Fermi gases and dilute neutron matter with low-momentum interactions

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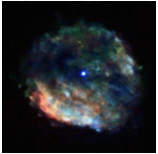
Outline

- Neutron stars, dilute neutron matter, and ultracold atoms
- Usual regularization procedure for a contact interaction
- Low-momentum interactions with zero range
- Hartree-Fock-Bogoliubov with perturbative corrections
- Results for cold atoms
- Results for neutron matter
- Conclusions and outlook

More details: M.U. and S. Ramanan, Phys. Rev. A **103**, 063306 (2021).

Motivation: neutron stars

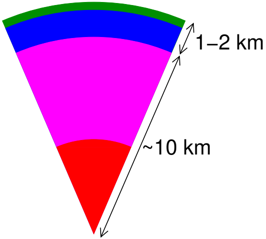


- Produced in core-collapse supernova explosions of intermediate-mass ($8 - 30 M_\odot$) stars
  [figure: supernova remnant RCW103 seen by Chandra in X-ray]

- Very compact: $M \sim 1 - 2 M_\odot$ ($2 - 4 \times 10^{30}$ kg) in a radius of $R \sim 10$ km
  $\rho >$ nuclear saturation density $\rho_0 = 2.7 \times 10^{14}$ g/cm^3, $n_0 = 0.16$ fm^{-3}

- Typical temperatures $T \approx 10^6 - 10^9$ K $\approx 0.1 - 100$ keV
  $T \ll E_F \rightarrow \; T = 0$ formalism sufficient for many purposes

- Complex inner structure:



outer crust: Coulomb lattice of neutron rich nuclei in a degenerate electron gas

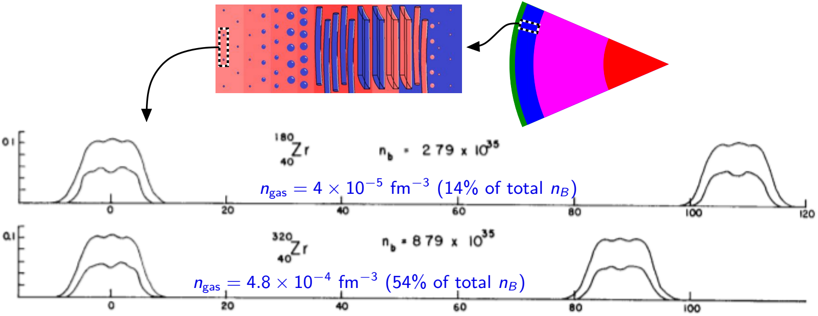
inner crust: unbound neutrons form a superfluid neutron gas between the nuclei (clusters)

outer core: homogeneous matter (n, p, e^-)

inner core: densities up to a few times ρ_0, new degrees of freedom: hyperons? quark matter?

What is “dilute” neutron matter?

- Upper layers of the inner crust (close to neutron-drip density $\sim 2.5 \times 10^{-4}$ fm^{-3})



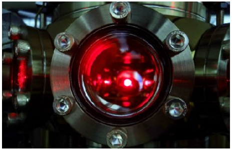
[Negele and Vautherin, NPA 207 (1973); similar results by Baldo et al., PRC 76 (2007)]

- In spite of its “low” density (still $\rho \gtrsim 10^{11}$ g/cm^3), the neutron gas is relevant because it occupies a much larger volume than the clusters

- Deeper in the crust: n_{gas} increases up to $\sim n_0/2 = 0.08$ fm^{-3}

Ultracold Fermi gases

- 2004: trapped Fermions cooled down to superfluidity
- Interaction between the atoms:
  $R \sim 10^{-9}$ m $\ll d \sim 1/k_F \sim 10^{-6}$ m
  $\rightarrow$ practically contact interaction
- Pauli principle: interaction (s wave) only between atoms of opposite "spin" $(\uparrow, \downarrow)$
- Interaction strength is characterized by the scattering length a
- Feshbach resonance: scattering length a can be tuned experimentally by changing the magnetic field B
- BEC-BCS crossover: ground state (GS) evolves from BEC of dimers ($a > 0$) to a BCS superfluid ($a < 0$)
- The case $a \rightarrow \infty$ is called the unitary limit



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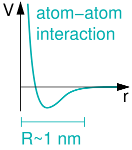
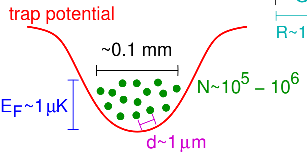
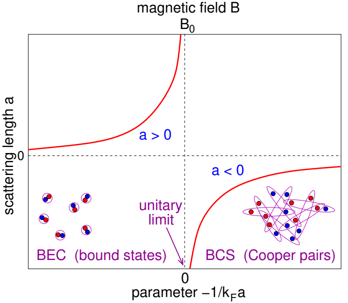
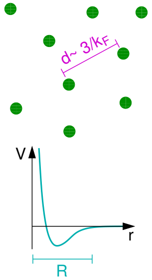






Comparison: dilute neutron matter vs unitary Fermi gas

	neutron gas	unitary Fermi gas
n	$4 \times 10^{-5} \dots 0.08\ \text{fm}^{-3}$	$\sim 1\ \mu\text{m}^{-3}$
$d = n^{-1/3}$	$30 \dots 2.3$ fm	$\sim 1\ \mu\text{m}$
$k_F = (3\pi^2 n)^{1/3}$	$0.1 \dots 1.3\ \text{fm}^{-1}$	$\sim 1\ \mu\text{m}^{-1}$
$E_F = k_F^2/2m$	$0.2 \dots 35$ MeV	$\sim 1\ \mu\text{K} \sim 10^{-10}$ eV
a	-18 fm	∞
R	2.5 fm	~ 1 nm
$1/k_F a$	$-0.5 \cdots -0.07$	0
$k_F R$	$0.25 \dots 3$	10^{-3}

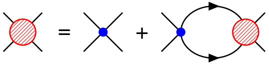


- Although the absolute scales are completely different, dimensionless quantities $1/k_F a$ and $k_F R$ are comparable
- Effective range R not negligible in neutron matter at relevant densities
- The neutron gas is close to the crossover regime but not in the unitary limit

Standard regularization procedure for a contact interaction

- Scattering length for coupling constant $g < 0$ and cutoff Λ $\quad (\epsilon_k = \frac{k^2}{2m})$

$$\frac{4\pi a}{m} = g + g \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{-2\epsilon_k} \frac{4\pi a}{m}$$



- Express g in terms of a, e.g. in the gap equation $\quad (E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta^2})$

$$\Delta = -g \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2E_k} \quad \Leftrightarrow \quad \Delta = -\frac{4\pi a}{m} \int^{\Lambda} \frac{d^3k}{(2\pi)^3} \Big(\frac{\Delta}{2E_k} - \frac{\Delta}{2\epsilon_k} \Big)$$

$\Rightarrow$ now the cutoff can be removed

- Coupling constant vanishes for $\Lambda \to \infty$: $\quad \frac{1}{g} = \frac{m}{4\pi a} - \frac{m\Lambda}{2\pi^2}$
- Keeping Λ finite would induce a finite effective range: $r_{\text{eff}} = \frac{4}{\pi\Lambda}$
- For cold atoms one should take the limit $\Lambda \to \infty$
- No Hartree field: $U_\sigma = g n_{-\sigma} \xrightarrow{\Lambda \to \infty} 0$
- In order to get the simplest weak-coupling correction $\frac{4\pi a}{m} n_\uparrow n_\downarrow$ to the GS energy, resummation of ladder diagrams is necessary

Low-momentum interactions with zero range

- In nuclear physics: "soft" $V_{\text{low-}k}$ or SRG interactions reproduce exactly the low-momentum scattering phase shifts of the full NN interaction below the cutoff

- Is it possible to reproduce the scattering amplitude of a contact interaction for $k < \Lambda$ with a finite cutoff Λ? $\rightarrow$ **Yes!**

- Explicit construction [Tabakin 1969] of a separable s-wave interaction $\left(V = 4\pi V_0, F(0) = 1\right)$

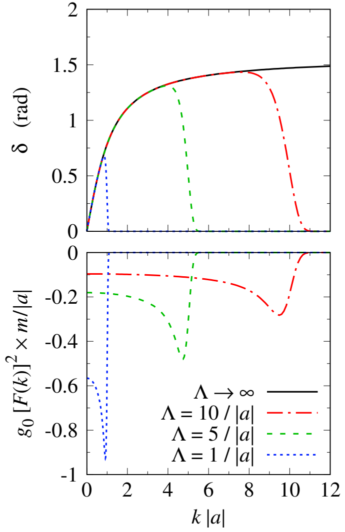
$$V_0(k, k') = g_0 F(k) F(k')$$

that gives scattering phase shifts

$$\delta(q) = R\left(\frac{k}{\Lambda}\right) \operatorname{arccot}\left(-\frac{1}{ka}\right)$$

- We use a smooth regulator $R(x) = \exp(-x^{20})$

- In the unitary limit and with a sharp regulator, i.e., $\delta(k) = \frac{\pi}{2}\theta(\Lambda - k)$, analytic expressions exist [Köhler 2007, Ruiz Arriola et al. 2017]



Hartree-Fock-Bogoliubov (HFB)

- In nuclear physics: hard core of “realistic” potentials requires explicit inclusion of short-range correlations, and nuclei are not bound in HF(B) approximation
- Soft interactions ($V_{\text{low-}k}$, SRG) much better suited for perturbative methods
- HFB with perturbative corrections can give good results for open-shell nuclei [e.g., Tichai et al. 2019] $\rightarrow$ try this method for cold atoms
- Momentum dependent mean field U_k and gap Δ_k: ($\Lambda' > \Lambda$ because of smooth cutoff)

$$U_k = \int \frac{d^3p}{(2\pi)^3} 4\pi V_0\Big(\frac{\vec{p}-\vec{k}}{2}, \frac{\vec{p}-\vec{k}}{2}\Big)\, v_p^2\,, \qquad \Delta_k = -\frac{2}{\pi}\int_0^{\Lambda'} dp\, p^2\, V_0(k,p)\, u_p v_p$$

with the usual definitions

$$u_k = \sqrt{\frac{1}{2} + \frac{\xi_k}{2E_k}}\,, \quad v_k = \sqrt{\frac{1}{2} - \frac{\xi_k}{2E_k}}\,, \quad \xi_k = \frac{k^2}{2m} + U_k - \mu\,, \quad E_k = \sqrt{\xi_k^2 + \Delta_k^2}$$

- Choice of the cutoff Λ: as small as possible to make the interaction perturbative, but without cutting the physically relevant states
- Hartree-Fock requires $\Lambda \geq k_F$, with pairing we need somewhat higher cutoff

Cutoff dependence of HFB results

(a) Weak coupling ($1/k_F a = -5$):

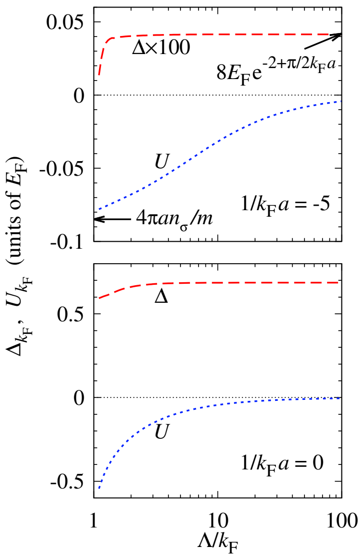
- Hartree shift $U_{k_F} \approx \frac{4\pi a}{m} n_\sigma$ for $\Lambda \to k_F$, but $U_{k_F} \to 0$ for $\Lambda \to \infty$
- Gap Δ_{k_F} reaches rapidly (for $\Lambda \gtrsim 1.5 k_F$) the usual BCS result $8E_F \exp(-2 + \frac{\pi}{2k_F a})$

(b) At unitarity ($1/k_F a = 0$):

- Hartree shift $U_{k_F} \sim -0.5 E_F$ at small Λ, but again $U_{k_F} \to 0$ for $\Lambda \to \infty$
- Gap Δ_{k_F} less cutoff dependent but reaches asymptotic value at larger Λ ($\sim 3k_F$) than in weak coupling

Physical quantities should be cutoff independent!

- If perturbative corrections to HFB converge in a range of cutoffs, the corrected results should be cutoff independent in this range



Bogoliubov Many-Body Perturbation Theory (BMBPT)

- Express $\hat{K} = \hat{H} - \mu\hat{N}$ in terms of quasiparticle (QP) operators

$$\beta_{\vec{k}\uparrow} = u_k\, a_{\vec{k}\uparrow} - v_k\, a^\dagger_{-\vec{k}\downarrow}\,, \quad \beta_{\vec{k}\downarrow} = u_k\, a_{\vec{k}\downarrow} + v_k\, a^\dagger_{-\vec{k}\uparrow}$$

- With the HFB solution for u_k and v_k, we can write

$$\hat{K} = \mathcal{E}_{\text{HFB}} + \sum_{\vec{k}\sigma} E_k\, \beta^\dagger_{\vec{k}\sigma}\beta_{\vec{k}\sigma} + :\hat{V}:$$

$$:\hat{V}: = V_{04}\,\beta\beta\beta\beta + V_{13}\,\beta^\dagger\beta\beta\beta + V_{22}\,\beta^\dagger\beta^\dagger\beta\beta + V_{31}\,\beta^\dagger\beta^\dagger\beta^\dagger\beta + V_{40}\,\beta^\dagger\beta^\dagger\beta^\dagger\beta^\dagger$$

- Treat $:\hat{V}:$ as a perturbation

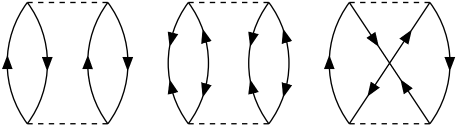
- Example: leading correction to GS energy is second order

$$\mathcal{E}_2 = -\frac{1}{4!}\sum_{ijkl}\frac{|\langle ijkl|\hat{V}_{40}|\text{HFB}\rangle|^2}{E_i + E_j + E_k + E_l} \qquad \text{with} \qquad |ijkl\rangle = \beta^\dagger_i\beta^\dagger_j\beta^\dagger_k\beta^\dagger_l|\text{HFB}\rangle$$

BMBPT at second and third order

- In practice, $\mathcal{E}_2$ has three terms corresponding to three diagrams:

upwards going line	u_k^2
downwards going line	v_k^2
anomalous line	$u_k v_k$
horizontal dashed line	$V_0(q, q')$

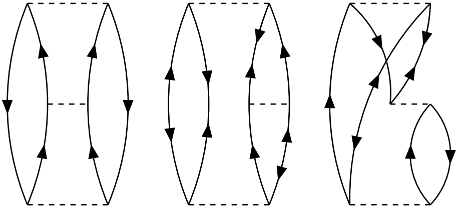


- Summation over intermediate 4 QP states:
  12 (four momenta) -3 (momentum conservation) -3 (rotational invariance)
  $=$ 6 dimensional integral, evaluated with MC integration with importance sampling

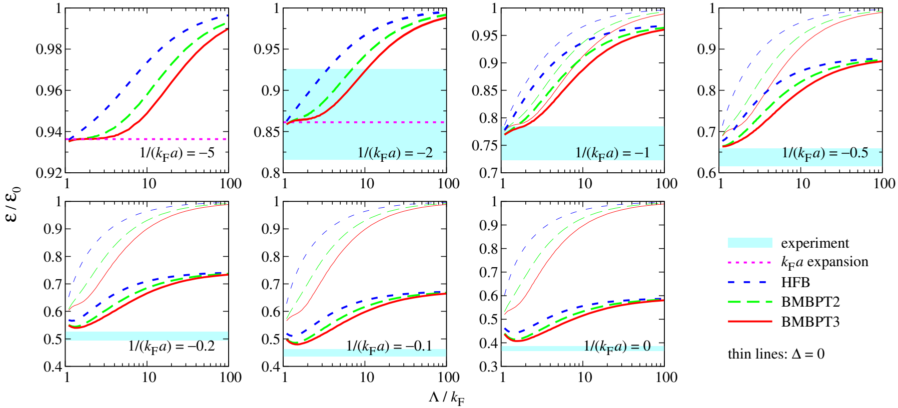
- Third-order correction $\mathcal{E}_3$ has 27 terms (only three examples shown):

- Use Mathematica to automatically generate all terms

- Number of integrations:
  18 (six momenta)
  -6 (momentum conservation)
  -3 (rotational invariance)
  $=$ 9 dimensions



Cutoff dependence of HF(B)+(B)MBPT GS energy



[exp.: Horikoshi et al. (2017), $k_F a$ expansion: Wellenhofer et al. (2021)]

- Approximate cutoff independence reached in a region of small cutoffs ($\Lambda \leq 3k_F$) at weak coupling
- Inclusion of pairing (thick vs thin lines) very important at stronger coupling
- For $\Lambda \simeq 1.5 - 2k_F$, results are close to experimental ones

Discussion

- BMBPT3 weakens cutoff dependence but is not enough to remove it

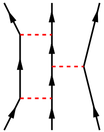
What is missing?

- Higher orders of BMBPT

  Is it efficient to expand about the HFB GS although we know that screening reduces the gap?

- Induced three-body force (3BF) and higher-body forces:

  even if there is no 3BF in the limit $\Lambda \to \infty$, at finite Λ there will be an effective 3BF to compensate for the contributions of loop momenta above Λ in diagrams like this one



- BMBPT expands the perturbed GS in terms of fermionic QP states:

  approaching the BEC regime, bosonic degrees of freedom (Bogoliubov-Anderson mode) become progressively more important which require resummation of (Q)RPA diagrams

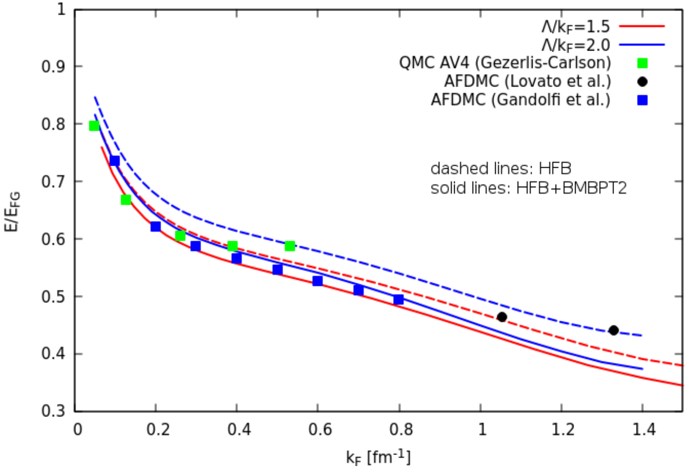
Differences between cold atoms and neutron matter

The nn interaction is more complicated:

- Even at the lowest relevant densities, the finite range of the nn interaction is not negligible
- Not only *s*-wave, but also higher partial waves:

  in practice we include waves up to $L = 6$
- Coupling between different L due to tensor force
- We use $V_{\text{low-}k}$ matrix elements [Bogner et al. 2007] generated from AV18 or chiral interactions (both give practically the same results)
- Although it is relatively weak in pure neutron matter, the 3BF (neglected here) could play a role at higher densities

Neutron-matter energy (in units of $\mathcal{E}_0$) as fct. of k_F



- In analogy with the cold atoms case, we expect best results for $\Lambda \simeq 1.5 - 2k_F$
- So far only BMBPT2 results available, BMBPT3 is work in progress
- Energies too low at high densities: missing 3BF?

Conclusions

- HFB+BMBPT with low-momentum interactions successfully used in nuclear structure calculations (e.g. by the Saclay group)
- In infinite matter, one can scale the cutoff Λ with k_F
- Low-momentum interactions give a better HF field and hence better results already at the mean-field (HFB) level and corrections are perturbative

Outlook

- BMBPT3 for neutron matter: work in progress
- Missing ("genuine" and "induced") 3BF: in-medium SRG method?
- Can IMSRG also help to solve the screening problem?
- Contribution of collective modes: work in progress
- Long-term objective: include also protons (neutron-star core)

Tabakin's formula for the separable interaction

$$V_0(q,q) = -\frac{\sin\delta(q)}{mq}\exp\left(\frac{2}{\pi}\mathcal{P}\int_0^\infty dq'\,\frac{q'\delta(q')}{q^2-q'^2}\right)$$

$$g_0 = V_0(0,0)\,, \quad F(q) = \sqrt{V_0(q,q)/g_0}$$