac{1}{\Gamma_1}\frac{\delta p}{p} - \boldsymbol{\xi}\cdot\boldsymbol{A},$$

$$\text{where}\quad \boldsymbol{A} = \frac{\nabla\rho}{\rho} - \frac{1}{\Gamma_1}\frac{\nabla p}{p}\quad\text{and}\quad \Gamma_1 = \left(\frac{\partial\ln p}{\partial\ln\rho}\right)_{\mathrm{ad}}$$

- Equilibrium tide solution (*ignoring rotation*):

$$\delta p = -\rho(\delta\Phi + U),$$

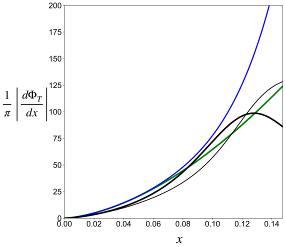
$$\delta\rho = \frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\delta\Phi + U}{g},$$

$$\xi_r = -\frac{\delta\Phi + U}{g},\quad \xi_h = \frac{1}{l(l+1)r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\xi_r\right),\quad \nabla\cdot\boldsymbol{\xi} = 0,$$

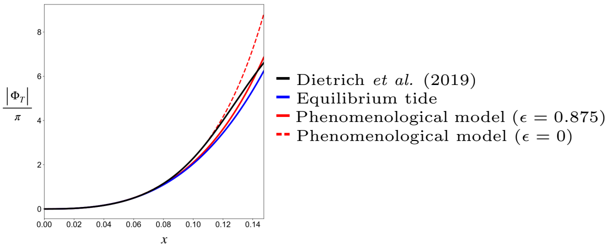
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\delta\Phi}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2}\delta\Phi - 4\pi G\frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\delta\Phi + U}{g} = 0,$$

$$\text{where}\quad g = \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

- Simple and inexpensive alternative for GW data analysis
  - Numerical relativity simulations necessary to describe dynamics near merger...
  - ...but computationally costly to incorporate in detector templates
- Tidal contribution to GW phase: $\frac{\mathrm{d}\Phi_T}{\mathrm{d}x} = -\frac{65}{2^5}\frac{k_2 x^{3/2} f(x)}{(M/R)^5}$, $x = (M\omega)^{2/3}$ $\quad (c = G = 1)$



Comparison of various waveform models



Comparison of phenomenological relation with waveform model of Dietrich *et al.* (2019) PRD **100**, 044003

- Noticeable differences between different models
- Dynamical tide produces subradian change to GW phasing
- Detectable by 3G detectors

Abbott, B. P. et al. (2017). GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.* **119**, 161101. ArXiv: 1710.05832.

Andersson, N. and Pnigouras, P. (2020). Exploring the effective tidal deformability of neutron stars. *Phys. Rev. D* **101**, 083001. ArXiv: 1906.08982.

Andersson, N. and Pnigouras, P. (2021). The phenomenology of dynamical neutron star tides. *Mon. Not. R. Astron. Soc.* **503**, 533. ArXiv: 1905.00012.

Chan, T. K., Sham, Y.-H., Leung, P. T. and Lin, L.-M. (2014). Multipolar universal relations between f-mode frequency and tidal deformability of compact stars. *Phys. Rev. D* **90**, 124023. ArXiv: 1408.3789.

Dietrich, T., Samajdar, A., Khan, S., Johnson-McDaniel, N. K., Dudi, R. and Tichy, W. (2019). Improving the NRTidal model for binary neutron star systems. *Phys. Rev. D* **100**, 044003. ArXiv: 1905.06011.

Herbrik, M. and Kokkotas, K. D. (2017). Stability analysis of magnetized neutron stars – a semi-analytic approach. *Mon. Not. R. Astron. Soc.* **466**, 1330–1347. ArXiv: 1511.04290.

Hinderer, T., Taracchini, A., Foucart, F., Buonanno, A., Steinhoff, J., Duez, M., Kidder, L. E., Pfeiffer, H. P., Scheel, M. A., Szilagyi, B., Hotokezaka, K., Kyutoku, K., Shibata, M. and Carpenter, C. W. (2016). Effects of Neutron-Star Dynamic Tides on Gravitational Waveforms within the Effective-One-Body Approach. *Phys. Rev. Lett.* **116**, 181101. ArXiv: 1602.00599.

Passamonti, A., Andersson, N. and Pnigouras, P. (2021). Dynamical tides in neutron stars: the impact of the crust. *Mon. Not. R. Astron. Soc.* **504**, 1273–1293. ArXiv: 2012.09637.

Passamonti, A., Andersson, N. and Pnigouras, P. (2022). Dynamical tides in superfluid neutron stars. *Mon. Not. R. Astron. Soc.* **514**, 1494. ArXiv: 2202.05161.

Pnigouras, P., Gittins, F., Nanda, A., Andersson, N. and Jones, D. I. (2022). Rotating Love: The dynamical tides of spinning Newtonian stars. ArXiv: 2205.07577.

Steinhoff, J., Hinderer, T., Buonanno, A. and Taracchini, A. (2016). Dynamical tides in general relativity: Effective action and effective-one-body Hamiltonian. *Phys. Rev. D* **94**, 104028. ArXiv: 1608.01907.

Tsang, D., Read, J. S., Hinderer, T., Piro, A. L. and Bondarescu, R. (2012). Resonant Shattering of Neutron Star Crusts. *Phys. Rev. Lett.* **108**, 011102. ArXiv: 1110.0467.