

Numerical approach to calculate in-medium emissions

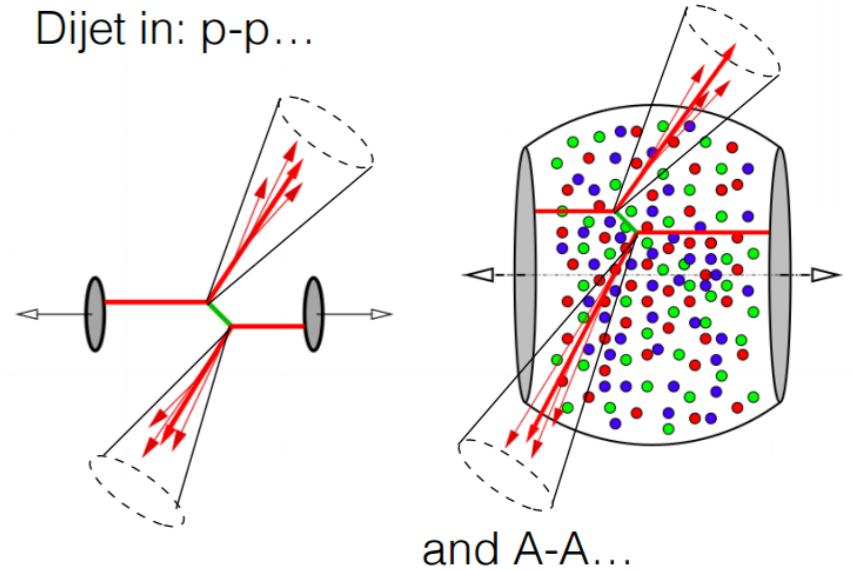
Carlota Andrés

CPhT, École polytechnique

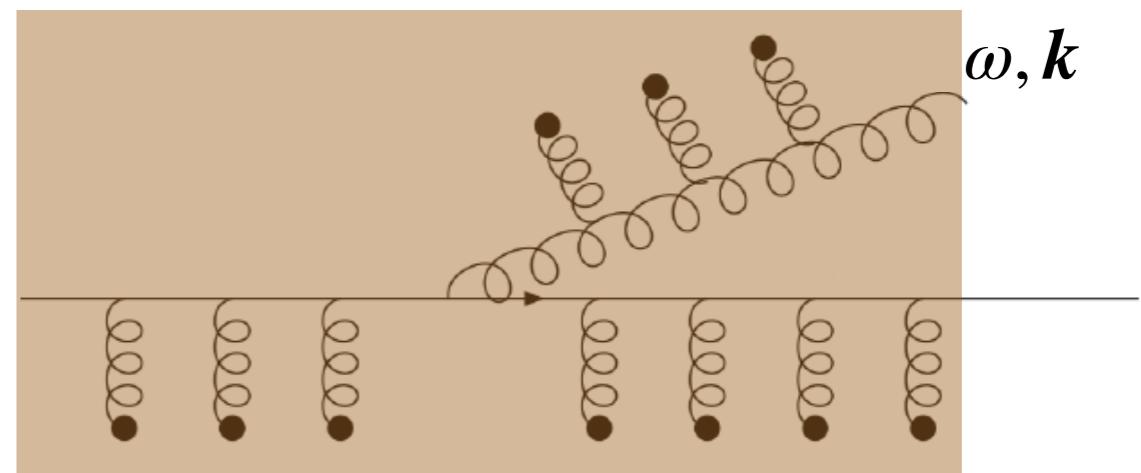
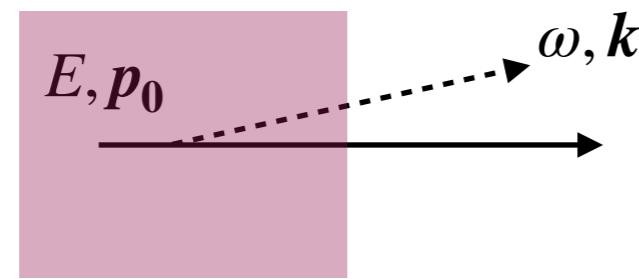
Jet quenching in the Quark-Gluon Plasma, ECT*, Italy

Energy loss

Dijet in: p-p...



- Jet quenching: high energy partons interact with the QGP losing energy
- How does a parton lose energy in a QCD medium?
 - Collisions - Important for heavy particles
 - **Radiation** - Extra gluon radiation induced by multiple scatterings with the medium
Dominant for light quarks and gluons (this talk)



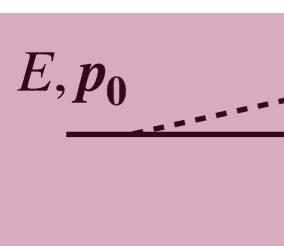
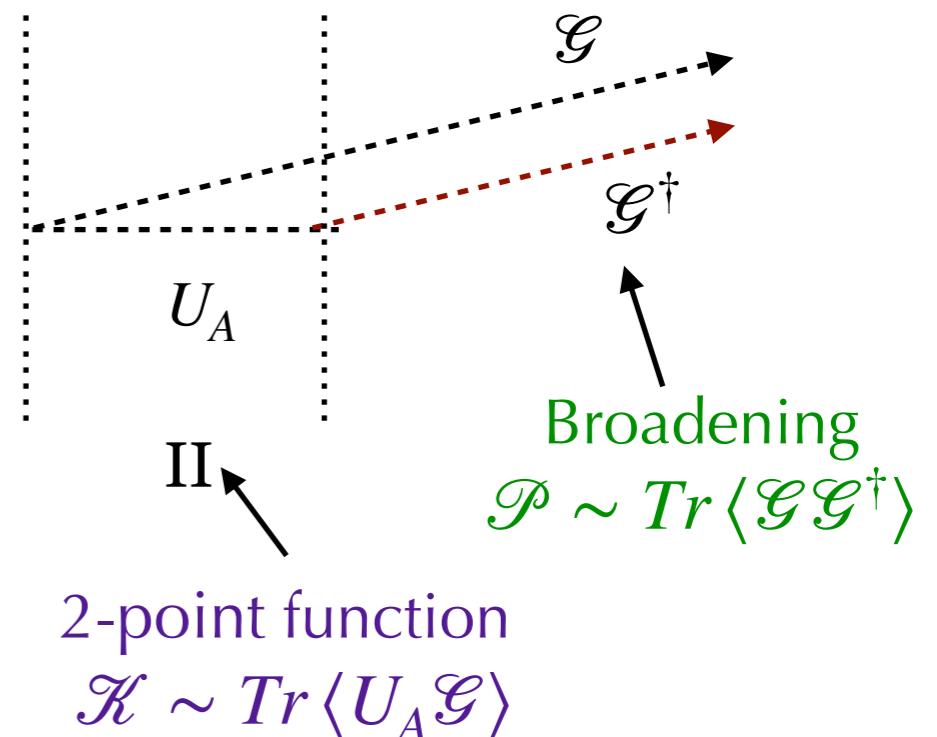
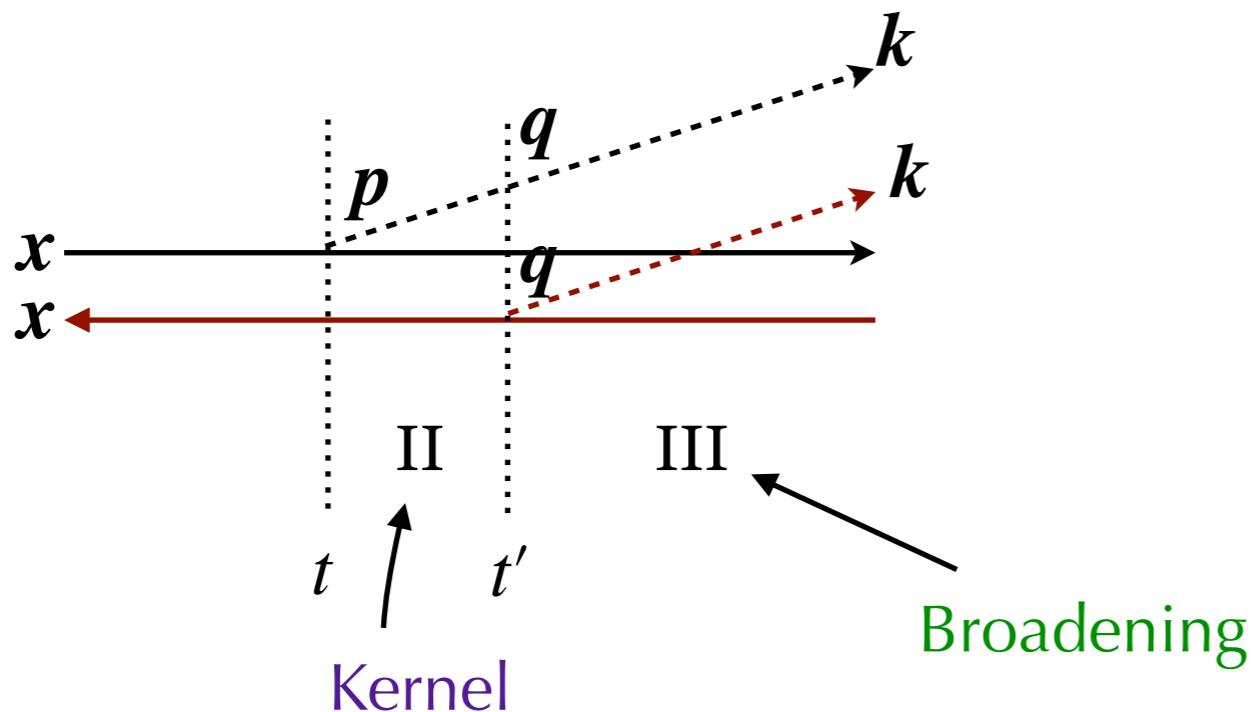
Medium-induced gluon spectrum

- For a **soft** emitted gluon ($\omega \ll E$)

Baier, Dokshitzer, Mueller, Peigné, Schiff (96)
Zaharov (97)

$$\omega \frac{dI}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega^2} \operatorname{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\mathbf{pq}} \mathbf{p} \cdot \mathbf{q} \tilde{\mathcal{K}}(t', \mathbf{q}; t, \mathbf{p}) \mathcal{P}(\infty, \mathbf{k}; t', \mathbf{q})$$

BDMPS-Z



For the spectrum beyond the soft approximation see Dominguez's talk. Today 11:00

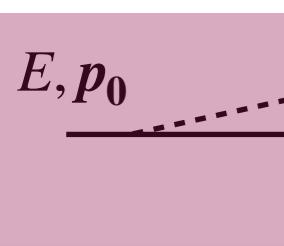
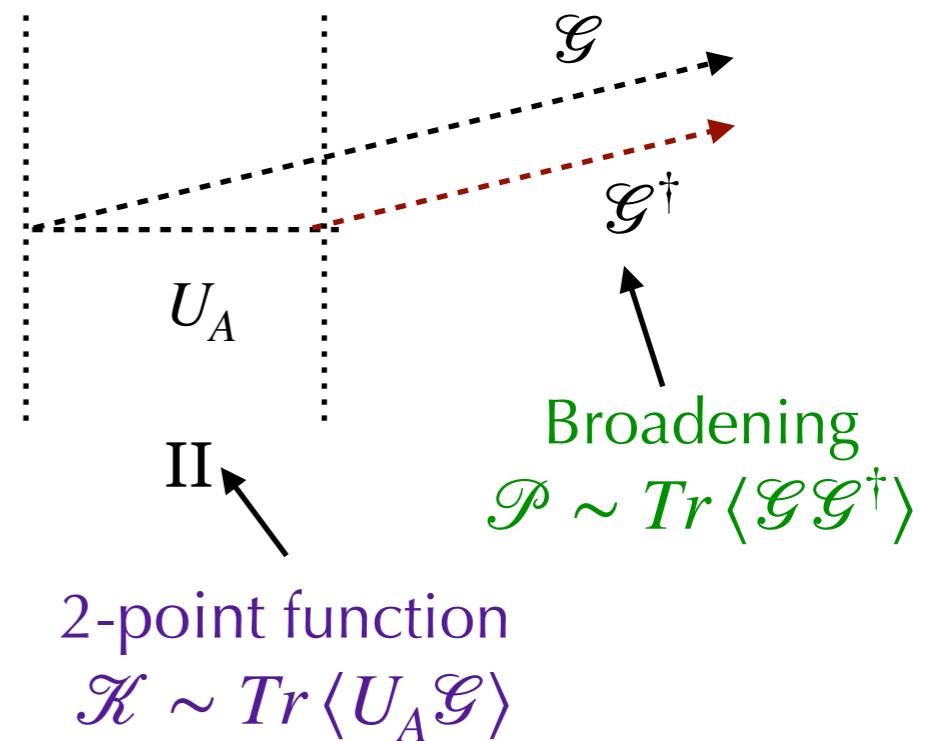
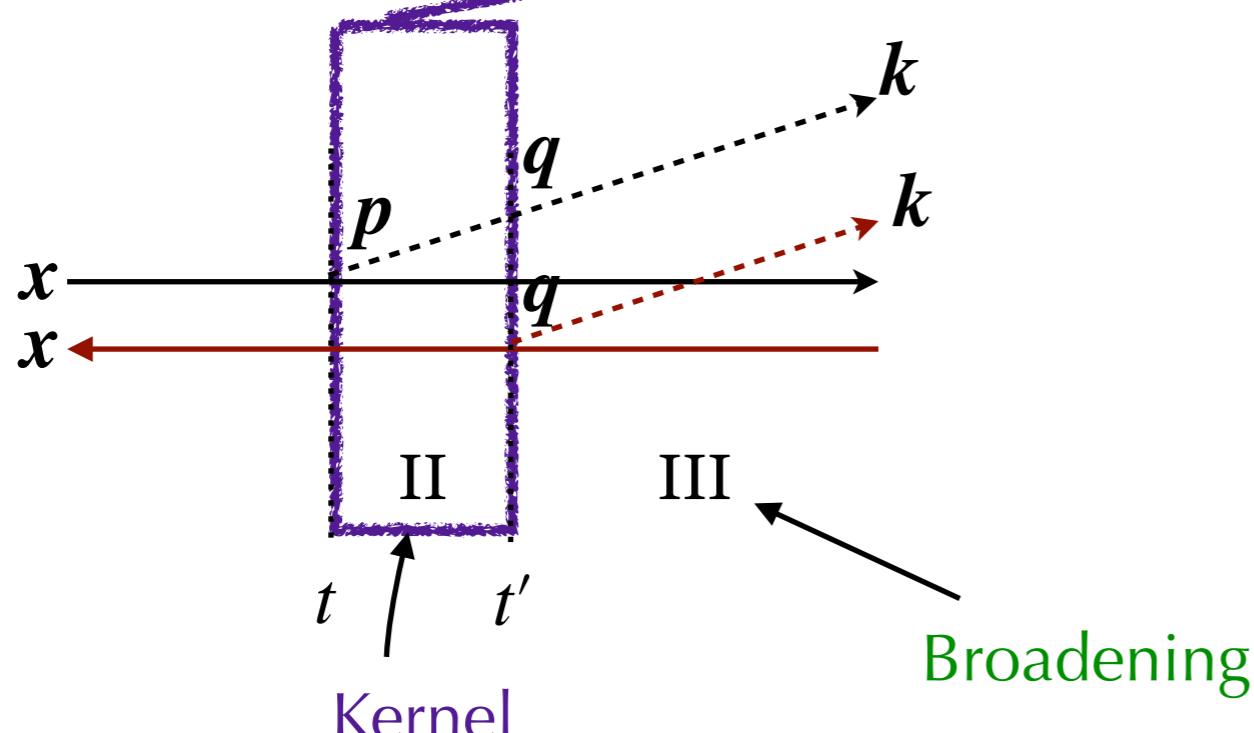
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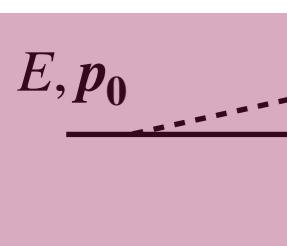
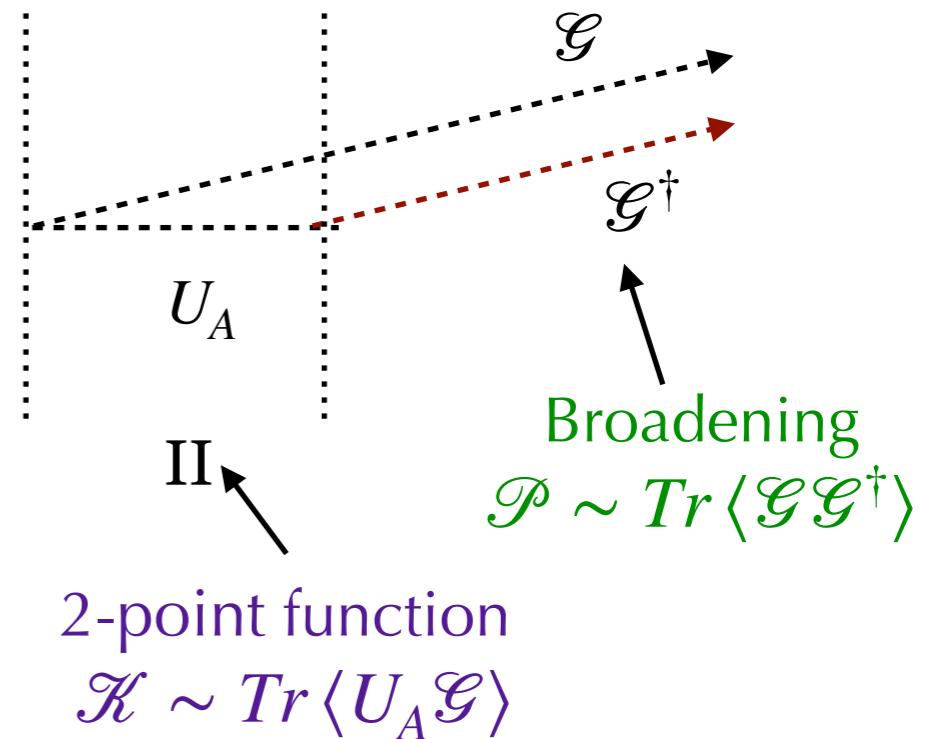
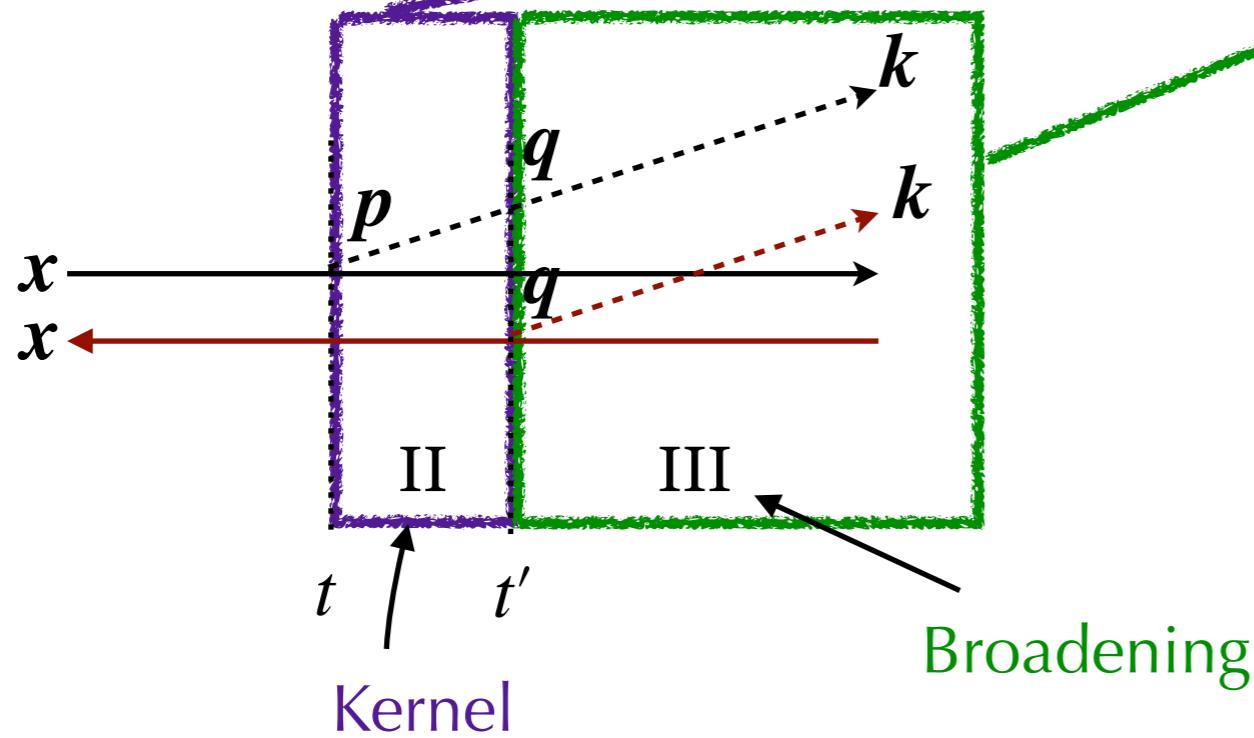
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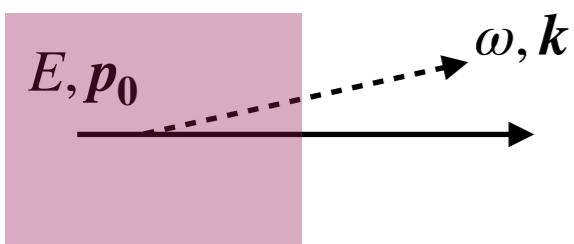
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BDMPS-Z

$$\mathcal{P}(t'', \mathbf{k}; t', \mathbf{q}) \equiv \int d^2\mathbf{z} e^{-i(\mathbf{k}-\mathbf{q}) \cdot \mathbf{z}} \exp \left\{ -\frac{1}{2} \int_{t'}^{t''} ds n(s) \sigma(z) \right\}$$

$$\begin{aligned} \mathcal{K}(t', \mathbf{z}; t, \mathbf{y}) &\equiv \int_{\mathbf{pq}} e^{i(\mathbf{q} \cdot \mathbf{z} - \mathbf{p} \cdot \mathbf{y})} \tilde{\mathcal{K}}(t', \mathbf{q}; t, \mathbf{p}) \\ &= \int_{\mathbf{r}(t)=\mathbf{y}}^{\mathbf{r}(t')=\mathbf{z}} \mathcal{D}\mathbf{r} \exp \left[\int_t^{t'} ds \left(\frac{i\omega}{2} \dot{\mathbf{r}}^2 - \frac{1}{2} \boxed{n(s)\sigma(\mathbf{r})} \right) \right] \end{aligned}$$

Medium information



Difficult to solve numerically for realistic $\sigma(\mathbf{r})$

The building block

- The in-medium spectrum is given by ($\omega \ll E$):

$$\omega \frac{dI}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega^2} \text{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\mathbf{pq}} \mathbf{p} \cdot \mathbf{q} \tilde{\mathcal{K}}(t', \mathbf{q}; t, \mathbf{p}) \mathcal{P}(\infty, \mathbf{k}; t', \mathbf{q})$$

BDMPS-Z

- It has been traditionally evaluated in many approximations (GLV, AMY, HO...)
- Several **new approaches go beyond the usual approximations**

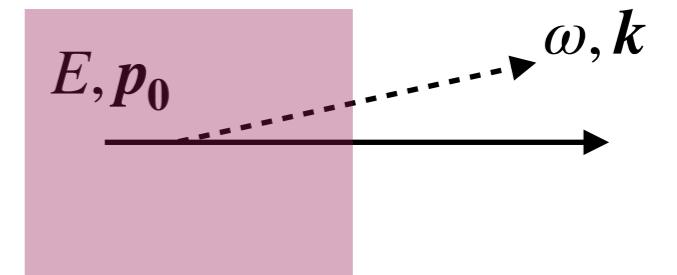
Finite length rates: Caron-Huot and Gale, [1006.2379](#)

IOE (expansion around the HO): Mehtar-Tani, Barata, Soto-Ontoso,
Tywoniuk, [1903.00506](#), [2106.07402](#)

Fully resummed spectrum: CA, Apolinario, Martinez, Dominguez, [2002.01517](#), [2011.06522](#)

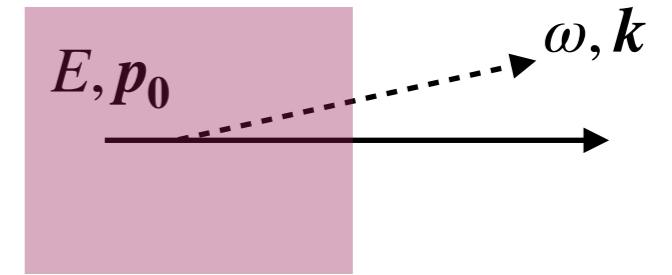
Finite length rates + non-perturbative potential: Schlichting, Soudi, [2111.13731](#)
See Soudi's talk. Today 13:30

IOE & Resummed opacity expansion (ROE) Isaksen, Takacs and Tywoniuk [arXiv:2206.0281](#)
See Isaksen's talk. Today 11:30



The building block

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$$\omega \frac{dI}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega^2} \operatorname{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\mathbf{pq}} \mathbf{p} \cdot \mathbf{q} \tilde{\mathcal{K}}(t', \mathbf{q}; t, \mathbf{p}) \mathcal{P}(\infty, \mathbf{k}; t', \mathbf{q})$$

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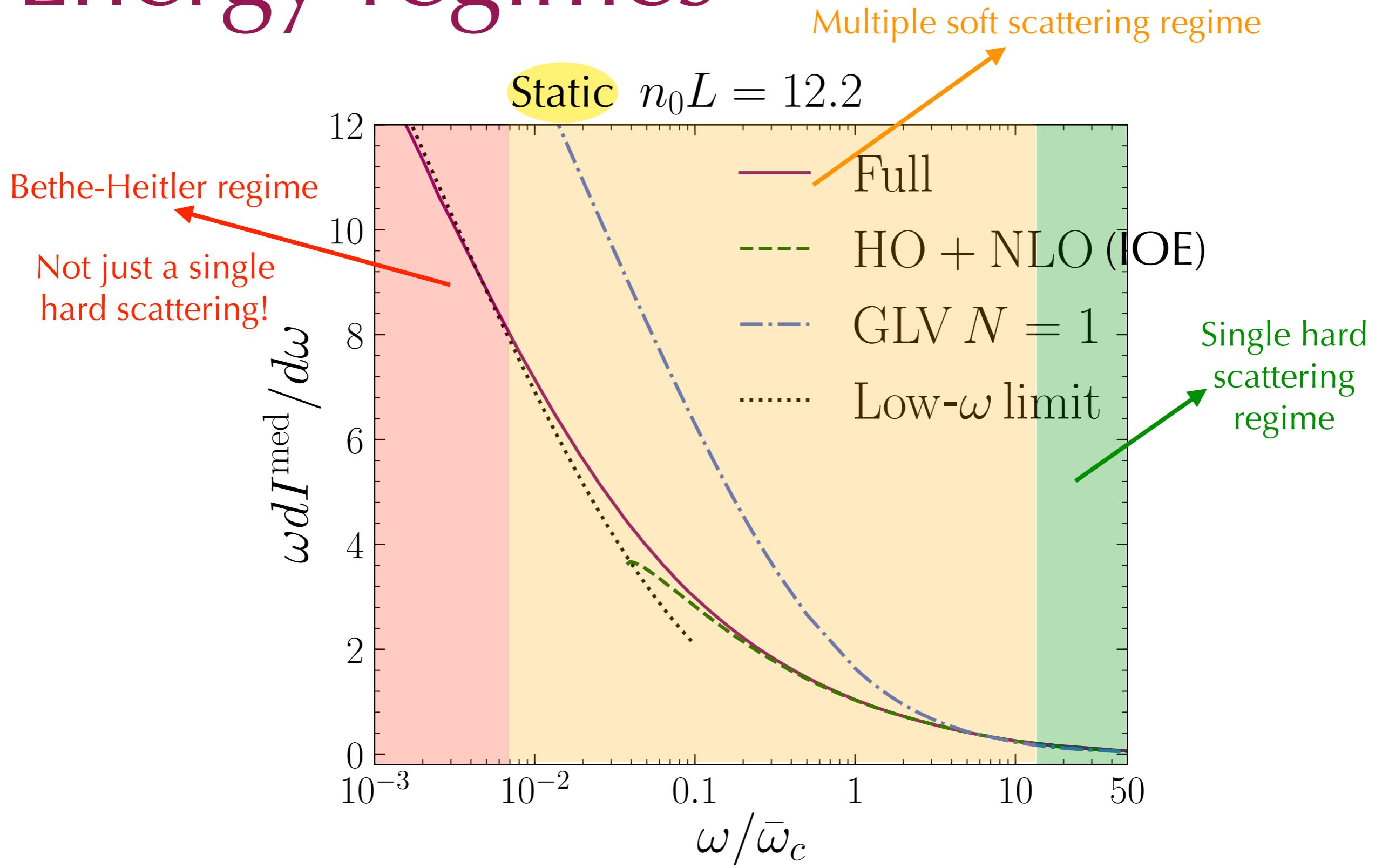
$$\mathcal{K}(s, \mathbf{q}; s, \mathbf{p}) = (2\pi)^2 \delta^{(2)}(\mathbf{q} - \mathbf{p})$$

$$\partial_t \mathcal{K}(s, \mathbf{q}; t, \mathbf{p}) = \frac{i\mathbf{p}^2}{2\omega} \mathcal{K}(s, \mathbf{q}; t, \mathbf{p}) + \frac{1}{2} n(t) \int_{\mathbf{k}'} \sigma(\mathbf{k}' - \mathbf{p}) \mathcal{K}(s, \mathbf{q}; t, \mathbf{k}')$$

$$\sigma(\mathbf{q}) \equiv -V(\mathbf{q}) + (2\pi)^2 \delta^2(\mathbf{q}) \int_l V(\mathbf{l})$$

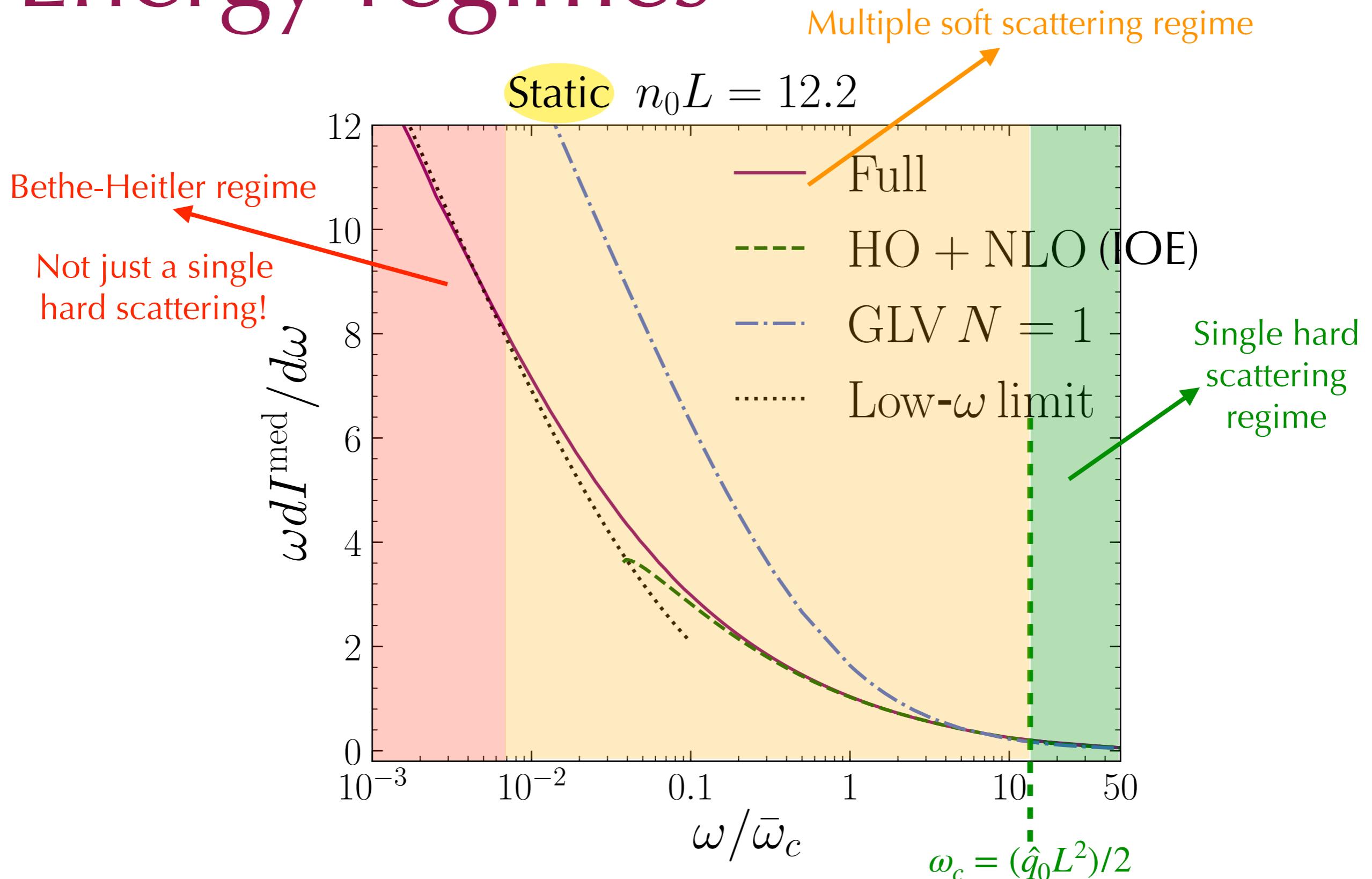
$$V(\mathbf{q}) = \frac{8\pi\mu^2}{(\mathbf{q}^2 + \mu^2)^2}$$

Energy regimes



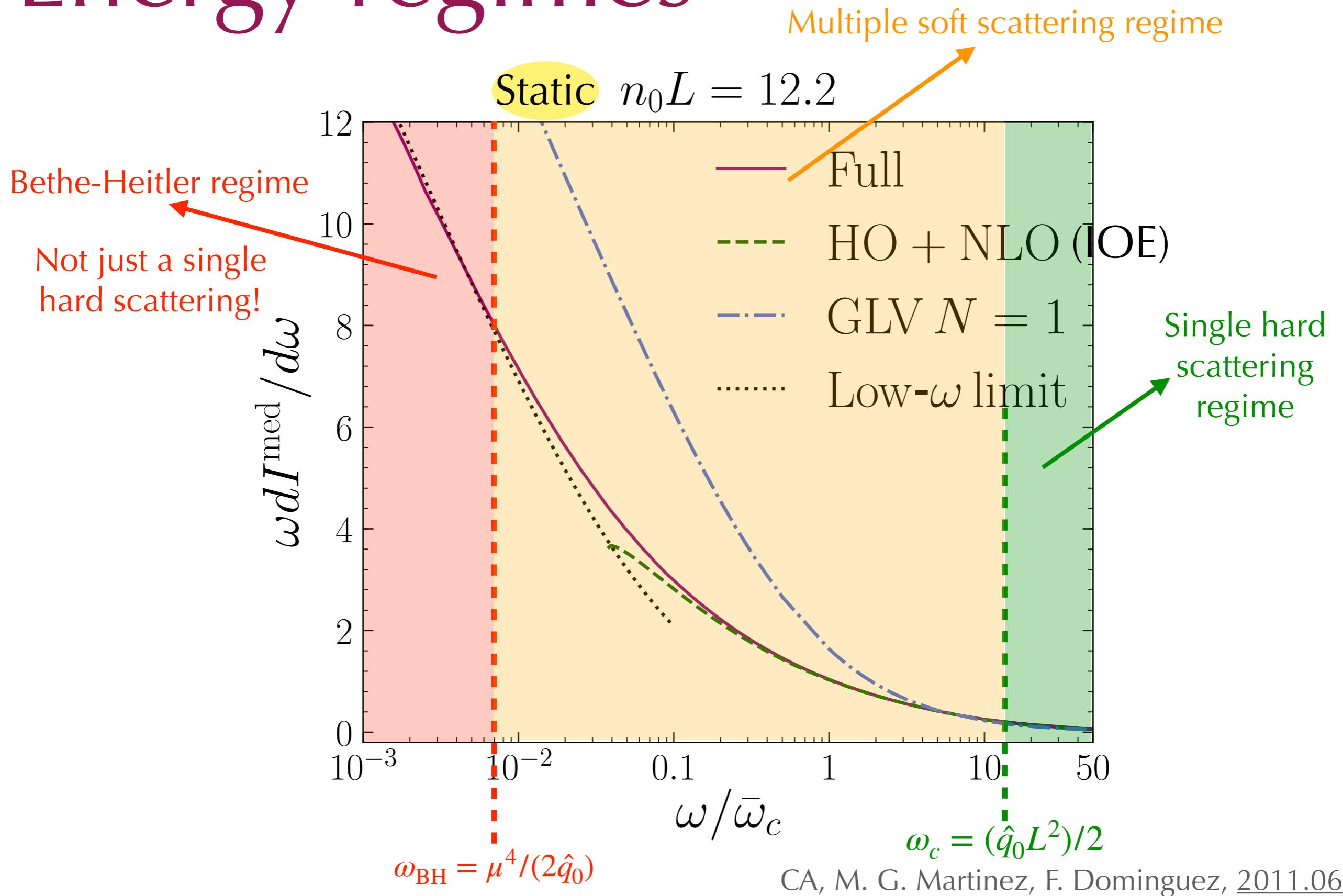
CA, M. G. Martinez, F. Dominguez, [2011.06522](#)

Energy regimes

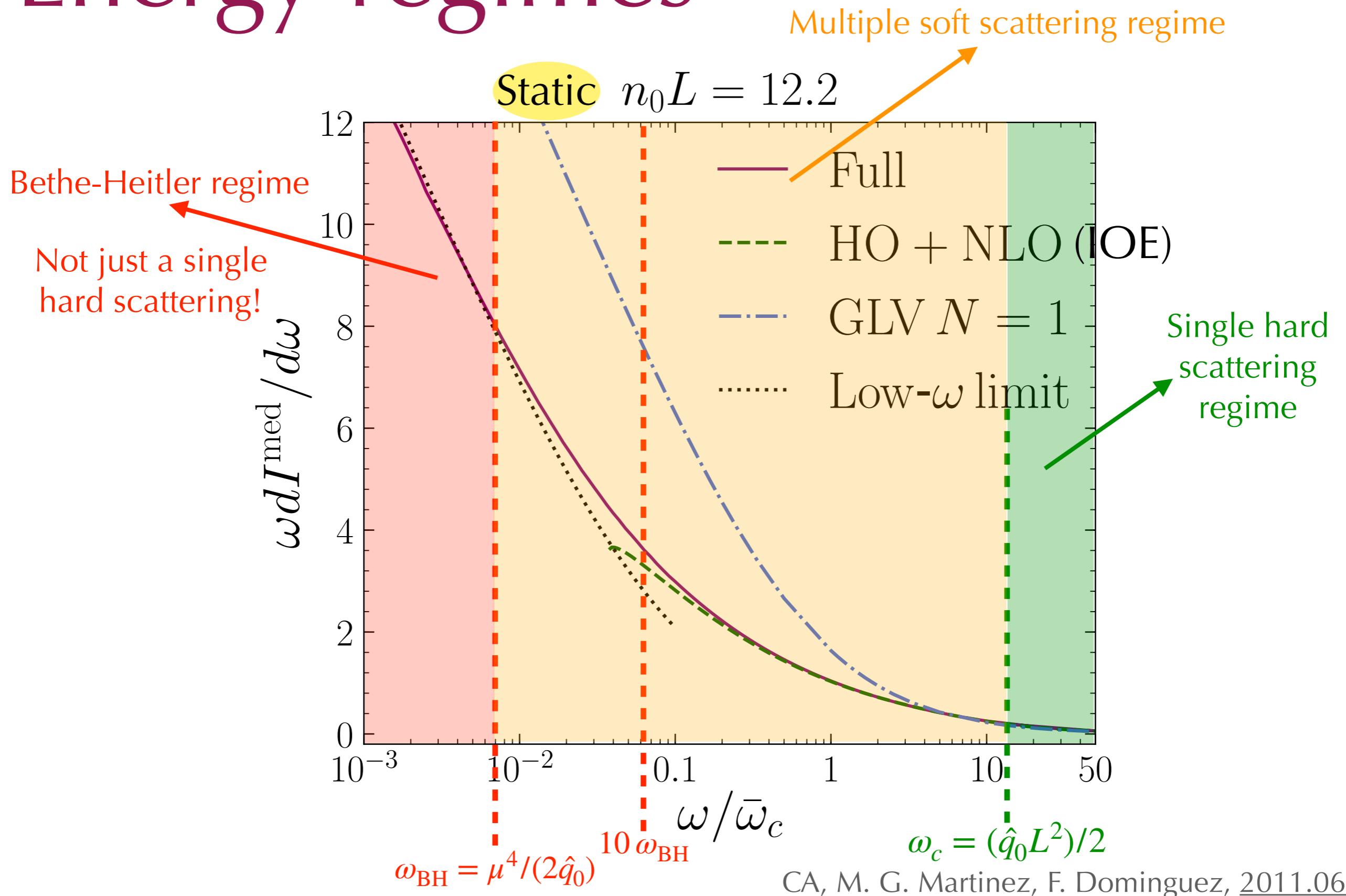


CA, M. G. Martinez, F. Dominguez, 2011.06522

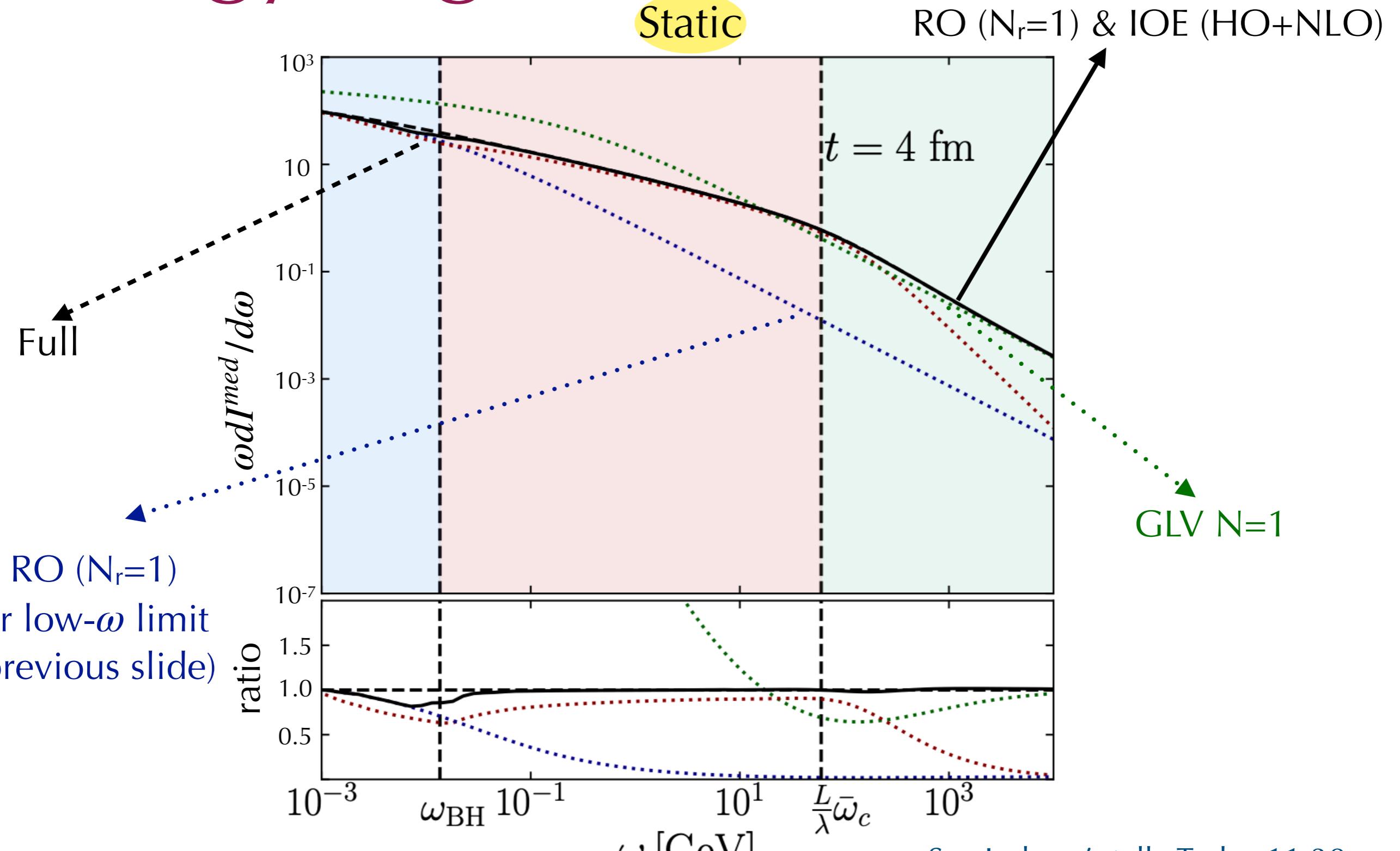
Energy regimes



Energy regimes



Energy regimes



How do we move to a longitudinal evolving medium?

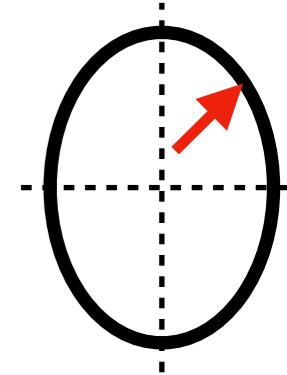
See also Souvik's talk. Tuesday 9:30

For **transverse** flow see Sadofyev's talk. Wed 11:30

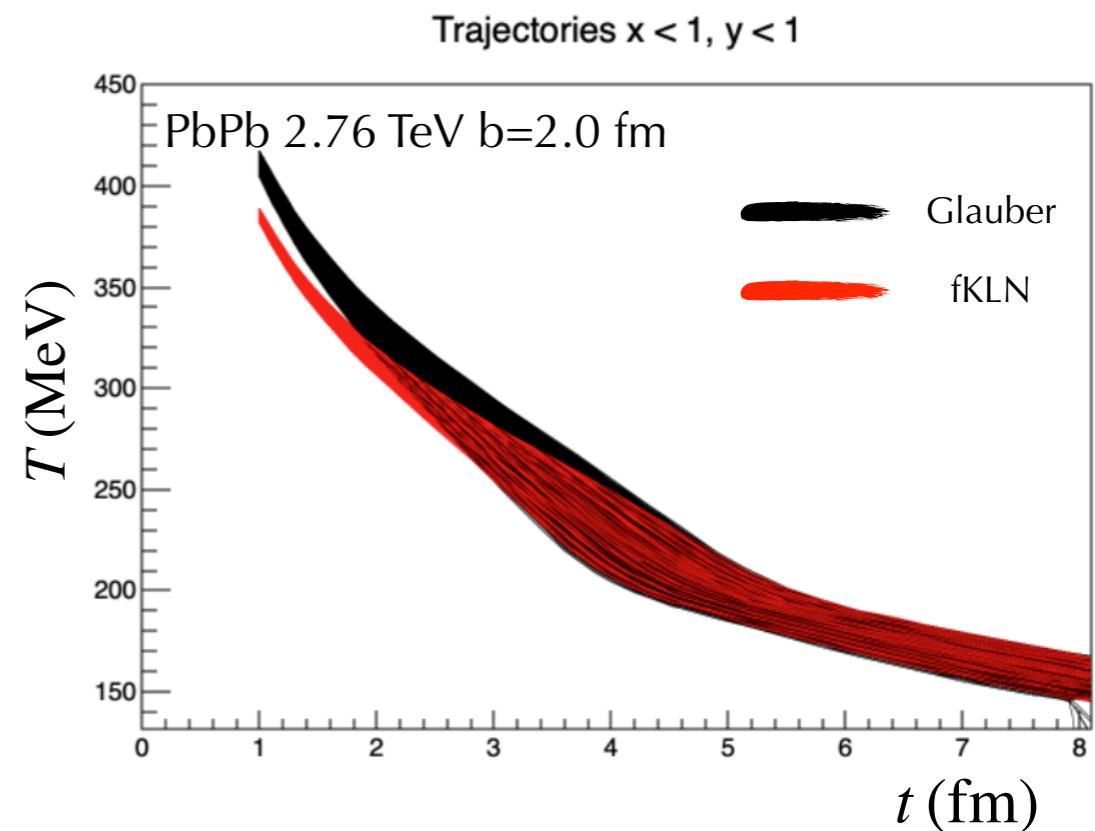
For **transverse inhomogeneities** see Barata's talk. Wed 12:00

Beyond the brick

*with multiple scatterings



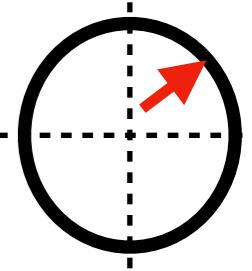
- The goal is to compute the energy lost by a hard parton along **its trajectory** within an **evolving** media
- One should read the medium properties (for instance, the temperature T) from the hydro at each point of the path
- Then, obtain the medium parameters entering the spectrum:
 $n(T(t))$, $\mu(T(t)) \dots$
- And feed them to the code and compute the spectrum **along the trajectory**



The spectrum depends on the full trajectory, there is no “per-point spectrum”

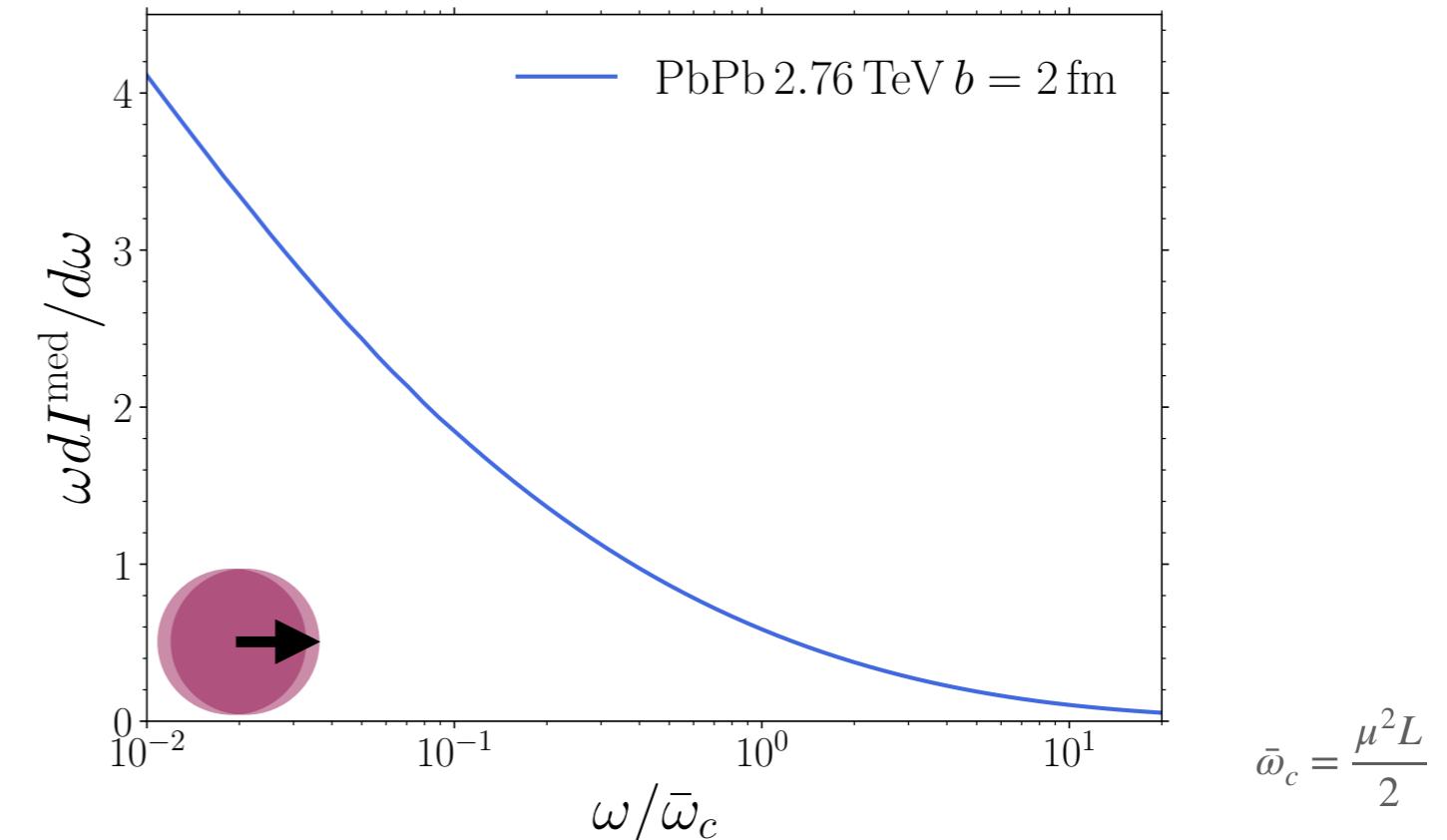
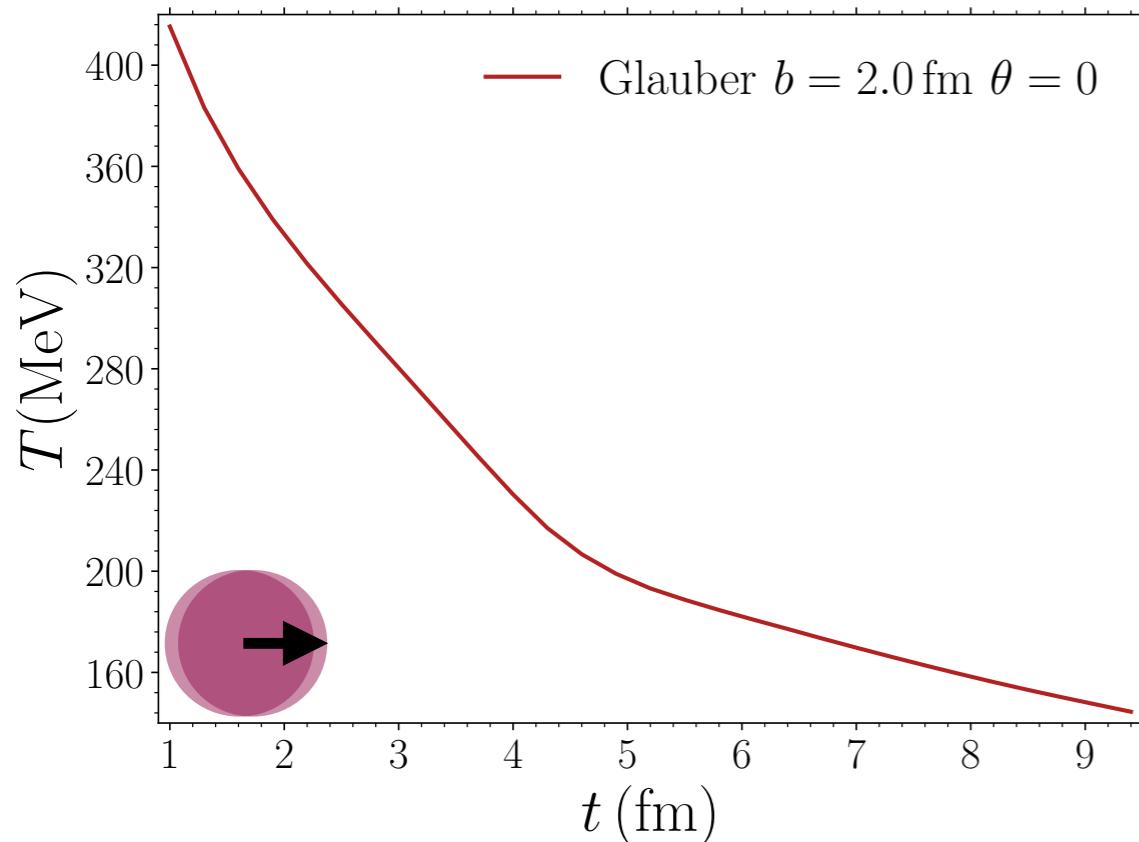
Beyond the brick II

*with multiple scatterings



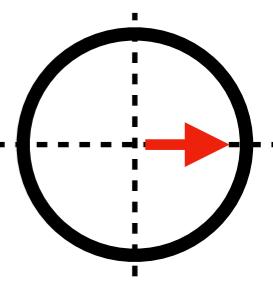
- We can compute the spectrum with time-dependent variables along a path

$$n_{hydro}(t) = k_1 T(t), \quad \mu_{hydro}^2(t) = k_2 T^2(t)$$



- But this is computationally demanding. Currently, it seems too costly to do it for every trajectory *on the fly*
- Pre-tabulate it? **How do we know a priori how the medium parameters will behave along all possible paths?**

How?



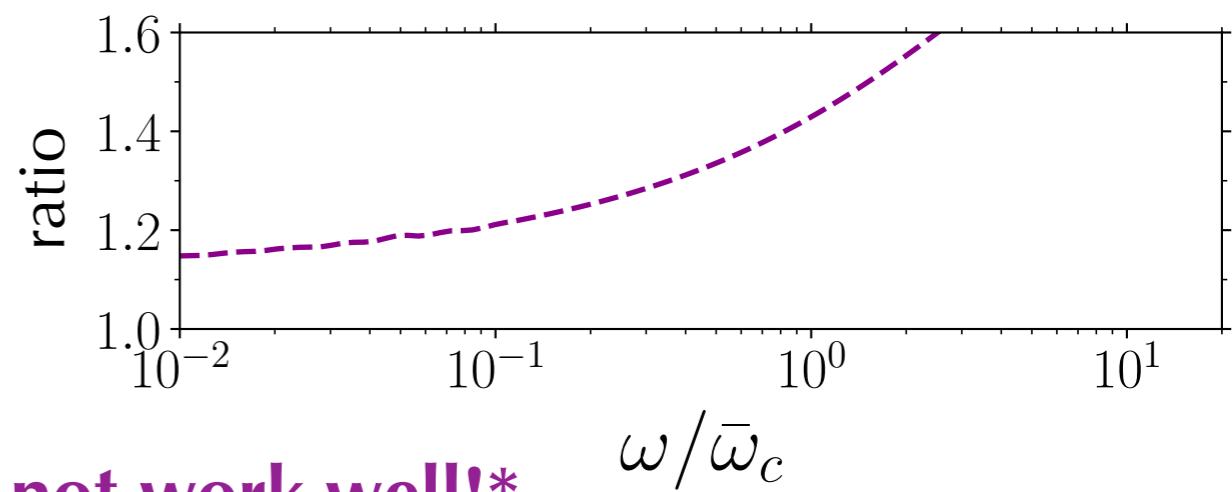
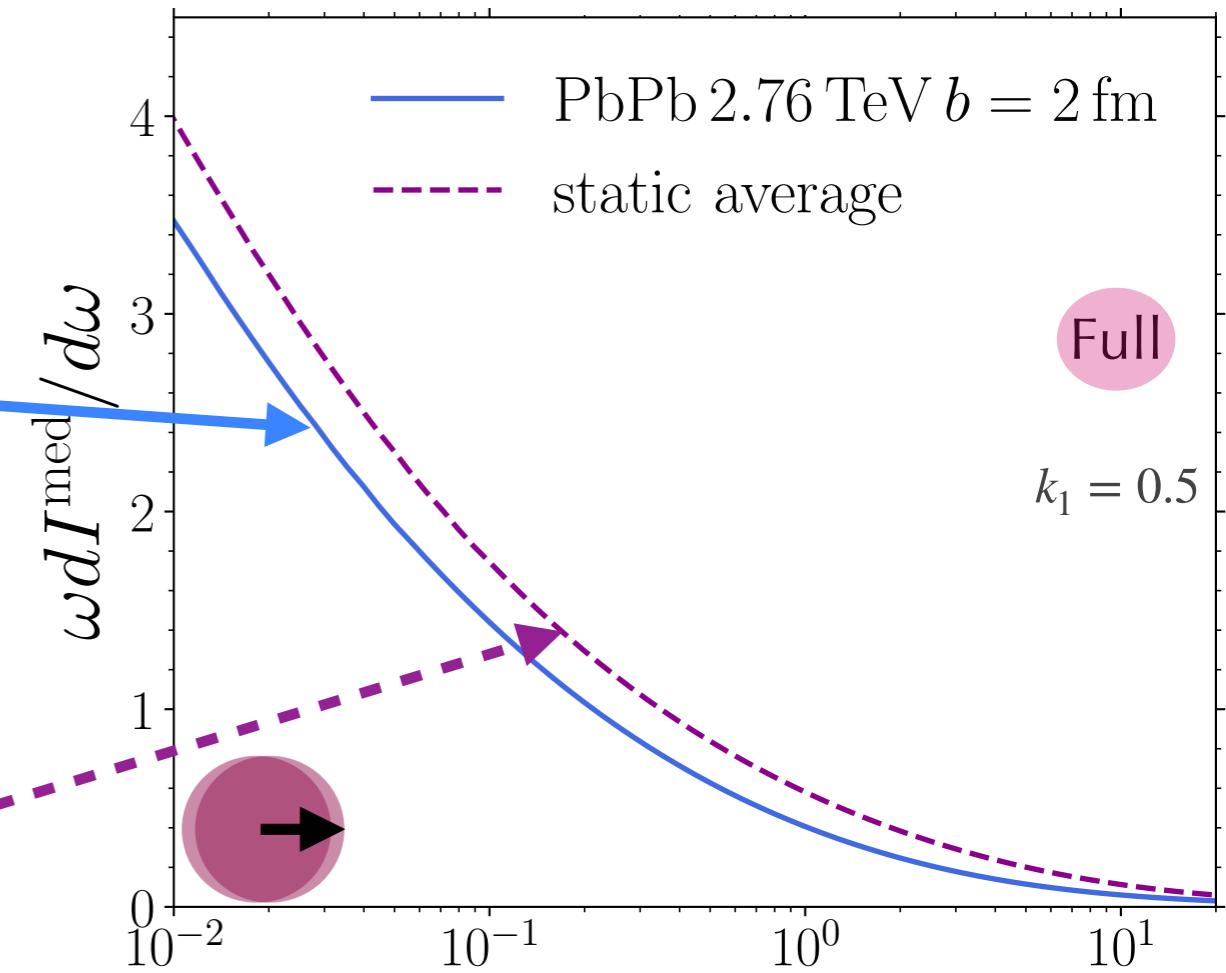
- Using average values?
- Compute the full solution along a path thorough a hydro

$$n_{hydro}(t) = k_1 T(t) \quad \mu_{hydro}^2(t) = k_2 T^2(t)$$

- Compare to the static where the medium parameters are given by their average along the path:

$$n_0 L \Big|_{\text{static}} = k_1 \langle T \rangle L'$$

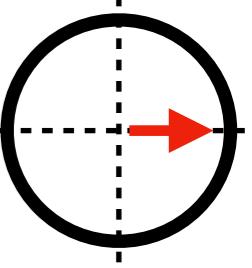
$$\bar{\omega}_c \Big|_{\text{static}} = \frac{\mu^2 L}{2} \Big|_{\text{static}} = \frac{1}{2} k_2 \langle T \rangle^2 L'$$



Using average values does not work well!*

$$\bar{\omega}_c = \bar{\omega}_c \Big|_{\text{static}}$$

* And this is something we know since 2003



Scaling laws?

- The idea is to **find an equivalent static scenario**

Find the values of the parameters that best approximate the dynamic spectrum along the path

Salgado, Wiedemann, [0302184](#)

Priyam Adhya, Salgado, Spousta, Tywoniuk, [1911.12193](#)

- Compute the **full solution along a path** thorough a hydro

$$n_{hydro}(t) = k_1 T(t) \quad \mu_{hydro}^2(t) = k_2 T^2(t)$$

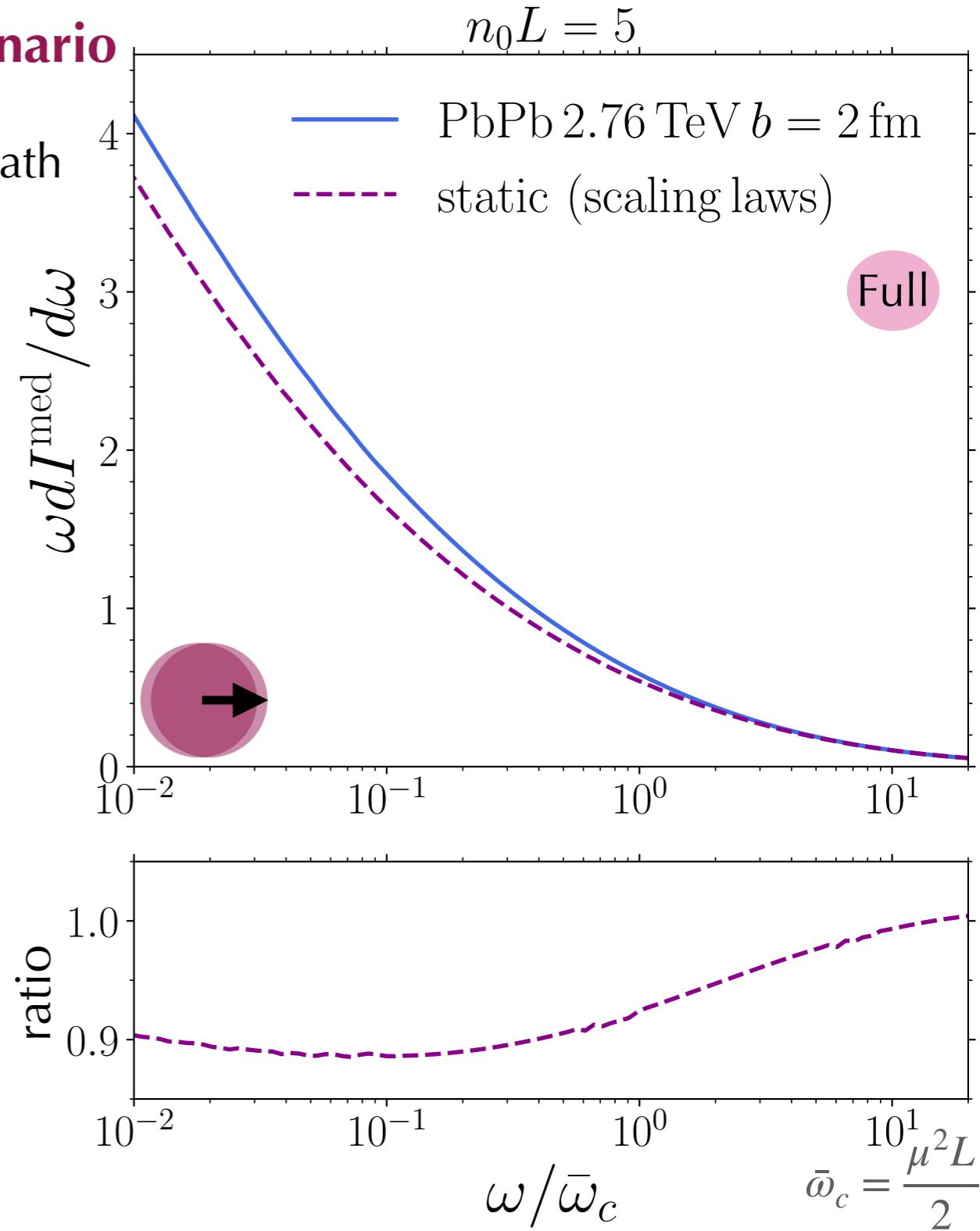
- Find the values of the parameters of the **static scenario** we want to compare to:

$$n_0 L \Big|_{\text{static}} = \int_0^{L'} dt n_{hydro}(t)$$

$$\frac{n_0 \mu^2 L^2}{2} \Big|_{\text{static}} = \int_0^{L'} dt t n_{hydro}(t) \mu_{hydro}^2(t)$$

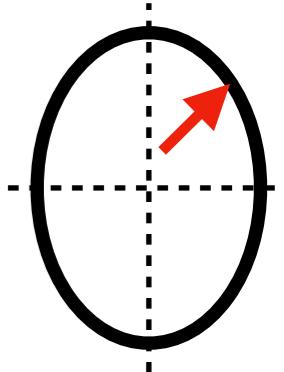
- It works better than using average values, but the **errors go up to ~10-20%**

Hydro: Luzum and Romatschke, [0901.4588](#)



Scaling laws w.r.t. a power-law case?

- Why the pre-tabulated spectrum needs to be the **static** one?



- The idea is to **find an equivalent scenario given by a power-law**

Goal: find matching relations between the spectrum in the real world (along a path throughout a hydro) and a pre-tabulated power-law spectrum

- For instance:

Compute the **spectrum along a path** throughout a hydro:

$$n_{hydro}(t) = k_1 T(t) \quad \mu_{hydro}^2(t) = k_2 T^2(t)$$

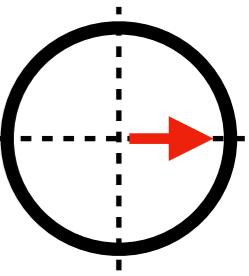
Compare to a **power-law** spectrum for a profile given by:

$$n(t) = \frac{n'_0}{(t + t_0)^\alpha} \quad \mu^2(t) = \frac{\mu'^2}{(t + t_0)^{2\alpha}}$$

with a **scaling law** given by

$$\int_0^{L_1} dt n(t) = \int_0^{L_2} dt n_{hydro}(t)$$

$$\int_0^{L_1} dt t n(t) \mu^2(t) = \int_0^{L_2} dt t n_{hydro}(t) \mu_{hydro}^2(t)$$



Scaling laws w.r.t. a power-law case

- Select a trajectory in central PbPb 2.76 TeV collisions

- Compute the **full spectrum along this path**

$$n_{hydro}(t) = k_1 T(t)$$

$$\mu_{hydro}^2(t) = k_2 T^2(t)$$

- Compare to a **power-law spectrum** given by

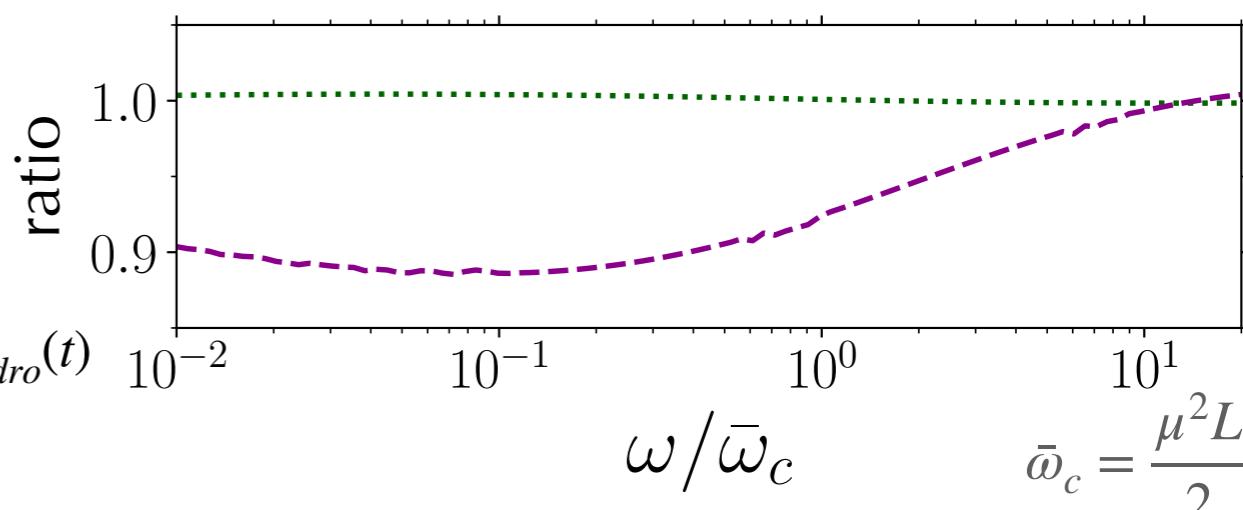
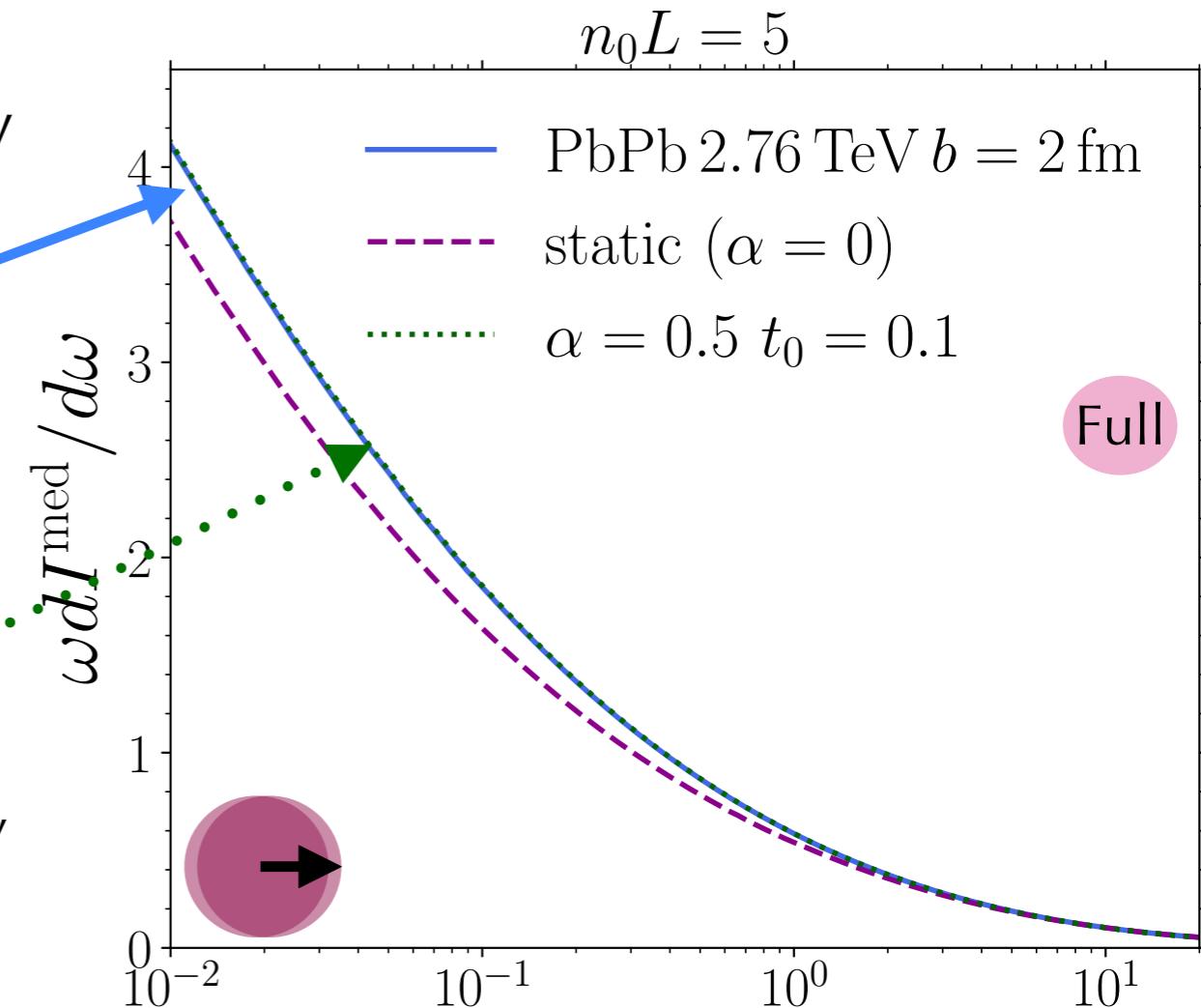
$$n(t) = \frac{n'_0}{(t + t_0)^\alpha}$$

$$\mu^2(t) = \frac{\mu'^2}{(t + t_0)^{2\alpha}}$$

$$\alpha = 0.5$$

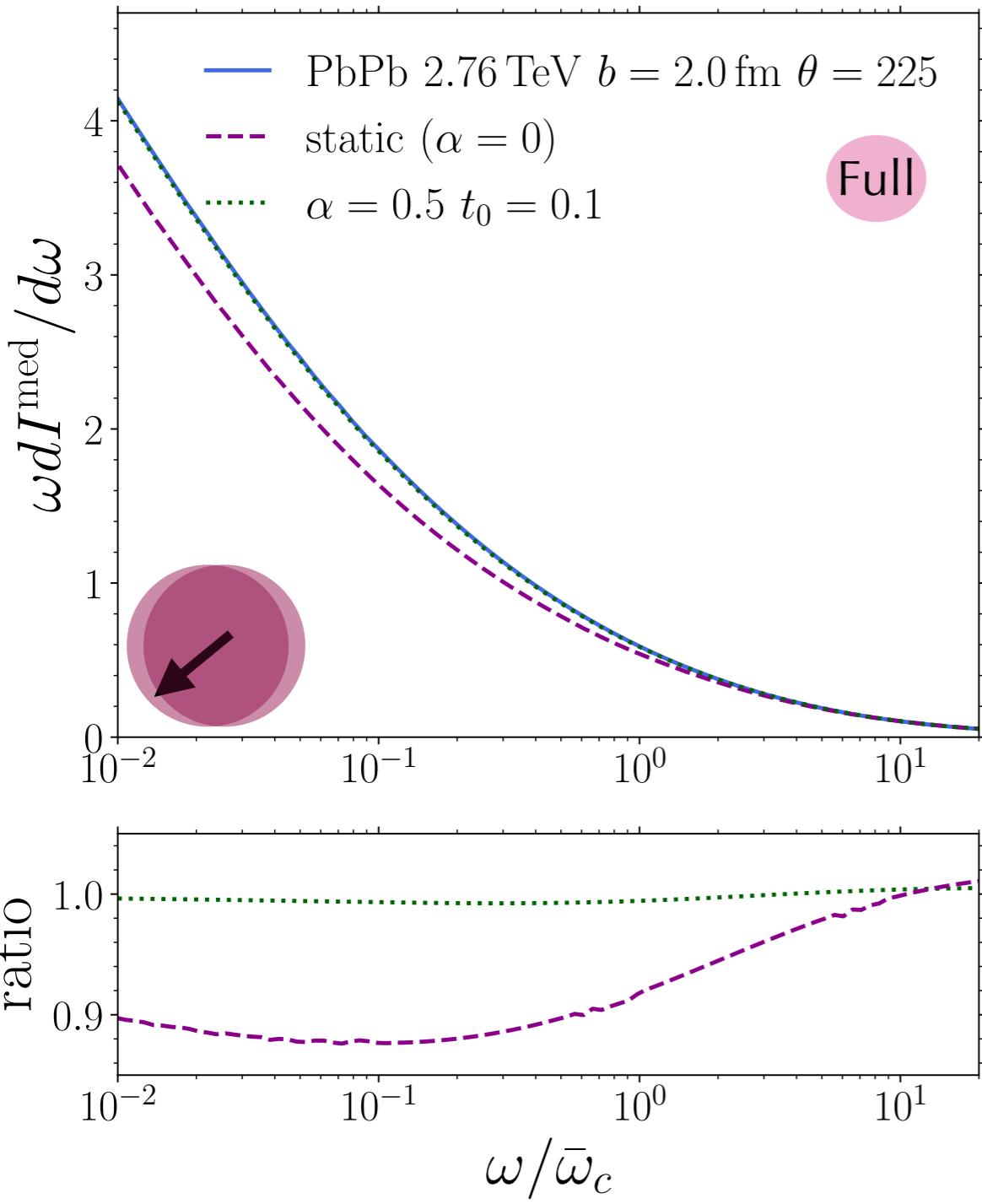
with a **scaling law** given by

$$\int_0^{L_1} dt n(t) = \int_0^{L_2} dt n_{hydro}(t) \quad \int_0^{L_1} dt t n(t) \mu^2(t) = \int_0^{L_2} dt t n_{hydro}(t) \mu_{hydro}^2(t)$$

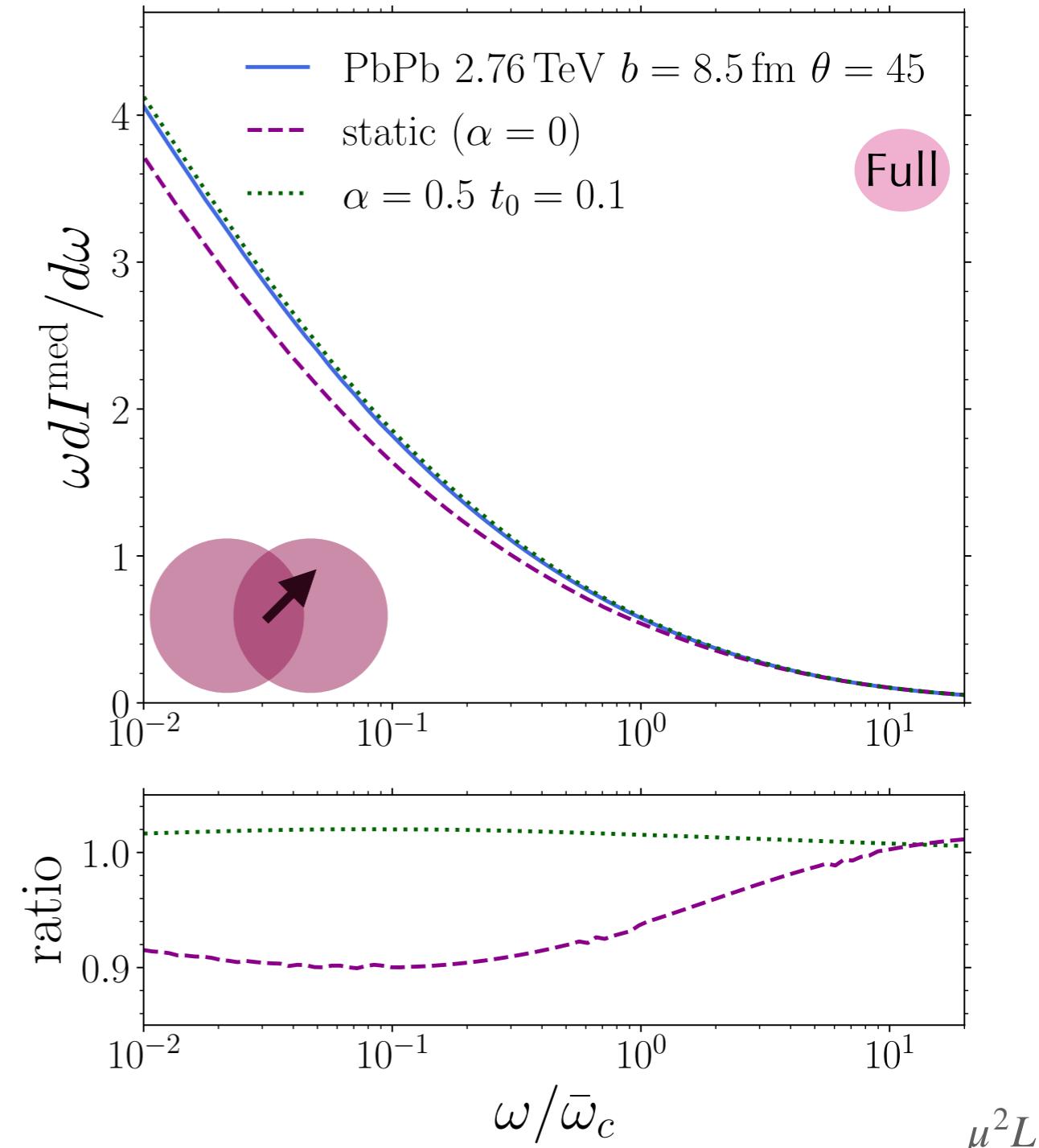


Scaling laws w.r.t. a power-law case

- PbPb 2.76 TeV 0-5%



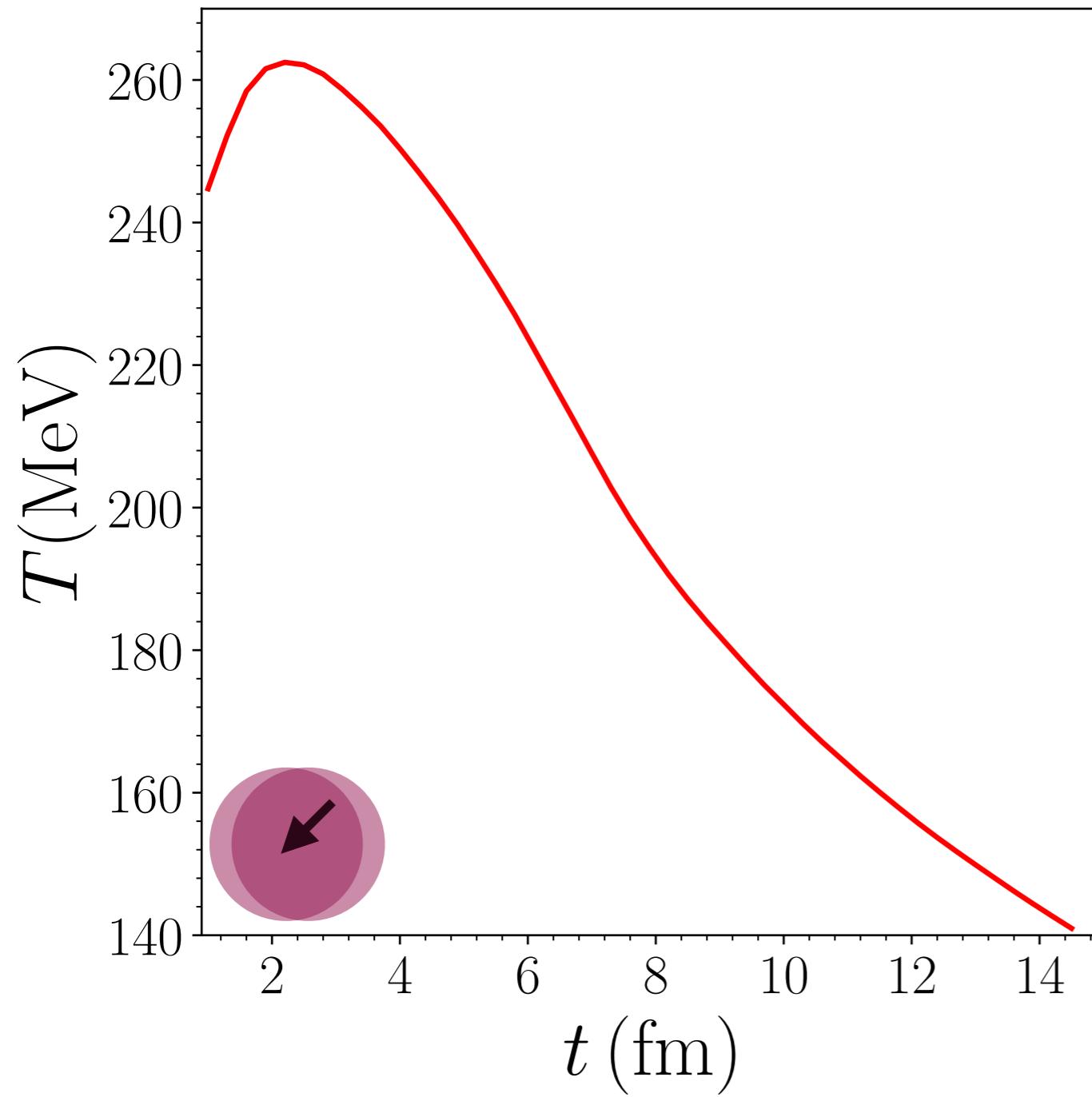
- PbPb 2.76 TeV 30-40%



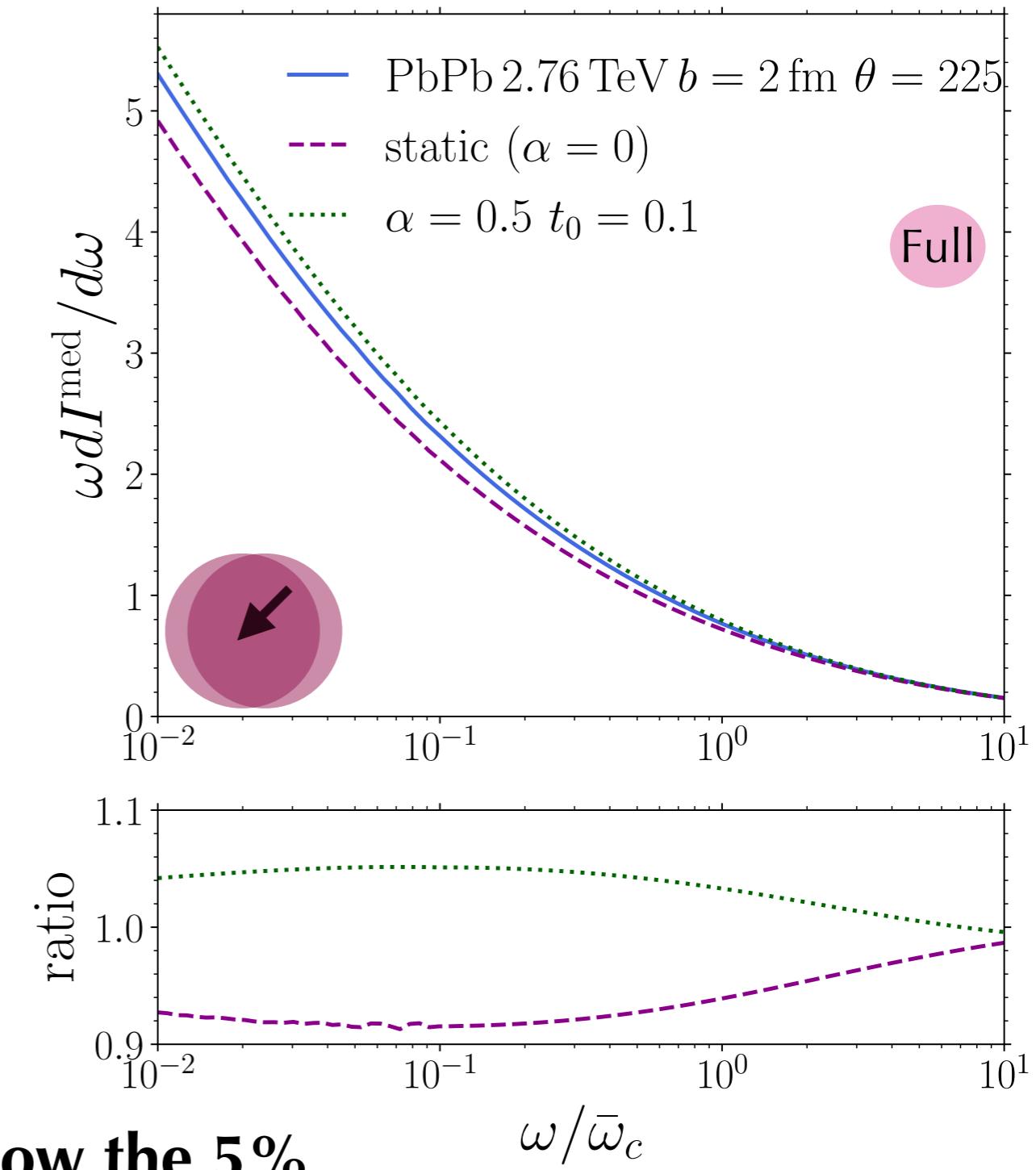
$$\bar{\omega}_c = \frac{\mu^2 L}{2}$$

Rare cases

PbPb 2.76 TeV 0-5%



Errors below the 5%



Hydro: Luzum and Romatschke, [0901.4588](#)

$$\bar{\omega}_c = \frac{\mu^2 L}{2}$$

Towards phenomenology

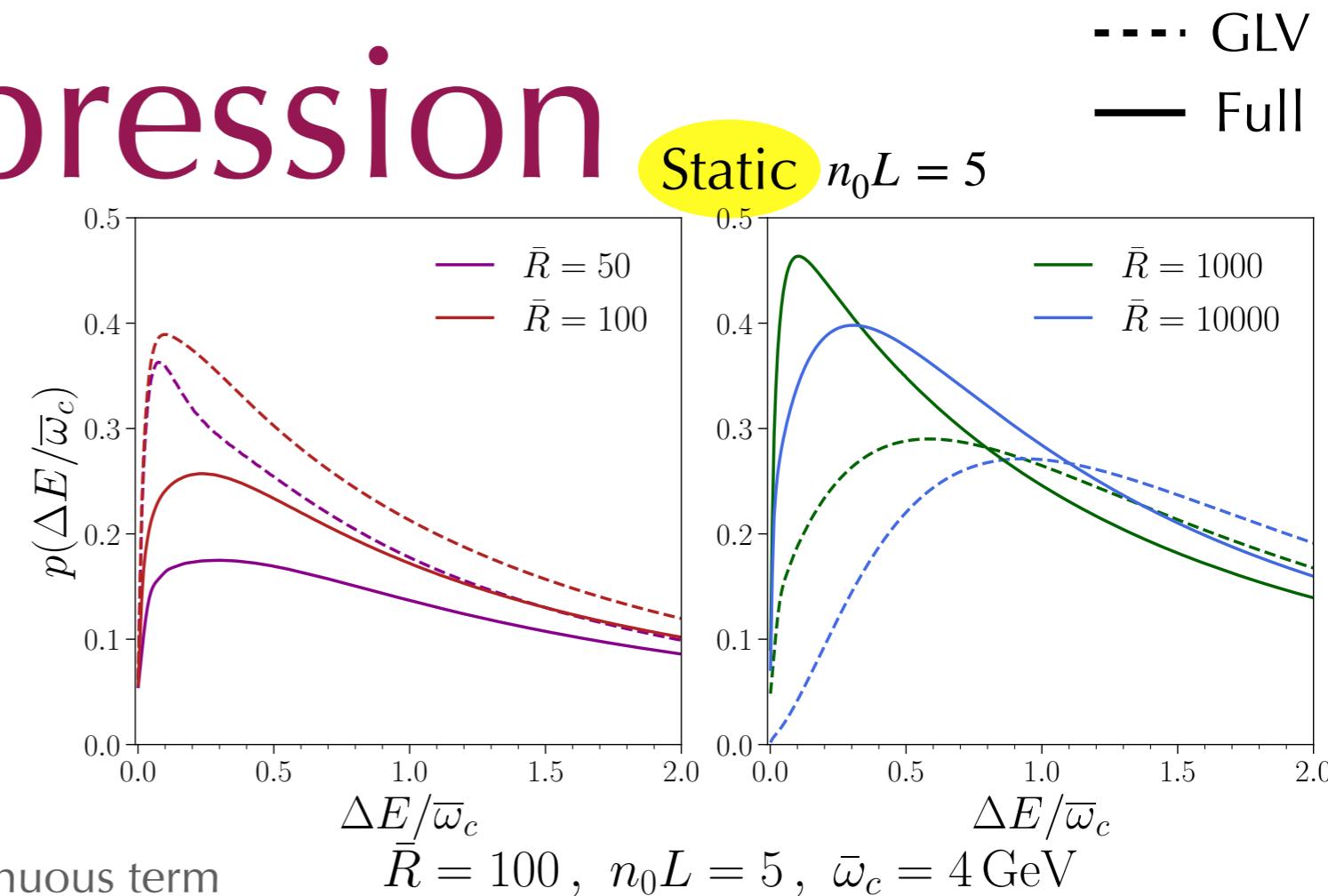
Hadron suppression

Quenching weights ($P(\Delta E)$)

- Probability distribution of the hard parton of losing medium-induced energy ΔE

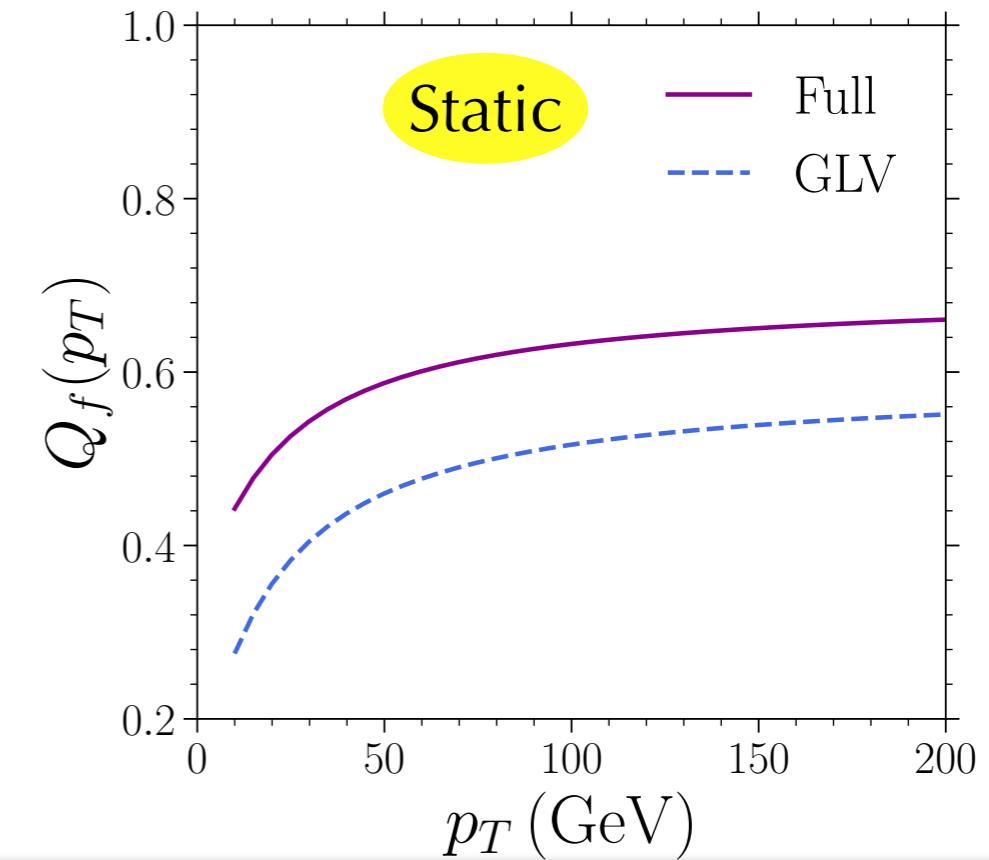
(assuming gluons emitted independently)

*Plotting just their continuous term

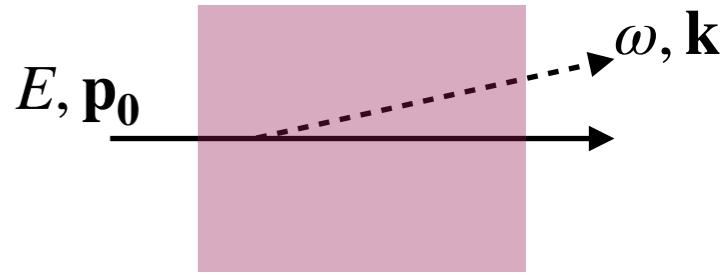


Quenching factor (Q_f)

$$Q_f(p_T) = \frac{d\sigma^{med}(p_T)/dp_T}{d\sigma^{vac}(p_T)/dp_T} \sim \int d\Delta E P(\Delta E) \left(\frac{p_t}{p_T + \Delta E} \right)^n$$



Energy distribution



$$\omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega} \text{Re} \int_0^\infty ds n(s) \int_0^s dt \int_{\mathbf{pql}} i \mathbf{p} \cdot \left(\frac{\mathbf{l}}{l^2} - \frac{\mathbf{q}}{\mathbf{q}^2} \right) \sigma(\mathbf{l} - \mathbf{q}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) \mathcal{P}(\infty, \mathbf{k}; s, \mathbf{l})$$

BDMPS-Z (vacuum subtracted) $k < \omega$

- Up to now: we have removed this kinematic constraint:

$$\omega \frac{dI^{\text{med}}}{d\omega} = \int_0^\infty \omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2$$

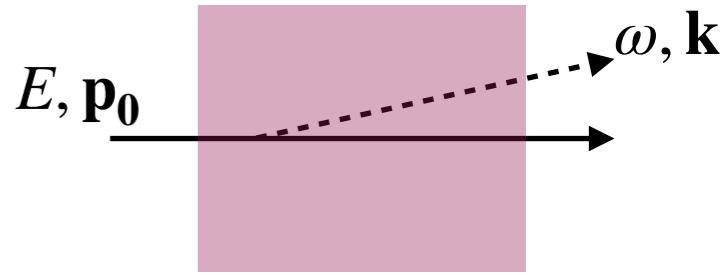
So we get rid of the broadening

$$\omega \frac{dI^{\text{med}}}{d\omega} = \frac{2\alpha_s C_R}{\omega} \text{Re} \int_0^\infty ds n(s) \int_0^s dt \int_{\mathbf{pql}} i \frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{q}^2} \sigma(\mathbf{q} - \mathbf{l}) \tilde{\mathcal{K}}(s, \mathbf{l}; t, \mathbf{p})$$

- Integrate in transverse momentum with the constraint:

$$\omega \frac{dI^{\text{med}}}{d\omega} = \int_0^{\omega^2} \omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2$$

Energy distribution



$$\omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega} \text{Re} \int_0^\infty ds n(s) \int_0^s dt \int_{\mathbf{pql}} i \mathbf{p} \cdot \left(\frac{\mathbf{l}}{l^2} - \frac{\mathbf{q}}{\mathbf{q}^2} \right) \sigma(\mathbf{l} - \mathbf{q}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) \mathcal{P}(\infty, \mathbf{k}; s, \mathbf{l})$$

BDMPS-Z (vacuum subtracted) $k < \omega$

- Up to now: we have removed this kinematic constraint:

$$\omega \frac{dI^{\text{med}}}{d\omega} = \int_0^\infty \omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2$$

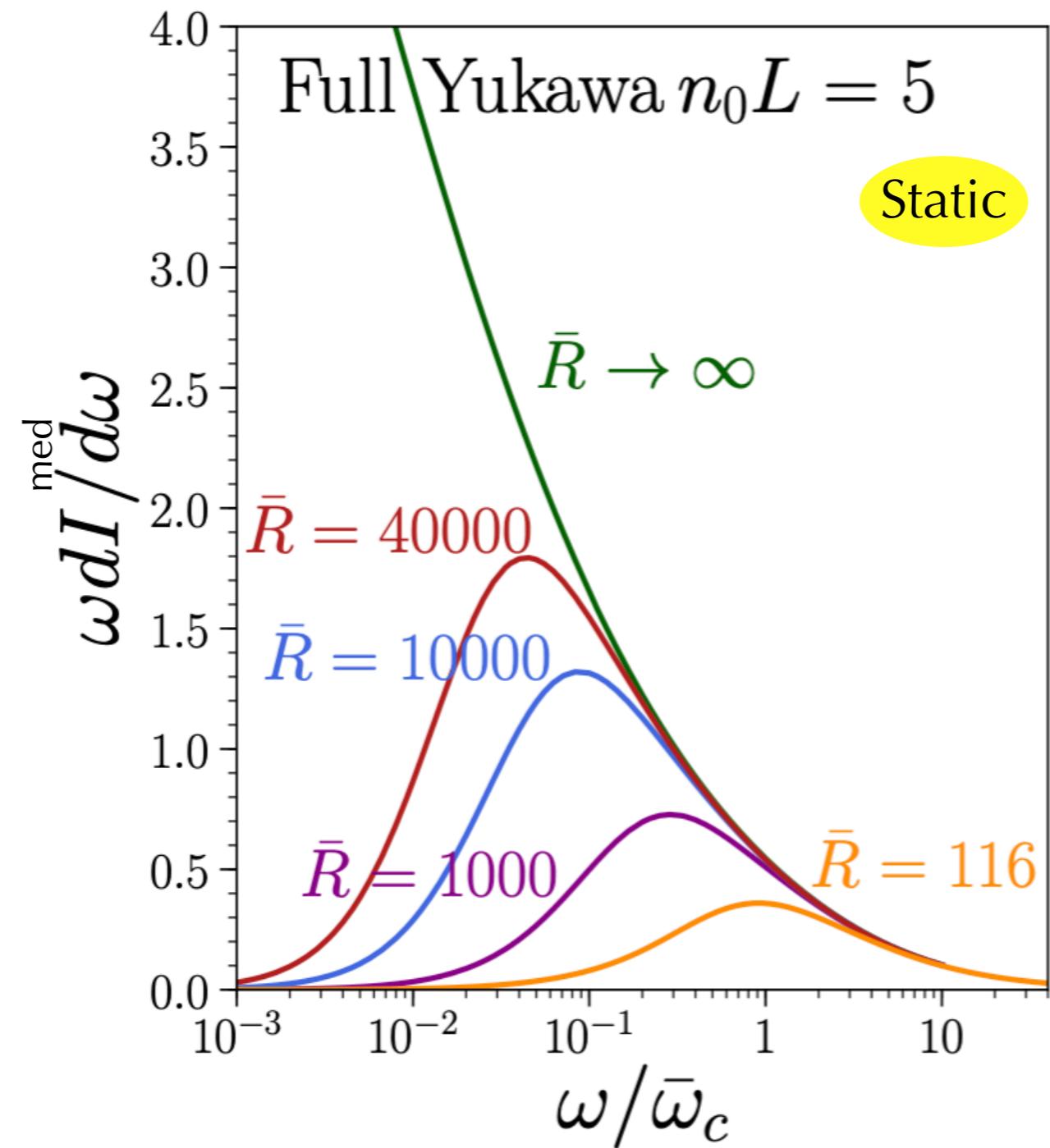
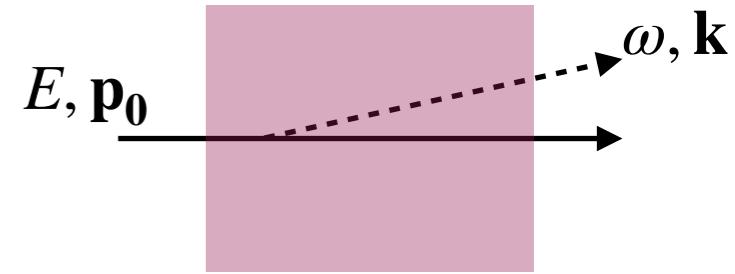
So we get rid of the broadening

$$\omega \frac{dI^{\text{med}}}{d\omega} = \frac{2\alpha_s C_R}{\omega} \text{Re} \int_0^\infty ds n(s) \int_0^s dt \int_{\mathbf{pql}} i \frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{q}^2} \sigma(\mathbf{q} - \mathbf{l}) \tilde{\mathcal{K}}(s, \mathbf{l}; t, \mathbf{p})$$

- Integrate in transverse momentum with the constraint:

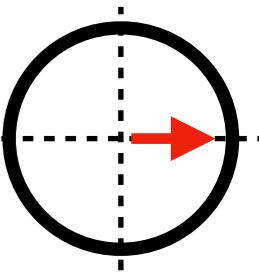
$$\omega \frac{dI^{\text{med}}}{d\omega} = \int_0^{\omega^2} \omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}^2} d\mathbf{k}^2 \quad \xrightarrow{\bar{R} = \bar{\omega}_c L}$$

Energy distribution



$$\bar{\omega}_c = \frac{n_0 L}{\mu^2 L} = \frac{\bar{R}}{2}$$

Scaling laws? Full



- Compute the **full solution** along a path thorough a hydro

$$n_{hydro}(t) = k_1 T(t)$$

$$\mu_{hydro}^2(t) = k_2 T^2(t)$$

- Compare to a **power-law spectrum** given by

$$n(t) = \frac{n'_0}{(t + t_0)^\alpha}$$

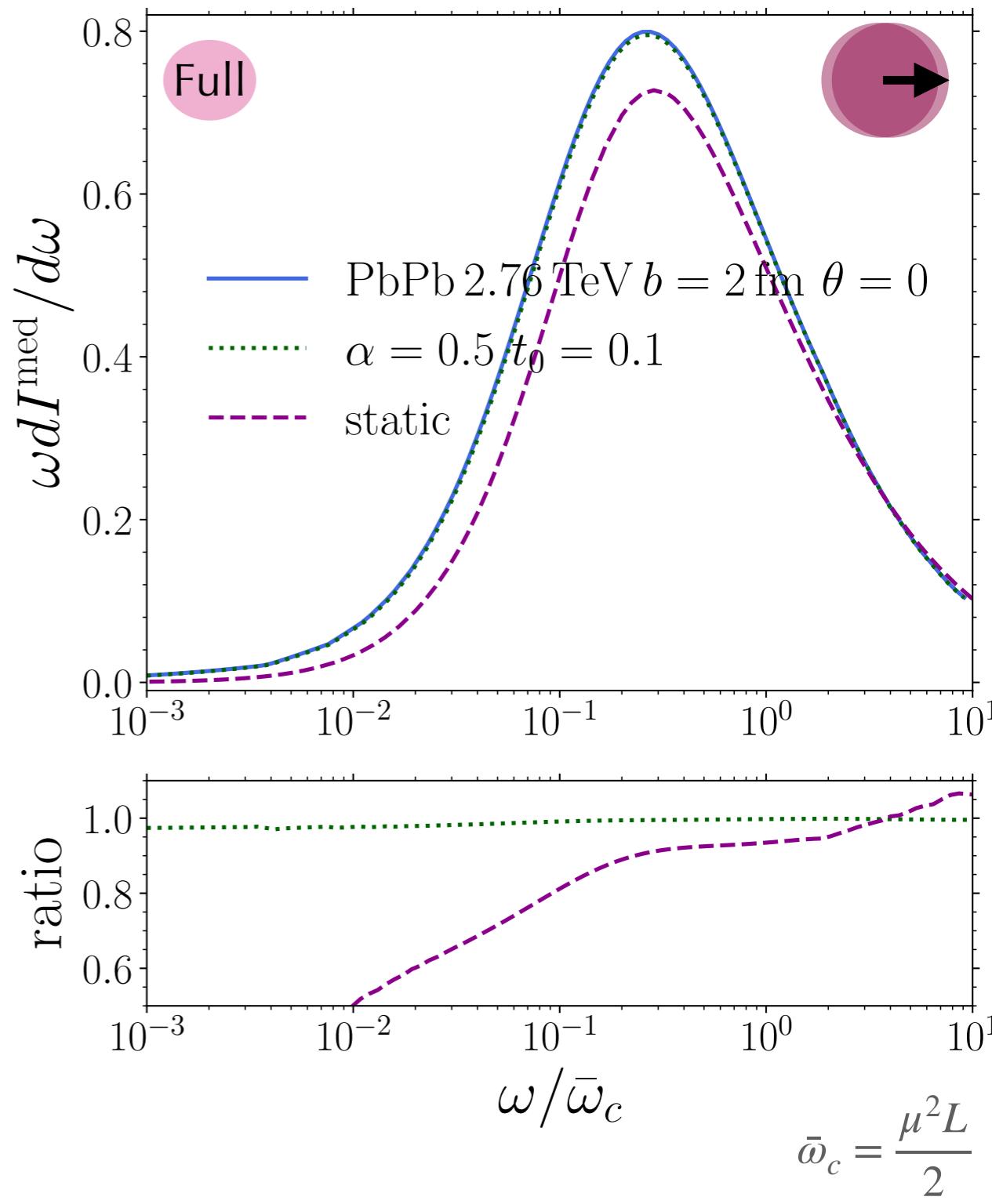
$$\mu^2(t) = \frac{\mu'^2}{(t + t_0)^{2\alpha}}$$

- Using the following scaling laws

$$\int_0^{L_1} dt n(t) = \int_0^{L_2} dt n_{hydro}(t)$$

$$\int_0^{L_1} dt t n(t) \mu^2(t) = \int_0^{L_2} dt t n_{hydro}(t) \mu_{hydro}^2(t)$$

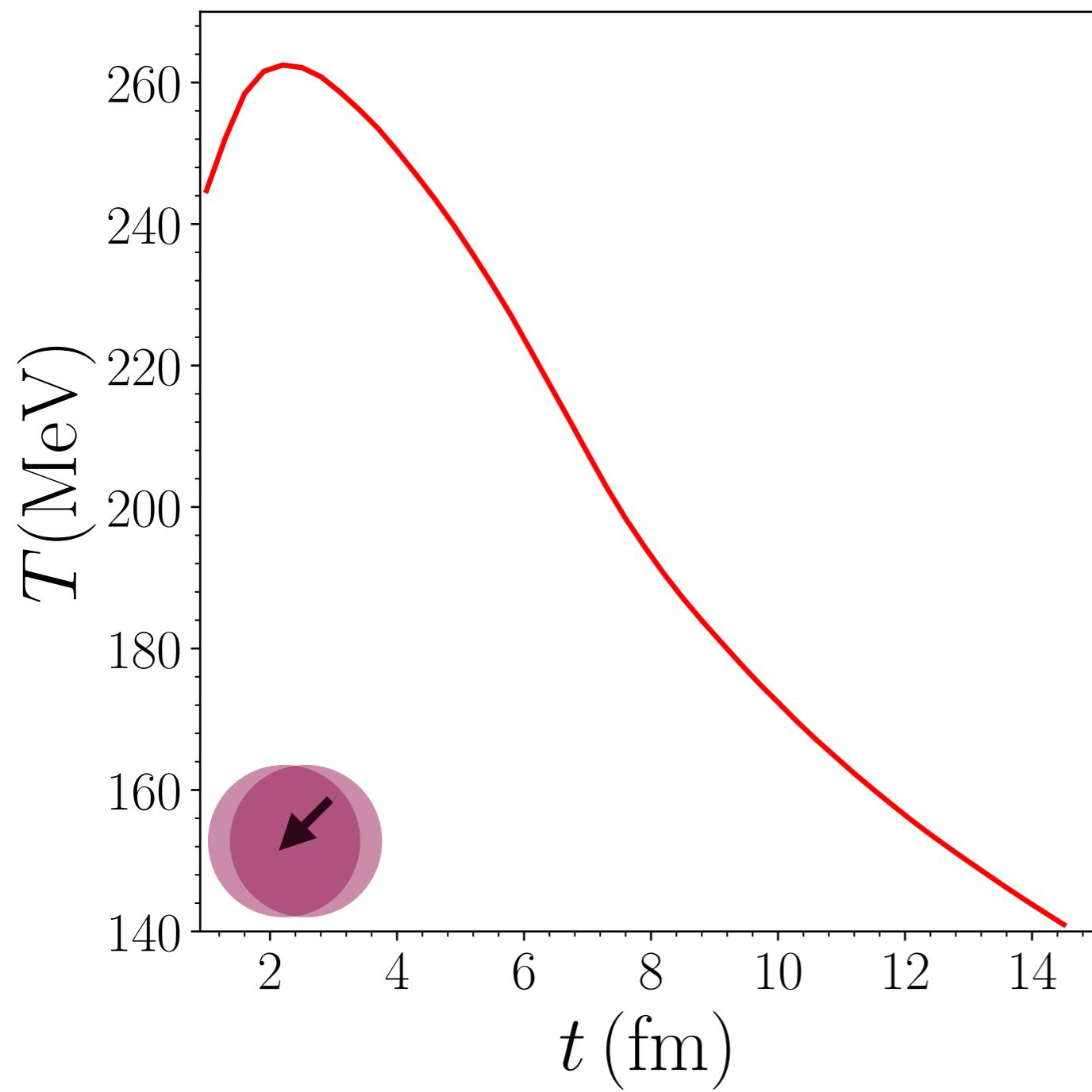
$$\int_0^{L_1} dt t^2 n(t) \mu^2(t) = \int_0^{L_2} dt t^2 n_{hydro}(t) \mu_{hydro}^2(t)$$



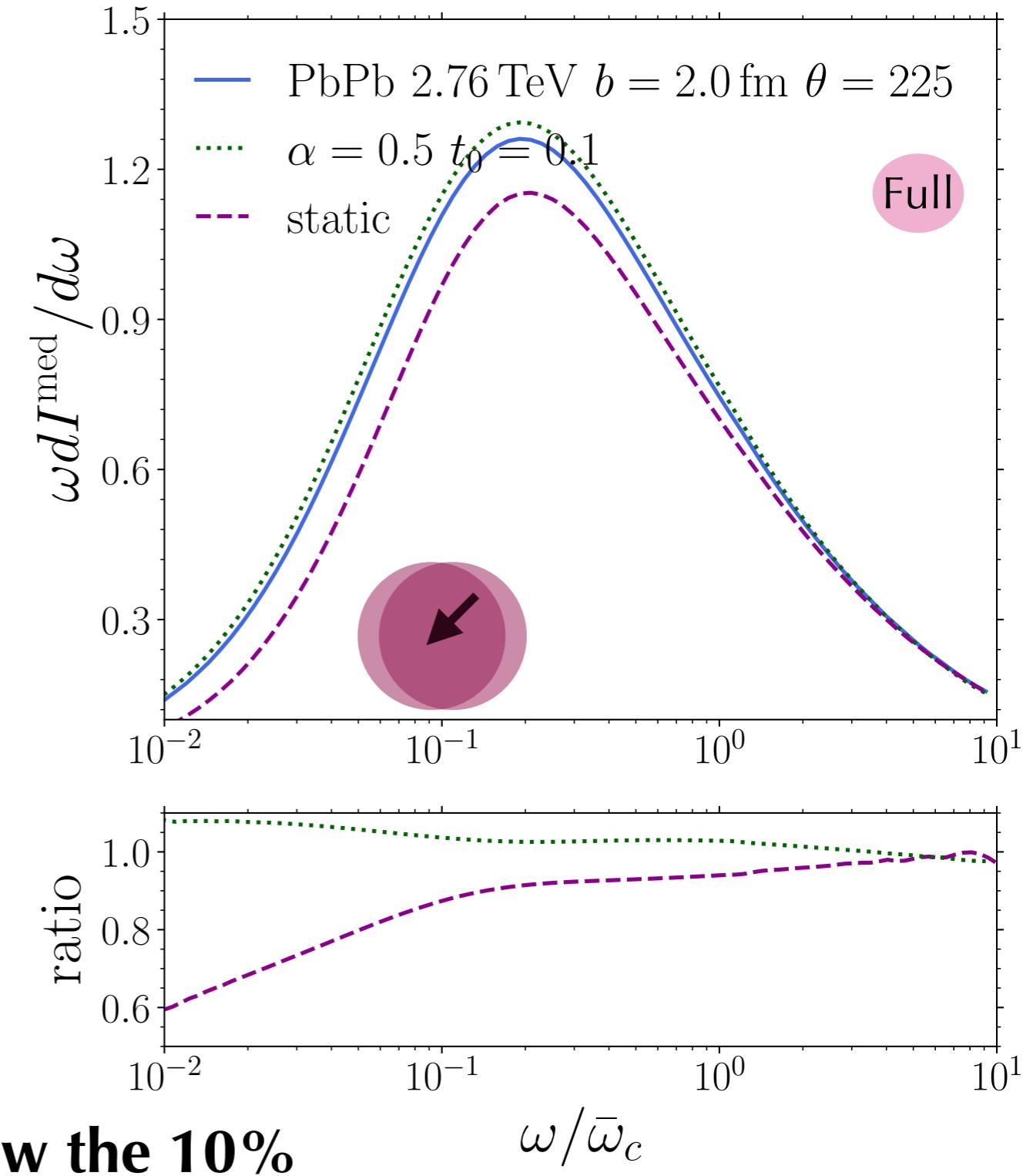
$$\bar{\omega}_c = \frac{\mu^2 L}{2}$$

Rare case

PbPb 2.76 TeV 0-5%



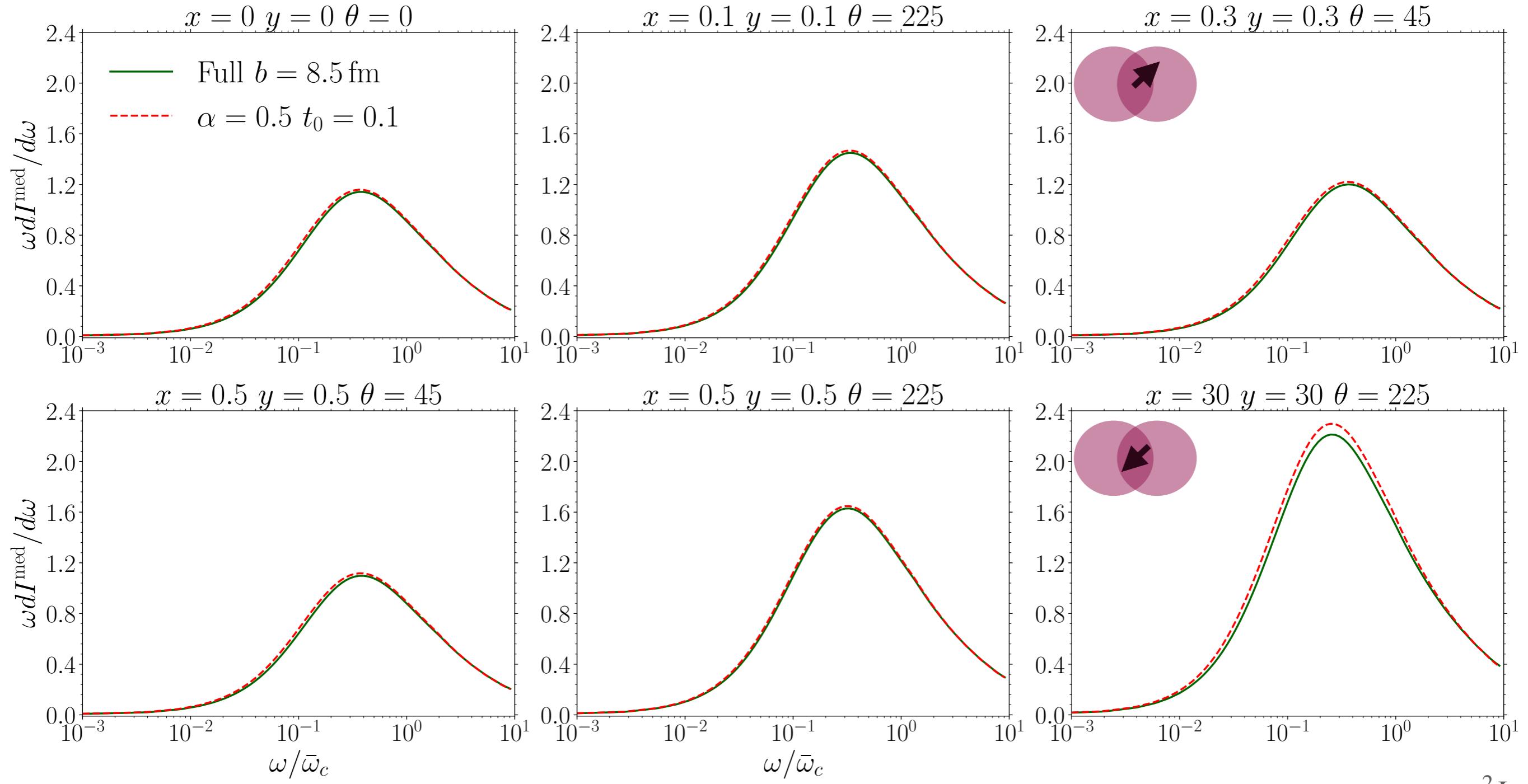
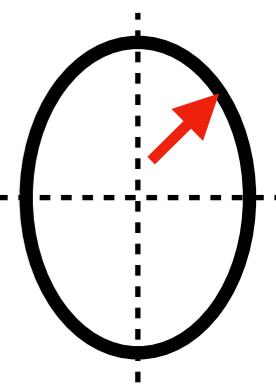
Errors below the 10%



$$\bar{\omega}_c = \frac{\mu^2 L}{2}$$

Summary

PbPb 2.76 TeV 30-40%



Hydro: Luzum and Romatschke, [0901.4588](#)

$$\bar{\omega}_c = \frac{\mu^2 L}{2}$$

Conclusions

- Many **new developments** in the computation of the **medium-induced radiation spectrum** (in the **brick**)
- These numerical approaches **allow to compute the spectrum along a path in realistic media** (given by a hydro)
 - But it is computationally demanding
 - So we want to use **pre-compute the spectra** for a set of given profiles approximating realistic conditions (scaling laws)
 - Using **power-law profiles reduces the errors substantially**
- We can think of other approaches
 - For instance, MC approach to mimic the Caron-Huot rates
 - Park et al. HP2016 proceedings [1612.06754](#)

Whatever the approach/approximation used, we can quantify the errors!

Thanks

The HO as an example

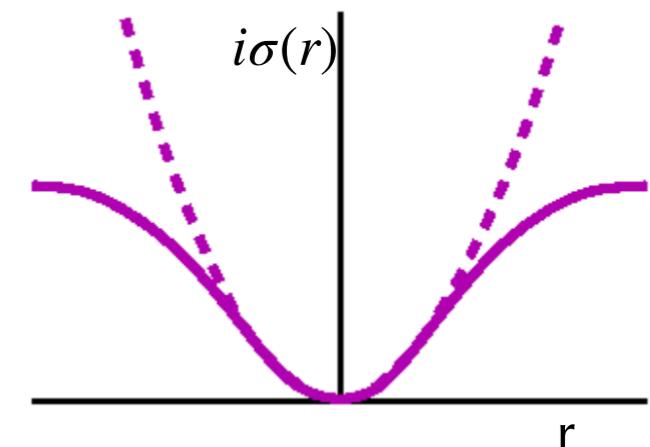
Harmonic, \hat{q} or Gaussian approximation

- Small \mathbf{r} approximation of the dipole cross section

$$n(s)\sigma(\mathbf{r}) \approx \frac{1}{2}\hat{q}(s)\mathbf{r}^2 + \mathcal{O}(\mathbf{r}^2 \ln \mathbf{r}^2)$$

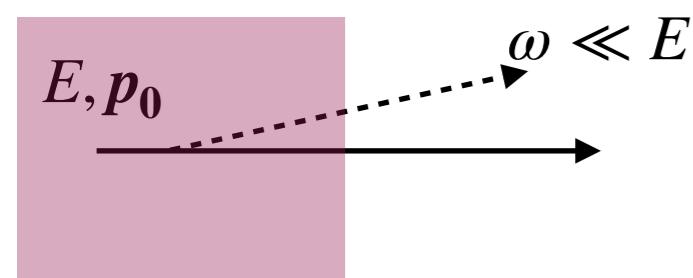
↓

For a static medium: the kernel can be computed analytically



Note: $\hat{q} = 4iV''(0)$

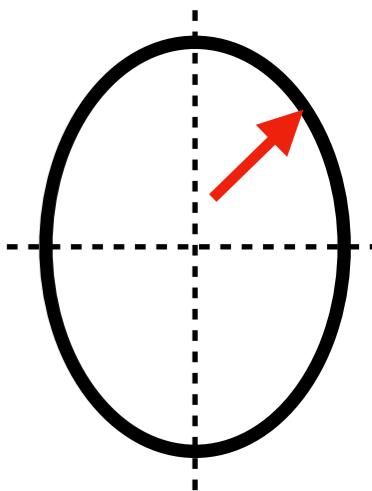
- The energy spectrum for the **brick** (no cut-off $\omega \frac{dI}{d\omega} = \int_0^\infty d^2\mathbf{k} \omega \frac{dI}{d\omega d^2\mathbf{k}}$) Stolen from Peter Arnold slides at the INT jet's workshop



$$\omega \frac{dI^{\text{HO}}}{d\omega} = \frac{2\alpha_s C_R}{\pi} \ln \left| \cos \left((1-i) \sqrt{\frac{\hat{q}L^2}{4\omega}} \right) \right|$$

Baier, Dokshitzer, Mueller, Peigné, Schiff (96)
Zakharov (97)
Wiedemann, Salgado (2003)

The HO as an example



- HO spectrum in an evolving media

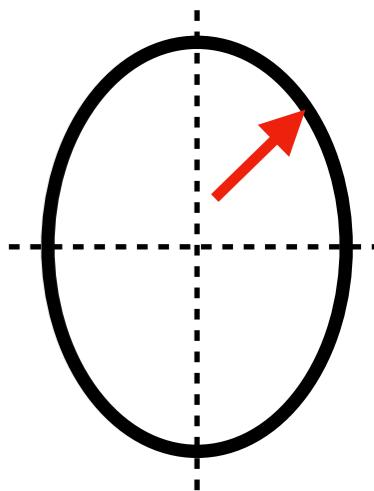
$$\omega \frac{dI^{\text{HO}}}{d\omega} = \frac{2\alpha_s C_R}{\pi} \ln |c(0)|$$

$$\frac{d^2 c(t)}{dt} = -\omega_0^2(t) c(t)$$

$$\omega_0(t) = (1 - i)\sqrt{\frac{\hat{q}(t)}{4\omega}}$$

Arnold, [0808.2767](#)

The HO as an example



- HO spectrum in an evolving media

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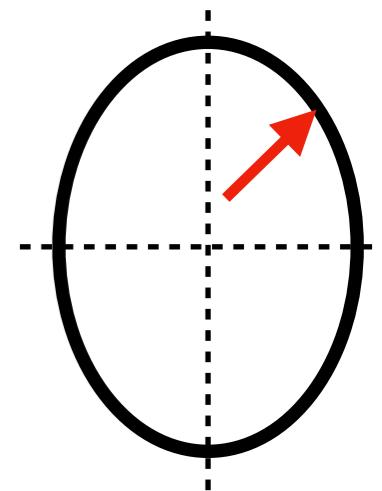
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Arnold, [0808.2767](#)

Evaluated at the position of the parent parton at time t

The HO as an example



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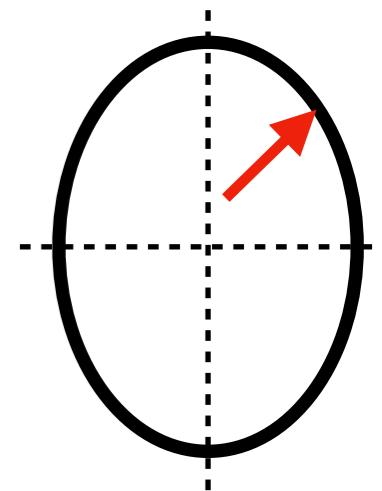
$$\omega_0(t) = (1 - i) \sqrt{\frac{\hat{q}(t)}{4\omega}}$$

Arnold, [0808.2767](#)

Evaluated at the position of the parent parton at time t

Need to solve this differential equation for each trajectory

The HO as an example



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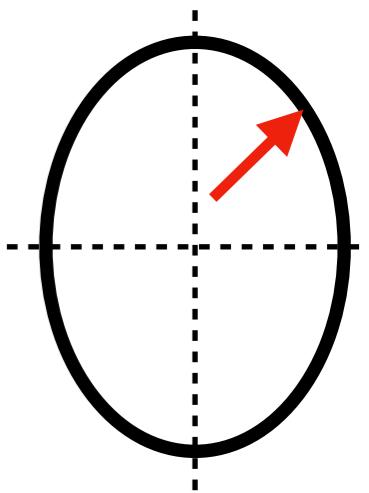
Arnold, [0808.2767](#)

Evaluated at the position of the parent parton at time t

Need to **solve this differential equation for each trajectory**

- This is a conceptual issue: we cannot know a priori how $\hat{q}(t)$ will behave along the path

The HO as an example



- HO spectrum in an evolving media

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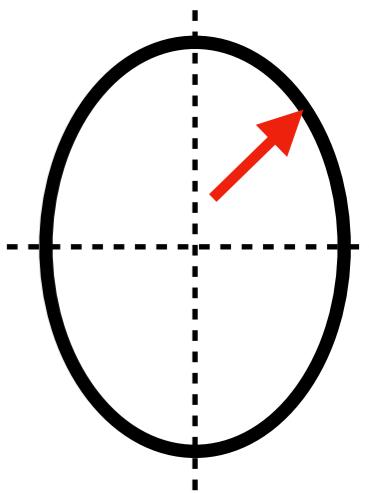
Arnold, [0808.2767](#)

Evaluated at the position of the parent parton at time t

Need to **solve this differential equation for each trajectory**

- This is a conceptual issue: we cannot know a priori how $\hat{q}(t)$ will behave along the path
- In the IOE framework one also needs to solve this differential equation

The HO as an example



- HO spectrum in an evolving media

$$\omega \frac{dI^{\text{HO}}}{d\omega} = \frac{2\alpha_s C_R}{\pi} \ln |c(0)|$$

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$$\omega_0(t) = (1 - i) \sqrt{\frac{\hat{q}(t)}{4\omega}}$$

Arnold, [0808.2767](#)

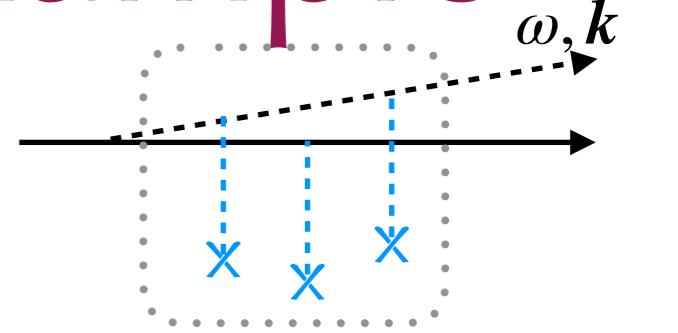
Evaluated at the position of the parent parton at time t

Need to **solve this differential equation for each trajectory**

- This is a conceptual issue: we cannot know a priori how $\hat{q}(t)$ will behave along the path
- In the IOE framework one also needs to solve this differential equation
- What do people do?

The full solution as an example

- Full one (soft) gluon emission in-medium calculation



$$\omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega} \text{Re} \int_0^L ds n(s) \int_0^s dt \int_{\mathbf{pql}} i\mathbf{p} \cdot \left(\frac{\mathbf{l}}{\mathbf{l}^2} - \frac{\mathbf{q}}{\mathbf{q}^2} \right) \sigma(\mathbf{l} - \mathbf{q}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) \mathcal{P}(L, \mathbf{k}; s, \mathbf{l})$$

BDMPS-Z
(Vacuum subtracted)

- Broadening

$$\partial_\tau \mathcal{P}(\tau, \mathbf{k}; s, \mathbf{l}) = -\frac{1}{2} n(\tau) \int_{\mathbf{k}'} \sigma(\mathbf{k} - \mathbf{k}') \mathcal{P}(\tau, \mathbf{k}'; s, \mathbf{l})$$

- Emission Kernel

$$\partial_t \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) = \frac{i\mathbf{p}^2}{2\omega} \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) + \frac{1}{2} n(t) \int_{\mathbf{k}'} \sigma(\mathbf{k}' - \mathbf{p}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{k}')$$

Screening mass

$$\sigma(\mathbf{q}) \equiv -V(\mathbf{q}) + (2\pi)^2 \delta^2(\mathbf{q}) \int_l V(l) \quad V(\mathbf{q}) = \frac{8\pi\mu^2(t)}{(\mathbf{q}^2 + \mu^2(t))^2}$$

The full solution as an example

- Full one (soft) gluon emission in-medium calculation

$$\omega \frac{dI^{\text{med}}}{d\omega d^2\mathbf{k}} = \frac{2\alpha_s C_R}{(2\pi)^2 \omega} \text{Re} \int_0^L ds n(s) \int_0^s dt \int_{\mathbf{pql}} i\mathbf{p} \cdot \left(\frac{\mathbf{l}}{\mathbf{l}^2} - \frac{\mathbf{q}}{\mathbf{q}^2} \right) \sigma(\mathbf{l} - \mathbf{q}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) \mathcal{P}(L, \mathbf{k}; s, \mathbf{l})$$

BDMPS-Z
(Vacuum subtracted)

- Broadening

$$\partial_\tau \mathcal{P}(\tau, \mathbf{k}; s, \mathbf{l}) = -\frac{1}{2} n(\tau) \int_{\mathbf{k}'} \boxed{\sigma(\mathbf{k} - \mathbf{k}') \mathcal{P}(\tau, \mathbf{k}'; s, \mathbf{l})}$$

Medium information

- Emission Kernel

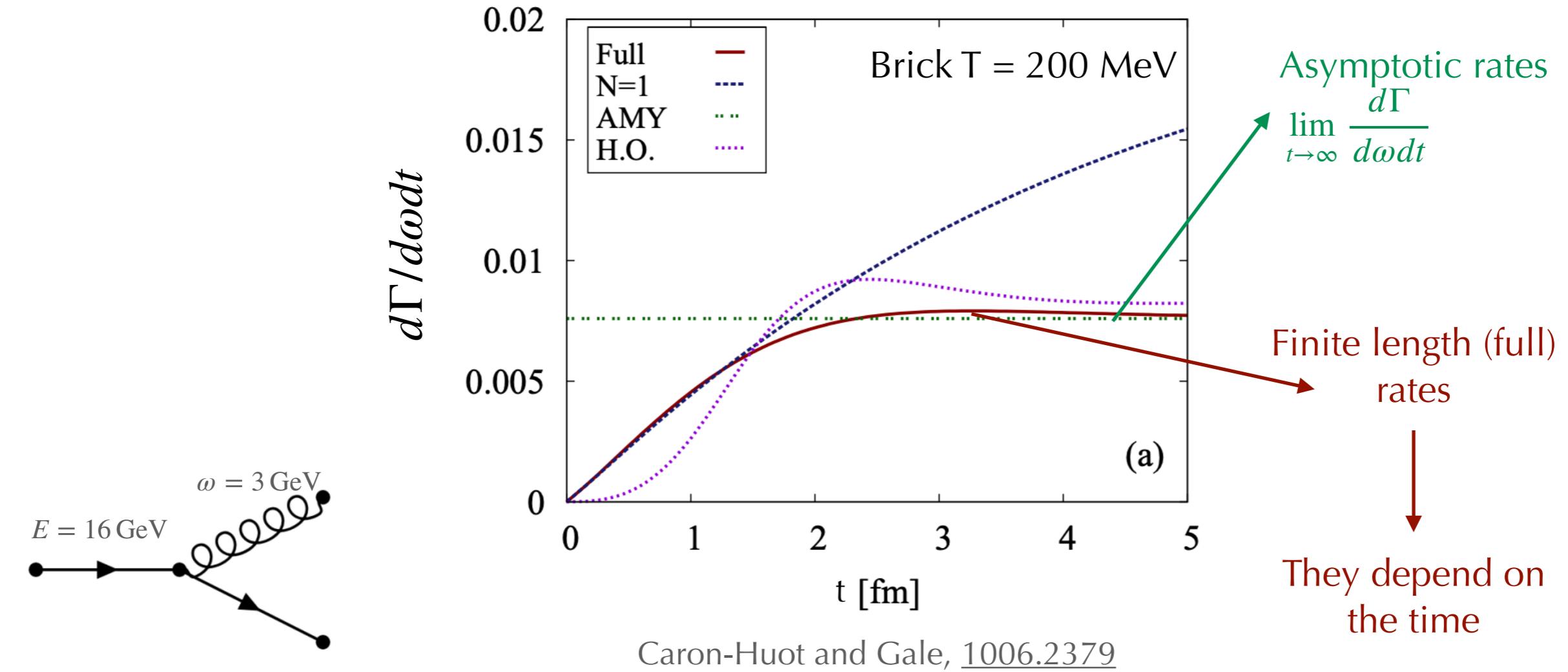
$$\partial_t \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) = \frac{i\mathbf{p}^2}{2\omega} \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{p}) + \frac{1}{2} n(t) \int_{\mathbf{k}'} \boxed{\sigma(\mathbf{k}' - \mathbf{p}) \tilde{\mathcal{K}}(s, \mathbf{q}; t, \mathbf{k}')}}$$

$$\sigma(\mathbf{q}) \equiv -V(\mathbf{q}) + (2\pi)^2 \delta^2(\mathbf{q}) \int_l V(l)$$

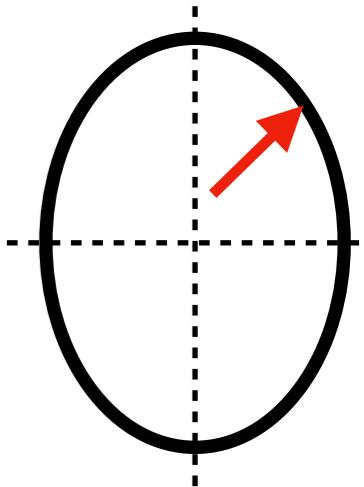
Screening mass

$$V(\mathbf{q}) = \frac{8\pi \mu^2(t)}{(\mathbf{q}^2 + \mu^2(t))^2}$$

Why don't we use the rates?



Using rates



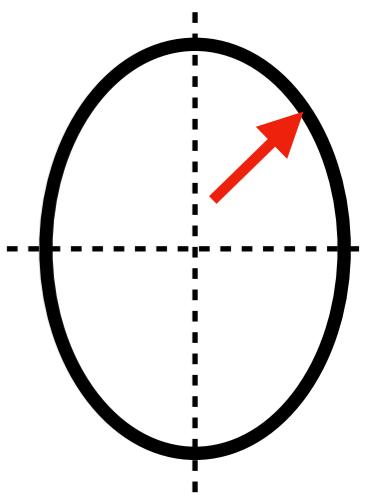
- The spectrum is the integral of the rates over the trajectory

$$\omega \frac{dI}{d\omega} = \int_0^L dt \omega \frac{d\Gamma}{d\omega dt}$$

- The **rates** are **only** sensitive to a region of the medium of the **size of the formation time**
- If the formation time is small, computing the rates for a brick is a good approximation (i.e. we can assume a constant temperature during the emission)
- So, one can **pre-compute the static rates** for all medium temperatures and at each point of the path just read the rate corresponding to the temperature at that point

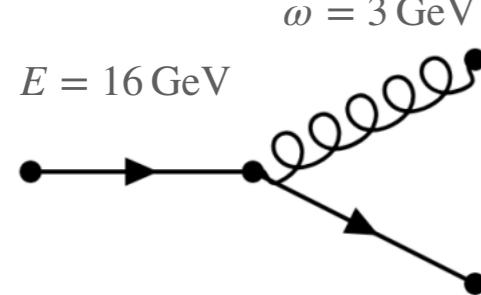
This is how MARTINI implements the AMY (infinite length) rates

Asymptotic rates

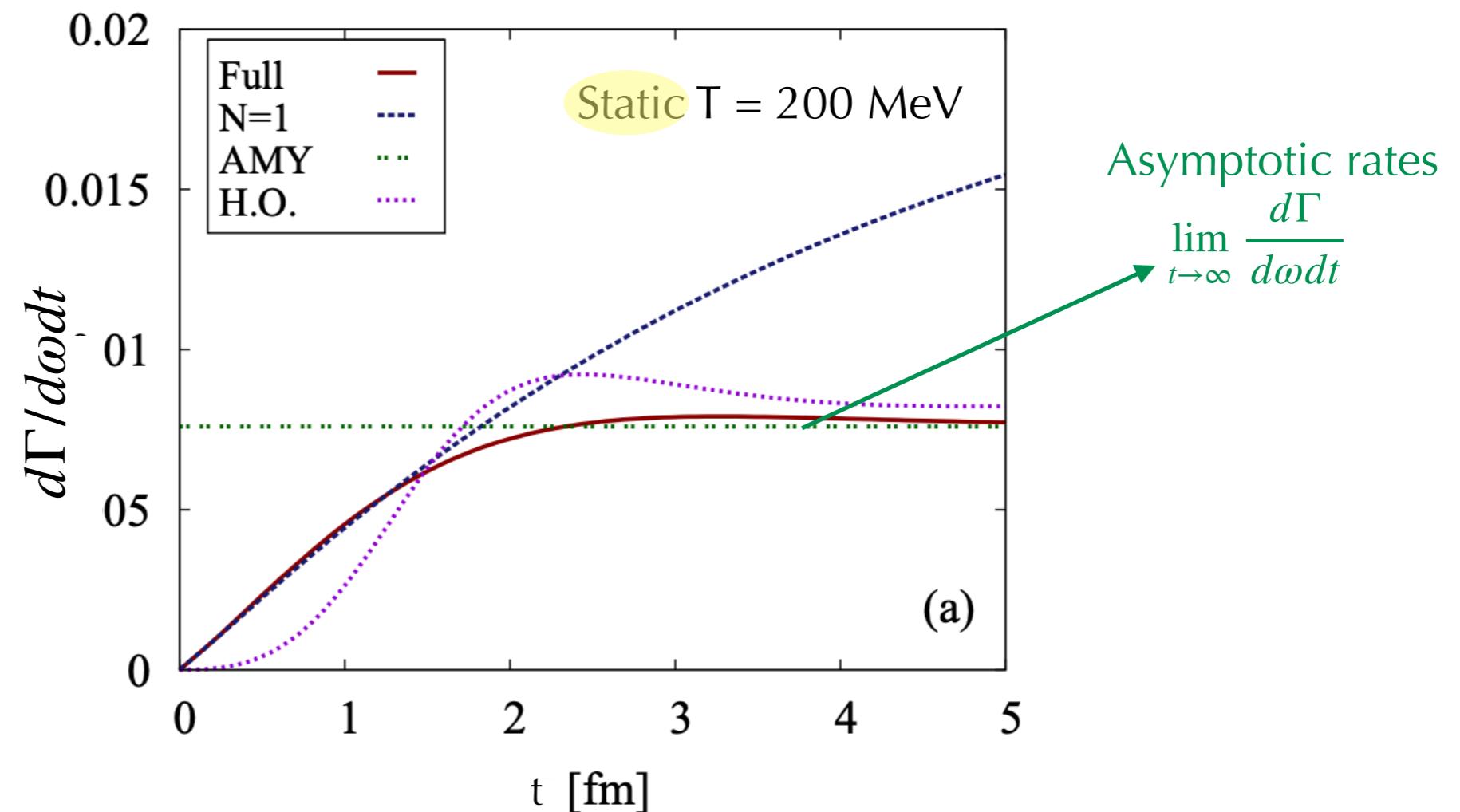


- The **asymptotic rates** are the rates for a medium of **infinite length**:

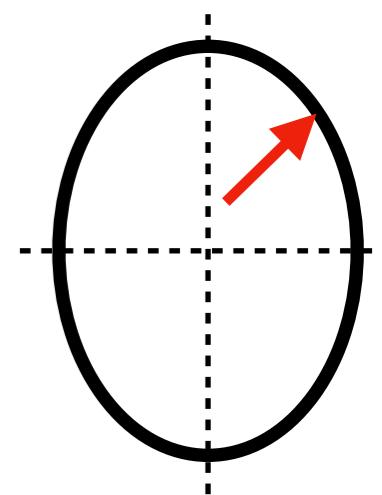
$$\lim_{t \rightarrow \infty} \frac{d\Gamma}{d\omega dt}$$



$$V(q) \propto \frac{1}{q^2(q^2 + m_D^2)}$$

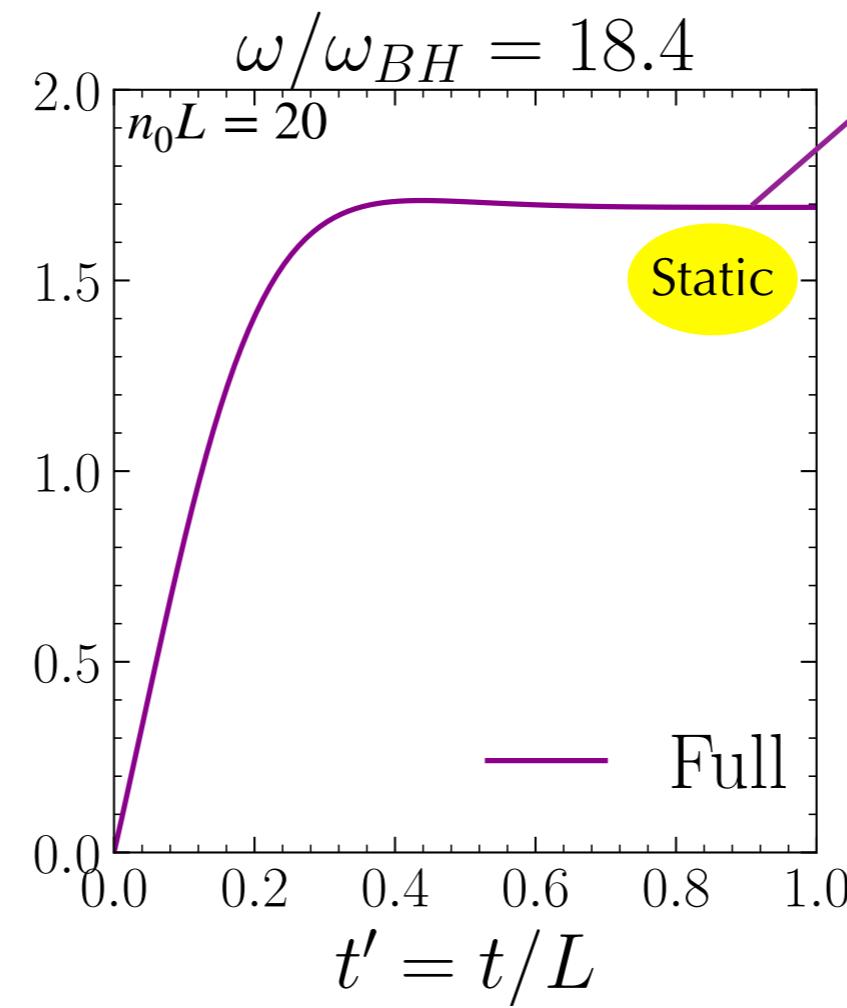
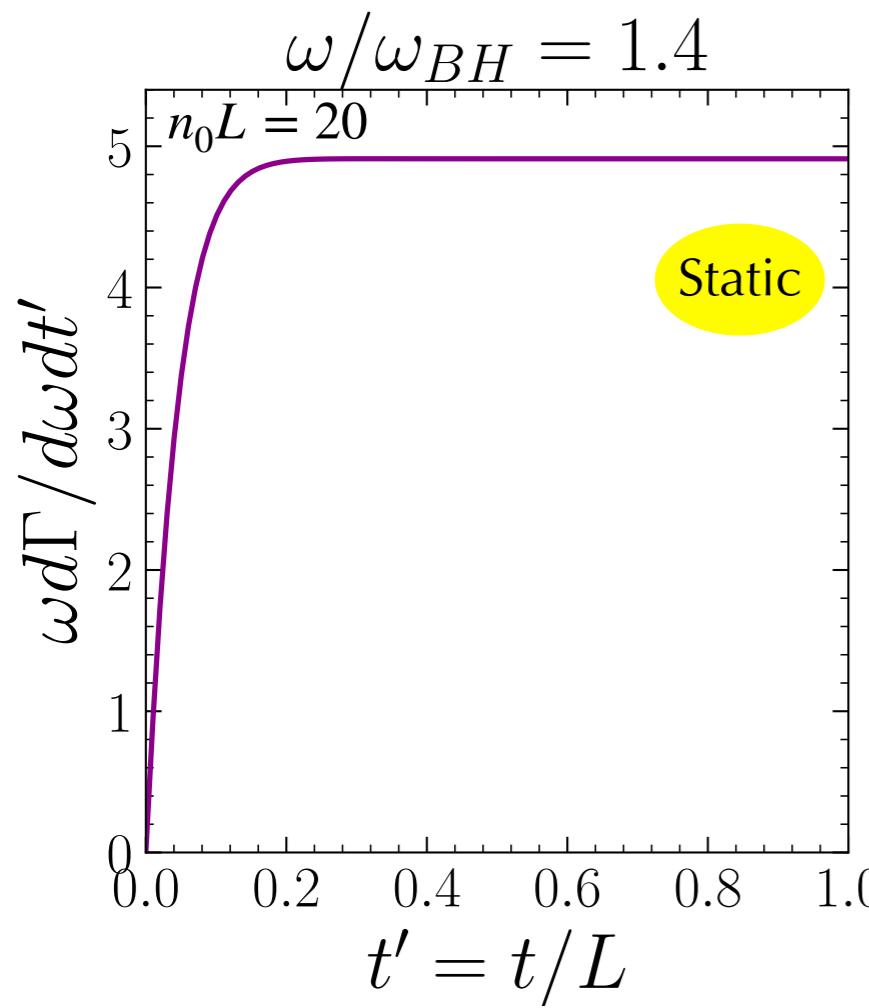


Asymptotic rates

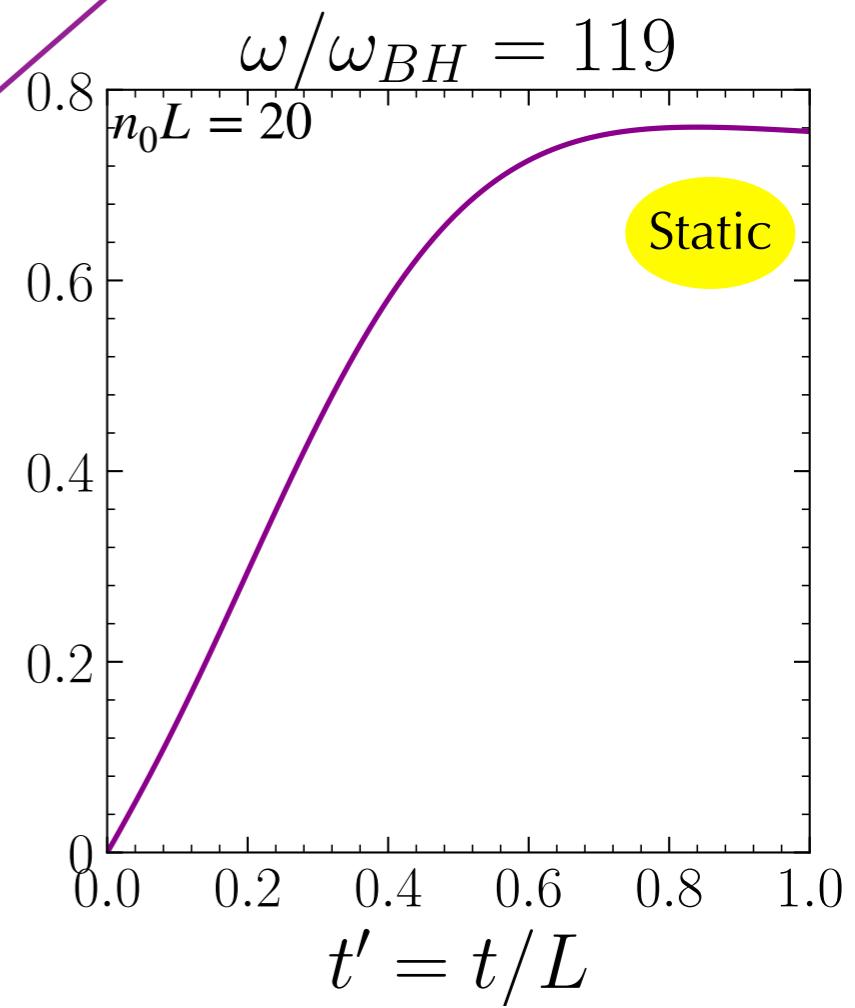


- Asymptotic rates for a Yukawa parton-medium interaction $V(\mathbf{q}) = \frac{8\pi\mu^2}{(\mathbf{q}^2 + \mu^2)^2}$

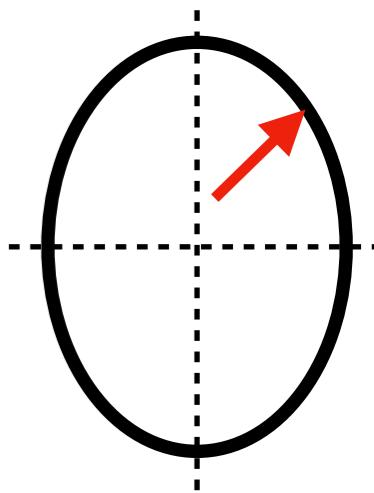
$$\omega_{BH} = \frac{\mu^2}{2n_0}$$



Asymptotic rates
(soft limit of AMY
result)



Use of asymptotic rates



- Obtain the full spectrum along the path for $n(t) = \frac{n'_0}{(t + t_0)^\alpha}$
- At each point of the trajectory read the value of $n(t)$ and compute the static asymptotic rate $\lim_{t \rightarrow \infty} \frac{d\Gamma}{d\omega dt}$ for that value

$$\lim_{t \rightarrow \infty} \omega \frac{d\Gamma}{d\omega dt} = 4\alpha_s C_R \Re \int_p i \mathbf{p} \cdot G_\omega(\mathbf{p})$$

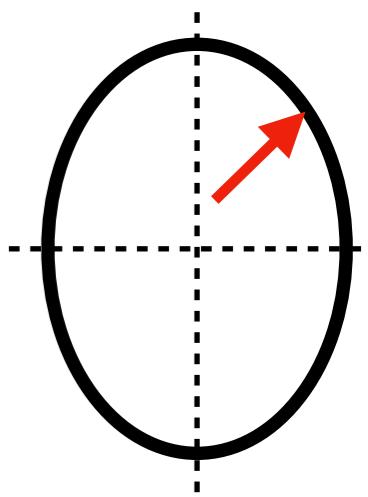
$$n_0 \int_{\mathbf{q}} \frac{\mathbf{q}}{\mathbf{q}^2} \sigma(\mathbf{q} - \mathbf{p}) = \frac{i p^2}{2\omega} G_\omega(\mathbf{p}) + \frac{n_0}{2} \int_{\mathbf{q}} \sigma(\mathbf{q} - \mathbf{p}) G_\omega(\mathbf{q})$$

Easy to evaluate!

(Soft limit of AMY equation)

- Integrate the asymptotic (static) rates along the trajectory to obtain the spectrum

Use of asymptotic rates



Dynamic spectrum: spectrum along the path

$$\text{for } n(t) = \frac{n'_0}{(t + t_0)^\alpha}$$

Asymptotic rates:

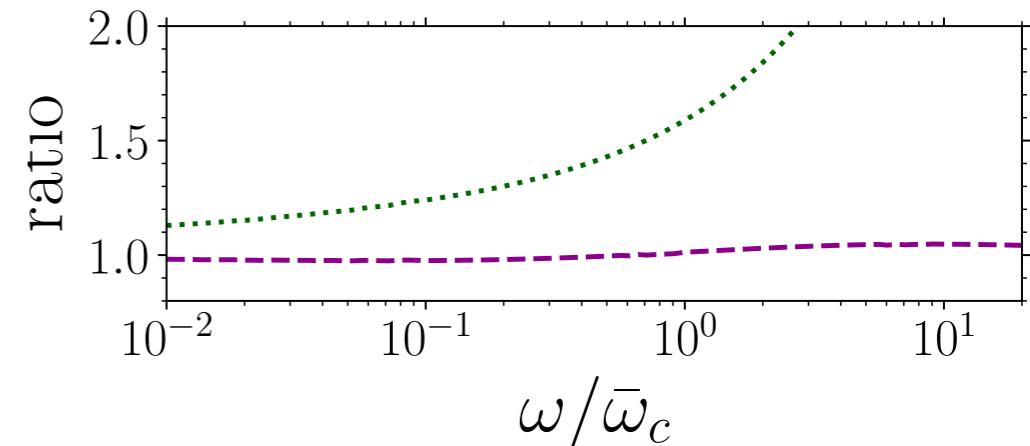
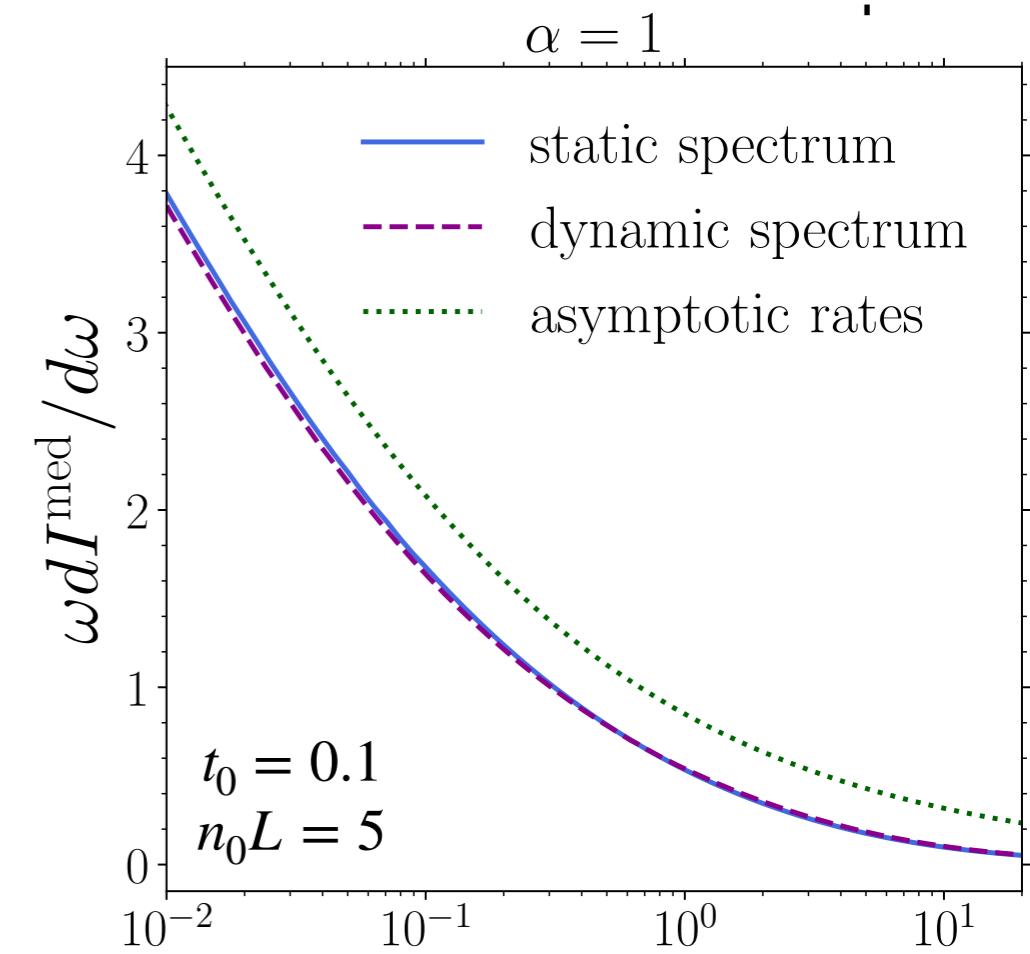
spectrum obtained from integrating
the asymptotic (static) rates

Static spectrum:

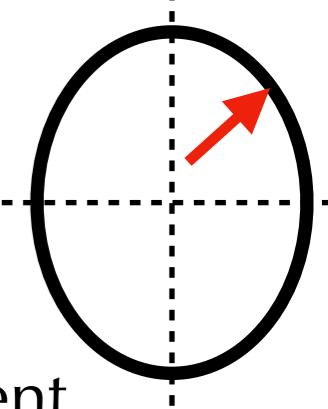
static spectrum where the
values of the parameters are set by the scaling laws

- Asymptotic rates do not work well,
especially at large energies

$$t_f \propto \frac{\omega}{k^2}$$



Use of full rates



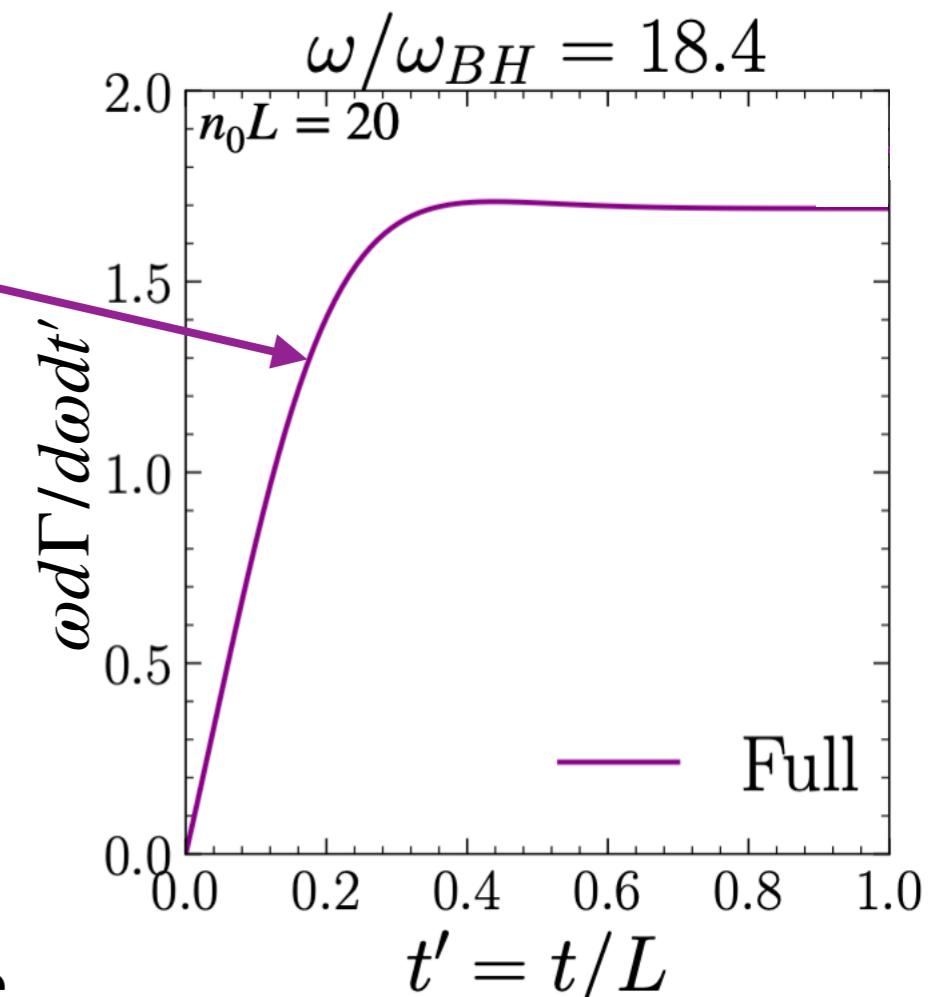
- For each point of the trajectory read the value of $n(t)$ and find the equivalent **full static rate**

- But the full rates depend on the time

At each point of the trajectory we need to know which time to look at

- How fast the rates reach the asymptote depends on how dense the medium is

The static rate must be evaluated at an *effective* time



$$n_0 \tau(t) = \int_0^t dt' n(t')$$

Scaling laws at the level of the rates (at each point of the path)

Beyond the soft approximation

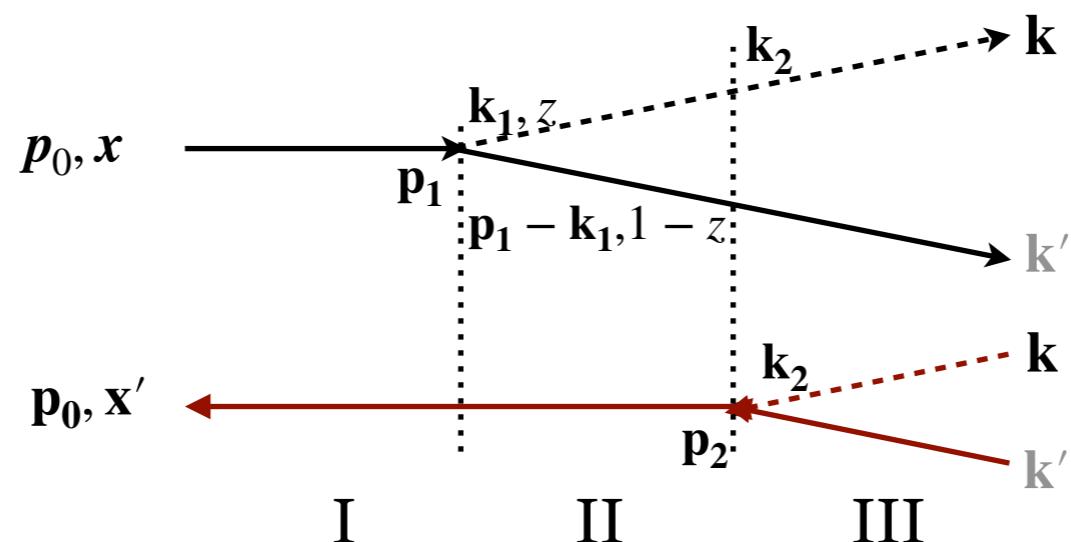
Beyond the soft approximation

- The emitted gluon carries a fraction of energy z of the energy of the parent quark

$$\begin{aligned} z \frac{dI}{dz d^2\mathbf{k}} = & \frac{\alpha_s P_{g \leftarrow q}(z)}{(2\pi)^2 z (1-z)^2 E^2} \operatorname{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\mathbf{p}_1 \mathbf{p}_2 \mathbf{k}_1 \mathbf{k}_2} \mathcal{P}_q(t, \mathbf{p}_1; 0, \mathbf{p}_0) \\ & \times (\mathbf{k}_1 - z\mathbf{p}_1) \cdot (\mathbf{k}_2 - z\mathbf{p}_2) \tilde{\mathcal{K}}^{(3)}(t', \mathbf{k}_2 - z\mathbf{p}_2; t, \mathbf{k}_1 - z\mathbf{p}_1; \mathbf{p}_2 - \mathbf{p}_1) \\ & \times \mathcal{P}_g(\infty, \mathbf{k}; t', \mathbf{k}_2) \end{aligned}$$

Blaizot, Dominguez, Iancu, Mehtar-Tani arXiv:1209.4585

Apolinario, Armesto, Milhano, Salgado arXiv:1407.0599



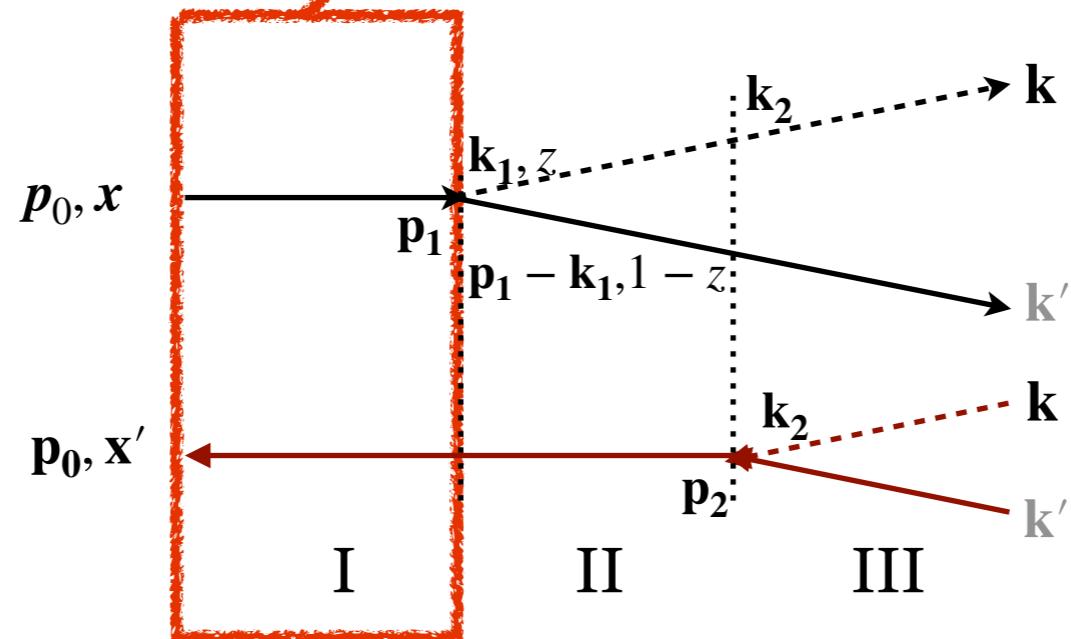
Beyond the soft approximation

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Blaizot, Dominguez, Iancu, Mehtar-Tani arXiv:1209.4585

Apolinario, Armesto, Milhano, Salgado arXiv:1407.0599



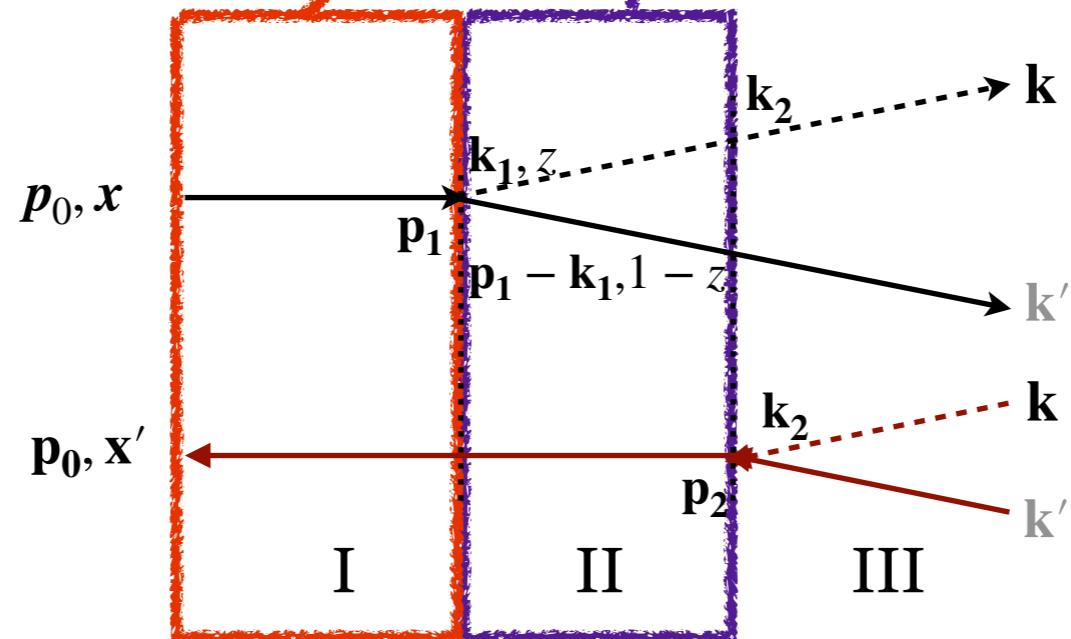
Beyond the soft approximation

- The emitted gluon carries a fraction of energy z of the energy of the parent quark

$$z \frac{dI}{dz d^2\mathbf{k}} = \frac{\alpha_s P_{g \leftarrow q}(z)}{(2\pi)^2 z (1-z)^2 E^2} \text{Re} \int_0^\infty dt' \int_0^{t'} dt \int_{\mathbf{p}_1 \mathbf{p}_2 \mathbf{k}_1 \mathbf{k}_2} \mathcal{P}_q(t, \mathbf{p}_1; 0, \mathbf{p}_0) \\ \times (\mathbf{k}_1 - z\mathbf{p}_1) \cdot (\mathbf{k}_2 - z\mathbf{p}_2) \tilde{\mathcal{K}}^{(3)}(t', \mathbf{k}_2 - z\mathbf{p}_2; t, \mathbf{k}_1 - z\mathbf{p}_1; \mathbf{p}_2 - \mathbf{p}_1) \\ \times \mathcal{P}_g(\infty, \mathbf{k}; t', \mathbf{k}_2)$$

Blaizot, Dominguez, Iancu, Mehtar-Tani arXiv:1209.4585

Apolinario, Armesto, Milhano, Salgado arXiv:1407.0599



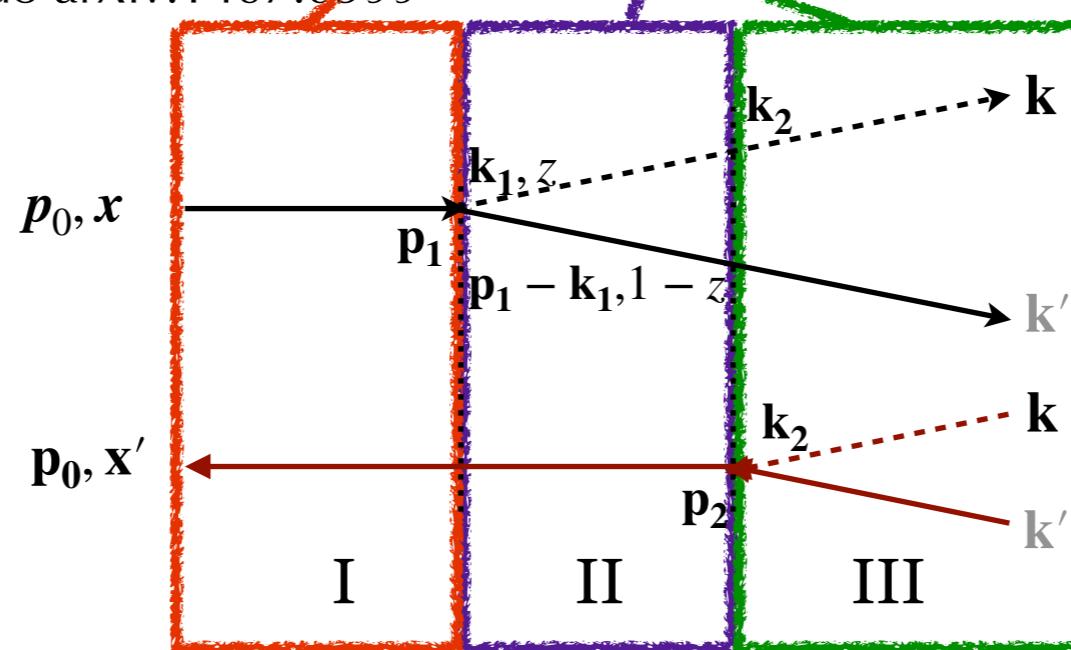
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- The emission kernel is

$$\tilde{\mathcal{K}}^{(3)}(t_2, \mathbf{q}_2; t_1, \mathbf{q}_1; \mathbf{l}) = \int_{\mathbf{u}_1 \mathbf{u}_2 \mathbf{v}} e^{-i(\mathbf{u}_2 \cdot \mathbf{q}_2 - \mathbf{u}_1 \cdot \mathbf{q}_1 + \mathbf{v} \cdot \mathbf{l})} \\ \times \int_{\mathbf{u}(t_1) = \mathbf{u}_1}^{\mathbf{u}(t_2) = \mathbf{u}_2} \mathcal{D}\mathbf{u} \exp \left\{ \frac{iz(1-z)E}{2} \int_{t_1}^{t_2} d\xi \dot{\mathbf{u}}^2 - \frac{1}{2} \int_{t_1}^{t_2} d\xi n(\xi) \sigma_3(\mathbf{u}(\xi), \mathbf{v}; z) \right\}$$

$$\sigma_3(\mathbf{u}, \mathbf{v}; z) = \frac{1}{2} \sigma(\mathbf{u}) + \frac{1}{2} \sigma((1-z)\mathbf{u} + \mathbf{v}) - \frac{1}{2N_c^2} \sigma(-z\mathbf{u} + \mathbf{v}).$$

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- The emission kernel is

$$\tilde{\mathcal{K}}^{(3)}(t_2, \mathbf{q}_2; t_1, \mathbf{q}_1; \mathbf{l}) = (2\pi)^4 \delta^{(2)}(\mathbf{l}) \delta^{(2)}(\mathbf{q}_2 - \mathbf{q}_1) e^{-i \frac{\mathbf{q}_2^2}{2z(1-z)E} (t_2 - t_1)} \\ - \frac{1}{2} \int_{t_1}^{t_2} ds n(s) e^{-i \frac{\mathbf{q}_2^2}{2z(1-z)E} (t_2 - s)} \int_{\mathbf{q}' \mathbf{l}'} \tilde{\sigma}_3(\mathbf{q}_2 - \mathbf{q}', \mathbf{l} - \mathbf{l}'; z) \tilde{\mathcal{K}}^{(3)}(s, \mathbf{q}'; t_1, \mathbf{q}_1; \mathbf{l}')$$

$$\tilde{\sigma}_3(\mathbf{q}, \mathbf{l}; z) = \int_{\mathbf{u} \mathbf{v}} e^{-i(\mathbf{u} \cdot \mathbf{q} + \mathbf{v} \cdot \mathbf{l})} \sigma_3(\mathbf{u}, \mathbf{v}; z) \\ = (2\pi)^2 \frac{1}{2} \left[\delta^{(2)}(\mathbf{l}) \sigma(\mathbf{q}) + \delta^{(2)}((1-z)\mathbf{l} - \mathbf{q}) \sigma(\mathbf{l}) - \frac{1}{N_c^2} \delta^{(2)}(\mathbf{q} + z\mathbf{l}) \sigma(\mathbf{l}) \right]$$

Energy spectrum without kinematic constraint

- Integrating in \mathbf{k} over the full transverse plane ($\bar{R} \rightarrow \infty$)

$$z \frac{dI}{dz} = \frac{\alpha_s P_{g \leftarrow q}(z)}{(1-z)E} \operatorname{Re} \int_0^L ds n(s) \int_0^s dt \int_{\mathbf{q}_1 \mathbf{q}_2 \mathbf{l}} i \mathbf{q}_1 \cdot \left(\frac{\mathbf{l}}{\mathbf{l}^2} - \frac{\mathbf{q}_2}{\mathbf{q}_2^2} \right) \\ \times \tilde{\sigma}'(\mathbf{l} - \mathbf{q}_2; z) \tilde{\mathcal{K}}'(s, \mathbf{q}_2; t, \mathbf{q}_1)$$

- The emission kernel is

$$\tilde{\mathcal{K}}'(s, \mathbf{q}_2; t, \mathbf{q}_1) = \int_{\mathbf{l}} \tilde{\mathcal{K}}^{(3)}(s, \mathbf{q}_2; t, \mathbf{q}_1; \mathbf{l}) = \int_{\mathbf{u}_1 \mathbf{u}_2} e^{-i(\mathbf{u}_2 \cdot \mathbf{q}_2 - \mathbf{u}_1 \cdot \mathbf{q}_1)} \\ \times \int_{\mathbf{u}(t)=\mathbf{u}_1}^{\mathbf{u}(s)=\mathbf{u}_2} \mathcal{D}\mathbf{u} \exp \left\{ \frac{iz(1-z)E}{2} \int_t^s d\xi \dot{\mathbf{u}}^2 - \frac{1}{2} \int_t^s d\xi n(\xi) \sigma_3(\mathbf{u}(\xi), 0; z) \right\}$$

$$\tilde{\sigma}'(\mathbf{q}; z) = \frac{1}{2} \sigma(\mathbf{q}) + \frac{1}{2(1-z)^2} \sigma \left(\frac{\mathbf{q}}{1-z} \right) - \frac{1}{2N_c^2 z^2} \sigma \left(\frac{\mathbf{q}}{z} \right)$$

Results

