



Quantum induced anomalous diffusion in QCD matter

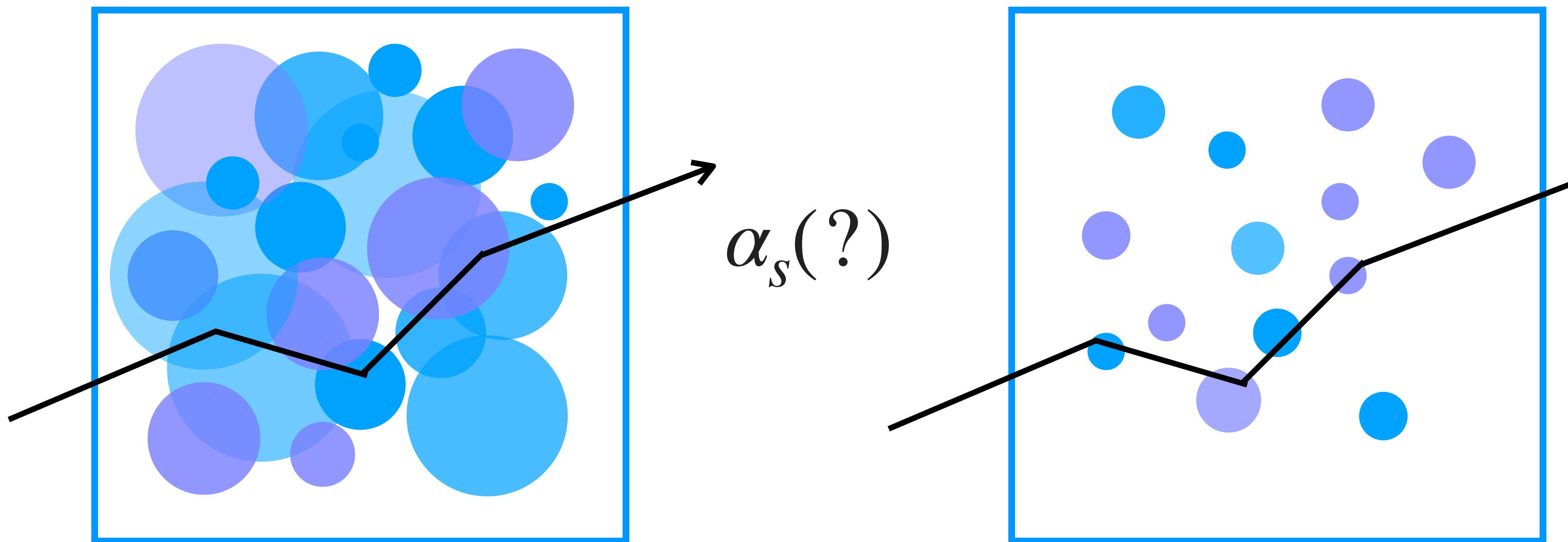
Yacine Mehtar-Tani

In collaboration with Paul Caucal

2109.12041 [hep-ph] 2203.09407 [hep-ph]

Jet Quenching In The Quark-Gluon Plasma @ ECT*, Trento
June 13-17 , 2022

- How do Jets couple to the QGP?
- What is the nature of scatterers in the QGP?



→ Investigate transverse momentum diffusion

Anomalous diffusion (generalities)

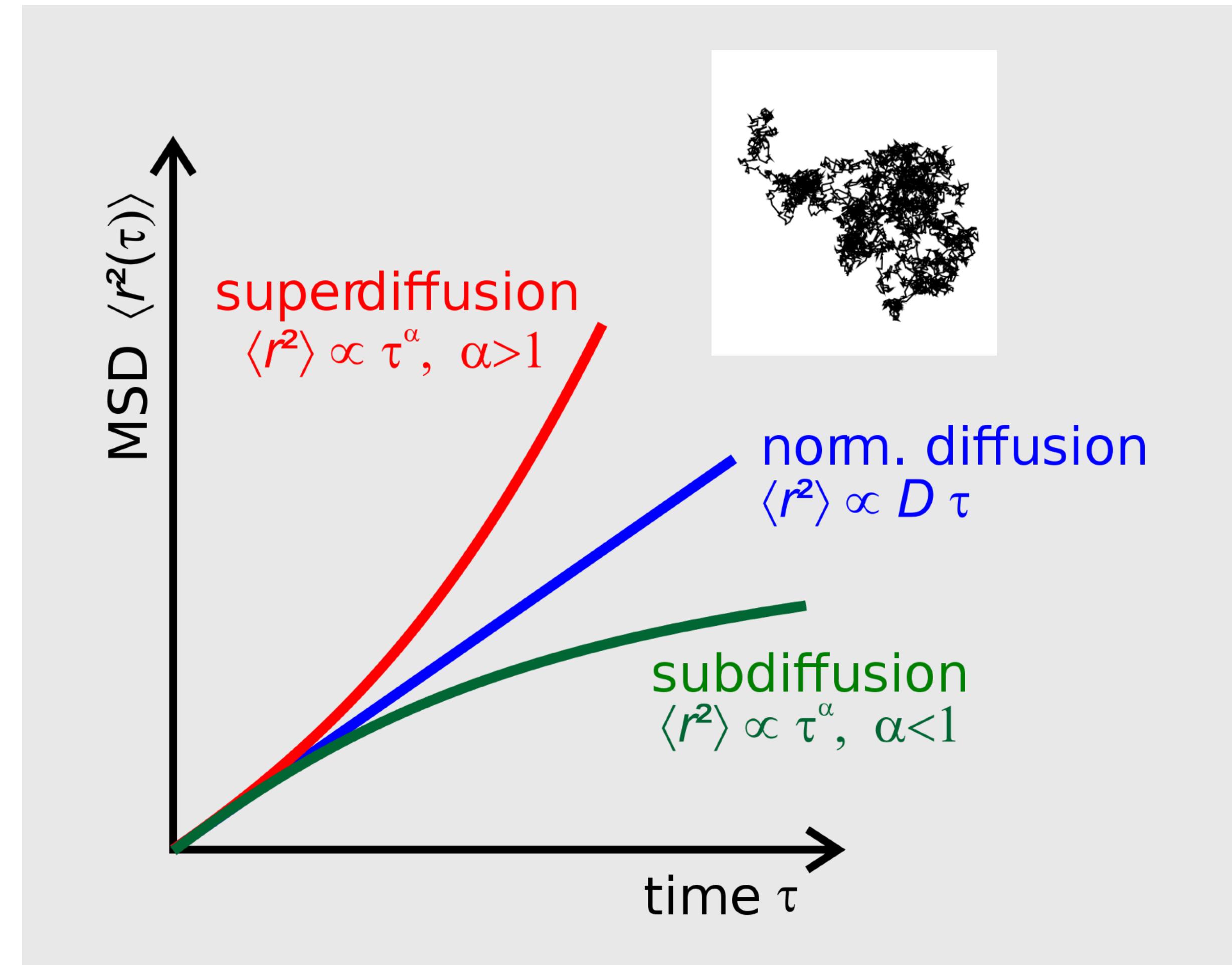
- Mean squared displacement (MSD) in brownian motion

$$\langle r^2 \rangle = D t$$

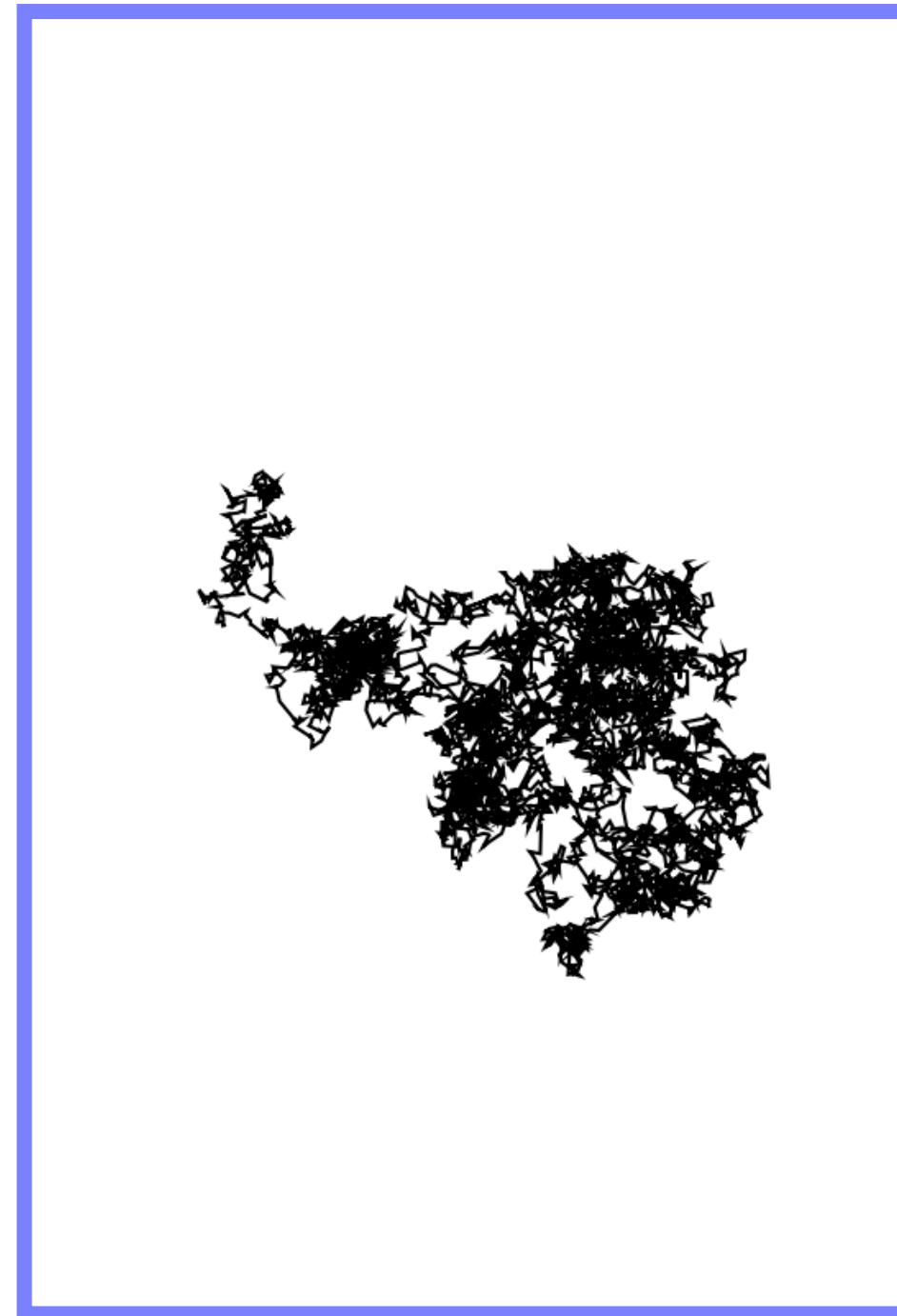
- Anomalous diffusion $\alpha \neq 1$:

$$\langle r^2 \rangle \propto t^\alpha$$

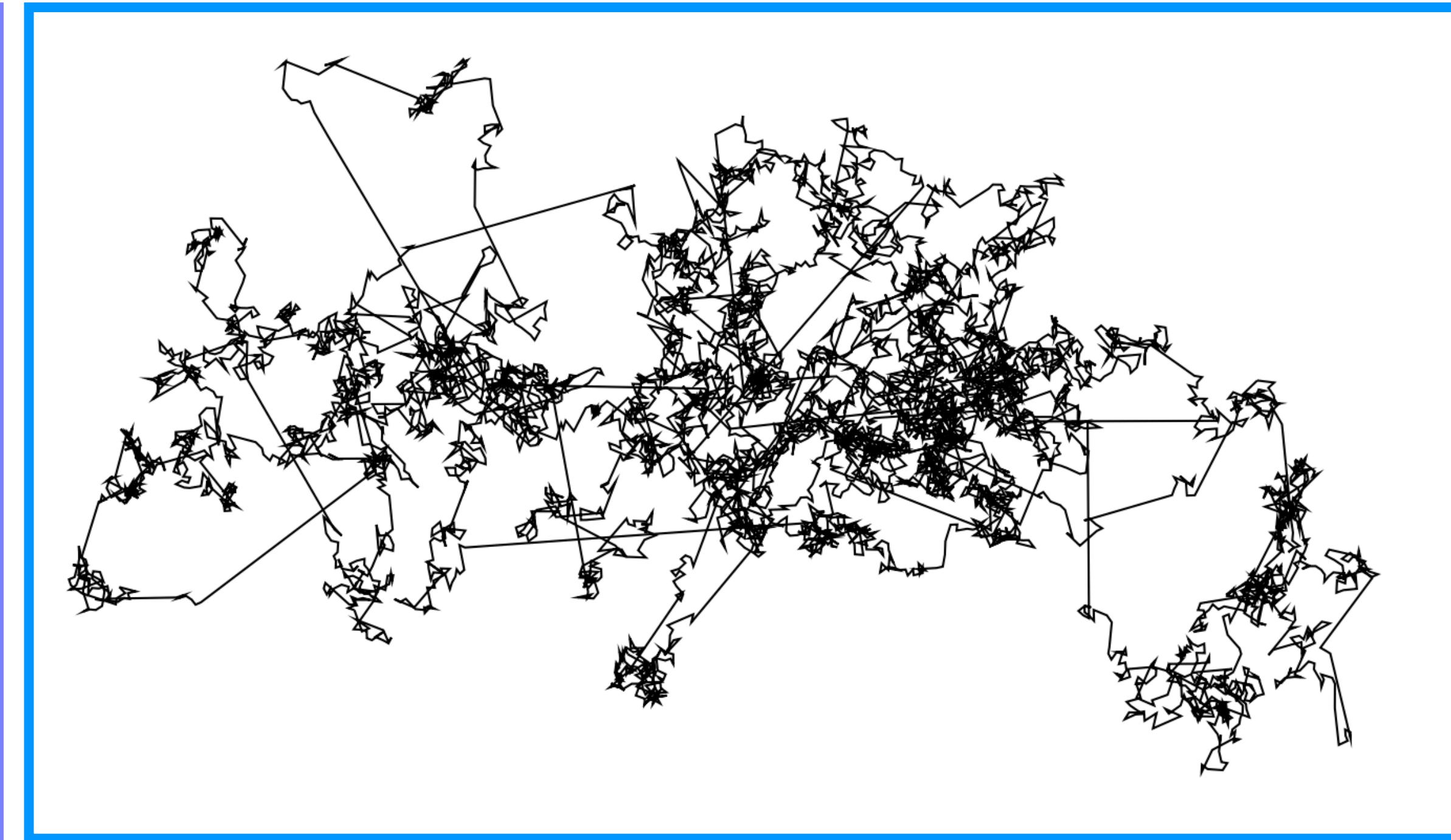
→ biological systems, molecular chemistry, optical lattice, turbulent diffusion and polymer transport theory



Normal diffusion
(Brownian motion)



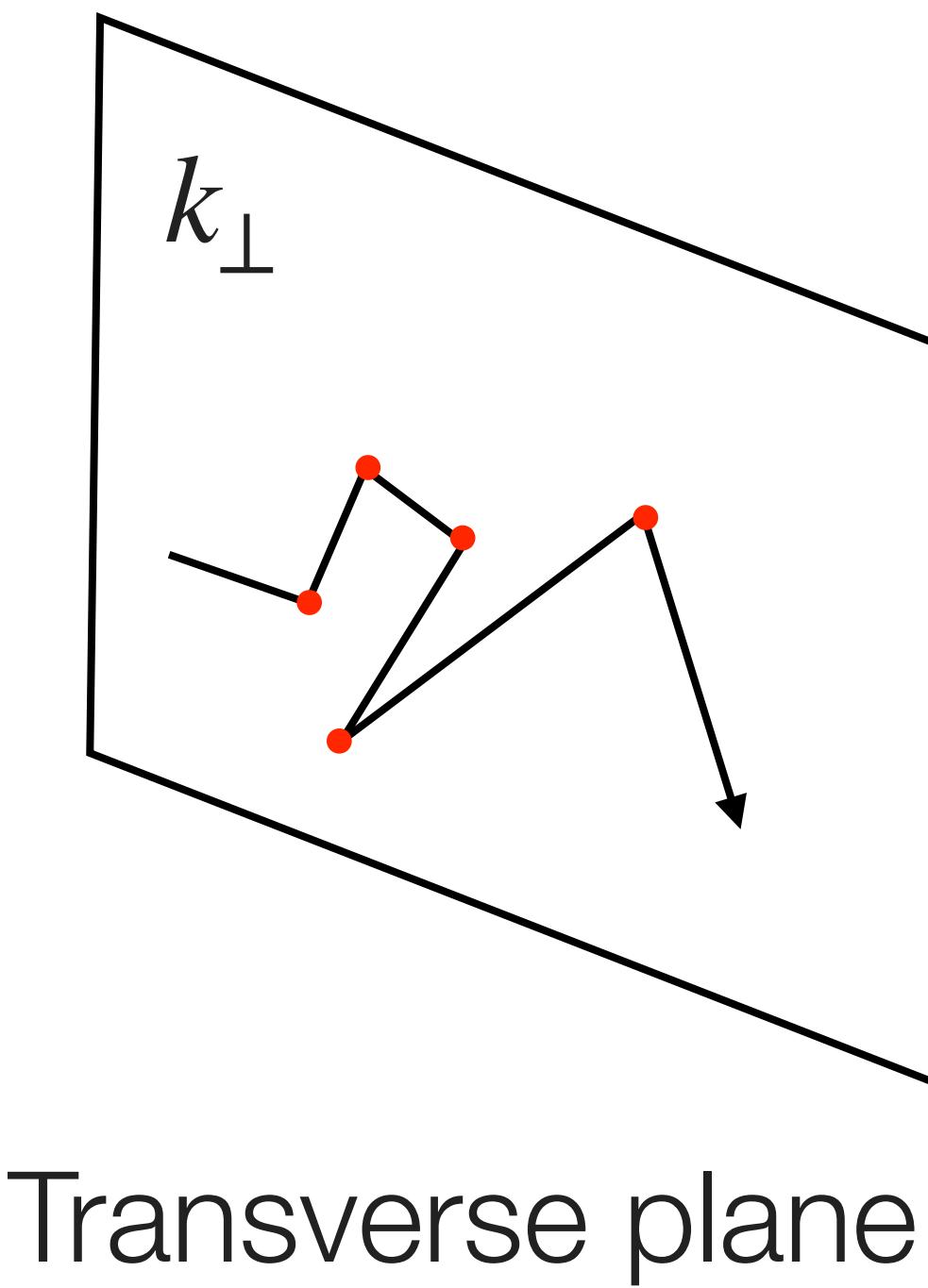
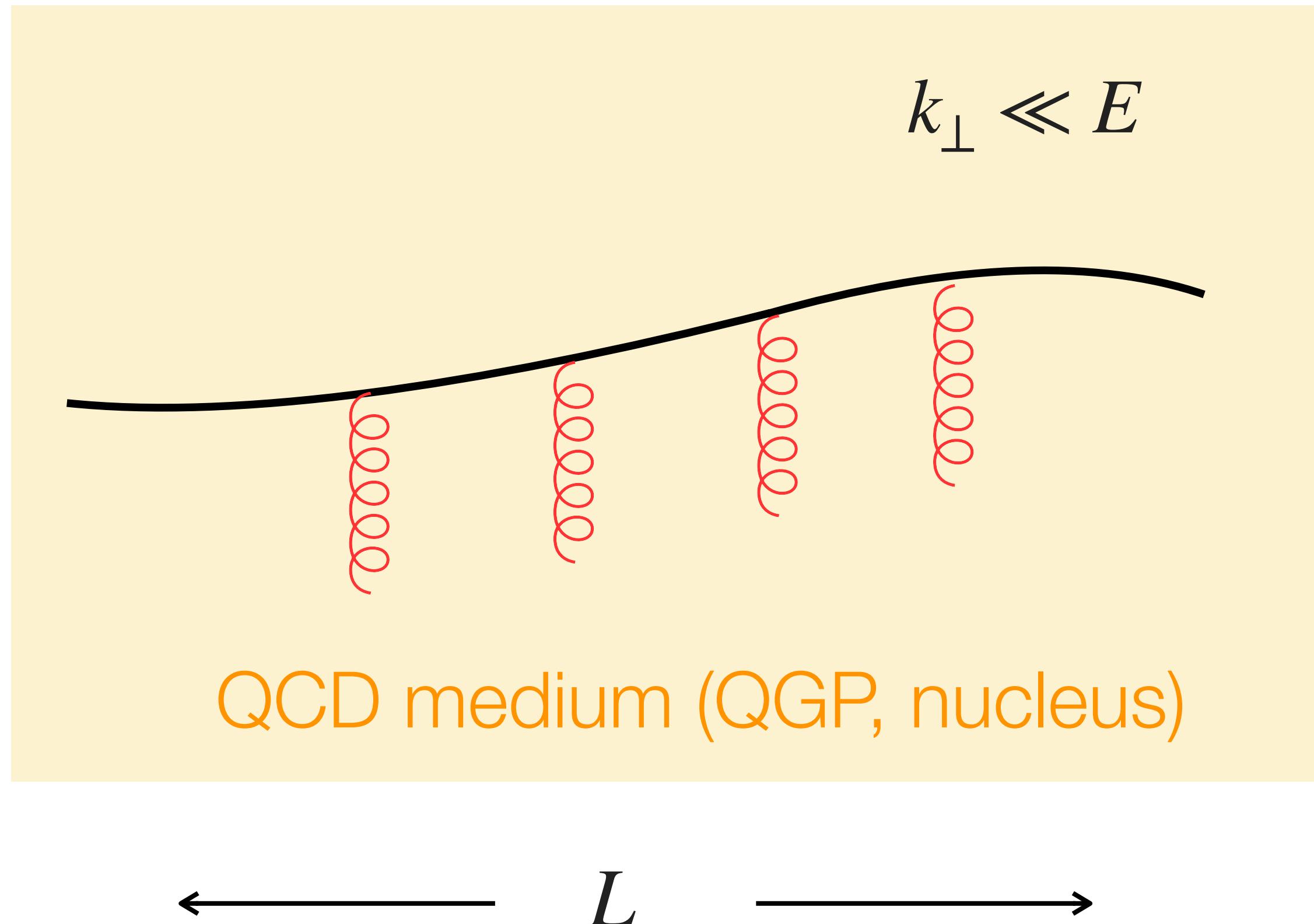
Super-diffusion
(Lévy random walks)



Large step length

Transverse momentum broadening (TMB)

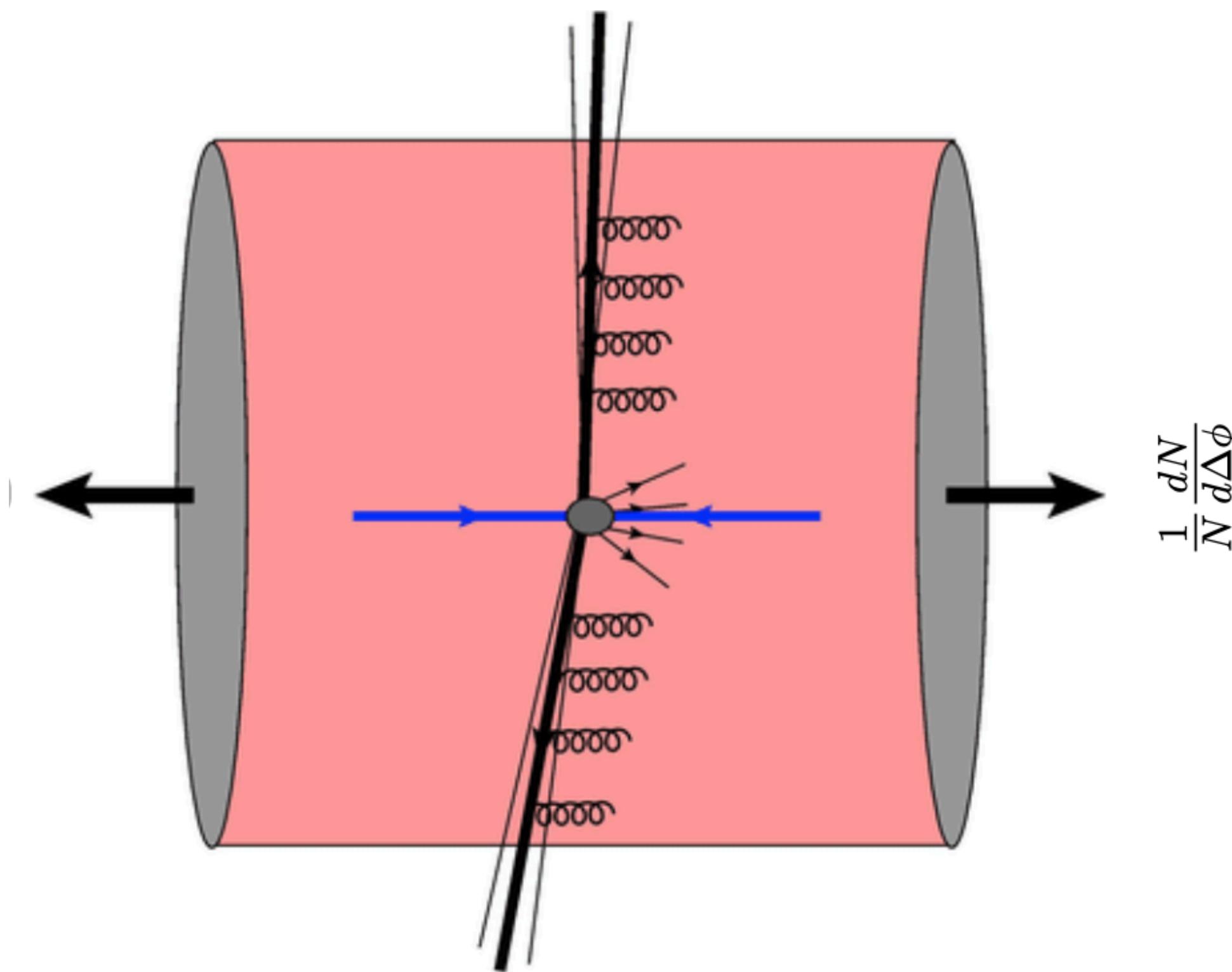
High energy partons experience random kicks in hot or cold nuclear matter that cause their transverse momentum to increase over time



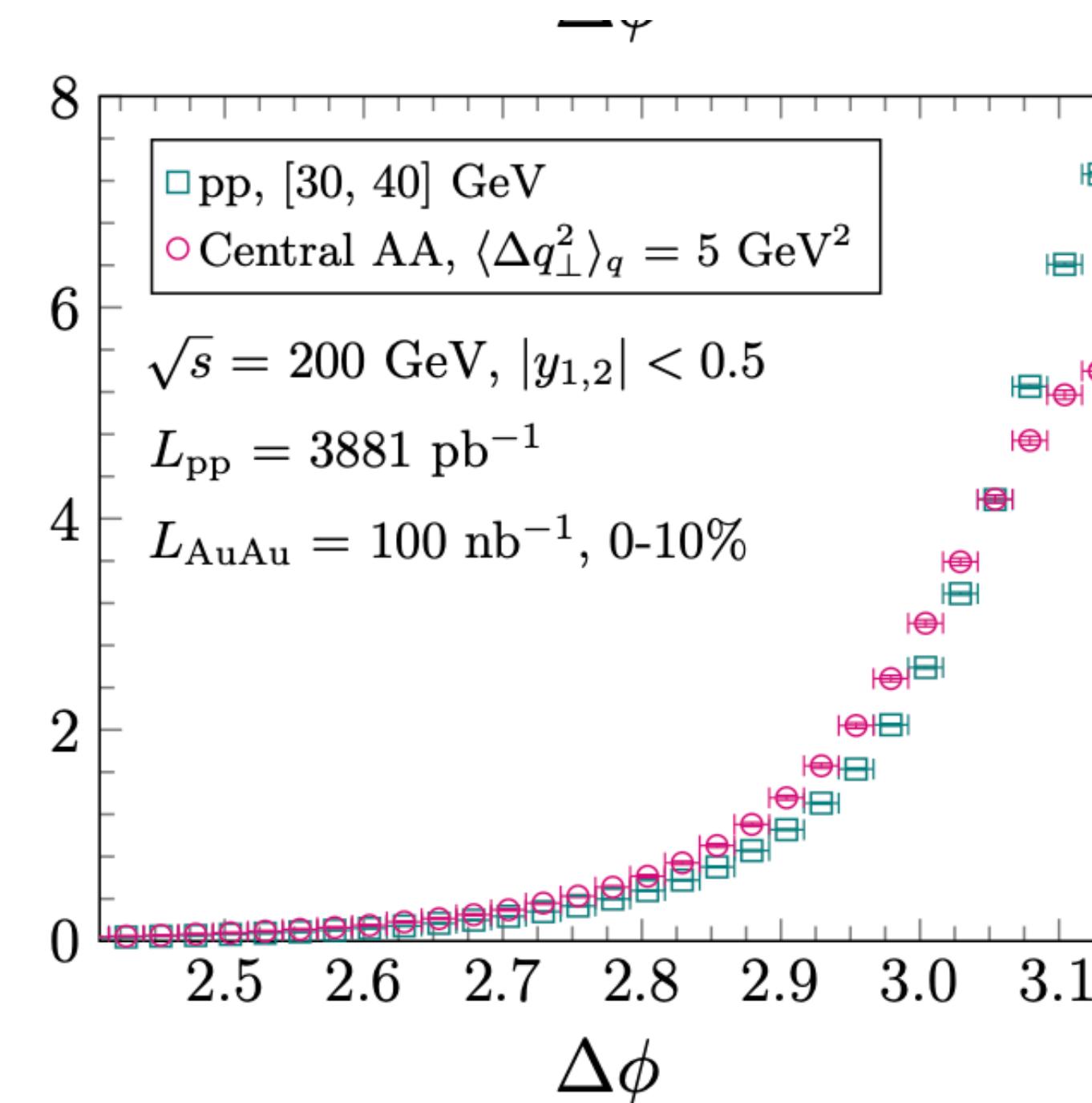
TMB in various processes

- Probe the QGP in Heavy Ion Collisions: *dijet azimuthal de-correlation*, *Molière scattering*, *jet quenching* ...

Mueller, Wu, Xiao, Yuan (2016)

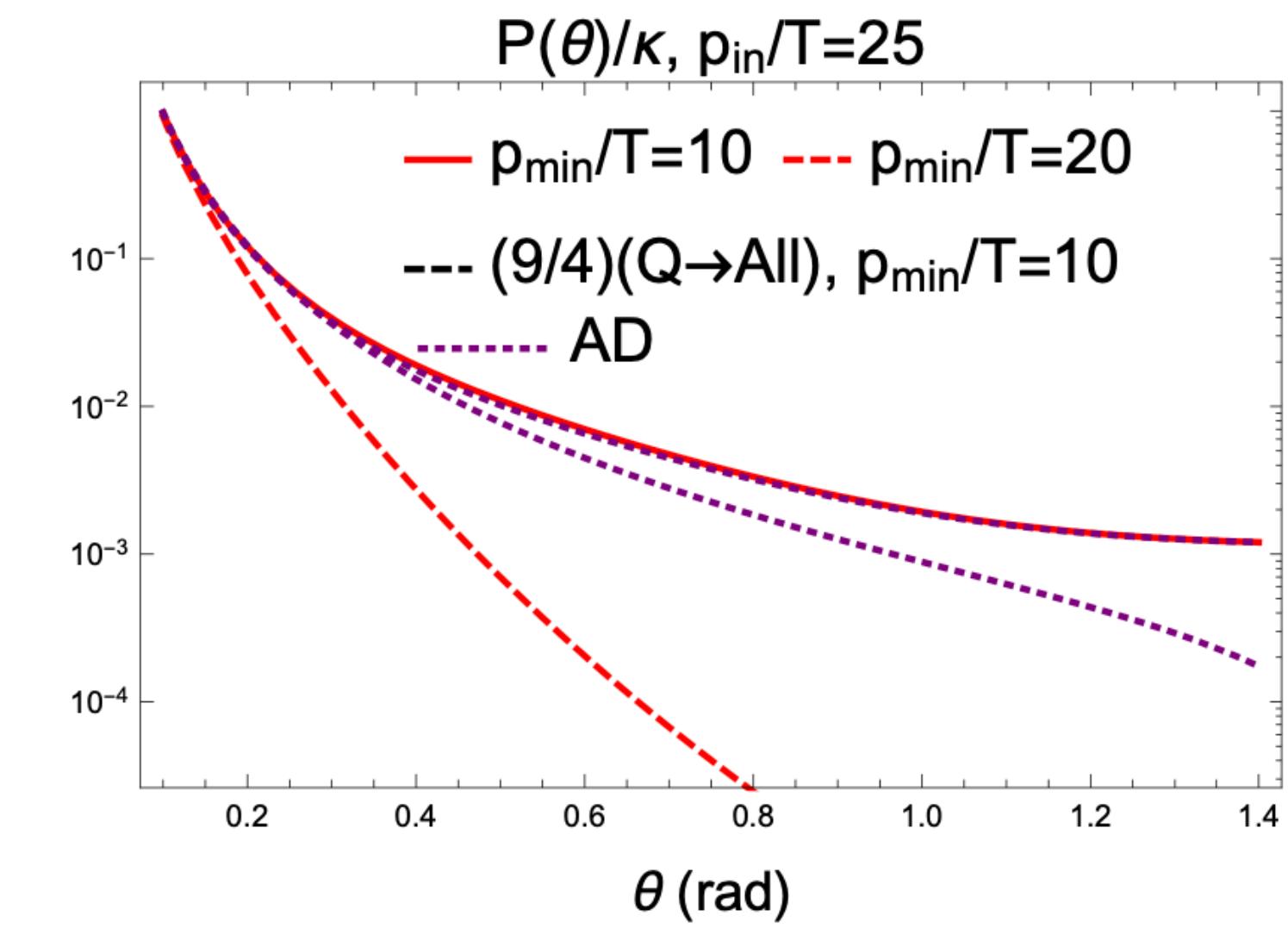


Jia, Xiao, Yuan (2019)



Molière scattering

D'Eramo, Lekaveckas,
Liu, Rajagopal (2012)
D'Eramo, Rajagopal, Yin
(2019)



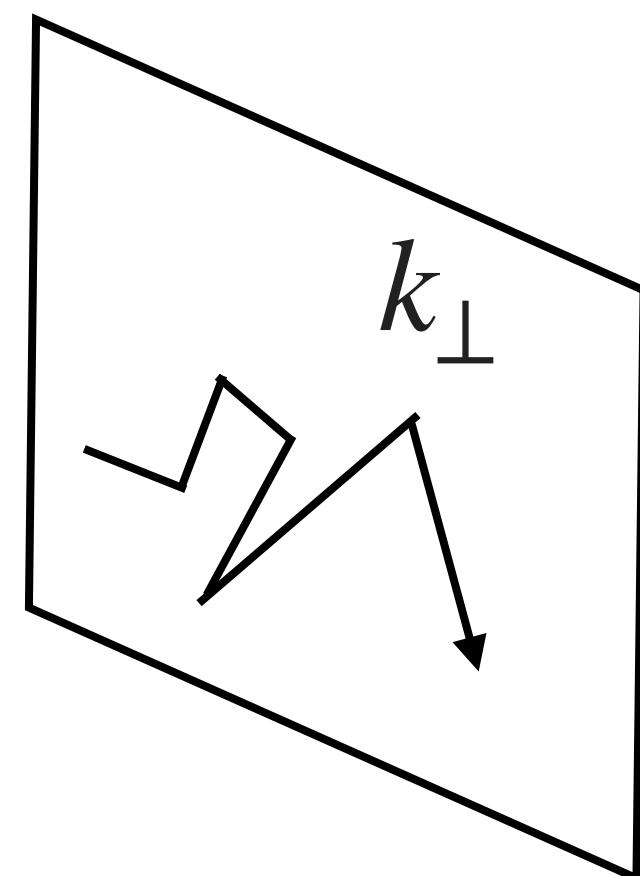
Transverse momentum diffusion

- Independent multiple scattering approximation: $m_D^{-1} \ll \ell_{\text{mfp}} \ll L$
- Probability distribution for a high energy particle to acquire a transverse momentum k_\perp after a time t in the plasma obeys a Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(k_\perp) = \frac{1}{4} \hat{q} \nabla_\perp^2 P(k_\perp)$$

- Transverse momentum broadening: normal diffusion

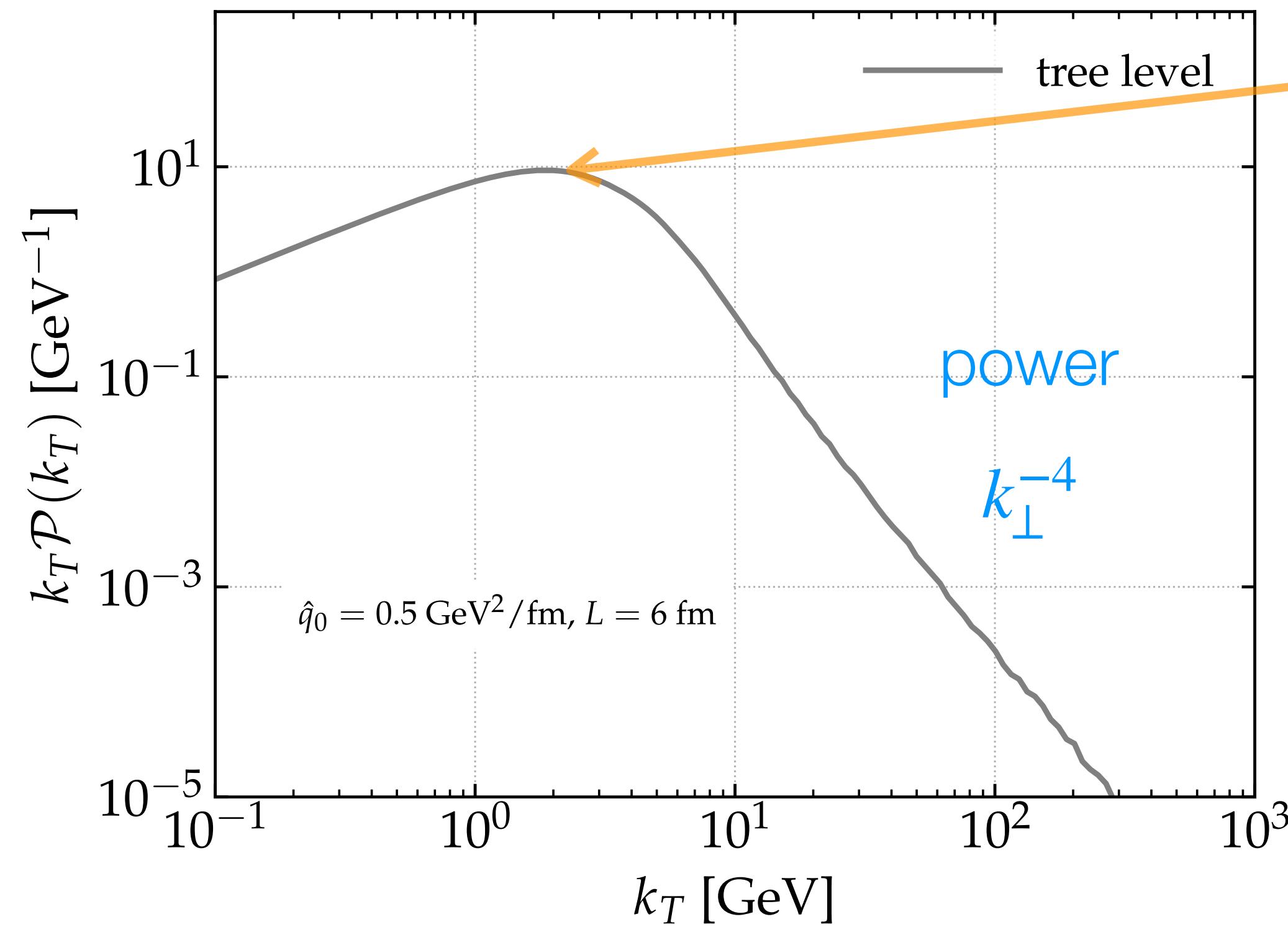
$$\langle k_\perp^2 \rangle_{\text{typ}} \propto \hat{q} t$$



Normal diffusion at tree level

TMB distribution at leading order:

Gaussian for $k_{\perp} < Q_s \sim \hat{q}L$ and exhibits the power law tail k_{\perp}^{-4} for $k_{\perp} > Q_s$



$$P(k_{\perp}) = \frac{4\pi}{\hat{q}L} e^{-\frac{k_{\perp}^2}{\hat{q}L}}$$

Normal diffusion

$$\langle k_{\perp}^2 \rangle_{\text{typ}} \propto L$$

L : Medium size

Jet quenching parameter at tree level
(2 to 2 matrix element)

$$\hat{q} = C_R \int_{q_{\perp}} q_{\perp}^2 \frac{d^2\sigma}{d^2q_{\perp}} \simeq 4\pi\alpha_s^2 n \log \frac{q_{\max}}{\mu^2}$$

Q: What is the effect of quantum corrections on transverse momentum broadening?

Quantum corrections to \hat{q}

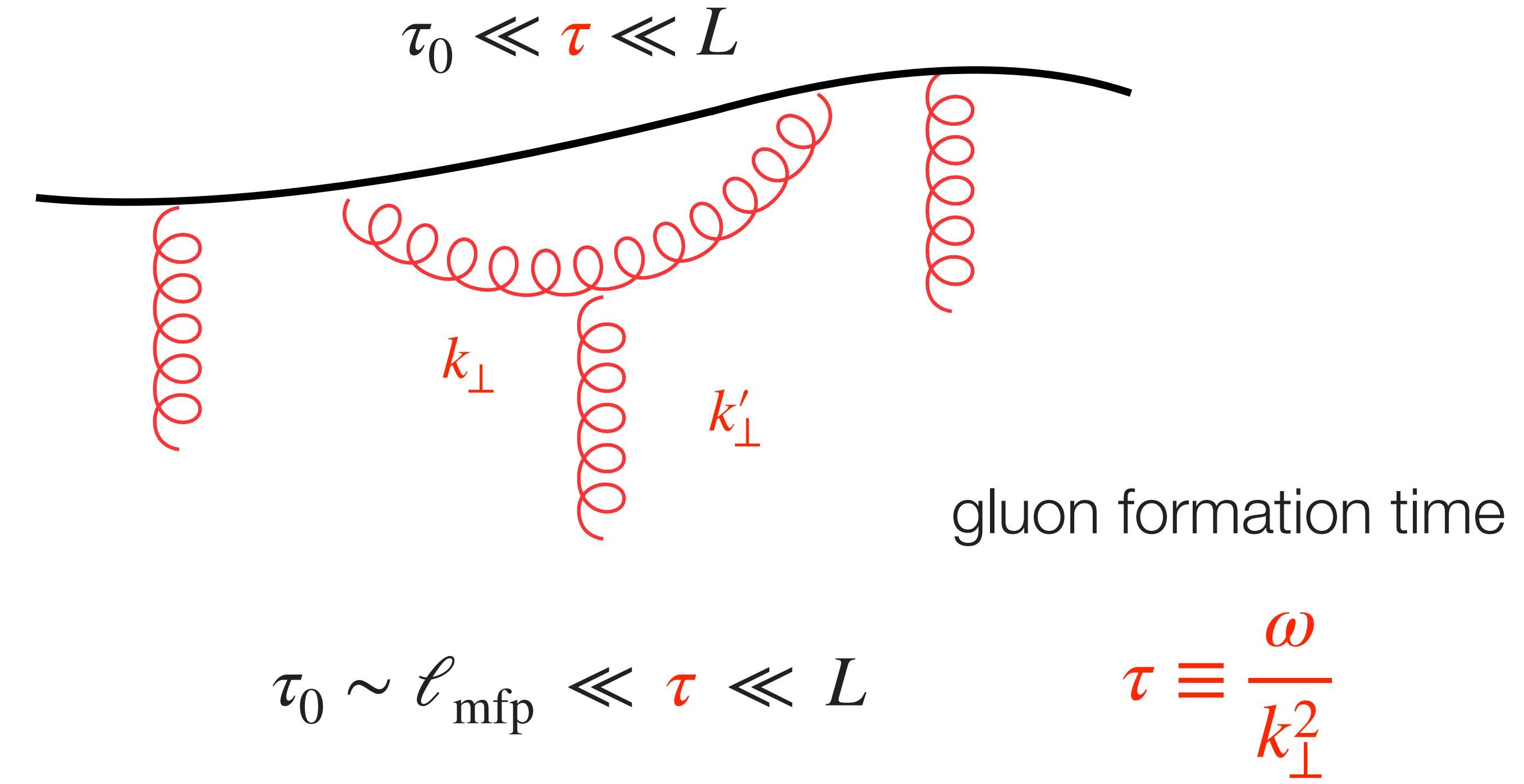
- Potentially large double logs (DL) in transverse momentum broadening at NLO

$$\text{NLO} \sim \bar{\alpha} \int \frac{dk_{\perp}^2}{k_{\perp}^2} \int \frac{d\tau}{\tau}$$

[Liou, Mueller, Wu (2013)

Blaizot, Dominguez, Iancu, MT (2014)]

$$\langle k_{\perp}^2 \rangle = \hat{q}_0 L \left(1 + \frac{\bar{\alpha}}{2} \log^2 \frac{L}{\tau_0} \right)$$



- Not the standard DGLAP double log: the factor 1/2 reflects the presence of multiple scattering constraint $k_{\perp} > \hat{q} \tau \equiv Q_s^2$

Nonlinear evolution of \hat{q}

- All orders DL resummed and absorbed in a **redefinition of the jet quenching parameter**
- double logs (DL) \rightarrow single scattering

$$\frac{\partial}{\partial \ln \tau} \hat{q}(k_\perp, \tau) = \bar{\alpha} \int_{Q_s^2(\tau)}^{k_\perp^2} \frac{dk'^2_\perp}{k'^2_\perp} \hat{q}(k'_\perp, \tau)$$

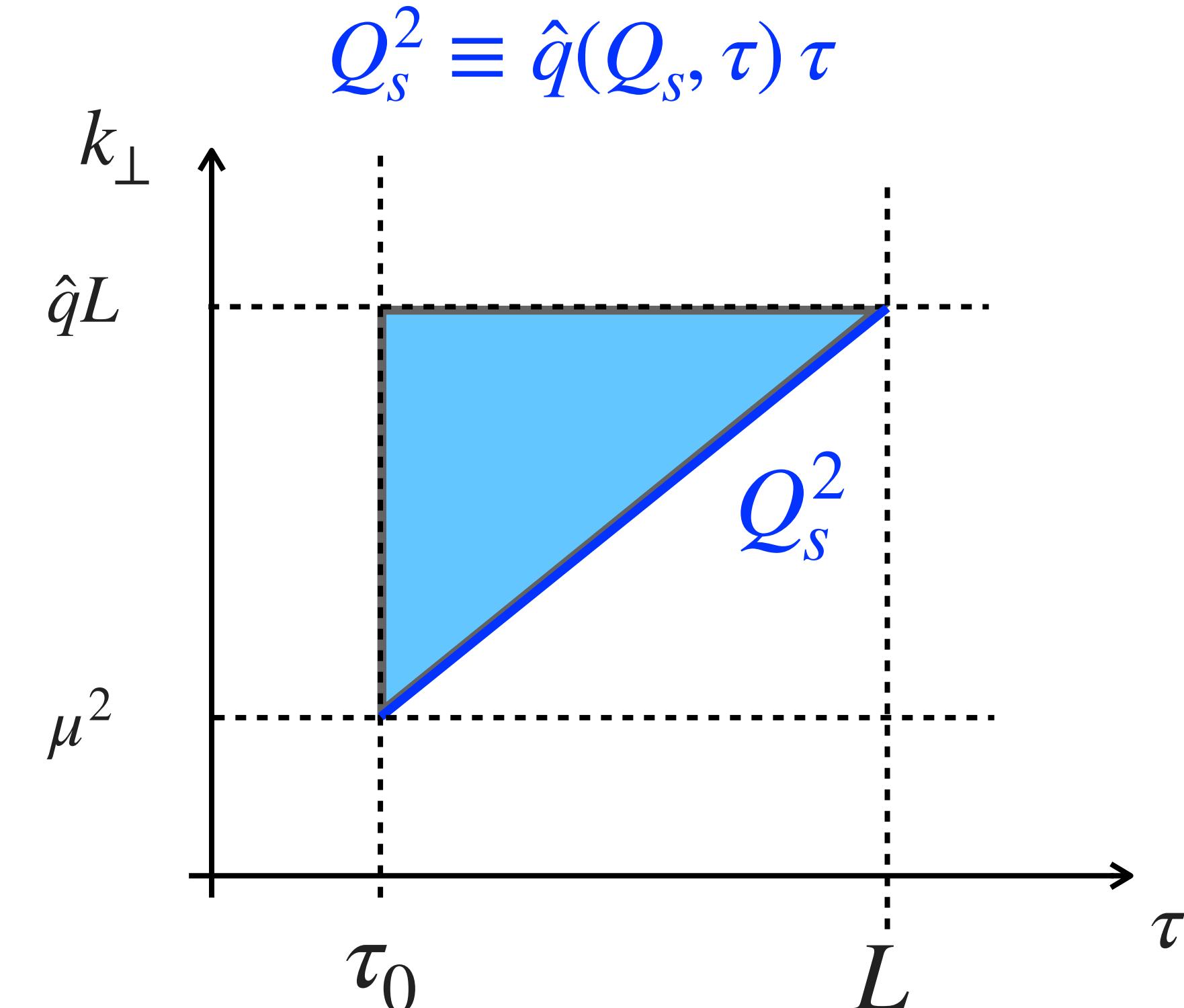
\rightarrow DLA limit of DGLAP/BFKL

[Blaizot, MT (2014), Iancu (2014)]

\rightarrow reminiscent of Color-Glass-Condensate

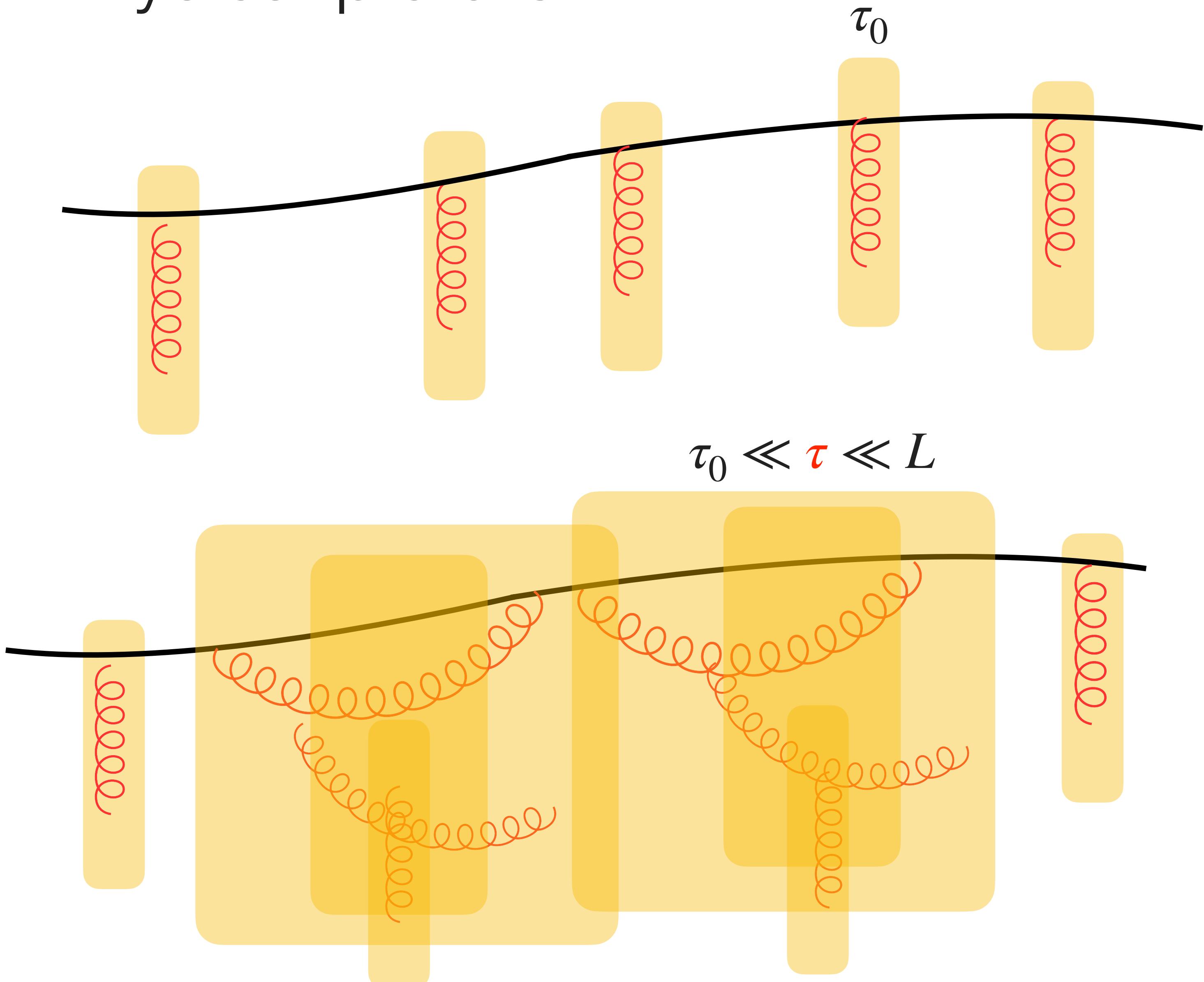
[Mueller, Qiu, (1986), Venugopalan, McLerran (1993),
Balitsky, Kovchegov, Jallilian-Marian, Iancu,
McLerran, Weigert, Leonidov, Kovner (1996-2001)]

Multiple-scattering boundary: saturation line
(screening of mass singularity)

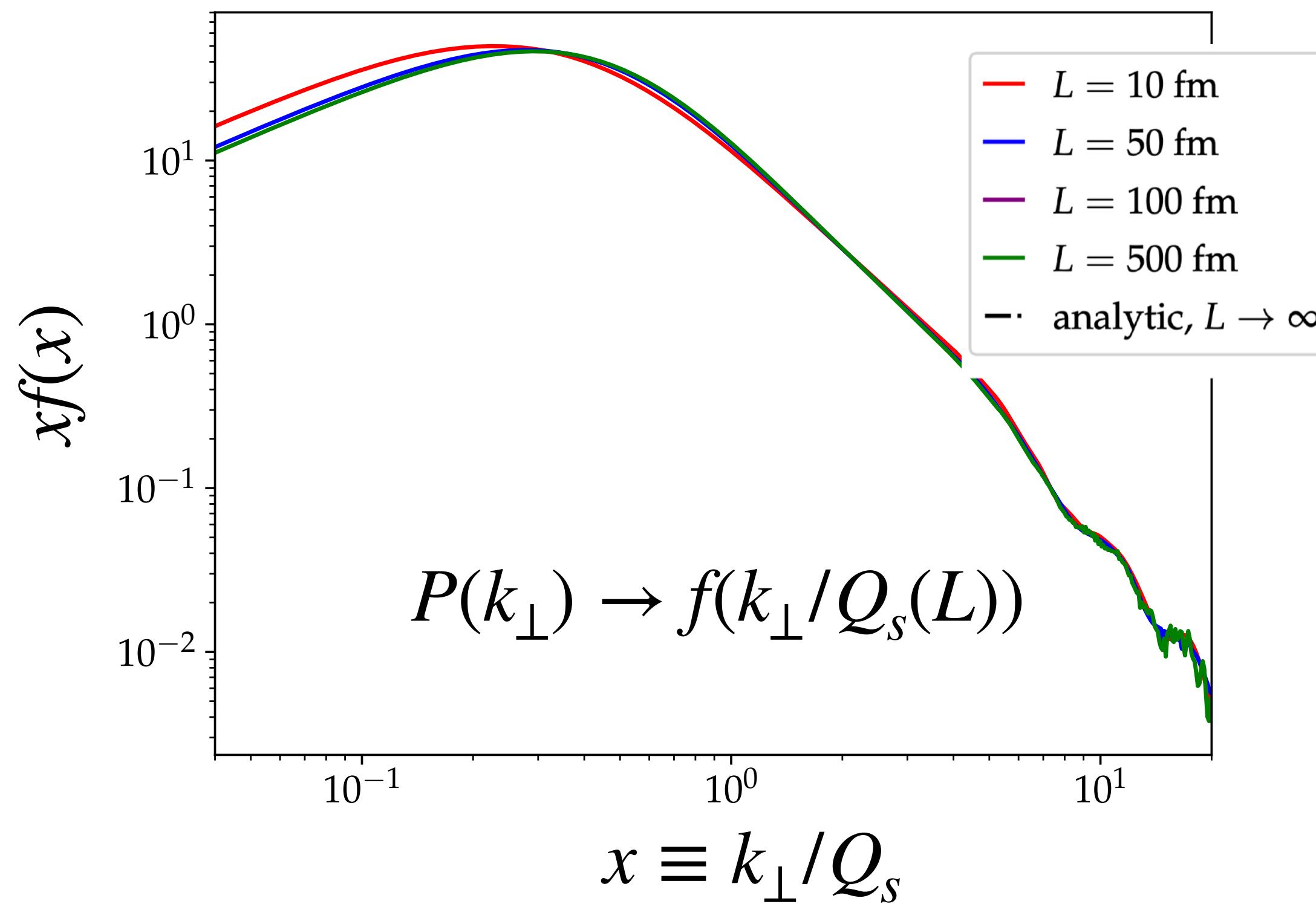


Physical picture

- LO: local/instantaneous interactions
- DLA + saturation: quasi-local interactions
- Exponentiation of the double logs with adequate phase space constraints



Geometric scaling in TMB



YMT, P. Caucal 2109.12041 [hep-ph]



Brownian motion

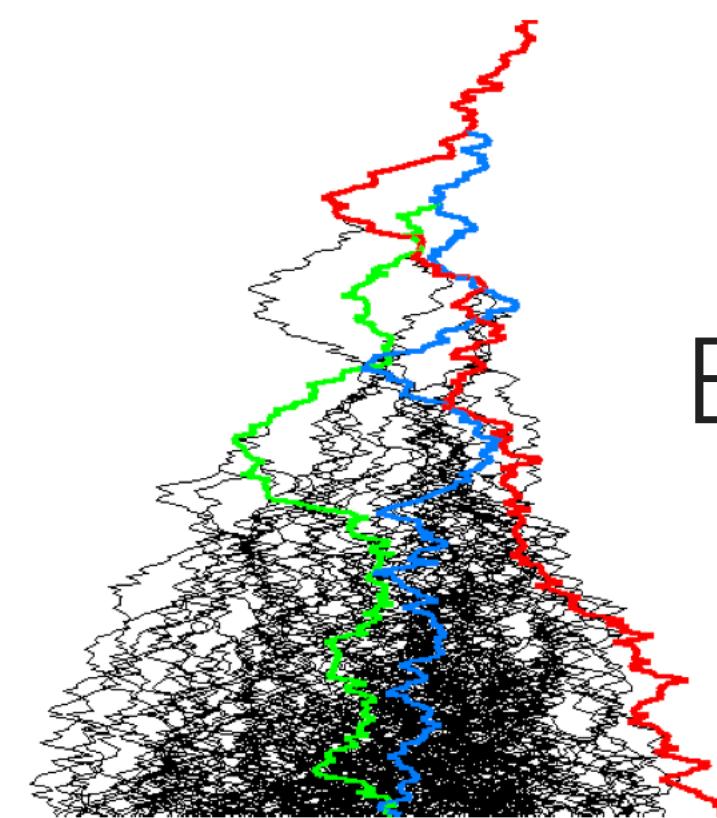
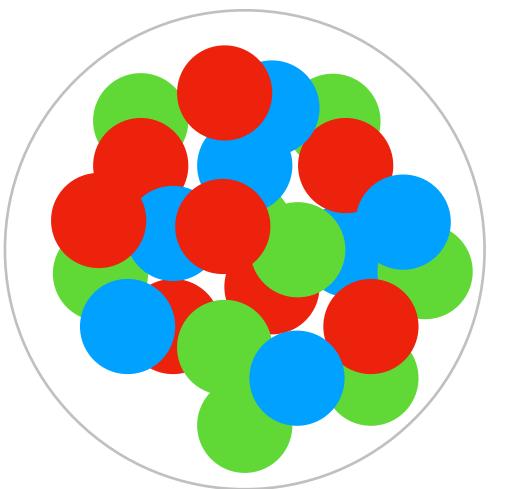
Lévy flight

- Nonlinear effects yield Geometric Scaling (GS) → self-similar dynamics

studied in the context of saturation physics in DIS: Mueller, Triantafyllopoulos, Iancu, McLerran, Itakura, Munier, Peschanski (2002-2004)

Traveling wave solutions

- Other QCD context: GS solutions to [Balitsky-Kovchegov \(1997\)](#) (and BFKL) equations near the saturation line [Mueller, Triantafyllopoulos \(2002\)](#) ([Iancu, Itakura, McLerran \(2002\)](#))
- Connexion to traveling wave physics by [Munier and Peschanski \(2003\)](#)
- **This talk:** derive sub-asymptotic behavior. We follow [Brunet and Derrida's \(1988\)](#) and [U. Ebert and W. van Saarloos \(2000\)](#) approaches to [FKPP](#) equation (Fisher-Kolmogorov-Petrovsky, Piskunov)
- (population growth, wave propagation, etc)



branching

Brownian motion

$$\frac{\partial}{\partial t} u = D \frac{\partial^2}{\partial^2 x} u + u(1 - u)$$

Exact scaling solution for large media $L \rightarrow \infty$ (fixed coupling)

Scaling solution:

$$\hat{q}(r_\perp, L) L \equiv Q_s^2(L) g\left(x = \log \frac{1}{r_\perp^2 Q_s^2(L)}\right)$$

$$x \equiv \log \frac{1}{Q_s^2 r_\perp^2} \quad \beta \simeq \sqrt{\bar{\alpha}}$$

$$\begin{aligned} g(x) &= e^{\beta x} (1 + \beta x) & x > 0 \\ g(x) &= e^{2\beta x} & x < 0 \end{aligned}$$

TMB relates to $q\bar{q}$ dipole S-matrix by a Fourier transform

$$P(k_\perp) \equiv \int d^2 x_\perp S(x_\perp) e^{-i x_\perp \cdot k_\perp}$$

Asymptotically: Lévy distribution

$$S(r_\perp, L) \rightarrow e^{-(r_\perp^2 Q_s^2(L))^{1-2\sqrt{\bar{\alpha}}}}$$

→ non-Gaussian

Anomalous scaling:
super diffusive process

$$Q_s^2(L) \propto L^{1+2\sqrt{\bar{\alpha}}}$$

Universal pre-asymptotic solutions for \hat{q}

- Position of the wave front:

$$\rho_s(Y) = \log Q_s^2(Y) = \textcolor{blue}{c}Y + \textcolor{orange}{b} \log Y + \text{const.}$$

- Shape of the wave front :

YMT, P. Caucal 2109.12041 [hep-ph]

$$\frac{\hat{q}(Y, k_\perp)L}{Q_s^2} = \exp\left(\textcolor{blue}{\beta}x - \frac{\beta x^2}{4cY}\right) \left[1 + \beta x + \frac{bx}{c^2Y} \left(1 + \frac{\beta(c+4)x}{6} \right) + \mathcal{O}(Y^{-2}) \right]$$

Blue: asymptotic limit. Orange: pre-asymptotic $\mathcal{O}(1/Y)$

Velocity of the wave front:

$$c = 1 + 2\sqrt{\bar{\alpha} + \bar{\alpha}^2} + 2\bar{\alpha} \simeq 1 + 2\sqrt{\bar{\alpha}}$$

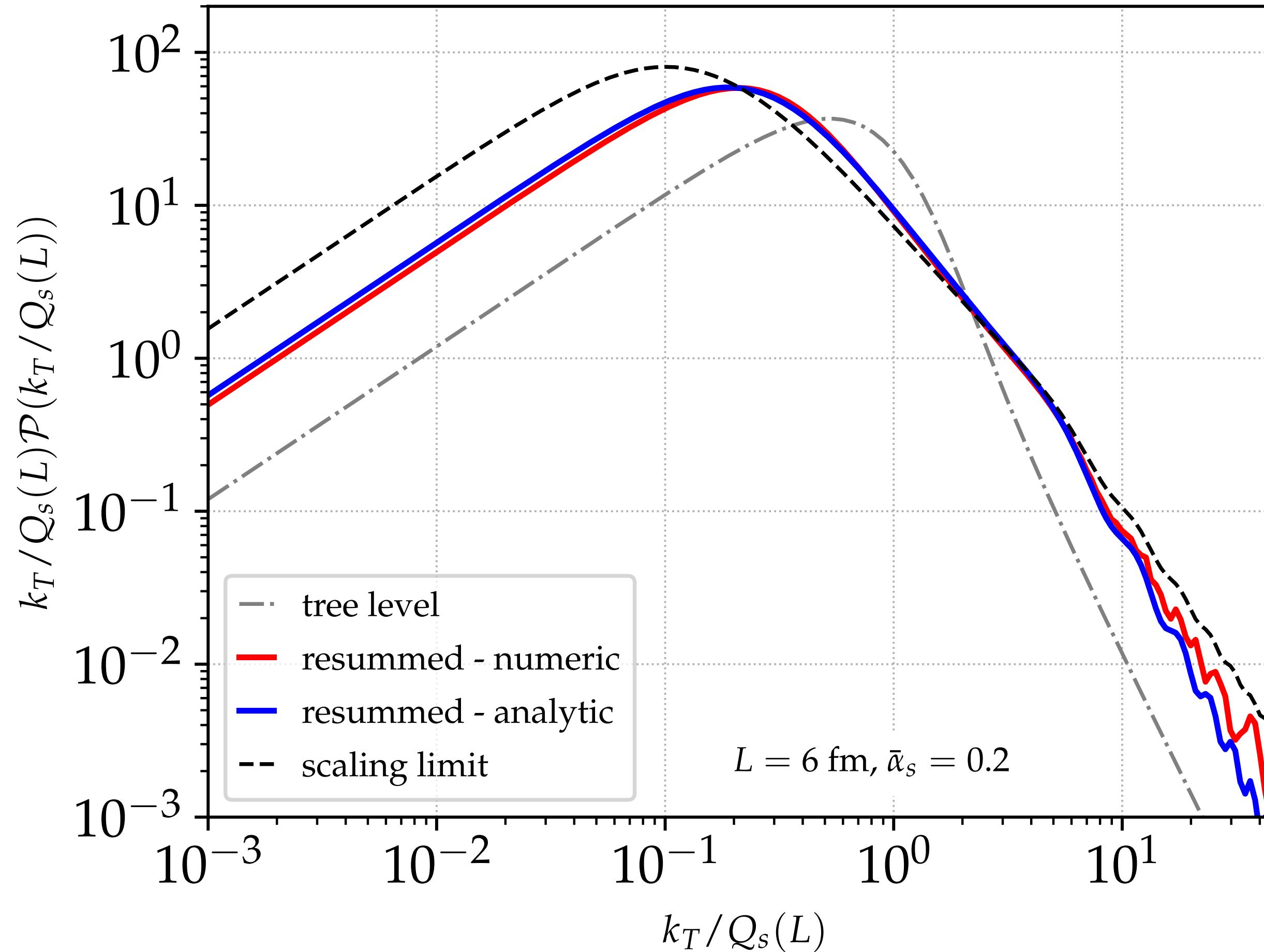
$$x \equiv \log \frac{k_\perp^2}{Q_s^2}$$

$$Y \equiv \log \frac{L}{\tau_0}$$

$$\beta = \frac{c-1}{2c}$$

$$b = -\frac{2}{3(1-\beta)}$$

Analytic vs numerics



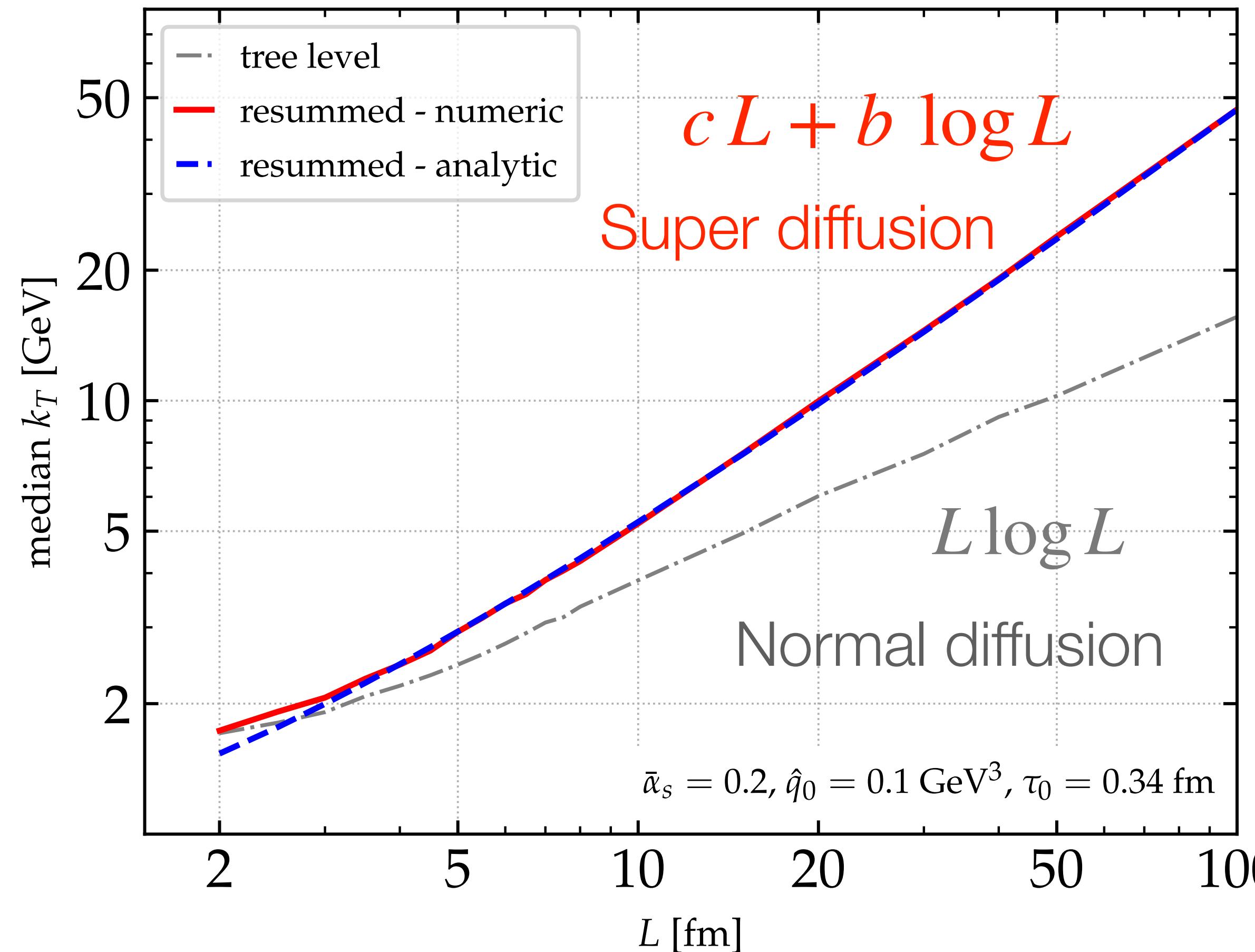
→ Universal pre-asymptotic solution provides a good description of numerical simulations for $L = 6 \text{ fm}$ and $\bar{\alpha} = 0.2$

→ wider distribution due to heavy Lévy tail

Time dependence of the typical transverse momentum

$$\rho_s(Y) = \log Q_s^2(Y) = \textcolor{blue}{c}Y + \textcolor{red}{b}\log Y + \text{const.}$$

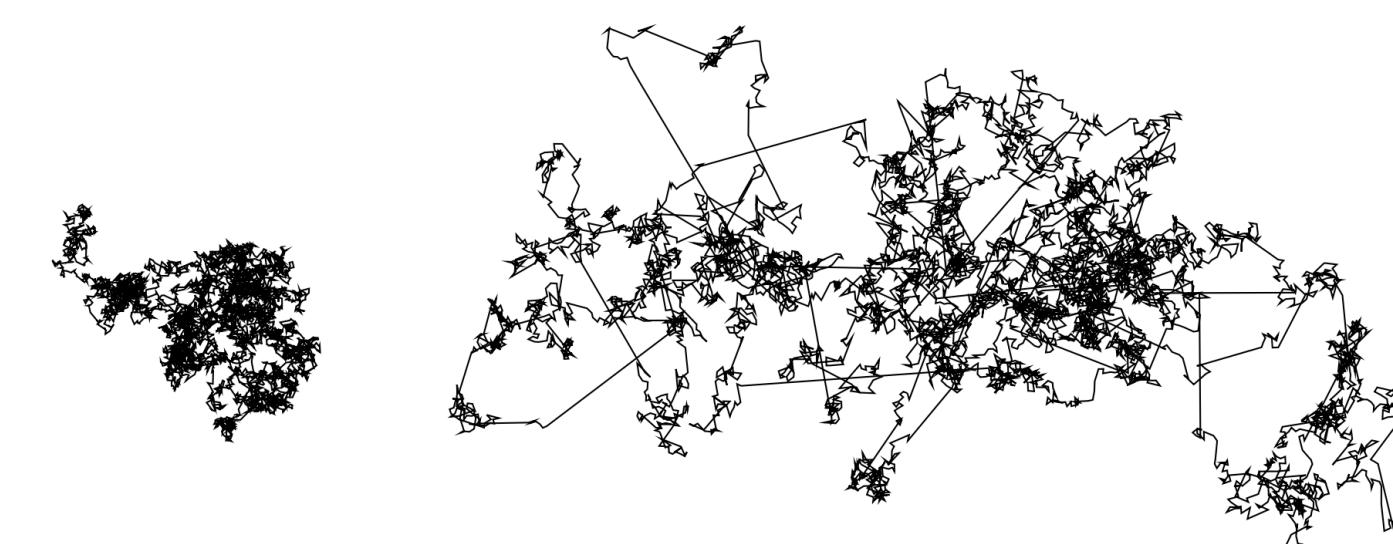
$$Y \equiv \log \frac{L}{\tau_0} \quad c \simeq 1 + 2\sqrt{\bar{\alpha}}$$



Nonlocal quantum corrections:
anomalous system size dependence
(super diffusion)

$$Q_s^2 \simeq \langle k_\perp^2 \rangle_{\text{median}} \propto L^{1+2\sqrt{\bar{\alpha}}}$$

Universal behavior observed down to $L \simeq 3 \text{ fm}$



Heavy tail - Lévy random walk

- Extended geometric scaling window at high k_{\perp}

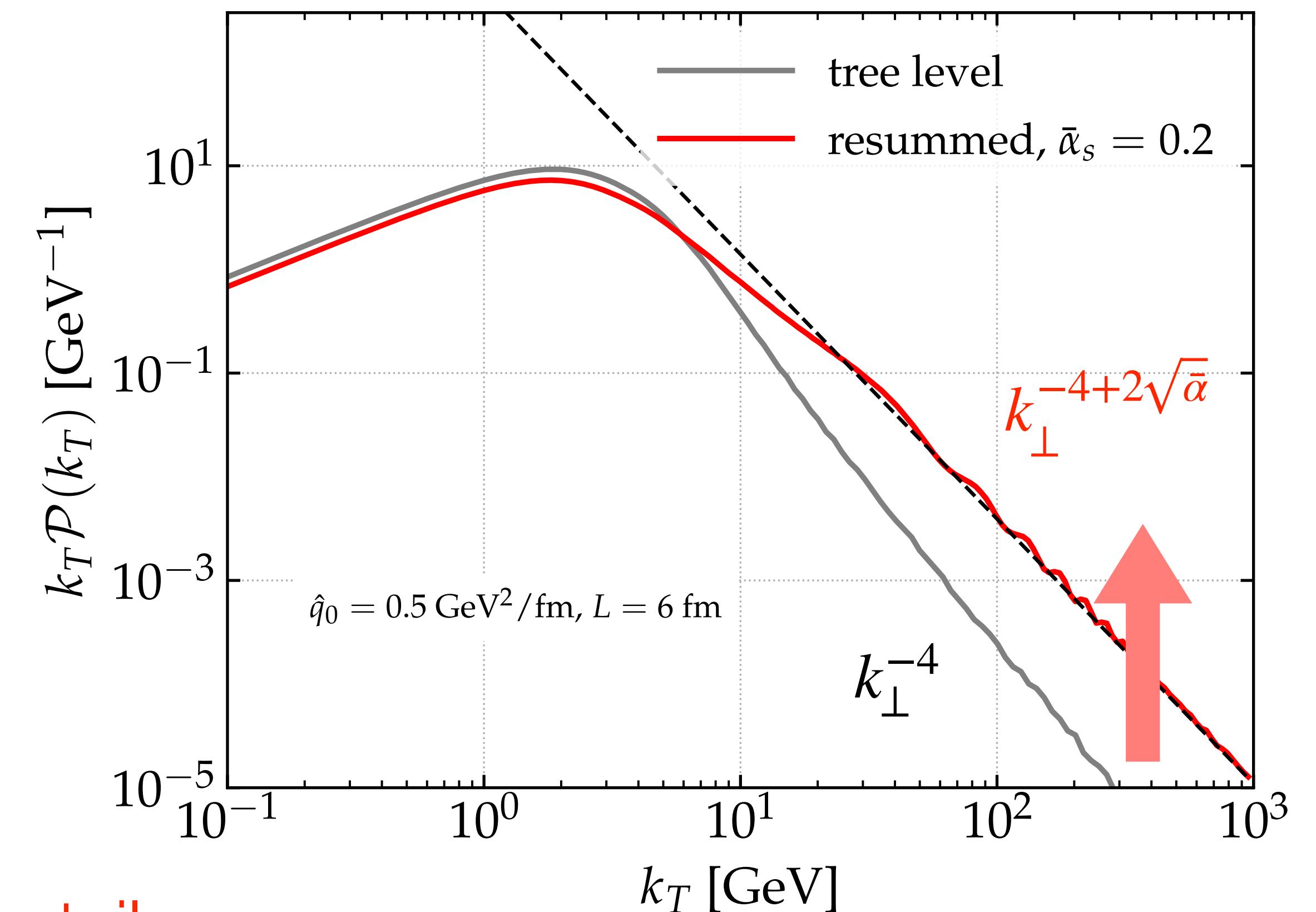
$$Q_s < k_{\perp} \ll \frac{Q_s^2}{\mu}$$

- Asymptotic behavior after resummation (heavy tail)

$$P(k_{\perp}) \rightarrow f(k_{\perp}/Q_s(L)) \simeq \left(\frac{Q_s(L)}{k_{\perp}} \right)^{4-2\sqrt{\bar{\alpha}}}$$

→ Quantum evolution yields heavy power law tail

→ Substantial enhancement w.r.t. the Molière spectrum at high k_{\perp}



D'Eramo, Lekaveckas,
Liu, Rajagopal (2012)

Running coupling case

- One-loop running coupling:

$$\bar{\alpha}(k_\perp) \simeq \frac{b}{\ln k_\perp^2 / \Lambda_{QCD}^2} \quad \text{with} \quad b = \frac{12N_c}{11N_c - 2N_f}$$

- Slower evolution w.r.t. $Y \equiv \log \frac{L}{\tau_0}$

- Modified scaling variable

$$x \equiv \log \frac{k_\perp^2}{Q_s^2} \rightarrow x \equiv \frac{\log \frac{k_\perp^2}{Q_s^2}}{\sqrt{Y}} \sim \sqrt{\bar{\alpha}(Y)} \log \frac{k_\perp^2}{Q_s^2}$$

Systematic approach for universal sub-leading terms

- Fixed coupling

Caucal, MT, 2109.12041 [hep-ph]

$$\rho_s(Y) = cY - \frac{3c}{1+c} \ln(Y) + \kappa - \frac{6c\sqrt{2\pi(c-1)}}{(1+c)^2} \frac{1}{\sqrt{Y}} + \mathcal{O}\left(\frac{1}{Y}\right)$$

- Running coupling

Caucal, MT, 2203.09407 [hep-ph]

$$\begin{aligned} \rho_s(Y) = & Y + 2\sqrt{4b_0Y} + 3\xi_1(4b_0Y)^{1/6} + \left(\frac{1}{4} - 2b_0\right) \ln(Y) + \kappa \\ & + \frac{7\xi_1^2}{180} \frac{1}{(4b_0Y)^{1/6}} + \xi_1 \left(\frac{5}{108} + 18b_0\right) \frac{1}{(4b_0Y)^{1/3}} + b_0 (1 - 8b_0) \frac{\ln(Y)}{\sqrt{4b_0Y}} + \mathcal{O}(Y^{-1/2}) \end{aligned}$$

First four terms conjectured by Iancu and Triantafyllopoulos (2015)

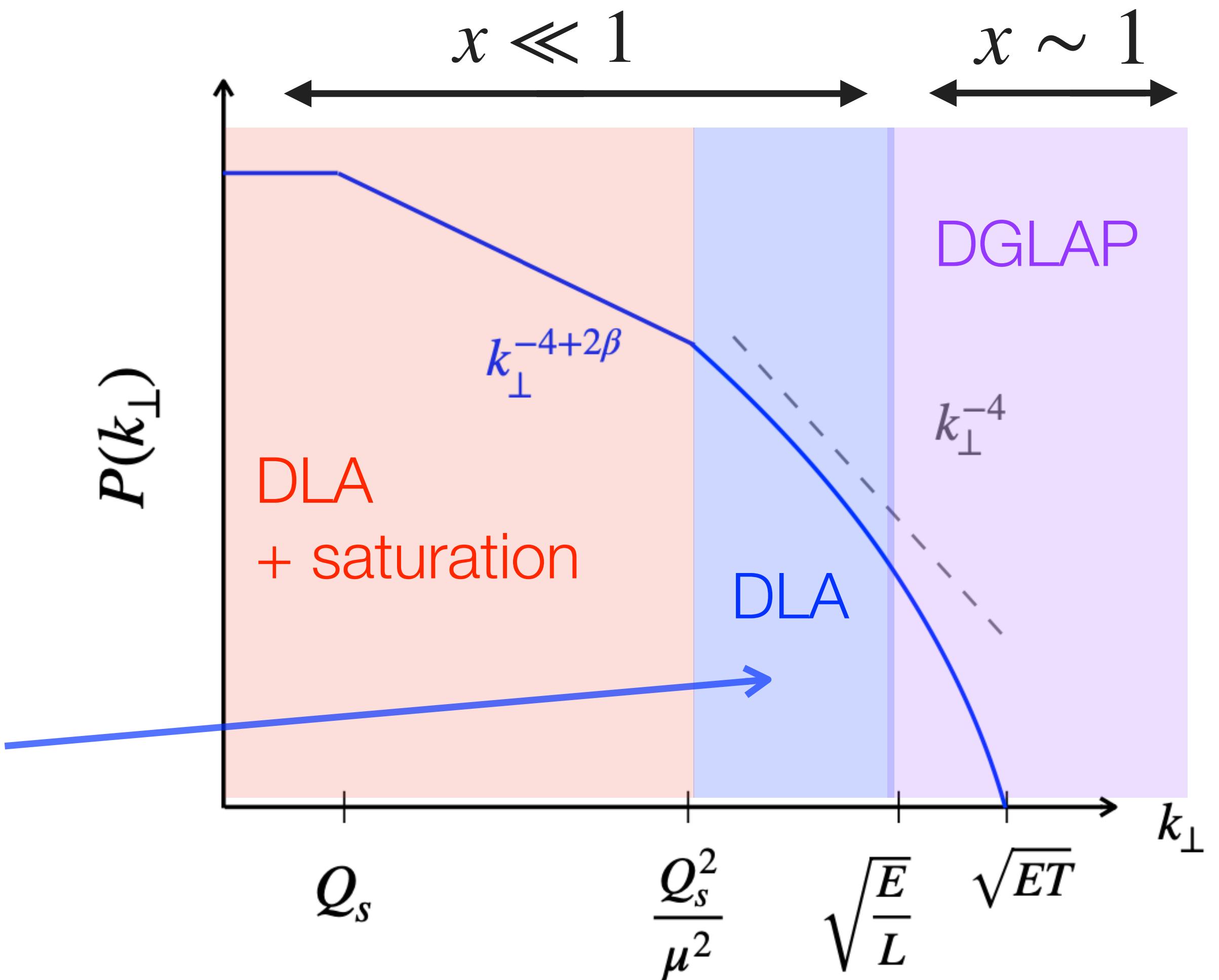
Larger momentum transfer

- Coulomb power law k_\perp^{-4} for $k_\perp \gg Q_s^2/\mu$

$$P(k_\perp) \simeq \frac{L}{k_\perp^4} \frac{\partial}{\partial \log k_\perp^2} \hat{q}(k_\perp^2, L)$$

- Hard scattering regime: gluon PDF of the plasma at DLA:

$$\hat{q}(k_\perp^2) \propto x g(x, k_\perp^2) \simeq \exp \left(\sqrt{\log \frac{k_\perp^2}{\mu^2} \log \frac{L}{\tau_0}} \right)$$



Caucal, MT
2203.09407 [hep-ph]

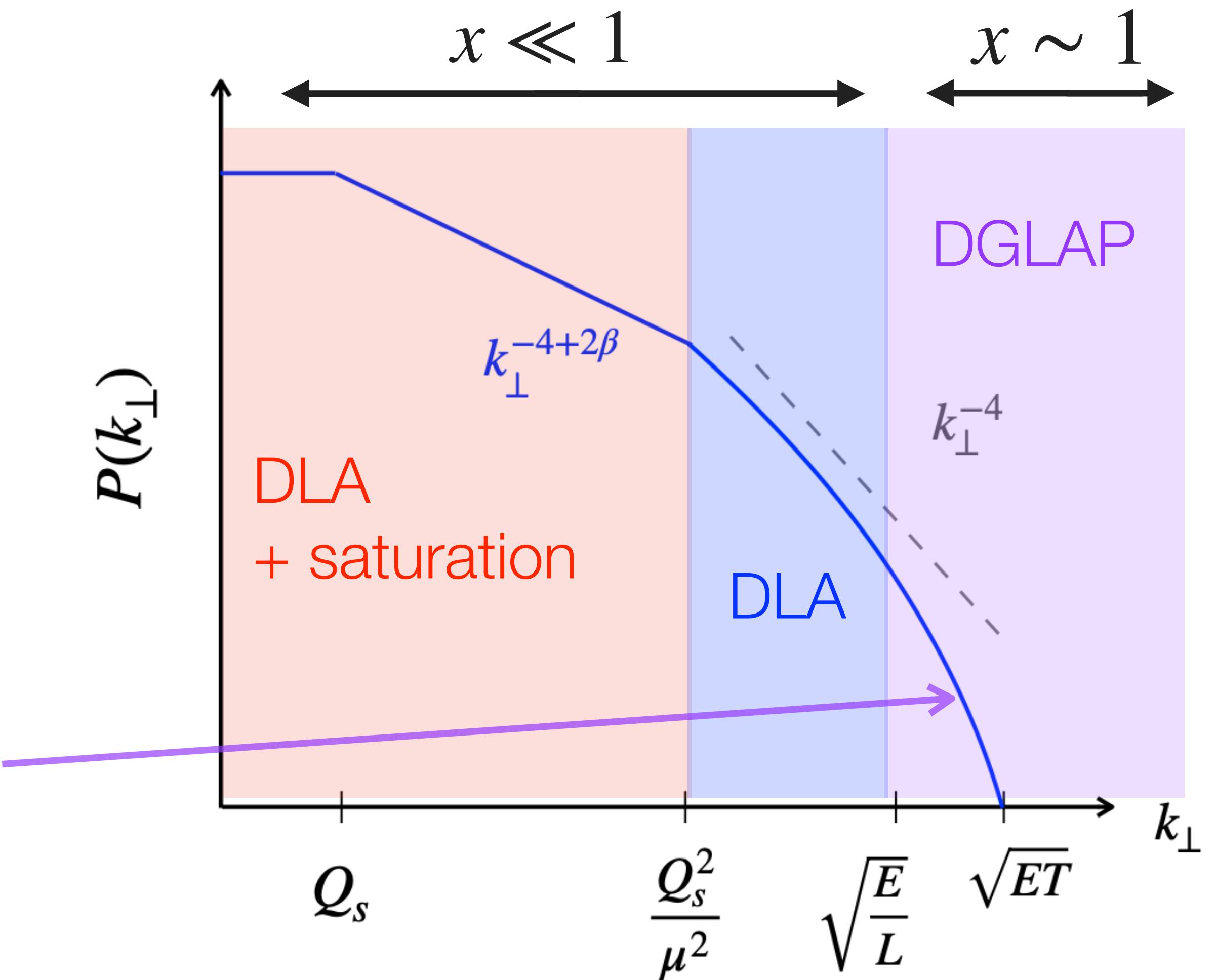
Larger momentum transfer

- Coulomb power law k_\perp^{-4} for $k_\perp \gg Q_s^2/\mu$

$$P(k_\perp) \simeq \frac{L}{k_\perp^4} \frac{\partial}{\partial \log k_\perp^2} \hat{q}(k_\perp^2, L)$$

- Hard scattering regime: gluon PDF of the plasma at DLA:

$$\hat{q}(k_\perp^2) \propto x g(x, k_\perp^2) \simeq \exp \left(\sqrt{\log \frac{k_\perp^2}{\mu^2} \log \frac{ET}{k_\perp^2}} \right)$$



- $k_\perp > \sqrt{E/L}$, expect energy dependence through $x \sim k_\perp^2/ET < 1/LT$

Wang, Casalderrey-Solana (2007) Kumar, Majumder, Chen (2019)

Caucal, MT
2203.09407 [hep-ph]

Summary

- TMB in QCD is a super-diffusive process (non-gaussian) due to logarithmically enhanced quantum corrections → anomalous system size dependence
- Exhibits geometric scaling and heavy tails akin to Lévy random walks → Substantial departure from the LO Molière scattering
- Systematic approach for computing universal asymptotic and pre-asymptotic solutions for transverse momentum broadening
- **Outlook:**
 - i) investigate anomalous diffusion in QCD matter, departure from Molière scattering (point-like scatterers) in HIC
 - ii) next-to-leading-log
 - iii) application to jet quenching observables and system size dependence

Backup

TMB and Wilson lines

- TMB is related to the scattering of color dipole off a **strong background** field A^μ

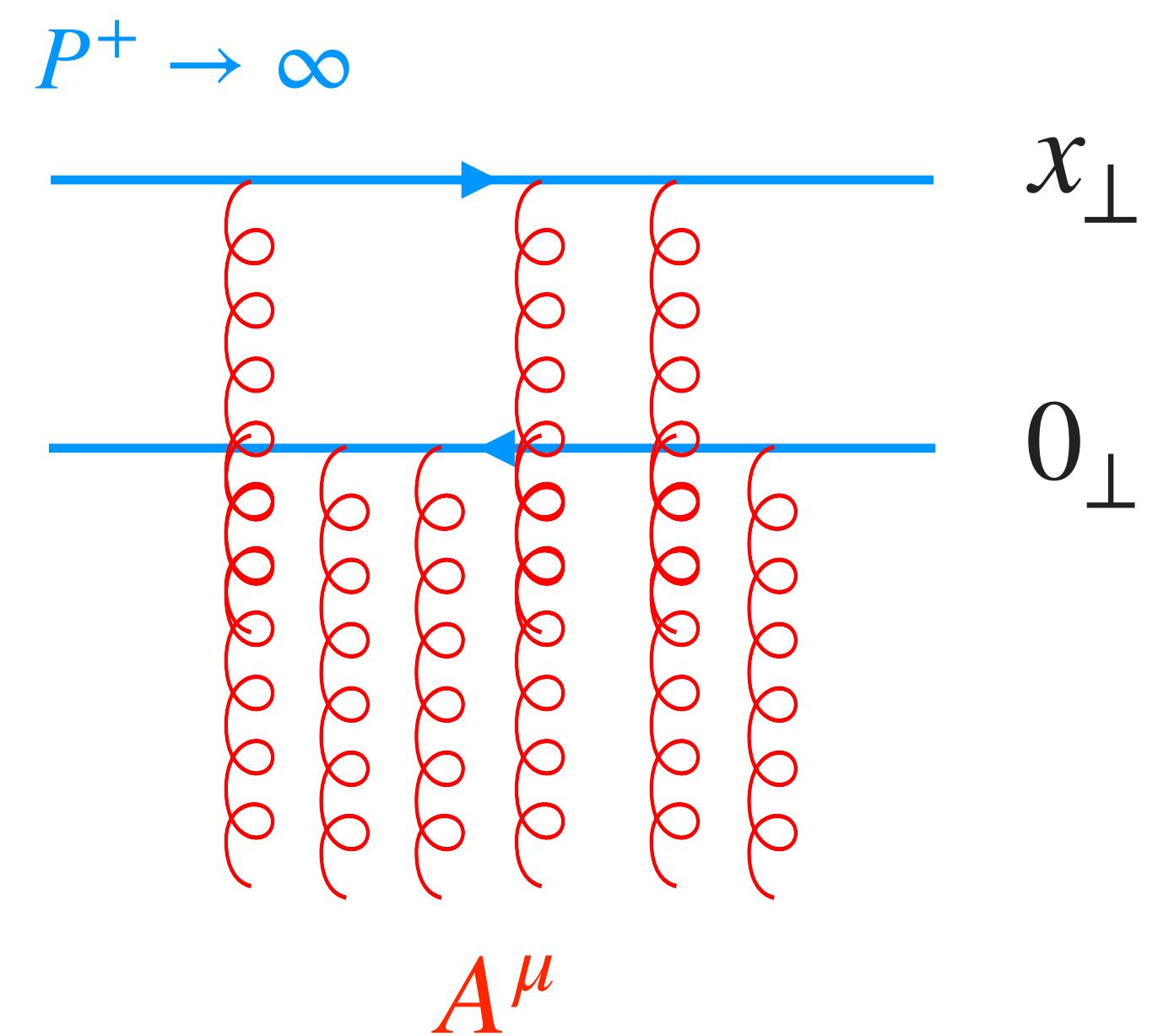
$$\mathcal{P}(\mathbf{k}_\perp) \equiv \frac{dN}{d^2\mathbf{k}_\perp} = \int d^2\mathbf{x}_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} S(\mathbf{x}_\perp).$$

- Dipole S-matrix

$$S(\mathbf{x}_\perp) \equiv \frac{1}{N_c} \text{Tr} \langle U(\mathbf{x}_\perp) U^\dagger(\mathbf{0}) \rangle,$$

Path ordered Wilson line:

$$U(\mathbf{x}_\perp) \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dx^+ t^a A_a^-(x^+, \mathbf{x}_\perp) \right]$$

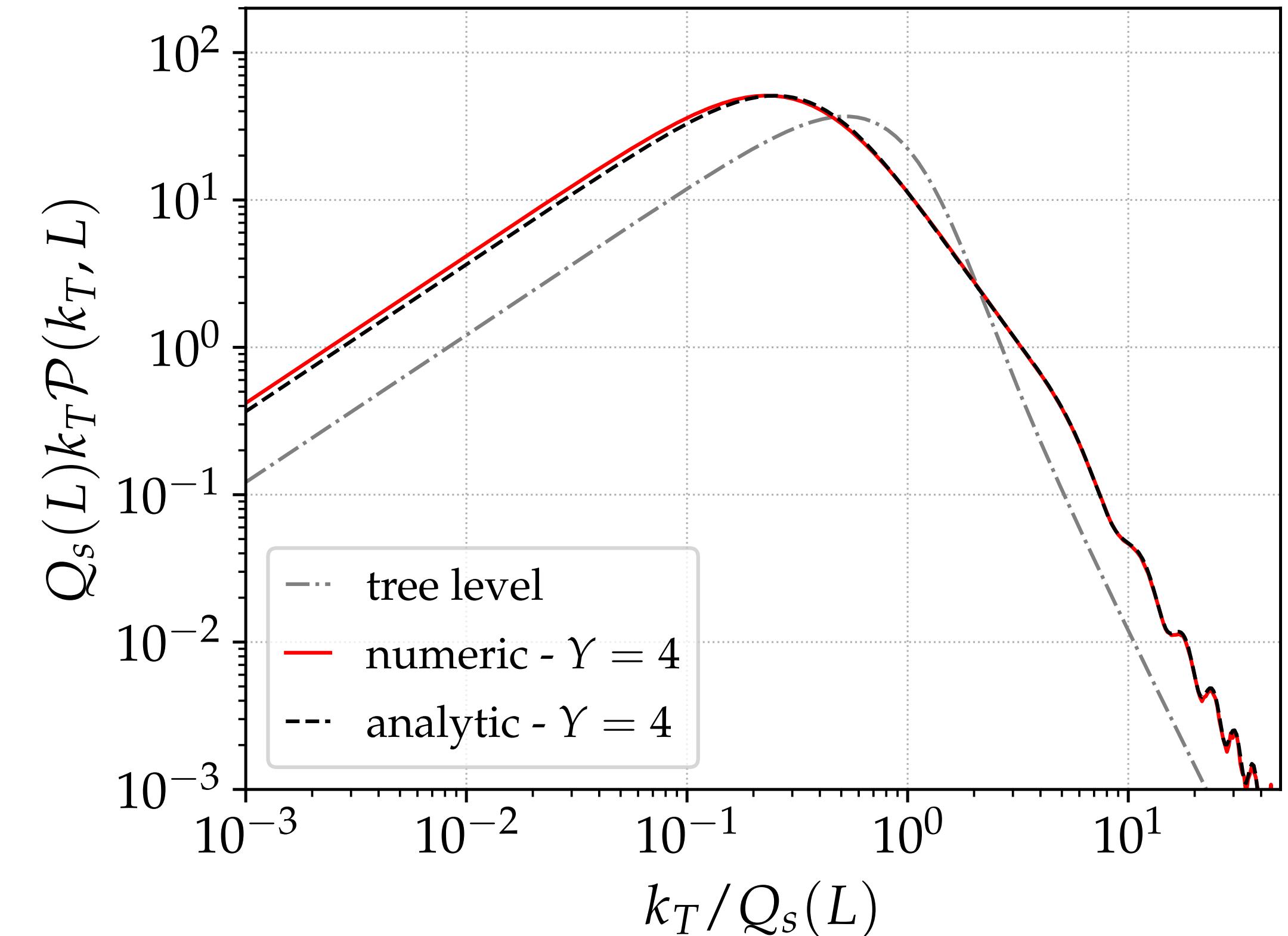


Running coupling case (vs numerics)

- At large Y (leading contribution):

$$G(\zeta) = \frac{2^{1/3} b_0^{1/6}}{\text{Ai}'(\xi_1)} \text{Ai} \left[\xi_1 + 2^{-1/3} b_0^{1/3} \zeta \right]$$

- Small Y (leading edge expansion diverges). Instead expand around the saturation line

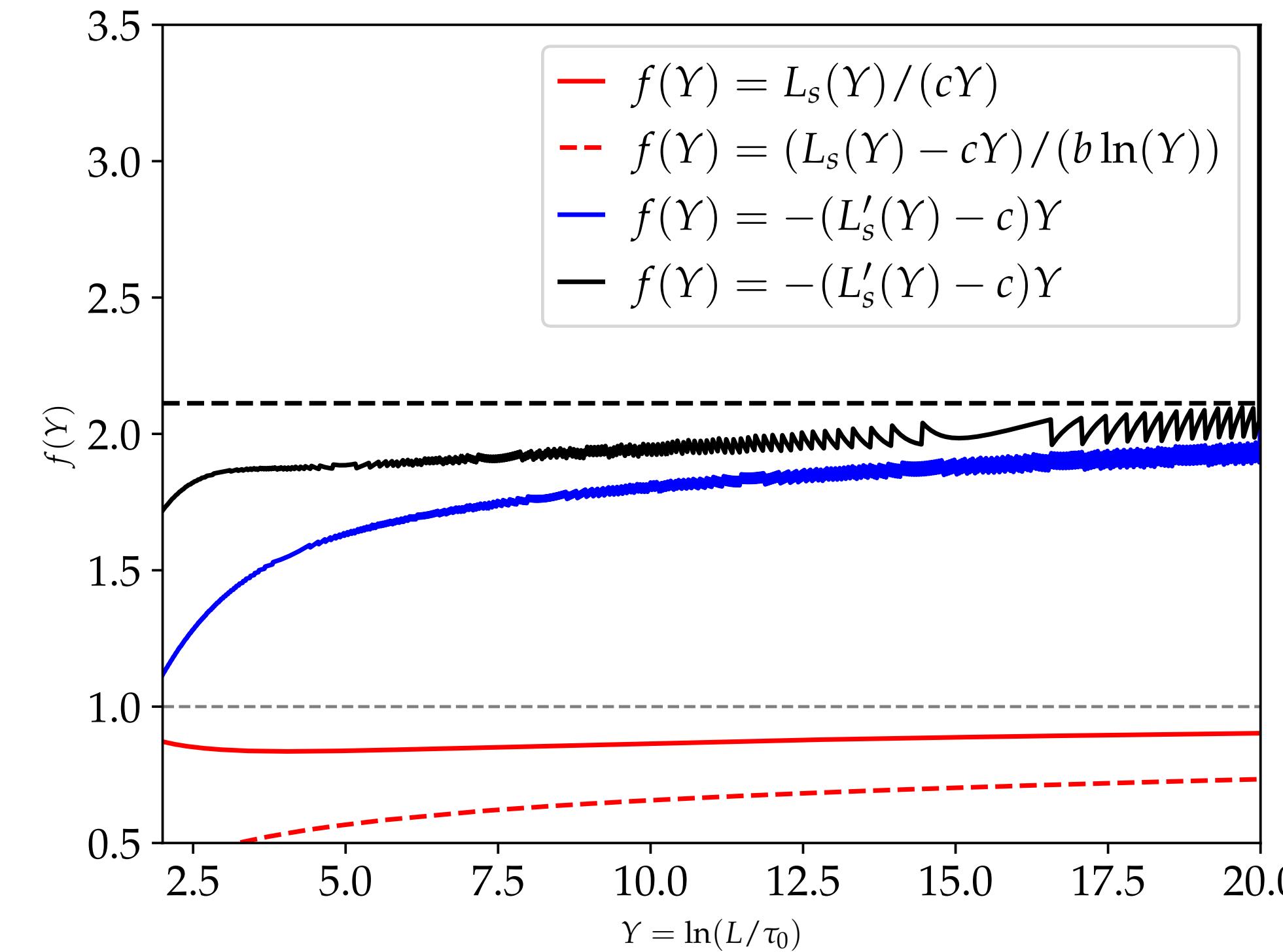


$$\hat{q}_<(Y, x) = \hat{q}_0 e^{\rho_s(Y)-Y} \exp \left(\frac{\dot{\rho}_s - 1}{\dot{\rho}_s} x + \frac{1}{2} \frac{\ddot{\rho}_s}{\dot{\rho}_s^3} x^2 + \mathcal{O}(x^3) \right)$$

Numerical cross-check for the sub-leading term

$$L_s(Y) = \log Q_s^2(Y) = \textcolor{blue}{c}Y + \textcolor{orange}{b}\log Y + \text{const.}$$

$$b = -\frac{2}{3(1-\beta)}$$



Compare to exact solution of linear evolution

Analytic solution for $Q_s(\tau) \sim \hat{q}_0 \tau$

Liou, Mueller, Wu (2013)
Iancu, Triantafyllopoulos (2015)

$$Q_s^2(L) \equiv \hat{q}(Y, Y)\tau = \hat{q}_0 \frac{I_0(2\sqrt{\bar{\alpha}}Y)}{\sqrt{\alpha Y}} \simeq L^{2\sqrt{\bar{\alpha}}}$$

$$Y \equiv \log \frac{L}{\tau_0}$$

$$\Rightarrow b_{\text{lin}} = \frac{3}{2} \neq b_{\text{non-lin}} = \frac{3}{2(1 - \beta)}$$

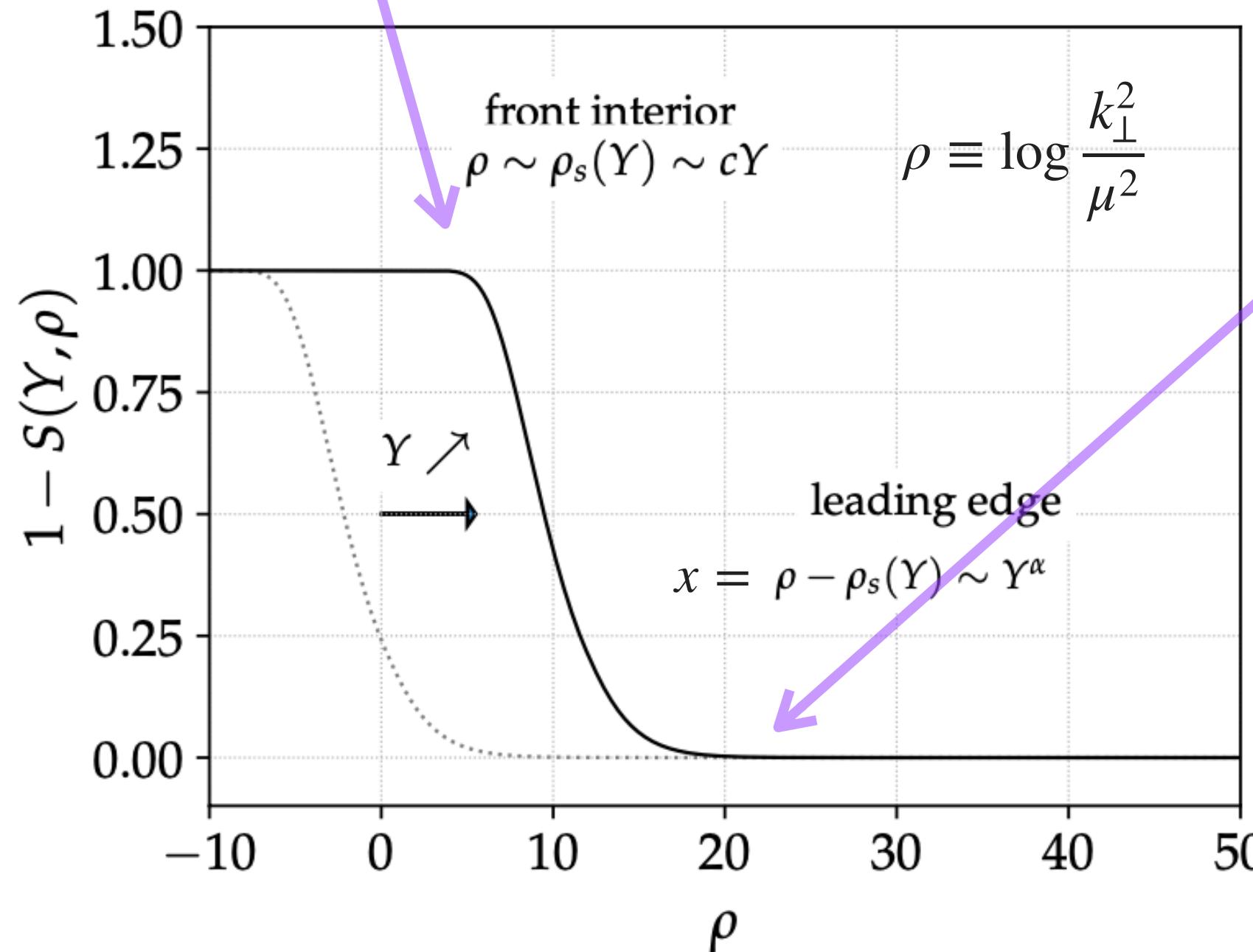
→ Also, only non-linear evolution yields exact geometric scaling

- Front interior

- Leading edge:** growth of perturbations around the unstable state $S = 0$ (diffusion of the wave front)

$$\frac{\hat{q}(Y, k_\perp) L}{Q_s^2} = e^{\beta x} \left[f_0(x) + \frac{1}{Y^{1/2}} f_1(x) + \frac{1}{Y} f_2(x) + \dots \right]$$

$$\frac{\hat{q}(Y, k_\perp) L}{Q_s^2} = e^{\beta x} \left[Y^{1/2} G_1\left(\frac{x}{Y^{1/2}}\right) + G_0\left(\frac{x}{Y^{1/2}}\right) + \frac{1}{Y^{1/2}} G_{-1}\left(\frac{x}{Y^{1/2}}\right) + \dots \right]$$



U. Ebert and W. van Saarloos (2000)

$f_n(x)$	x^j	x^0	x^1	x^2	x^3	\dots	x^j	x^{j+1}	x^{j+2}	\dots
$f_0(x)$		1	β	/	/	/	/	/	/	/
$f_1(x)$		0	0	0	/	/	/	/	/	/
$f_2(x)$		0	$\frac{\delta_1}{c^2}$	$\frac{\delta_1(c-1)(c+3)}{8c^3}$	$\frac{\delta_1(c-1)^2(c+1)}{48c^4}$	/	/	/	/	/
\dots		\dots	\dots	\dots	\dots	\dots	\dots	\dots	\dots	\dots
$f_n(x)$		0	f_n^1	f_n^2	f_n^3	G_0	f_n^n	f_n^{n+1}		
\dots		\dots	\dots	\dots	\dots	\dots	\dots	\dots	\dots	\dots

f_n : polynomials of degree n

- Leading edge expansion resums terms on the diagonals
- Boundary conditions at infinity constraint the saturation line: $Q_s(Y)$

Traveling waves (TW) solutions

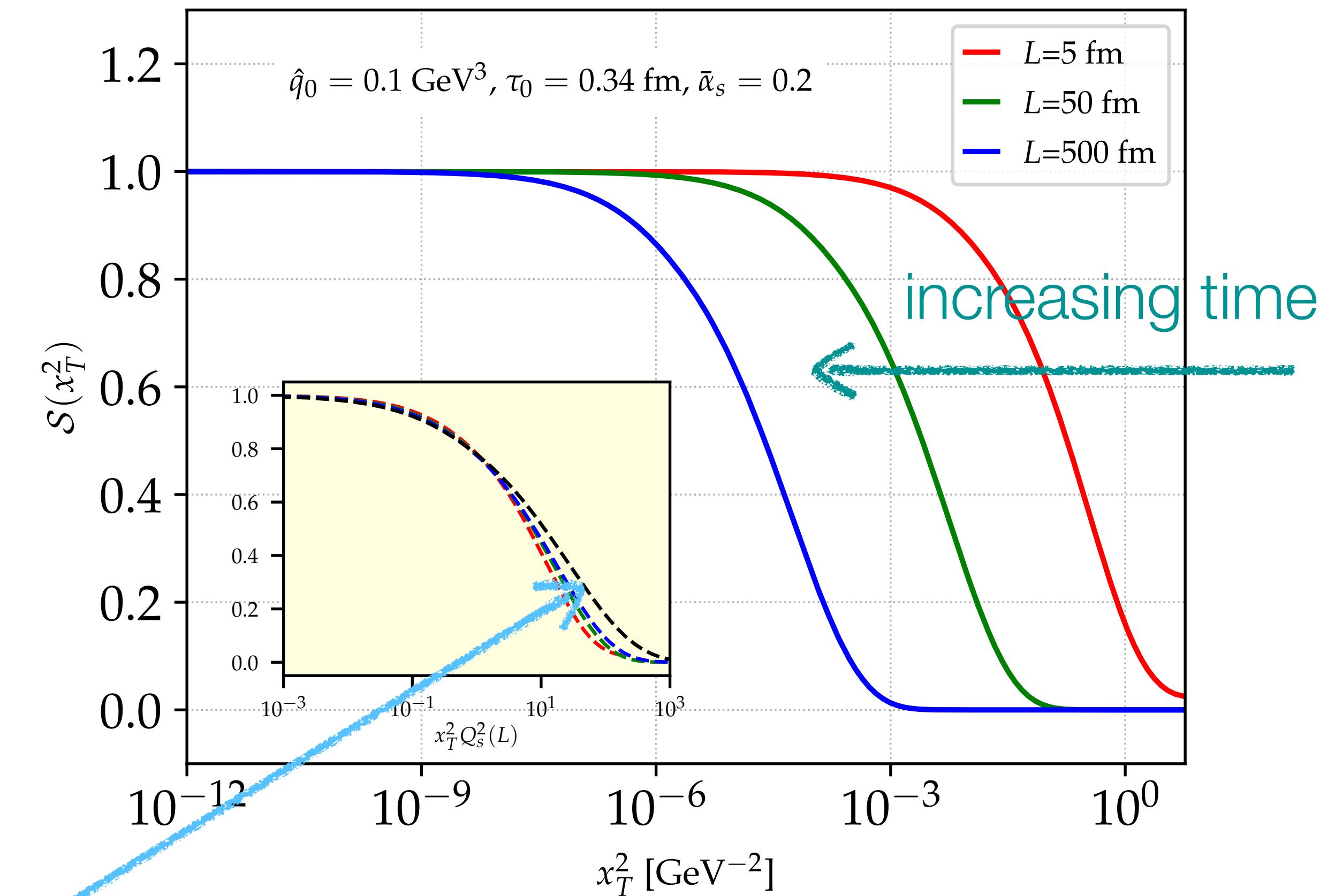
- TMB relates to $q\bar{q}$ dipole S-matrix by a Fourier transform

$$P(k_{\perp}) \equiv \int d^2x_{\perp} S(x_{\perp}) e^{-ix_{\perp}\cdot k_{\perp}}$$

Where

$$S(x_{\perp}) = e^{-\frac{1}{4}x_{\perp}^2 L \hat{q}(1/x_{\perp}, L)}$$

- Dipole S-matrix admits TW solutions
- Exact scaling solution (dashed black curve): are the deviations to the asymptotic limit universal?



Operator definition of \hat{q}

- \hat{q} is in the double log regime and can be accessed with OPE or kt-factorization

$$\hat{q} \equiv \frac{4\pi^2 \alpha_s n}{P^-} \int_0^\infty \frac{dx^+}{2\pi} e^{ixP^-x^+} \langle P | F^{i-}(x^+) [x^+, 0^+] F^{i-}(0^+) | P \rangle \Theta(x^+ < L)$$

- No x (coherence length) dependence when

$$(xP^-)^{-1} \gg L$$